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Cohomology of the Vector Fields Lie Algebra and Modules of Differential Operators on a Smooth Manifold

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Abstract. Let M be a smooth manifold, $\mathcal{S}$ the space of polynomial on fibers functions on T^*M (i.e., of symmetric contravariant tensor fields). We compute the first cohomology space of the Lie algebra, Vect(M), of vector fields on M with coefficients in the space of linear differential operators on $\mathcal{S}$. This cohomology space is closely related to the Vect(M)-modules, $\mathcal{D}_\lambda(M)$, of linear differential operators on the space of tensor densities on M of degree λ.

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1. Introduction and the Main Theorem

Let M be a smooth manifold and Vect(M) the Lie algebra of vector fields on M.

The main purpose of this article is to study the cohomology of Vect(M) with coefficients in the space of linear differential operators acting on tensor fields. This cohomology is, actually, a natural generalization of the Gelfand–Fuchs cohomology (i.e., of Vect(M)-cohomology with coefficients in the modules of tensor fields on M).

The problem of computation of such cohomology spaces naturally arises if one considers *deformations* of the Vect(M)-module structure on the space of tensor fields.

The general theory of deformations of Lie algebra modules is due to Nijenhuis and Richardson [13, 15]. Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module, then the problem of deformation of the $\mathfrak{g}$-module structure on V is related to the cohomology spaces: $\mathrm{H}^1(\mathfrak{g}; \mathrm{End}(V))$ and $\mathrm{H}^2(\mathfrak{g}; \mathrm{End}(V))$. More precisely, the first cohomology space classifies *infinitesimal* deformation, while the second one contains the obstructions to integrability of a given infinitesimal deformation.

The origin of our investigation is related to the space of scalar linear differential operators on M viewed as a module over Vect(M). It is quite clear a-priori that this module should be considered as a deformation of the corresponding module of *symbols* (i.e., of polynomial on fibers functions on T^*M). We are, therefore, led to study the first cohomology of Vect(M) with coefficients in the Vect(M)-module of operators on the space of symbols.

1.1. DIFFERENTIAL OPERATORS ON SYMMETRIC CONTRAVARIANT TENSOR FIELDS

Consider the space, $\mathcal{S}(M)$ (or $\mathcal{S}$ for short), of symmetric contravariant tensor fields on M (i.e., $\mathcal{S} = \Gamma(STM)$). As a Vect($M$)-module it is isomorphic to the space of smooth functions on T^*M polynomial on the fibers. Therefore, $\mathcal{S}$ is a Poisson algebra with a natural graduation given by the decomposition

$$\mathcal{S} = \bigoplus_{k=0}^{\infty} \mathcal{S}_k, \tag{1.1}$$

where $\mathcal{S}_k$ is the space of kth order tensor fields. Obviously, $\mathcal{S}_0$ is isomorphic to $C^\infty(M)$ and $\mathcal{S}_1$ to Vect(M). The Poisson bracket on $\mathcal{S}$ is usually called the (symmetric) Schouten bracket (see, e.g., [7]).

The action of $X \in \mathrm{Vect}(M)$ on $\mathcal{S}$ is given by the Hamiltonian vector field

$$L_X = \frac{\partial X}{\partial \xi_i}\frac{\partial}{\partial x^i} - \frac{\partial X}{\partial x^i}\frac{\partial}{\partial \xi_i}, \tag{1.2}$$

where (x, ξ) are local coordinates on T^*M (we identified X with the first-order polynomial $X = X^i\xi_i$; the summation over repeated indices is understood).

Let us introduce the space, $\mathcal{D}(\mathcal{S})$, of all linear differential operators on $\mathcal{S}$. This space is a Vect(M)-module with a filtration

$$\mathcal{D}^0(\mathcal{S}) \subset \mathcal{D}^1(\mathcal{S}) \subset \cdots \subset \mathcal{D}^r(\mathcal{S}) \subset \cdots, \tag{1.3}$$

where $\mathcal{D}^r(\mathcal{S})$ is the space of rth order differential operators.

In this article we compute the first cohomology space

$$\mathrm{H}^1(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S})). \tag{1.4}$$

of Vect(M) acting on $\mathcal{D}(\mathcal{S})$.

Note that for $M = S^1$ this computation has been done in [11, 1] see also [6] for the case of the Lie algebra of formal vector fields on $\mathbb{R}$.

1.2. MODULES OF DIFFERENTIAL OPERATORS ON TENSOR DENSITIES

Let $\mathcal{F}_\lambda(M)$ (or $\mathcal{F}_\lambda$ in short) be the space of tensor densities of degree λ on M (i.e. the space of sections of the line bundle $\Delta_\lambda(M) = |\Lambda^n T^*M|^{\otimes\lambda}$ over M). Clearly, $\mathcal{F}_0 \cong C^\infty(M)$ as a Vect(M)-module, any two Vect(M)-modules of tensor densities are nonisomorphic (see also [7]).

Denote $\mathcal{D}_\lambda$ the space $\mathcal{D}(\mathcal{F}_\lambda)$ of linear differential operators on $\mathcal{F}_\lambda$. This space is an associative (and, therefore, a Lie) algebra with the filtration by the order of differentiation:

$$\mathcal{D}^0_\lambda \subset \mathcal{D}^1_\lambda \subset \cdots \subset \mathcal{D}^k_\lambda \subset \cdots \tag{1.5}$$

The algebra $\mathcal{S}$ is naturally identified with the associated graded algebra $\mathrm{gr}(\mathcal{D}_\lambda)$ that is,

$$\mathcal{D}^k_\lambda/\mathcal{D}^{k-1}_\lambda \cong \mathcal{S}_k\,. \tag{1.6}$$

The corresponding projection $\sigma_k : \mathcal{D}^k_\lambda \to \mathcal{S}_k$ is called the (principal) *symbol.*

The associative algebra $\mathcal{D}_\lambda$ can be naturally interpreted as a nontrivial deformation of $\mathcal{S}$ and constitutes one of the main objects considered in *deformation quantization.*

We will be interested, however, only in the $\mathrm{Vect}(M)$-module structure on $\mathcal{D}_\lambda$ rather than in the whole associative (or Lie algebra) structure. The (tautological) Lie algebra embedding $\mathrm{Vect}(M) \hookrightarrow \mathcal{D}_\lambda$

$$X \longmapsto L^\lambda_X, \tag{1.7}$$

where L^λ_X is the Lie derivative on $\mathcal{F}_\lambda$, defines a $\mathrm{Vect}(M)$-module structure on $\mathcal{D}_\lambda$.

Remark 1.1. If M is oriented by a volume form Ω, then

$$L^\lambda_X = L_X + \lambda\, \mathrm{div}_\Omega X. \tag{1.8}$$

Moreover, $\mathcal{D}_\lambda$ and $\mathcal{D}_\mu$ are isomorphic associative algebras. However, as $\mathrm{Vect}(M)$-modules they are isomorphic if and only if $\lambda + \mu = 1$ [3, 10].

1.3. THE MAIN THEOREM

The space $\mathcal{D}(\mathcal{S})$ is decomposed, as a $\mathrm{Vect}(M)$-module, into the direct sum:

$$\mathcal{D}(\mathcal{S}) = \bigoplus_{k,\ell} \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell), \tag{1.9}$$

where $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell) \subset \mathrm{Hom}(\mathcal{S}_k, \mathcal{S}_\ell)$. It would then suffice to compute the cohomology (1.4) with coefficients in each of these modules. Our main result is the following

THEOREM 1.2. *If* $\dim M \geqslant 2$, *then*

$$\mathrm{H}^1(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)) = \begin{cases} \mathbb{R}, & \text{if} \quad k-\ell = 2, \\ \mathbb{R}, & \text{if} \quad k-\ell = 1, \ell \neq 0, \\ \mathbb{R} \oplus \mathrm{H}^1_{\mathrm{DR}}(M), & \text{if} \quad k-\ell = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1.10}$$

where $\mathrm{H}^1_{\mathrm{DR}}(M)$ *is the first space of the de Rham cohomology of* M.

The proof will be given in Section 4.

From now on we assume that $\dim M \geqslant 2$.

1.4. DIFFERENTIABILITY

As a first step towards the proof of Theorem 1.2, we will prove now that any 1-cocycle on Vect(M) with values in the space of differential operators $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ is locally differentiable. Due to the well-known Peetre Theorem [14], this means that for any $\gamma \in \mathrm{Z}^1(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell))$, the bilinear map $(X, P) \mapsto \gamma(X)(P)$, where $X \in \mathrm{Vect}(M)$ and $P \in \mathcal{S}_k$, is local:

$$\mathrm{Supp}\, \gamma(X)(P) \subset \mathrm{Supp}\, X \cap \mathrm{Supp}\, P \tag{1.11}$$

PROPOSITION 1.3. *Any 1-cocycle* γ *on* Vect(M) *with values in* $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ *is local.*

Proof. Let $U \subset M$ be open and $X \in \mathrm{Vect}(M)$ vanish on U. We have to show that $\gamma(X)_{|_U} = 0$. Let x_0 be any point in U. As is well known, there exists a neighborhood $V \subset U$ of x_0 and vector fields X_i, X'_i, $i = 1, \ldots, r$ on V such that

$$X = \sum_{1 \leqslant i \leqslant r} [X_i, X'_i] \quad \text{and} \quad X_{i|_V} = X'_{i\,|_V} = 0,$$

where r depends only on the dimension of M. One has, using the fact that γ is a 1-cocycle

$$\gamma(X)_{|_V} = \sum_{1 \leqslant i \leqslant r} (L_{X_i} \gamma(X'_i)_{|_V} - L_{X'_i} \gamma(X_i)_{|_V}) = 0. \qquad \square$$

2. Nontrivial Cohomology Classes

Let us now describe a natural basis of the above cohomology spaces (1.10).

2.1. CASE $k = \ell$

Since $\mathrm{Id} \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_k)$ is Vect(M)-invariant, $c \mapsto c\,\mathrm{Id}$ maps any cocycle c to a cocycle and thus induces a homomorphism $\mathrm{H}(\mathrm{Vect}(M); C^\infty(M)) \to \mathrm{H}(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_k))$. Theorem 1.2 states that it is an isomorphism in degree one.

Recall that $\mathrm{H}(\mathrm{Vect}(M); C^\infty(M))$ is well known (see [7]). In particular, given a covariant derivation ∇, the 1-cocycles are the maps

$$c_{a,\omega} : X \mapsto a\,\mathrm{div}_\nabla(X) + i_X\omega, \tag{2.1}$$

where $a \in \mathbb{R}$ and ω is a closed 1-form, div_∇ being the divergence associated to ∇. The cocycle (2.1) is a coboundary if and only if $a = 0$ and ω is exact.

2.2. CASE $k = \ell + 1$, $\ell \neq 0$

Consider the exact sequence of $\mathrm{Vect}(M)$-modules

$$0 \longrightarrow \mathcal{D}_\lambda^{k-1} \longrightarrow \mathcal{D}_\lambda^k \longrightarrow \mathcal{S}_k \longrightarrow 0. \tag{2.2}$$

Dividing out by $\mathcal{D}_\lambda^{k-2}$ leads to the exact sequence

$$0 \longrightarrow \mathcal{S}_{k-1} \longrightarrow \mathcal{D}_\lambda^k/\mathcal{D}_\lambda^{k-2} \longrightarrow \mathcal{S}_k \longrightarrow 0 \tag{2.3}$$

Assume $k \neq 1$ and $\lambda \neq 1/2$. The sequence (2.3) does not split [12]. Its cohomology class is a nonzero element in $\mathrm{H}^1(\mathrm{Vect}(M); \mathrm{Hom}(\mathcal{S}_k, \mathcal{S}_{k-1}))$ (see Appendix). This class admits a representative with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-1})$, since the $\mathrm{Vect}(M)$-actions in (2.3) are differential. It thus defines a nontrivial class in $\mathrm{H}^1(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-1}))$ which, by Theorem 1.2, is a basis of this space.

2.3. CASE $k = \ell + 2$

If $\lambda = 1/2$, it is shown in [12] that the sequence (2.3) is split and that the sequence

$$0 \longrightarrow \mathcal{S}_{k-2} \longrightarrow \mathcal{D}_{1/2}^k/\mathcal{D}_{1/2}^{k-3} \longrightarrow \mathcal{D}_{1/2}^k/\mathcal{D}_{1/2}^{k-2} \longrightarrow 0 \tag{2.4}$$

is not. Moreover, the splitting of (2.3) is given by differential projectors. Since (2.3) is split, the class $[\mathcal{D}_{1/2}^{k-1}, \mathcal{D}_{1/2}^k]$ of (2.2) belongs to $\mathrm{H}^1(\mathrm{Vect}(M); \mathrm{Hom}(\mathcal{S}_k, \mathcal{D}_{1/2}^{k-2}))$. Since (2.4) is not split, its projection $\sigma_{k-2\sharp}[\mathcal{D}_{1/2}^{k-1}, \mathcal{D}_{1/2}^k]$ is nonzero (see Lemma 6.2 from Appendix).

As in the previous case, this projection is easily seen to admit a representative with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-2})$. Hence, it provides a basis of $\mathrm{H}^1(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-2}))$.

Remark 2.1. In the above Subsections 2.1 and 2.3 we have associated nontrivial cohomology classes to the exact sequences (2.3) and (2.4). It is important to note that these classes are 'natural' in the following sense. For any open subset $U \subset M$ their restrictions to U are precisely the classes associated to the same sequences upon U.

3. Projectively Equivariant Cohomology

Throughout this section we put $M \cong \mathbb{R}^n$ and $n \geqslant 2$.

3.1. THE LIE ALGEBRA OF INFINITESIMAL PROJECTIVE TRANSFORMATIONS

The main idea of our proof of Theorem 1.2 is to use the filtration with respect to the Lie subalgebra

$$\mathrm{sl}(n+1, \mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n). \tag{3.1}$$

It is suggested by the fact that the exact sequence (2.2) that generate our cohomology is split as a sequence of $\mathrm{sl}(n+1, \mathbb{R})$-modules [12]. In some sense, this Lie subalgebra plays the same rôle in our approach as the linear subalgebra $\mathrm{gl}(n, \mathbb{R})$ in the traditional one (cf. [7]).

Recall that the standard action of the Lie algebra $\mathrm{sl}(n+1, \mathbb{R})$ on $\mathbb{R}^n$ is generated by the vector fields

$$X_i = \frac{\partial}{\partial x^i}, \qquad X_{ij} = x^i \frac{\partial}{\partial x^j}, \qquad \bar{X}_i = x^i \mathcal{E}, \tag{3.2}$$

where

$$\mathcal{E} = x^i \frac{\partial}{\partial x^i}. \tag{3.3}$$

Observe in particular that X_i and X_{ij} generate an action of the Lie algebra $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

3.2. COMPUTING THE RELATIVE COHOMOLOGY SPACE

In this section we will compute the first space of the so-called relative cohomology of $\mathrm{Vect}(\mathbb{R}^n)$, i.e. the cohomology of the complex of $\mathrm{Vect}(\mathbb{R}^n)$-cochains vanishing on the subalgebra $\mathrm{sl}(n+1, \mathbb{R})$. We will prove the following theorem:

THEOREM 3.1. *If $n \geqslant 2$, then*

$$\mathrm{H}^1(\mathrm{Vect}(\mathbb{R}^n), \mathrm{sl}(n+1, \mathbb{R}); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)) = \begin{cases} \mathbb{R}, & \text{if } \; k-\ell = 2, \\ \mathbb{R}, & \text{if } \; k-\ell = 1, \ell \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

3.3. EQUIVARIANCE PROPERTY

We begin the proof with a simple observation.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra and V a $\mathfrak{g}$-module. If $c : \mathfrak{g} \to V$ is a 1-cocycle such that $c_{|\mathfrak{h}} \equiv 0$, then it is *equivariant* with respect to $\mathfrak{h}$, i.e.

$$L_X(c(Y)) = c([X, Y]), \quad X \in \mathfrak{h}, \tag{3.5}$$

where L stays for the $\mathfrak{g}$-action on the module V.

Consequently, our strategy to compute the space of relative cohomology (3.4) consists, first, in classifying the $\mathrm{sl}(n+1, \mathbb{R})$-equivariant linear maps

$c : \mathrm{Vect}(\mathbb{R}) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ vanishing on $\mathrm{sl}(n+1, \mathbb{R})$ and, second, to isolate among them the 1-cocycles.

3.4. COMMUTANT OF THE AFFINE LIE ALGEBRA

Consider the space of polynomials $\mathbb{C}[x, \xi] = \mathbb{C}[x^1, \ldots, x^n, \xi_1, \ldots, \xi_n]$ as a submodule of $\mathcal{S}$ under the action of $\mathrm{sl}(n+1, \mathbb{R})$. We need to compute the *commutant* of the subalgebra $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, i.e. the algebra of differential operators on $\mathbb{C}[x, \xi]$ commuting with the $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$-action.

The differential operators on $\mathbb{C}[x, \xi]$ given by

$$\mathrm{E} = \xi_i \frac{\partial}{\partial \xi_i}, \qquad \mathrm{D} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i} \tag{3.6}$$

commute with the $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$-action. Let us recall the classical result of the Weyl invariant theory (see [17]).

PROPOSITION 3.2. *The algebra of differential operators on* $\mathbb{C}[x, \xi]$ *commuting with the action of the affine Lie algebra, is generated by* E *and* D.

We will call the operators (3.6) the Euler operator and the divergence operator, respectively. The eigenspaces of E are obviously consist of homogeneous polynomials in ξ.

COROLLARY 3.3. *The operator* $\mathrm{D}^{k-\ell}$ *is the unique (up to a constant)* $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$*-equivariant differential operator from* $\mathcal{S}_k$ *to* $\mathcal{S}_\ell$.

Proof. Any differential operator on $\mathcal{S}_k$ is indeed determined by its values on the subspace $\mathbb{C}[x, \xi]$. □

The Euler operator E is clearly equivariant with respect to the whole $\mathrm{Vect}(\mathbb{R}^n)$. We will need the commutation relations of the operator D with the quadratic generators of $\mathrm{sl}(n+1, \mathbb{R})$.

LEMMA 3.4. *For* $\bar{X}_i$ *as in (3.2), one has*

$$[L_{X_i}, D] = \big(2E + (n+1)\big) \circ \frac{\partial}{\partial \xi_i}\,, \tag{3.7}$$

Proof. Straightforward. □

3.5. BILINEAR $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$-INVARIANT OPERATORS

We also need to classify the bilinear $\mathrm{gl}(n, \mathbb{R}) \ltimes \mathbb{R}^n$-invariant differential operators. For that purpose, let us use a natural identification

$$\mathbb{C}[x, \xi] \otimes \mathbb{C}[y, \eta] \cong \mathbb{C}[x, \xi, y, \eta]. \tag{3.8}$$

There are, obviously, four invariant differential operators $\mathrm{D}_{(x,\xi)}, \mathrm{D}_{(y,\eta)}$ (the divergence operators with respect to the first and the second arguments) and $\mathrm{D}_{(x,\eta)}, \mathrm{D}_{(y,\xi)}$ (the operators of contraction in terms of tensors). Applying again [17] one gets the following

PROPOSITION 3.5. *Every bilinear differential operator*

$$\mathcal{S}_j \otimes \mathcal{S}_k \to \mathcal{S}_\ell \tag{3.9}$$

invariant with respect to the action of the affine Lie algebra, is a homogeneous polynomial in $\mathrm{D}_{(x,\xi)}, \mathrm{D}_{(x,\eta)}, \mathrm{D}_{(y,\xi)}$ *and* $\mathrm{D}_{(y,\eta)}$ *of degree* $j+k-\ell$.

We are now ready to start the proof of Theorem 3.1.

3.6. BILINEAR $\mathrm{sl}(n+1,\mathbb{R})$-EQUIVARIANT OPERATORS

In view of Section 3.3, we will now classify the $\mathrm{sl}(n+1,\mathbb{R})$-equivariant linear differential maps

$$c : \mathrm{Vect}(\mathbb{R}^n) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell) \tag{3.10}$$

vanishing on the subalgebra $\mathrm{sl}(n+1,\mathbb{R}) \subset \mathrm{Vect}(\mathbb{R}^n)$. We can, equivalently, consider the equivariant bilinear maps

$$C : \mathcal{S}_1 \otimes \mathcal{S}_k \to \mathcal{S}_{k-p}, \tag{3.11}$$

where $p = k - \ell$.

By Proposition 3.5, any such operator is of the form

$$\begin{aligned} C = \sum_{s=0}^{p+1} \Bigg(& \frac{\alpha_s}{s!(p-s+1)!}\, \mathrm{D}^s_{(x,\eta)} \mathrm{D}^{p-s+1}_{(y,\eta)} + \\ & + \frac{\beta_s}{(s-1)!(p-s+1)!}\, \mathrm{D}_{(x,\xi)} \mathrm{D}^{s-1}_{(x,\eta)} \mathrm{D}^{p-s+1}_{(y,\eta)} \Bigg) \Big|_{\substack{y=x\\ \eta=\xi}} + \\ & + \sum_{s=0}^{p} \frac{\gamma_s}{s!(p-s)!}\, \mathrm{D}_{(y,\xi)} \mathrm{D}^s_{(x,\eta)} \mathrm{D}^{p-s}_{(y,\eta)} \Big|_{\substack{y=x\\ \eta=\xi}}, \end{aligned} \tag{3.12}$$

where $\alpha_s, \beta_s, \gamma_s \in \mathbb{R}$.

Moreover,

$$\alpha_s = \beta_s = \gamma_s = 0 \quad \text{for} \quad s < 2, \tag{3.13}$$

since C vanishes on the affine subalgebra and

$$(k-p)\alpha_2 + (n+1)\beta_2 + (p-1)\gamma_2 = 0, \tag{3.14}$$

since C vanishes on the quadratic generators $\bar{X}_i$ of $\mathrm{sl}(n+1,\mathbb{R})$.

For $k = p$, we have not to take into account the coefficients α_s in the expression (3.12) because the corresponding terms vanish when applied to $\mathcal{S}_1 \otimes \mathcal{S}_k$.

It is quite easy, using (3.7) and analogous relations with the operators $\mathrm{D}_{(x,\eta)}$, $\mathrm{D}_{(y,\xi)}$ and $\mathrm{D}_{(y,\eta)}$, to obtain the necessary and sufficient condition for the coefficients in (3.12) for C to be equivariant. One gets the following recurrence relations:

$$(s-1)\,\alpha_{s+1} - (2k+n-p+s-1)\,\alpha_s - \gamma_s = 0, \tag{3.15}$$

$$(s-1)\,\beta_{s+1} - (2k+n-p+s-1)\,\beta_s - \gamma_s = 0, \tag{3.16}$$

$$(s-2)\,\gamma_s - (2k+n-p+s-1)\,\gamma_{s-1} = 0, \tag{3.17}$$

$$(k-p)\,\alpha_{s+1} + (n+1)\,\beta_{s+1} + (p-s)\,\gamma_{s+1} + (k-p+s)\,\gamma_s = 0, \tag{3.18}$$

where $2 \leqslant s \leqslant p$. (For $k = p$, Equation (3.15) has not to be taken into account.)

Now, to solve the system (3.14–3.18), we need the following technical

LEMMA 3.6. *If* $\alpha_s, \beta_s, \gamma_c$ *verify Equations (3.15)–(3.17) and (3.14), then* $\alpha_s, \beta_s, \gamma_c$ *verify Equation (3.18).*

(A similar result holds true when $k = p$.)

Proof. Check that for $s = 1$ Equation (3.18) coincides with (3.14), the result follows then by induction. □

It is now very easy to get the complete solution of the system (3.14–3.17). One has the following four cases.

(a) For $p = 0$ and for $(p = 1, k = 1)$ there is no solution.

(b) For $(p = 1, k \geqslant 2)$ the system has a one-dimensional space of solutions spanned by

$$C_1 = \frac{1}{2}\,\mathrm{D}^2_{(x,\eta)} + \frac{k-1}{n+1}\,\mathrm{D}_{(x,\xi)}\mathrm{D}_{(x,\eta)}\,, \tag{3.19}$$

which is, in fact, a just a solution of Equation (3.14).

Remark 3.7. One readily checks that the operator $c(X) \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-1})$ given by (3.19) coincides (up to a constant) with the operator of contraction with the tensor field

$$c_1(X) = \left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}(X^\ell) + \frac{2}{n+1}\,\delta^\ell_j\,\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^s}(X^s)\right)\mathrm{d}x^i\mathrm{d}x^j \otimes \xi_\ell \tag{3.20}$$

This expression is obviously a 1-cocycle. The expression (3.20) is known in the literature as the Lie derivative of a flat *projective connection* (cf., e.g., [8]).

(c) For $p \geqslant 2, k > p$, the system (3.15)–(3.17) under the condition (3.14), has a two-dimensional space of solutions parametrized by (α_2, β_2).

(d) For $p = k \geqslant 2$, Equation (3.15) should be discarded. The space of solutions is again one-dimensional.

3.7. PROJECTIVELY INVARIANT COCYCLES

We will now determine which of the $\mathrm{sl}(n+1, \mathbb{R})$-equivariant maps (3.10) classified in the preceding section are 1-cocycles. Let us examine separately the cases (b)–(d).

(b) In the simplest case, $p = 1$, one easily checks that the unique $\mathrm{sl}(n+1, \mathbb{R})$-equivariant map (3.19), indeed, defines a 1-cocycle on $\mathrm{Vect}(\mathbb{R}^n)$ with values in $\mathcal{D}^0(\mathcal{S}_k, \mathcal{S}_{k-1})$.

(c) The cocycle relation adds the equation $\beta_3 = 2\beta_2$ to the general system (3.15–3.17).

In the case $p = 2$, one checks by a straightforward computation, that the solutions are the constant multiples of the solution given by

$$\begin{aligned} &\alpha_2 = 2, \qquad \alpha_3 = 2k + n + 1, \\ &\beta_2 = 1, \qquad \beta_3 = 2, \\ &\delta_2 = -(2k + n - 3). \end{aligned} \tag{3.21}$$

In the case $p > 2$, the only solution of the system (3.15)–(3.17) together with the equation $\beta_3 = 2\beta_2$ is zero.

(d) If $k = p$, then the nontrivial solutions of the system are cocycles if and only if $k = p = 2$. This cocycle is precisely of the form (3.21) disregarding α_2 and α_3.

PROPOSITION 3.8. *The 1-cocycles on* $\mathrm{Vect}(\mathbb{R}^n)$ *defined by the formulæ (3.19) and (3.21) are nontrivial.*

Proof. This follows immediately from Sections 5.1, 2.3 and the fact that the sequence (2.2) is split when restricted to $\mathrm{sl}(n+1, \mathbb{R})$, see [12]. Let us also give an elementary proof.

Recall that a 1-cocycle on $\mathrm{Vect}(\mathbb{R}^n)$ with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ is a coboundary if it is of the form $X \mapsto [L_X, B]$ for some $B \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$. Moreover, the 1-cocycle vanishes on $\mathrm{sl}(n+1, \mathbb{R})$ if and only if B is $\mathrm{sl}(n+1, \mathbb{R})$-equivariant.

LEMMA 3.9. (cf. [9]). *If* $k \neq \ell$, *there is no* $\mathrm{sl}(n+1, \mathbb{R})$*-equivariant operators* $B \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ *different from zero.*

Proof. In virtue of Corollary 3.3, the property of $\mathrm{sl}(n+1, \mathbb{R})$-equivariance implies, in particular, that B has to be proportional to $\mathrm{D}^{k-\ell}$. Now, the commutation relation (3.7) shows that this operator can never be $\mathrm{sl}(n+1, \mathbb{R})$-equivariant. □

Proposition 3.8 follows. □

3.8. PROOF OF THEOREM 3.1

We have shown that there exist unique (up to a constant) 1-cocycles c_1 and c_2 on $\mathrm{Vect}(\mathbb{R}^n)$ with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-1})$ and $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-2})$ respectively, vanishing on $\mathrm{sl}(n+1, \mathbb{R})$. These cocycles define nontrivial classes of relative cohomology.

Theorem 3.1 is proven.

4. Proof of Theorem 1.2

Using the filtration with respect to the subalgebra $\mathrm{sl}(n+1, \mathbb{R})$, we will first prove Theorem 1.2 in the case when M is a vector space and then extend it to an arbitrary manifold. To that end, we need some more information about the cohomology of $\mathrm{sl}(n+1, \mathbb{R})$.

4.1. COHOMOLOGY OF $\mathrm{sl}(n+1, \mathbb{R})$

The cohomology of the Lie algebra $\mathrm{sl}(n+1, \mathbb{R})$ with coefficients in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ has been computed in [9].

THEOREM 4.1. *The space of cohomology* $\mathrm{H}(\mathrm{sl}(n+1, \mathbb{R}); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell))$ *is trivial for* $k \neq \ell$, *for* $k = \ell$ *it is isomorphic to the Grassman algebra of invariant functionals on* $\mathrm{gl}(n, \mathbb{R})$*:*

$$\mathrm{H}(\mathrm{sl}(n+1, \mathbb{R}); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_k)) = \left(\bigwedge \mathrm{gl}(n, \mathbb{R})^*\right)^{\mathrm{gl}(n,\mathbb{R})}. \tag{4.1}$$

In particular,

$$\mathrm{H}^1(\mathrm{sl}(n+1, \mathbb{R}); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)) = \begin{cases} \mathbb{R}, & k = \ell, \\ 0, & \text{otherwise} \end{cases} \tag{4.2}$$

and the class of the 1-cocycle $X \mapsto \mathrm{div}(X)\mathrm{Id}$ spans that space in the case $k = \ell$. (In fact, it corresponds to the invariant function $\mathrm{tr} : \mathrm{gl}(n, \mathbb{R}) \to \mathbb{R}$.) Note that this cocycle is just the restriction to $\mathrm{sl}(n+1, \mathbb{R})$ of the cocycle $c_{1,0}$, see (2.1).

4.2. THE CASE OF $\mathbb{R}^n$

The restriction of a 1-cocycle $c : \mathrm{Vect}(\mathbb{R}^n) \to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ to $\mathrm{sl}(n+1, \mathbb{R})$ is a 1-cocycle on $\mathrm{sl}(n+1, \mathbb{R})$. If $k \neq \ell$, then this restriction is trivial and, therefore, c is cohomological to a 1-cocycle on $\mathrm{Vect}(\mathbb{R}^n)$ vanishing on $\mathrm{sl}(n+1, \mathbb{R})$; if $k = \ell$, then the restriction of c to $\mathrm{sl}(n+1, \mathbb{R})$ is cohomological to $c_{1,0}$ and so $c - c_{1,0}$ is, again, cohomological to a 1-cocycle on $\mathrm{Vect}(\mathbb{R}^n)$ vanishing on $\mathrm{sl}(n+1, \mathbb{R})$. The result then follows from Theorem 3.1.

Theorem 1.2 is proven for the special case $M = \mathbb{R}^n$.

4.3. THE GENERAL CASE

Let us now prove Theorem 1.2 for an arbitrary manifold M. Consider a 1-cocycle c on Vect(M) with values in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$.

(a) If $k - \ell \neq 0, 1, 2$, then in any domain of chart $U \cong \mathbb{R}^n$, the restriction $c_{|_U}$ is a coboundary, that is $c(X)_{|_U} = L_X(S_U)$, where $S_U \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ is some operator on U. But, on $U \cap V$, one has $c(X)_{|_{U\cap V}} = L_X(S_U) = L_X(S_V)$ and so the operator $S_U - S_V$ is invariant. Lemma 3.9 implies $S_U - S_V = 0$. Therefore, the S_U's are the restrictions of some globally defined $S \in \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell)$ and c is its coboundary.

(b) If $k - \ell = 1$ or 2, it follows from Theorem 1.2 for $M = \mathbb{R}^n$ that the class of $c_{|_U}$ is determined up to a constant. In view of Remark 2.1, one has thus

$$c_{|_U} = \alpha_U \, \gamma_{|_U} + L_X(S_U), \tag{4.3}$$

for some $\alpha_U \in \mathbb{R}$ and S_U as above, where γ is a representative of one of the classes associated to the sequences (2.3) and (2.4), respectively. On $U \cap V$ one obviously has $\alpha_U = \alpha_V$ and $S_U = S_V$ since $(\alpha_U - \alpha_V)\gamma_{|_{U\cap V}} = \partial(S_U - S_V)$, $\gamma_{|_{U\cap V}}$ is nontrivial and, as above, $S_U - S_V$ is invariant.

(c) If $k - \ell = 0$, one has

$$c_{|_U} = \alpha_U \, c_{1,0}{}_{|_U} + L_X(S_U). \tag{4.4}$$

Once again, $\alpha_U = \alpha_V \,(:= a)$ and $S_U - S_V$ is invariant, but any invariant operator in $\mathcal{D}(\mathcal{S}_k, \mathcal{S}_k)$ is proportional to the identity so that $S_U - S_V = \beta_{UV}\,\mathrm{Id}$, where β_{UV} is a constant. It is clear that the β_{UV}'s define a Čech 1-cocycle. If now ω is a closed 1-form representing the corresponding de Rham class, one easily sees that c is cohomologous to $c_{a,\omega}$.

Theorem 1.2 is proven.

5. Cocycles Associated to a Connection

Using a torsion free covariant derivation ∇, it is possible to construct globally defined cocycles spanning $\mathrm{H}^1(\mathrm{Vect}(M); \mathcal{D}(\mathcal{S}_k, \mathcal{S}_\ell))$ for $k - \ell = 1, 2$.

5.1. LIE DERIVATIVE OF A CONNECTION

For each vector field X, the Lie derivative

$$L_X(\nabla) : (Y, Z) \longmapsto [X, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X, Z]$$

of ∇ is well-known to be a symmetric $(1, 2)$-tensor field. It yields a nontrivial 1-cocycle

$$X \longmapsto L_X(\nabla)$$

on Vect(M) with values in $\Gamma(\bigotimes_2^1 TM)$. Therefore, for $k \geqslant 2$, the contraction

$$\gamma_1^\nabla(X)(P) = \langle P, L_X(\nabla)\rangle, \qquad P \in \mathcal{S}_k, \tag{5.1}$$

defines a 1-cocycle on Vect(M) with values in $\mathcal{D}^0(\mathcal{S}_k, \mathcal{S}_{k-1})$.

5.2. SECOND-ORDER COHOMOLOGY CLASS AND THE Vey COCYCLE

The last case, $\ell = k - 2$, is directly related to deformation quantization.

For any symplectic manifold V, there exists a nonzero class in $\mathrm{H}^2(C^\infty(V); C^\infty(V))$. It is given by so-called *Vey cocycle* usually denoted S_Γ^3 (see [2] and [16] for explicit construction using a connection Γ on V).

In the particular, if $V = T^*M$ one can choose the connection so that S_Γ^3 is homogeneous of weight -3, namely, restricted to $\mathcal{S} \subset C^\infty(T^*M)$,

$$S_\Gamma^3 : \mathcal{S}_k \otimes \mathcal{S}_\ell \to \mathcal{S}_{k+\ell-3}, \tag{5.2}$$

see [3] (e.g. choosing Γ as a lift of ∇ to T^*M). It follows easily from (5.2) that the map Vect(M) $\to \mathcal{D}(\mathcal{S}_k, \mathcal{S}_{k-2})$ defined by

$$\gamma_2^\nabla(X)(P) = S_\Gamma^3(X, P), \qquad P \in \mathcal{S}_k. \tag{5.3}$$

is a 1-cocycle.

6. Appendix: Approximations of the Class of a Short Exact Sequence of Modules

6.1. CLASS OF A SHORT EXACT SEQUENCE OF $\mathfrak{g}$-MODULES

We will need some general information about short exact sequences of filtered modules.

Let $\mathfrak{g}$ be a Lie algebra. Consider an exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow A \stackrel{i}{\longrightarrow} B \stackrel{j}{\longrightarrow} C \longrightarrow 0 \tag{6.1}$$

It is characterized by an element of $\mathrm{H}^1(\mathfrak{g}; \mathrm{Hom}(C, A))$ (cf. [7], Sec. 1.4.5). It will be convenient to denote it $[A, B]$. Recall that if $\tau : C \to B$ is a section of j, then $[A, B]$ is the class of the 1-cocycle $\gamma^\tau : \mathfrak{g} \to \mathrm{Hom}(C, A)$ given by

$$\gamma^\tau(X)(T) = i^{-1}(X.\tau(T) - \tau(X.T)), \tag{6.2}$$

where $X \in \mathfrak{g}$ and $T \in C$ (this expression is well defined since

$$X.\tau(T) - \tau(X.T) \in \ker j).$$

Given a submodule V of A, one has the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & V & & & & & & \\
 & & \downarrow{\scriptstyle i_V} & & & & & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & A/V & \longrightarrow & B/V & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
\qquad (6.3)
$$

where i_V is the injection of V into A and π_A, π_B are the projections.

One has the relation $[A/V, B/V] = \pi_{A\sharp}[A, B]$. Moreover, the left vertical of (6.3) leads to the exact triangle

$$
\begin{array}{ccc}
cl\mathrm{H}(\mathfrak{g}; \mathrm{Hom}(C, V)) & \searrow^{i_{V\sharp}} & \\
\Delta \uparrow & & \mathrm{H}(\mathfrak{g}; \mathrm{Hom}(C, A)) \\
\mathrm{H}(\mathfrak{g}; \mathrm{Hom}(C, A/V)) & \swarrow_{\pi_{A\sharp}} &
\end{array}
\qquad (6.4)
$$

where Δ is the connecting homomorphism. One easily obtains the following proposition:

PROPOSITION 6.1.

(i) *The class* $[A/V, B/V]$ *vanishes if and only if* $[A, B] \in \mathrm{im}\, i_{V\sharp}$.
(ii) *If* $[A/V, B/V] = 0$ *then the class of the exact sequence*

$$0 \longrightarrow V \stackrel{i \circ i_V}{\longrightarrow} B \longrightarrow B/V \longrightarrow 0 \qquad (6.5)$$

is $[V, B] = [V, A] + [A, B]$ *and vanishes if and only if* $[V, A] = [A, B] = 0$.

6.2. CASE OF A FILTERED MODULE

Consider now a flag of filtered $\mathfrak{g}$-modules $A_0 \subset A_1 \subset \cdots \subset A_r \subset \cdots$ and put $S_r = A_r/A_{r-1}$. Let us study the classes $[A_r, A_{r+1}]$ of the sequences

$$0 \longrightarrow A_r \longrightarrow A_{r+1} \longrightarrow S_{r+1} \longrightarrow 0. \qquad (6.6)$$

The quotient by $V = A_{r-1}$, leads to its 'first approximation':

$$0 \longrightarrow S_r \longrightarrow A_{r+1}/A_{r-1} \longrightarrow S_{r+1} \longrightarrow 0. \tag{6.7}$$

If the sequence (6.7) is split, then $[A_r, A_{r+1}] \in \mathrm{H}^1(\mathfrak{g}; \mathrm{Hom}(S_{r+1}, A_{r-1}))$ and so

$$[A_{r-1}, A_{r+1}] = [A_{r-1}, A_r] + [A_r, A_{r+1}] \tag{6.8}$$

by Proposition 6.1.

The next approximation is a result of the quotient by A_{r-2}. Let $\pi_r : A_r \to A_r/A_{r-1}$ be the projection to the quotient-module.

LEMMA 6.2. *If the sequence (6.7) is split for all $r > 0$, but the sequences*

$$0 \longrightarrow S_{r-1} \longrightarrow A_{r+1}/A_{r-2} \longrightarrow A_{r+1}/A_{r-1} \longrightarrow 0, \tag{6.9}$$

for $r > 1$ are not split, then the class $\pi_{r-1\sharp}[A_r, A_{r+1}]$ does not vanish.

Proof. Since the sequence (6.7) is split, one has $\pi_{r-1\sharp}[A_{r-1}, A_r] = 0$. If, in addition, $\pi_{r-1\sharp}[A_r, A_{r+1}] = 0$, then by Proposition 6.1

$$[A_{r-1}/A_{r-2}, A_{r+1}/A_{r-2}] = \pi_{r-1\sharp}[A_{r-1}, A_{r+1}] = \pi_{r-1\sharp}([A_{r-1}, A_r] + [A_r, A_{r+1}]) = 0$$

and the sequence (6.9) is split. □

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References

1. Bouarroudj, S. and Ovsienko, V.: Three cocycles on Diff(S^1) generalizing the Schwarzian derivative, *Internat. Math. Res. Notices*, No. 1 (1998), 25–39.
2. De Wilde, M. and Lecomte, P.: Cohomologie 3-différentiable de l'algèbre de Poisson d'une variété symplectique, *Ann. Inst. Fourier* **33**(4) (1983), 83–94.
3. De Wilde, M. and Lecomte, P.: Star-products on cotangent bundles, *Lett. Math. Phys.* **7** (1983), 235–241.
4. Duval, C. and Ovsienko, V.: Space of second order linear differential operators as a module over the Lie algebra of vector fields, *Adv. in Math.* **132**(2) (1997), 316–333.
5. Duval, C. Lecomte, P. and Ovsienko, V.: Conformally equivariant quantization: Existence and uniqueness, *Ann. Inst. Fourier* **49**(6) (1999), 1999–2029.
6. Feigin, B. L. and Fuchs, D. B.: Homology of the Lie algebra of vector fields on the line, *Funct. Anal. Appl.* **14** (1980), 201–212.
7. Fuchs, D. B.: *Cohomology of Infinite-Dimensional Lie Algebras*, Consultants Bureau, New York, 1987.
8. Kobayashi, S. and Horst, C.: Topics in complex differential geometry, In: *Complex Differential Geometry*, Birkhäuser-Verlag, Basel, 1983, pp. 4–66.
9. Lecomte, P.: On the cohomology of $\mathrm{sl}(m+1, \mathbb{R})$ acting on differential operators and $\mathrm{sl}(m+1, \mathbb{R})$-equivariant symbol, Preprint Université de Liège, 1998.

10. Lecomte, P., Mathonet, P. and Tousset, E.: Comparison of some modules of the Lie algebra of vector fields, *Indag. Math.* **7**(4) (1996), 461–471.
11. Lecomte, P. and Ovsienko, V.: Projectively invariant symbol map and cohomology of vector fields Lie algebras intervening in quantization, dg-ga/9611006.
12. Lecomte, P. and Ovsienko, V.: Projectively invariant symbol calculus, *Lett. Math. Phys.* **49**(3) (1999), 173–196.
13. Nijenhuis, A. and Richardson, R. W.: Deformations of homomorphisms of Lie algebras, *Bull. Amer. Math. Soc.* **73** (1967), 175–179.
14. Peetre, J.: Une caractérisation abstraite des opérateurs différentiels, *Math. Scand.* **7** (1959), 211–218; **8** (1960), 116–120.
15. Richardson, R. W.: Deformations of subalgebras of Lie algebras, *J. Differential Geom.* **3** (1969), 289–308.
16. Roger, C.: Déformations algébriques et applications la physique, *Gaz. Math.* 1991, No. 49, 75–94.
17. Weyl, H.: *The Classical Groups,* Princeton Univ. Press, 1946.