Deformations of Poisson brackets and extensions of Lie algebras of contact vector fields

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Deformations of Poisson brackets and extensions of Lie algebras of contact vector fields

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Introduction

The Lie algebras considered in this paper lie at the juncture of two popular themes in mathematics and physics: Weyl quantization and the Virasoro algebra.

Quantization in the sense of Hermann Weyl [55] leads to deformations of an algebra of functions on a symplectic manifold. The first work on this theme is due to Moyal (see [41]), who proposed a deformation of the Poisson bracket on the space of functions on $\mathbb{R}^{2n}$. The operation that he introduced has been called the "Moyal bracket". A systematic study of the deformations of Poisson brackets in the case of an arbitrary symplectic manifold was begun in the classical works of Vey [54] and Lichnerowicz and his coworkers (see [11], [33], [34]). We point out only some of the works on this theme: [30], [31], [50], [51], [56], [57]. De Wilde and Lecomte proved [56], [57] the existence of a universal deformation of the Poisson bracket on an arbitrary symplectic manifold (see [31], [50]). This results in the most interesting examples of infinite-dimensional Lie algebras. (Here we note [52], in which Saveliev and Vershik studied, in particular, the algebraic structure of such an algebra in the case of a two-dimensional torus.)

The Virasoro algebra is a central extension of the Lie algebra of vector fields on the circle. It was discovered by Gel'fand and Fuks [14]. The Virasoro algebra (and its representations) have turned out to be related to various problems in geometry [3], [9], [10]. It is actively exploited in quantum field theory (see [5], for example). The problem of looking for analogues of the Virasoro algebra that are associated with manifolds of dimension greater than 1 has encountered the following obstructions. The Lie algebras of vector fields on a manifold different from S^1 do not have non-trivial central extensions (see [12]). The same thing is true for the Lie algebras of contact vector fields on contact manifolds. The known central extensions of the Lie algebras of Hamiltonian vector fields on a symplectic manifold do not possess a uniqueness property and do not extend to the corresponding Lie superalgebras, which was communicated to the authors by Kirillov. Many examples of infinite-dimensional Lie algebras are given in [24] (see also [14], [22], [26]).

Our goal is to connect these two subjects.

Contact geometry is the odd-dimensional double of symplectic geometry. The Lie algebra of contact vector fields on a contact manifold is the contact double of the Lie algebra of Hamiltonian vector fields on a symplectic manifold. Meanwhile there do not exist any non-trivial deformations of this Lie algebra [11]. The analogue of a deformation of the Poisson bracket is a series of extensions of the Lie algebra of contact vector fields [47] (see also [44]). These extensions are constructed using some modules of (generalized) tensor fields: modules of contact 1-forms, and so on. In dimension $4k+1$ some of the Lie algebras that arise have central extensions. In this way we define "contact Virasoro algebras", the main object of our study.

The general meaning of the process of quantization is the replacement of an algebra of functions on a symplectic manifold (the algebra of observables) by some algebra of self-adjoint operators in a Hilbert space. In this process the product of functions goes over to a product of operators and ceases to be commutative. Thus, we get *deformations* of the algebra of functions in the class of associative algebras. The "new" non-commutative multiplication of functions is usually called a $*$-product. In the case of the linear symplectic space $\mathbb{R}^{2n}$ Weyl quantization proposes replacing functions on $\mathbb{R}^{2n}$ by differential operators on $\mathbb{R}^n$ (and for the Hilbert space one usually takes $L^2(\mathbb{R}^n)$). In this simplest case the *-product is multiplication of differential operators. The corresponding associative deformation was computed by Moyal [41] and is usually called the Moyal product.

The most important property of the *-product is its compatibility with a deformation of the Poisson bracket: the Leibniz rule remains valid:

$$\{A, B *_t C\}_t = \{A, B\}_t *_t C + B *_t \{A, C\}_t,$$

where $*_t$ denotes *-product (associative deformation of multiplications of functions with respect to the parameter t), and $\{,\}_t$ is a deformation of the Poisson bracket. This property means that the deformed Poisson bracket defines a derivation of the *-algebra of functions.

On a contact manifold there does not exist a *-product in the usual ("deformation") sense [11]. The analogue of a *-algebra is a series of associative extensions of the algebra of functions on a contact manifold. We construct a series of "contact" *-algebras on an arbitrary contact manifold. The Leibniz rule will hold for each of these algebras. This allows us to give a meaning (in these terms) to the idea of "quantization of a contact manifold".

The associative algebras that have been constructed, just like Lie algebras, can have non-trivial central extensions.

This paper consists of three chapters and two appendices. We have tried to arrange the presentation so that each chapter (and if possible each section) can be read (or omitted) independently.

In Chapter I we consider the linear symplectic space $\mathbb{R}^{2n+2}$, its projectivization $\mathbb{R}P^{2n+1}$, and the standard contact sphere S^{2n+1}. In this simplest case the deformation quantization was constructed classically. We shall explicitly describe the extensions of the Lie algebras of contact vector fields on $\mathbb{R}P^{2n+1}$ (on S^{2n+1}) corresponding to the Moyal bracket, and the contact *-algebras associated with the Moyal product. In this chapter we give several examples. The simplest of them arise for $n = 0$ (on S^1). The Virasoro algebra occurs in our series of Lie algebras in a natural way. The other algebras of this series are non-trivial extensions of the Virasoro algebra.

Chapter II contains a survey of results on deformation quantization on an arbitrary symplectic manifold. It is also an introduction to the modern "cohomological technique" used in the theory of deformations of algebras. The main object of this theory is a graded Lie algebra (and its cohomology). An example of such an algebra is the Richardson–Nijenhuis algebra, the main tool for computing the cohomology of a Poisson Lie algebra. We give a method for constructing a universal deformation of the Poisson bracket and *-product on an arbitrary symplectic manifold.

In Chapter III we generalize the results of Chapter I to an arbitrary symplectic manifold.

The appendices contain some applications, computations, and questions.

It is a very pleasant task for us to thank V.I. Arnol'd, who encouraged us to write this paper. We also thank A.A. Kirillov who pointed out an inaccuracy in [44], [47] in the computation of the cocycle defining the central extensions (corrected in the present paper). Discussions with him in Luminy were very useful in preparing this work.

§1. Main theorems

In this section we state the main results. Below we give all the constructions and provide detailed definitions of all the concepts that we need.

Let M^{2n+1} be a smooth contact manifold. We denote by $\mathfrak{h}(M)$ the Lie algebra of all contact vector fields on M. (For brevity we shall often denote this Lie algebra by the symbol $\mathfrak{h}$.)

We consider the space of *tensor densities* on M. These are geometric objects ("geometric quantities", see [25]), which (locally) have the form

$$\varphi = f \cdot (dx_1 \wedge \cdots \wedge dx_{2n+1})^{\mu}, \tag{1.1}$$

where $f = f(x_1, \ldots, x_{2n+1})$ is a function on M, x_i are coordinates, and the number μ is called the *degree* of the tensor density φ. In other words, tensor densities of degree μ are sections of the bundle $(\wedge^{2n+1}T^*M)^{\mu}$ over M. Each space of tensor densities of a given degree on M is an $\mathfrak{h}(M)$-module (relative to the Lie derivative along the vector field).

We denote by $\mathcal{F}_\lambda = \mathcal{F}_\lambda(M)$ the space of tensor densities on M of degree

$$\mu = \frac{\lambda}{n+1}. \tag{1.2}$$

***Theorem* 1.1.** *There exists a chain of non-trivial extensions of Lie algebras*

$$\begin{gathered}
0 \to \mathcal{F}_1 \to \mathfrak{h}_1 \to \mathfrak{h} \to 0,\\
0 \to \mathcal{F}_3 \to \mathfrak{h}_2 \to \mathfrak{h}_1 \to 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
0 \to \mathcal{F}_{2m-1} \to \mathfrak{h}_m \to \mathfrak{h}_{m-1} \to 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered}$$

Remark 1.1. We shall leave it to the reader to verify that the space $\mathcal{F}_1$ as an $\mathfrak{h}(M)$-module is isomorphic to the space of contact $\mathfrak{h}$-forms on M. Thus, the Lie algebra $\mathfrak{h}_1$ is an extension of the Lie algebra of all contact vector fields on M by the module of all contact $\mathfrak{h}$-forms. This extension was first proposed by Lichnerowicz [34].

All the series of extensions can be represented in diagram form (Fig. 1).

***Theorem* 1.2.** *Suppose that the contact manifold M is compact and has dimension* $\dim M = 4k+1$. *Then the Lie algebra $\mathfrak{h}_m$ has a non-trivial central extension for $m > k$.*

Remark 1.2. For $k = 0$ we talk about extensions of the Lie algebra of all vector fields on S^1 (the contact structure is trivial). The central extension of $\mathfrak{h}_0 = \mathfrak{h}$ is unique and is called the *Virasoro algebra*. The Lie algebras obtained as a result of the central extensions $\mathfrak{h}_m$ form a whole series of extensions of the Virasoro algebra by modules of tensor fields on S^1. This series is considered in detail in §4.

We now consider the commutative associative algebra $C^\infty(M)$ of all functions on M relative to usual (pointwise) multiplication. We denote this algebra by $K = K(M)$.

We shall also describe the extensions of K using the modules of tensor densities on M. These extensions are no longer commutative, although they remain associative. The resulting associative algebras turn out to be modules over the Lie algebras $\mathfrak{h}_m$, where the action of $\mathfrak{h}_m$ on them is given by derivations (that is, they satisfy the Leibniz identity).

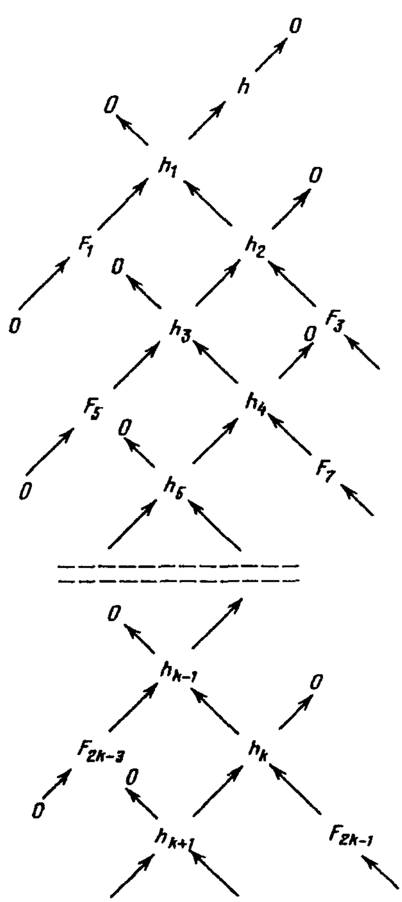

Fig. 1

***Theorem* 1.3.** (i) *There exists a chain of non-trivial extensions of the algebra $K(M)$ in the class of associative algebras*

$$\begin{gathered} 0 \to \mathcal{F}_1 \to K_1 \to K \to 0, \\ 0 \to \mathcal{F}_2 \to K_2 \to K_1 \to 0, \\ \dots\dots\dots\dots\dots\dots \\ 0 \to \mathcal{F}_m \to K_m \to K_{m-1} \to 0, \\ \dots\dots\dots\dots\dots\dots \end{gathered}$$

(ii) *There are well-defined inclusions $\mathfrak{h}_m \subset \mathrm{Diff}(K_{2m+1})$ of the Lie algebras $\mathfrak{h}_m$ in the Lie algebras of all derivations of the associative algebras K_{2m+1}.*

It is likely that the inductive limit $K_\infty = \varprojlim K_m$ should be taken as the analogue of a *-algebra on the contact manifold M.

***Theorem* 1.4.** *On a compact contact manifold M^{2n+1} the associative algebra K_m has a non-trivial central extension for $m > n$.*

Remark 1.3. The existence of central extensions for the associative algebras K_m (in contrast to Lie algebras) does not impose restrictions on the dimension of the manifold. There actually exists a construction of Lie algebras that are related to contact manifolds of dimension $4k+3$, which have non-trivial central extensions, but these algebras cannot be realised as a chain of non-trivial extensions of $\mathfrak{h}(M)$.

For Lie algebras of vector fields on manifolds there is the concept of local [11], [12] (diagonal, see [12]) cohomology. (All the cocycles that are considered in relation to this are defined using differential operators.) The analogous definitions also exist for associative algebras of functions (see §6.2 for more details). We shall prove a "uniqueness theorem" in a weak form (in the class of extensions associated with local cohomology).

***Proposition* 1.1.** *In the case of the standard compact sphere S^{2n+1} (and also in the projective space $\mathbb{R}P^{2n+1}$) all the extensions are unique in the class of local extensions.*

Remark 1.4. We assume that the uniqueness remains valid if we remove the locality condition. On the other hand, in the case of more complicated contact manifolds the analogy with the symplectic case forces us to suspect the existence of other non-trivial extensions even in the class of local extensions. In either case the extensions that we construct will be connected with the so-called universal deformation (the most interesting deformation) of the Poisson bracket (and also of the algebra of functions) on the symplectization of the contact manifold.

CHAPTER I

ALGEBRA

§2. Moyal deformations of the Poisson bracket and *-product on $\mathbb{R}^{2n}$

2.0. Linear symplectic space.

Consider the linear space $\mathbb{R}^{2n}$, and fix a non-degenerate skew-symmetric bilinear form ω on $\mathbb{R}^{2n}$: $\omega(v, w) = -\omega(w, v)$. Such a form is called *symplectic*, and the pair $(\mathbb{R}^{2n}, \omega)$ is called a *linear symplectic space.* The linear operators that preserve the symplectic form are called *linear canonical transformations.* They form a subgroup of $GL(2n, \mathbb{R})$ which is called the symplectic group and is denoted by $\mathrm{Sp}(2n, \mathbb{R})$.

The coordinates in which the form ω is given by the matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ are called *Darboux coordinates.* They are written $(p_1, ..., p_n, q_1, ..., q_n)$. In Darboux coordinates the form ω can be written in the form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

(the linear space $\mathbb{R}^{2n}$ is identified with its tangent space and the form ω turns into a differential form). The basis in $\mathbb{R}^{2n}$ that is dual to the Darboux coordinates is denoted by $\partial_{p_1}, \ldots, \partial_{p_n}, \partial_{q_1}, \ldots, \partial_{q_n}$.

The form ω defines an operator $\mathbb{R}^{2n} \to \mathbb{R}^{2n*}$ such that a vector v is mapped to the linear functional $\omega(\cdot, v)$. The inverse operator $\mathbb{R}^{2n} \to \mathbb{R}^{2n*}$ is given by the bivector

$$P = \sum_{i=1}^{n} \partial_{p_i} \wedge \partial_{q_i} = \omega^{-1}.$$

The bilinear operator $\{F, G\}$ on a space of functions on some manifold is called a *Poisson bracket* if it is skew-symmetric:

$$\{F, G\} = -\{G, F\} \tag{2.1}$$

and satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{2.2}$$

and the Leibniz identity

$$\{F, GH\} = \{F, G\}H + G\{F, H\}. \tag{2.3}$$

The identities (2.1) and (2.2) mean that a Lie algebra structure is defined on that space of functions. Therefore an operation satisfying these identities (but not necessarily (2.3)) is called a *Lie structure.*

The bivector P defines a Poisson bracket on $\mathbb{R}^{2n}$:

$$\{F, G\} := P(dF, dG). \tag{2.4}$$

In Darboux coordinates the bracket (2.4) takes the following form:

$$\{F,G\} = \sum_{i=1}^{n} (F_{p_i} G_{q_i} - F_{q_i} G_{p_i}) . \tag{2.5}$$

The operation (2.5) defines a symplectic form on $(\mathbb{R}^{2n})^*$ (the Poisson bracket of two linear functions on $\mathbb{R}^{2n}$ is a constant). The Lie algebra of all functions on $\mathbb{R}^{2n}$ is denoted by $\mathfrak{G}$.

The space of all homogeneous quadratic polynomials in p_i, q_i is closed under the Poisson bracket and forms a finite-dimensional Lie algebra. It is called the *symplectic Lie algebra* and is denoted by $\mathrm{sp}(2n, \mathbb{R})$.

With each function H on a manifold with a Poisson bracket one associates a vector field $C(H)$, called the *Hamiltonian vector field* with Hamiltonian H. The field $C(H)$ is defined by its action on a function:

$$L_{C(H)} F := \{F, H\}.$$

Here $L_{C(H)}$ is the Lie derivative along the vector field $C(H)$. It follows from formula (1.5) that in Darboux coordinates on $\mathbb{R}^{2n}$

$$C(H) = \sum_{i=1}^{n} H_{q_i} \partial_{p_i} - H_{p_i} \partial_{q_i} . \tag{2.6}$$

The Hamiltonian vector fields on $(\mathbb{R}^{2n}, \omega)$ preserve the symplectic form: $L_{C(H)}\omega = 0$. (This follows, in particular, from the Jacobi identity.)

The linear Hamiltonian vector fields on $\mathbb{R}^{2n}$ correspond to quadratic functions. Hence, the symplectic Lie algebra $\mathrm{sp}(2n, \mathbb{R})$ is isomorphic to the Lie algebra of all linear vector fields on $\mathbb{R}^{2n}$ that preserve the form ω.

2.1. Higher Poisson brackets on a linear symplectic space.
Two operations on $(\mathbb{R}^{2n}, \omega)$ are defined invariantly: the product of functions and the Poisson bracket. (This means that it is not necessary to use coordinates in their definitions.) We define the operations $\{F, G\}_m$, depending on the m-jets of the functions F and G. Darboux coordinates will occur in the definition. And indeed, these operations will not be preserved under arbitrary symplectic transformations, but will be invariant under linear canonical transformations and therefore will be invariantly defined as operations on linear symplectic space.

The bivector P defines a linear operator on the tensor product of the space of functions with itself:

$$\mathcal{P} : F \otimes G \mapsto P(F,G) = \sum_{i=1}^{n} (F_{p_i} \otimes G_{q_i} - F_{q_i} \otimes G_{p_i}) .$$

We denote by tr the operator that maps the tensor product of two functions into the usual product: $\mathrm{tr}(F \otimes G) = FG$. The operator tr is the operator of restriction to the diagonal $\mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ (the tensor product of functions on $\mathbb{R}^{2n}$ is identified with a function on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$).

We shall call the operations

$$(2.7) \qquad \{F,G\}_m = \operatorname{tr}(\mathcal{P}^m(F \otimes G))$$

higher Poisson brackets.

Examples. $\{F, G\}_0 = FG$ is the usual product of functions, $\{F,G\}_1 = \{F, G\}$ is the Poisson bracket. On $\mathbb{R}^2$ the higher Poisson brackets have the form of a "differential Newton binomial":

$$\{F,G\}_2 = F_{pp}G_{qq} - 2F_{pq}G_{pq} + F_{qq}G_{pp},$$
$$\{F,G\}_3 = F_{ppp}G_{qqq} - 3F_{ppq}G_{qqp} + 3F_{pqq}G_{ppq} - F_{qqq}G_{ppp}$$

and so on.

In the general case of $\mathbb{R}^{2n}$ the formula is analogous. For example,

$$\{F,G\}_2 = F_{p_ip_j}G_{q_iq_j} - 2F_{p_iq_j}G_{p_jq_i} + F_{q_iq_j}G_{p_ip_j}.$$

Here we have used the notational convention, widely used in physics, of summing over repeated indices. The indices i and j are independent and vary from 1 to n.

***Assertion* 2.1.** *The general formula for the operations* (2.7) *has the following form*:

$$(2.8) \qquad \{F,G\}_m = \sum_{|i|+|j|=m} (-1)^{|i|+1} \binom{m}{|i|} F_{p_iq_j}G_{p_jq_i},$$

where i and j are multi-indices, all the indices i_s, j_t vary from 1 *to n and are independent of each other.*

***Assertion* 2.2.** *Up to proportionality the operation* (2.8) *is a unique differential operator of order $2m$ on the space of functions on $\mathbb{R}^{2n}$, and this operator is invariant under the action of the group of linear canonical transformations* $\mathrm{Sp}(2n, \mathbb{R})$.

Remark 2.1. The operations (2.8) have a number of important properties. They are connected with the classical theory of invariants [17], [18] and variational problems [43]. Using them is a convenient way of describing invariant differential operators (see, for example, [45]). They are often called transvectants and hyper-Jacobians (this is also related to a number of their virtues).

Remark 2.2. In the case of $\mathbb{R}^2$ the higher Poisson brackets turn out to be related to the second Hamiltonian structure of Gel'fand and Dikii (Dickey) (see [45]).

2.2. Deformation of the Poisson Lie algebra.

***Theorem* 2.1** (see [11], [33], [54]). *The operation*

$$(2.9)\qquad \{F,G\}_t = \{F,G\}_1 + \frac{t}{6}\{F,G\}_3 + \cdots + \frac{t^k}{(2k+1)!}\{F,G\}_{2k+1} + \cdots$$

defines a Lie algebra structure on the space of functions on $\mathbb{R}^{2n}$, and this algebra structure is isomorphic to the algebra of functions on $\mathbb{R}^{2n}$ relative to the Poisson bracket.

(In order not to get involved with questions of the convergence of the series (2.9) it is possible to restrict, say, to functions that are polynomial in the p_i.)

This operation was discovered by Moyal. It is called the *Moyal bracket*. The operation is skew-symmetric and satisfies the Jacobi identity but not the Leibniz identity. Therefore it is a Lie bracket but not a Poisson bracket. The theorem means that the Moyal bracket is a non-trivial deformation of the Poisson bracket in the class of Lie structures (it is obvious that as $t \to 0$ it becomes the Poisson bracket).

Lie structures that depend on a bounded number of jets of functions were studied in a paper by Kirillov [21], in which he proved, in particular, that in this case a non-degenerate Lie structure on an even-dimensional manifold must be a Poisson structure. In the class of Poisson structures any deformation of the bracket (2.5) is trivial (Darboux's theorem). All this does not contradict Theorem 2.1, since the Moyal bracket depends on unboundedly high jets of the functions F and G.

Corollary (of Theorem 2.1). *The higher Poisson brackets satisfy the following identities* (of Jacobi type):

$$(2.10)\qquad \sum_{k=0}^{m} \frac{1}{(2k+1)!\,(2m-2k+1)!} \circlearrowleft \{F, \{G,H\}_{2k+1}\}_{2m-2k+1} = 0,$$

where the symbol $\circlearrowleft$ denotes cyclic permutation of the functions F, G, H.

Examples. For $m = 0$ the identity (2.10) coincides with the usual Jacobi identity. For $m = 1$ it can be written in the form

$$(2.10')\qquad \circlearrowleft\{F,\{G,H\}_1\}_3 + \circlearrowleft\{F,\{G,H\}_3\}_1 = 0,$$

and for $m = 2$ in the form

$$(2.10'')\qquad \frac{3}{10}\circlearrowleft\{F,\{G,H\}_1\}_5 + \circlearrowleft\{F,\{G,H\}_3\}_3 + \frac{3}{10}\circlearrowleft\{F,\{G,H\}_5\}_1 = 0.$$

The identities (2.10) follow immediately from Theorem 2.1 (namely, from the Jacobi identity for operation (2.9)). These identities have a simple meaning in cohomological language.

***Theorem* 2.2** (see [11], [54]). *The Moyal bracket is the unique* (*up to isomorphism*) *non-trivial deformation of the Poisson bracket in the class of Lie structures.*

2.3. Connection with Weyl quantization.

It is perfectly clear that, while significant for elegance, a unique deformation of a Poisson structure cannot have a physical interpretation. The Moyal bracket can be understood as a "quantum Poisson bracket".

The problem of quantization is the problem of associating self-adjoint operators in a Hilbert space with functions on $\mathbb{R}^{2n+2}$. In doing this it is required that the linear functions p_i, q_i be associated with operators $\widehat{p}_i, \widehat{q}_i$ that satisfy the (Heisenberg) commutation relations:

$$[\widehat{p}_i, \widehat{q}_j] = \hbar\delta_{ij}, \qquad [\widehat{p}_i, \widehat{p}_j] = [\widehat{q}_i, \widehat{q}_j] = 0,$$

where $\hbar$ is a parameter. By the Stone-von Neumann theorem quantization is unique in the sense that (up to conjugation) the operators $\widehat{p}_i$ and $\widehat{q}_i$ are differential operators in $L_2(\mathbb{R}^{n+1})$:

$$\widehat{q}_i = q_i \quad \text{is the operator of multiplication by a function,}$$

$$\widehat{p}_i = \hbar\frac{\partial}{\partial q_i}.$$

The product of operators defines an associative operation $*_\hbar$ on the space of functions via the condition $\widehat{F *_\hbar G} = \widehat{F}\widehat{G}$, which should be a deformation of the usual product of functions.

Weyl quantization solves this problem in the following way. With each polynomial $F(p_i, q_i)$ one associates a symmetric polynomial $\widehat{F}(\widehat{p}_i, \widehat{q}_i)$ of the operators p_i, q_i according to the rule

$$\widehat{F}(\widehat{p}_i, \widehat{q}_i) := \operatorname{Sym} F(\widehat{p}_i, \widehat{q}_i).$$

Assertion **2.3.** $\frac{1}{\hbar}(F *_\hbar G - G *_\hbar F) = \{F, G\}_{\hbar^2}$.

The resulting operation on the space of functions is called a *-product. The explicit expression for this operation has the following form:

$$(2.11) \quad F *_\hbar G = FG + \hbar\{F, G\}_1 + \frac{\hbar^2}{2}\{F, G\}_2 + \cdots + \frac{\hbar^k}{k!}\{F, G\}_k + \dots .$$

Formula (2.11) is often written symbolically in the following equivalent form (emphasizing the similarity with the exponential series):

$$F *_\hbar G = \exp\left(i\hbar\left(\sum_{i=1}^{n+1} \partial_{p_i} \wedge \partial_{q_i}\right)\right)(F \otimes G),$$

and formula (2.9) in the form

$$\{F, G\}_t = \frac{s\hbar\left(t\left(\sum_{i=1}^{n+1} \partial_{p_i} \wedge \partial_{q_i}\right)\right)}{t}(F \otimes G).$$

The operation (2.11) is called the *Moyal product*.

Thus, the algebra of functions on $\mathbb{R}^{2n+2}$ simultaneously admits a deformation as a Lie algebra and as an associative algebra. It is a remarkable fact that the Leibniz identity holds as before:

$$\{F, G *_{\hbar} H\}_{-\hbar^2} = \{F,G\}_{-\hbar^2} *_{\hbar} H + G *_{\hbar} \{F,H\}_{-\hbar^2}.$$

Weyl quantization is probably the best known and most popular method of quantizing classical mechanical systems. We shall not go into details, but restrict ourselves to a reference to a paper by Lichnerowicz [33] (see also [11]), in which an "axiomatic" approach to quantum mechanics is proposed using the language of deformations of Poisson algebras.

§3. Algebraic construction

3.1. Projectivization of a symplectic space.
The symplectic structure on the standard symplectic space $\mathbb{R}^{2n+2}$ defines a *standard contact structure* on $\mathbb{R}P^{2n+1}$ (and also on S^{2n+1}). We give a simple description of the Lie algebra of contact vector fields on $\mathbb{R}P^{2n+1}$ (and S^{2n+1}) and of all the modules $\mathcal{F}_\lambda$.

Algebraic definition. We consider affine coordinates on $\mathbb{R}P^{2n+1}$:

$$x_i = \frac{p_i}{q_{n+1}}, \quad y_i = \frac{q_i}{q_{n+1}}, \quad i = 1, \ldots, n, \qquad z = \frac{p_{n+1}}{q_{n+1}}, \tag{3.1}$$

where $p_1, \ldots, p_{n+1}, q_1, \ldots, q_{n+1}$ are Darboux coordinates on $\mathbb{R}^{2n+2}$. The *standard contact structure on* $\mathbb{R}P^{2n+1}$ is the *conformal class* $\langle\alpha\rangle$ *of the* 1*-form*

$$\alpha = \sum_{i=1}^{n} \frac{x_i dy_i - y_i\, dx_i}{2} - dz \tag{3.2}$$

(that is, the whole space of 1-forms of the shape $f \cdot \alpha$, where f is a function).

A *contact vector field* on $\mathbb{R}P^{2n+1}$ is a vector field $v \in \mathrm{Vect}(\mathbb{R}^{2n+1})$ that preserves the class $\langle\alpha\rangle$. In other words, the Lie derivative of α along v multiplies α by some function:

$$L_v(\alpha) = m_v \cdot \alpha.$$

The inadequacy of this definition is the necessity of verifying that it does not depend on the choice of Darboux coordinates (p, q) (which, however, is quite simple). In order to give another definition, we note that all the 1-forms of the class $\langle\alpha\rangle$ have common hyperplanes in the tangent space $T\mathbb{R}P^{2n+1}$ on which the form α vanishes.

Geometric definition. We consider an arbitrary point $p \in \mathbb{R}P^{2n+1}$ and the corresponding one-dimensional subspace $V_p \subset \mathbb{R}^{2n+2}$. The skew-orthogonal complement $V_p^{\perp}$ is the subspace of $\mathbb{R}^{2n+2}$ that is orthogonal to V_p relative to the (bilinear skew-symmetric) form ω. (The space $V_p^{\perp}$ has dimension $2n+1$

and contains V_p.) The projectivization of $V_p^\perp$

$$\begin{array}{ccccc} V_p \setminus \{0\} & \subset & \mathbb{R}^{2n+2} \setminus \{0\} & \supset & V_p^\perp \setminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ p & \in & \mathbb{R}P^{2n+1} & \supset & \mathbb{R}P_p^{2n} \end{array}$$

of dimension $2n$ is denoted by $\mathbb{R}P_p^{2n}$. Let Γ_p be a hyperplane in the tangent space $T_p(\mathbb{R}P^{2n+1})$, tangent to $\mathbb{R}P_p^{2n}$. The distribution Γ in the tangent bundle $T(\mathbb{R}P^{2n+1})$ is called the *standard contact structure on* $\mathbb{R}P^{2n+1}$, and the plane Γ_p is called the *contact hyperplane* at the point p.

The connection between the two definitions is that Γ_p is the same hyperplane on which the form α vanishes (check this!). Thus, the standard contact structure on $\mathbb{R}P^{2n+1}$ is the projectivization of the symplectic space $\mathbb{R}^{2n+2}$ (in the case of the sphere S^{2n+1} the definition is analogous).

General definition. An odd-dimensional manifold M^{2n+1} is called a *contact manifold* if a distribution of hyperplanes in TM is fixed on M that is non-degenerate at each point.

A local contact structure can be defined as a distribution of hyperplanes on which some 1-form α vanishes. Such 1-forms are called contact. All contact manifolds are locally isomorphic to each other. There exist coordinates (contact Darboux coordinates) in which the contact structure is defined by the 1-form (3.2). The non-degeneracy of the distribution means that locally the 1-form α can be chosen so that the form $\alpha \wedge (d\alpha)^n$ is non-degenerate.

Remark. There may exist many different contact structures on the same manifold. The classical example is the sphere S^3: there are infinitely many contact three-dimensional spheres. Martinet's theorem [39] asserts that in each homotopy class of distributions of hyperplanes on S^3 there is a contact structure (nowhere non-degenerate). These classes are labelled by the integers. Moreover, there are more contact structures on S^3: the so-called "non-standard" contact structure of Bennequin [4] is homotopic to the standard one in the class of distributions but not homotopic in the class of contact structures. (Recently Ya. Eliashberg proved that there are no other such structures on S^3.) On the other hand, a contact structure exists on any oriented manifold of dimension 3 (a classical theorem, proved by Lutz [36] and Martinet [39]). A contact structure exists in every homotopy class of two-dimensional distributions on an arbitrary three-dimensional manifold.

Thus, the class of three-dimensional contact manifolds is substantially greater than simply the class of (ordinary) three-dimensional manifolds. In higher dimensions the question is still more complicated and is less studied. All the Lie algebras of contact vector fields on different contact manifolds are not isomorphic to each other.

The Lie algebras that we shall discuss below are connected with the standard contact spheres S^{2n+1} (or $\mathbb{R}P^{2n+1}$). These algebras will be our fundamental example (the general case will be considered in Chapter III).

3.2. The Lie algebra of contact vector fields and modules of tensor densities on $\mathbb{R}P^{2n+1}$ (and S^{2n+1}).

For the linear symplectic space $\mathbb{R}^{2n+2}$ we consider the homogeneous functions (with singularities at the origin):

$$(3.3) \qquad F(cr) = c^{\nu} F(r), \qquad \nu, c \in \mathbb{R} \setminus \{0\}, r \in \mathbb{R}^{2n+2} \setminus \{0\}.$$

We denote by $\mathcal{F}_{\lambda}$ the *space of homogeneous functions of degree* $\nu = -2\lambda$.

It is clear that the Poisson bracket preserves homogeneity:

$$(3.4) \qquad \{\mathcal{F}_{\lambda}, \mathcal{F}_{\mu}\} \subset \mathcal{F}_{\lambda+\mu+1}.$$

Thus the space $\mathcal{F}_{-1}$ of homogeneous functions of degree 2 on $\mathbb{R}^{2n+2} \setminus \{0\}$ forms a *Lie subalgebra.* The space $\mathcal{F}_{\lambda}$ is a module over $\mathcal{F}_{-1}$.

***Proposition* 3.1.** (i) *The Lie algebra $\mathfrak{h}(\mathbb{R}P^{2n+1})$ of all contact vector fields on $\mathbb{R}P^{2n+1}$ is isomorphic to $\mathcal{F}_{-1}$.*

(ii) *The module of all tensor densities on $\mathbb{R}P^{2n+1}$ of degree λ is isomorphic to the $\mathcal{F}_{-1}$-module $\mathcal{F}_{\lambda}$.*

(For the case S^{2n+1} the analogous assertion holds if in definition (3.3) we impose the restriction $c > 0$, which "doubles" the space of homogeneous functions.)

Proof (see §10). Every homogeneous function of degree 2 corresponds to a homogeneous Hamiltonian field of degree 1 on $\mathbb{R}^{2n+2} \setminus \{0\}$. Such a vector field is projected onto $\mathbb{R}P^{2n+1}$ and defines a contact vector field. The converse: every contact field on $\mathbb{R}P^{2n+1}$ extends uniquely to a homogeneous Hamiltonian field of degree 1 on $\mathbb{R}^{2n+2}$; this is also easy to verify. The second part of the assertion, relating to $\mathcal{F}_{-1}$-modules, is proved in §10 in a more general case.

Corollary. *The module of contact 1-forms on $\mathbb{R}P^{2n+1}$ is isomorphic to the $\mathcal{F}_{-1}$-module $\mathcal{F}_1$ (of homogeneous functions of degree -2 on $\mathbb{R}^{2n+2} \setminus \{0\}$), and hence, is also isomorphic to the module of tensor densities of degree $1/(n+1)$ on $\mathbb{R}P^{2n+1}$.*

In fact $\mathcal{F}_1$ is the dual space to $\mathcal{F}_{-1}$ over $\mathcal{F}_0$, the space of functions on $\mathbb{R}P^{2n+1}$ (see details in §10).

Definition. The operation $\{,\}$ (satisfying the property (3.4)) is defined on the space of tensor densities on $\mathbb{R}P^{2n+1}$ (and also S^{2n+1}), and this operation is mapped to the Poisson bracket on $\mathbb{R}^{2n+2}$ under the isomorphism between tensor densities and functions. In Darboux coordinates this operation has the form

$$(3.5) \qquad \{f, g\}_1 = f_x g_y - f_y g_x + f_z(\mu g - \mathcal{E} g) + g_z(\lambda f - \mathcal{E} f),$$

where $\mathscr{E} = x\partial_x + y\partial_y$ is the *Euler field.* We shall call this operation the *Lagrange bracket.* The same isomorphism maps higher Poisson brackets onto the space of tensor densities. They satisfy the condition

$$\{\mathcal{F}_\lambda, \mathcal{F}_\mu\}_m \subset \mathcal{F}_{\lambda+\mu+m}. \tag{3.6}$$

Remark. For $\lambda = \mu = -1$ the Lagrange bracket (3.5) becomes the usual bracket on a contact manifold, defined by contact Hamiltonians (see [1], [2]). It is curious that thus the *contact Hamiltonian* is a tensor density of degree $-1/(n+1)$ (or an object inverse to a 1-form). For more details see §10.

3.3. Extensions of Lie algebras.

Let $\mathfrak{A}$ be a Lie algebra, and V a module over $\mathfrak{A}$. We denote by $C^q(\mathfrak{A}, V)$ the space of skew-symmetric q-linear functions on $\mathfrak{A}$ with values in V. The operator $d = dq : C^q(\mathfrak{A}, V) \to C^{q+1}(\mathfrak{A}, V)$ defined by

$$\begin{aligned} dc(a_1, \dots, a_{q+1}) = &\sum_{1\le r\le q+1} (-1)^r a_r c(a_1, \dots, \widehat{a_r}, \dots, a_{q+1}) \\ &+ \sum_{1\le r<s\le q+1} (-1)^{r+s-1} c([a_r, a_s], a_1, \dots, \widehat{a_r}, \dots, \widehat{a_s}, \dots, a_{q+1}) \end{aligned} \tag{3.7}$$

is called a *differential.* A functional c is called a *cocycle* if $dc = 0$. It is easy to check that $d^2 = 0$. Therefore $\operatorname{Im} d_{q-1} \subset \operatorname{Ker} d_q$. The quotient space $H^q(\mathfrak{A}, V) := \operatorname{Ker} d_q / \operatorname{Im} d_{q-1}$ is called the space of *q-dimensional cohomology.*

Consider a functional $c \in C^2(\mathfrak{A}, V)$. We define the following operation on the space $\mathfrak{A} \oplus V$:

$$[(x, v), (y, w)] = ([x, y]_{\mathfrak{A}}, x(w) - x(v) + \alpha c(x, y)), \tag{3.8}$$

where α is a parameter. This operation satisfies the Jacobi identity and defines a Lie algebra structure on $\mathfrak{A} \oplus V$ if and only if c is a cocycle.

The sequence of homomorphisms of Lie algebras

$$0 \to V \to \mathfrak{A} \oplus V \to \mathfrak{A} \to 0$$

is called an extension of the Lie algebra $\mathfrak{A}$ by the $\mathfrak{A}$-module V. If $c = 0$, then the Lie algebra (3.8) is called a semidirect product and is denoted by $\mathfrak{A} \ltimes V$. Two extensions defined by cocycles c_1 and c_2 are isomorphic if and only if their cohomology classes (that is, the projections of c_1 and c_2 in the space $H^2(\mathfrak{A}, V)$) coincide. In particular, if $c \in \operatorname{Im} d_1$, the extension is said to be trivial; in this case the Lie algebra (3.8) is isomorphic to $\mathfrak{A} \ltimes V$.

An extension is said to be *central* if V is a trivial $\mathfrak{A}$-module; in other words, the subspace V is contained in the centre of the Lie algebra (3.8) (that is, $[(x, 0), (0, v)] = 0$ for any $x \in \mathfrak{A}$, $v \in V$). Central extensions of Lie algebras are very popular in both mathematics and physics. They arise naturally in the classification of homogeneous symplectic manifolds (see [20], [29], [53]).

3.4. Extensions of a Poisson Lie algebra.
The construction that we shall describe is based on the following simple, general fact: *any formal deformation of a Lie algebra defines an infinite series of successive extensions of it.* We construct a series of extensions of the algebra of functions on a symplectic space, corresponding to the Moyal deformation (2.9).

Let $\mathcal{F}$ be the Lie algebra of smooth functions on $\mathbb{R}^{2n+2}\setminus\{0\}$ (a special choice of the class of functions will be explained later). The Moyal bracket defines a Lie algebra structure on the space $\mathcal{F}[[t]]$ of formal series in the variable t with coefficients from $\mathcal{F}$:

$$\{Ft^m, Gt^l\} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\{F,G\}_{2k+1}t^{m+l+k}$$

(this fact is a general one for a deformation of a Lie algebra: check that the Jacobi identity in the Lie algebra $\mathcal{F}[[t]]$ is equivalent to the series of "higher Jacobi identities" (2.10)).

We define a series of Lie algebras

$$\mathfrak{G}_k = \mathcal{F}[[t]]/\langle t^{k+1} = 0\rangle$$

(as quotients of the Lie algebra $\mathcal{F}[[t]]$ by a chain of ideals, embedded in each other). Every Lie algebra $\mathfrak{G}_k$ is an extension of the previous one:

$$0 \to \mathcal{F} \to \mathfrak{G}_k \to \mathfrak{G}_{k-1} \to 0. \tag{3.9}$$

(The $\mathfrak{G}_{k-1}$-module structure on $\mathcal{F}$ is defined by the action of the first term. Formally, if $F = F_0 + F_1t + \ldots + F_{k-1}t^{k-1}$ is an element of $\mathfrak{G}_{k-1}$, then $F(G) := \{F_0, G\}$.) It is easy to compute the cocycle defining this extension explicitly:

$$C_k(Ft^i, Gt^j) = \frac{t^{i+j+l}}{(2l+1)!}\{F,G\}_{2l+1}, \tag{3.10}$$

where the numbers i, j, k, l are related by the condition

$$i + j + l - k = 1. \tag{3.11}$$

The resulting extensions (3.9) are non-trivial, since the Moyal deformation has infinite rank (see §2 for more details) (that is, the series (2.9) cannot be represented as a polynomial in t). One has as a result a projective chain of homomorphisms of Lie algebras:

$$\ldots \mathfrak{G}_k \to \mathfrak{G}_{k-1} \to \ldots \mathfrak{G}_2 \to \mathfrak{G}_1 \to \mathcal{F} \to 0.$$

Definition of the Lie algebras $\mathfrak{h}_k$. In contrast to the Moyal deformation the chain of extensions of the Poisson algebra $\mathcal{F}$ can be restricted to the subalgebra $\mathcal{F}_{-1} = \mathfrak{h} \subset \mathcal{F}$ (of all homogeneous functions of degree 2), isomorphic to the Lie algebra of contact vector fields on $\mathbb{R}P^{2n+1}$. Then we obtain a series of Lie subalgebras $\mathfrak{h}_k \subset \mathcal{F}$. As a linear space each Lie algebra $\mathfrak{h}_k$ has the form

$$\mathfrak{h}_k \cong \mathcal{F}_{-1} \oplus \mathcal{F}_1 \oplus \mathcal{F}_3 \oplus \cdots \oplus \mathcal{F}_{2k-1}.$$

In fact the cocycle C_1,

$$C_1(F,G) = \frac{1}{6}\{F,G\}_3,$$

restricted to $\mathcal{F}_{-1}$, takes values in the $\mathcal{F}_{-1}$-module $\mathcal{F}_1$ (of functions on $\mathbb{R}^{2n+2}\setminus\{0\}$ that are homogeneous of degree -2), and so on. Using the conditions (3.6), we deduce by induction that the cocycle C_k, restricted to the Lie algebra $\mathfrak{h}_{k-1}$, takes values in $\mathcal{F}_{2k+1}$.

§4. Central extensions

The exact sequence of homomorphisms of Lie algebras

$$0 \to \mathbb{R} \to \mathfrak{A} \oplus \mathbb{R} \to \mathfrak{A} \to 0,$$

where the $\mathfrak{A}$-module structure on $\mathbb{R}$ is trivial, is called a *one-dimensional central extension of the Lie algebra* $\mathfrak{A}$.

Central extensions are the simplest extensions of Lie algebras. They appear both in geometry and in physics. Thus, they play an important role in symplectic geometry [20], [29] and in various versions of quantization (see also [23]).

We shall describe the construction of central extensions of the Lie algebras $\mathfrak{h}_m(\mathbb{R}P^{4k+1})$ for $m > k$.

4.1. Residue.

On the space $\mathcal{F}_{n+1}$ of functions on $\mathbb{R}^{2n+2}$ that are homogeneous of degree $-2n-2$, we define the linear functional

$$\mathrm{res} : \mathcal{F}_{n+1} \to \mathbb{R}.$$

Let $F \in \mathcal{F}_{n+1}$ and set

$$\mathrm{res}(F) = \int_{S^{2n+2}} F\alpha \wedge \omega^n, \tag{4.1}$$

where

$$\omega = \sum_{i=1}^{n+1} dp_i \wedge dq_i$$

is the symplectic form on $\mathbb{R}^{2n+2}$, and

$$\alpha = \frac{1}{2}\sum_{i=1}^{n+1}(p_i dq_i - q_i dp_i)$$

its primitive. The functional (4.1) is called the residue.

***Assertion* 4.1.** *The residue is the unique linear functional on the space $\mathcal{F}_{n+1}$ that is invariant under the action of the Lie algebra $\mathfrak{h}$.*

Remark 4.1. The space $\mathcal{F}_{n+1}$ is isomorphic as an $\mathfrak{h}$-module to the space of differential forms on the contact sphere S^{2n+1} of degree $2n+1$. In this language the residue is the integral of a differential form over the sphere.

4.2. Cocycles on the Lie algebra $\mathfrak{h}_m(\mathbb{R}P^{4k+1})$.
We recall that an element of the Lie algebra $\mathfrak{h}_m$ has the form

$$F = \sum_{i=0}^{k} F_i t^i,$$

where $F \in \mathcal{F}_{2i-1}$.

As in §3.4, we consider the Lie algebra $\mathcal{F}[[t]]$ of formal power series in t, whose coefficients are functions on $\mathbb{R}^{2n+2}$. The Moyal bracket makes $\mathcal{F}[[t]]$ into a Lie algebra.

First we define a 1-cocycle

$$\lambda : \mathcal{F}[[t]] \to \mathcal{F}[[t]].$$

We consider the radius-function on $\mathbb{R}^{2n+2}$,

$$r(x,y) = \sum_{i=1}^{n+1} (x_i^2 + y_i^2)$$

and set

$$\lambda(F) := \{F, \log r\}_t = \sum_{i=1}^{n+1} \{F_i, \log r\}_t \cdot t^i \tag{4.2}$$

for $F = \sum_{i=1}^{n+1} F_i t^i$.

***Proposition* 4.1.** *$\lambda(F)$ is a non-trivial cocycle on the Lie algebra $\mathcal{F}[[t]]$ with values in $\mathcal{F}[[t]]$.*

Now, using the 1-cocycle (4.2) we define a 2-cocycle on the Lie algebra $\mathfrak{h}_m$:

$$c : \mathfrak{h}_m \otimes \mathfrak{h}_m \to \mathbb{R}.$$

We set

$$c(F,G) = \operatorname{res}\left(F \cdot \{G, \log r\}_t\right), \tag{4.3}$$

where the residue is taken of the projection of $F \cdot \{G, \log r\}_t$ onto $\mathcal{F}_{n+1}$. In other words, in components

$$c(F_i t^i, G_j t^j) = \sum_{i+j+l-1=n/2} \frac{1}{(2l+1)!} \int_{S^{2n+1}} F_i \{G_j, \log r\}_{2l+1} \alpha \wedge \omega. \tag{4.4}$$

***Theorem* 4.1.** *The functional c defines a non-trivial 2-cocycle on the Lie algebra $\mathfrak{h}_m$.*

Proof. We shall first show that the functional c is skew-symmetric.

***Lemma* 4.1.** *For arbitrary functions $F \in \mathcal{F}_\lambda$, $G \in \mathcal{F}_\mu$, where $\lambda+\mu-1 = n-m$,*

$$\operatorname{res}(\{F, G\}_m) = 0.$$

Proof of the lemma. We fix an affine plane $\Gamma \subset \mathbb{R}^{2n+2}$ by the condition $q_{n+1} = 1$.

We choose affine coordinates on Γ:

$$x_i = \frac{p_i}{q_{n+1}}, \quad y_i = \frac{q_i}{q_{n+1}}, \quad i = 1, \ldots, n, \qquad z = \frac{p_{n+1}}{q_{n+1}}.$$

Let Ω denote the restriction of the $(2n+1)$-form $\alpha \wedge \omega^n$ to Γ:

$$\Omega = \alpha \wedge \omega^n\big|_\Gamma.$$

The form Ω is the standard volume form on the plane Γ:

$$\Omega = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \wedge dz.$$

Suppose that the homogeneous functions F, G satisfy the condition $\{F, G\}_m \in \mathcal{F}_{n+1}$ (this means that $F \in \mathcal{F}_\lambda$, $G \in \mathcal{F}_\mu$, and $\lambda+\mu+m = n+1$). Then the $(2n+1)$-form $\{F, G\}_m \cdot \alpha \wedge \omega^n$ is homogeneous of degree 0, and hence

$$\int_{S^{2n+1}} \{F, G\}_m \alpha \wedge \omega^n = 2 \int_\Gamma \{F, G\}_m \Omega$$

(restriction: $\{F, G\}_m|_\Gamma$ is a function that decreases like r^{-2n-2} on Γ and hence the integral on the right-hand side is well defined).

In order to show that this integral is identically zero (independently of the choice of the functions F and G), we use formula (2.8). We note first that all the terms in the integral

$$\int_\Gamma \{F, G\}_m \Omega$$

that contain derivatives with respect to $p_1, \ldots, p_n, q_1, \ldots, q_n$ vanish immediately (this follows directly from integration by parts). Secondly we note that this integral does not depend on the choice of the plane Γ. We fix a plane Γ' by the condition $q_1 = 1$. Then

$$\int_\Gamma \{F, G\}_m \Omega = \int_{\Gamma'} \{F, G\}_m \Omega.$$

But all the terms in the second integral that contain derivatives with respect to $p_2, \ldots, p_{n+1}, q_2, \ldots, q_{n+1}$ vanish. Thus, the whole expression is equal to zero and the lemma has been proved.

Now the skew-symmetry of the cocycle c can be easily established using the Leibniz identity.

We shall prove that the function c is a 2-cocycle:

$$c(F, \{G, H\}_t) + c(G, \{H, F\}_t) + c(H, \{F, G\}_t) = 0. \tag{4.5}$$

In fact, it is obvious that the functional λ defined by formula (4.2) is a 1-cocycle. Therefore

$$\lambda(\{F,G\}_t) = \{F,\lambda(G)\}_t - \{G,\lambda(F)\}_t.$$

We recall that $c(F, G) = \operatorname{res}(F, \lambda(G))$, from which (4.5) follows immediately.

It is obvious that the cocycle c is non-trivial, since there does not exist a homogeneous function G on $\mathbb{R}^{2n+2}$ such that $\lambda(F) = \{F, G\}_t$. This proves the theorem.

4.3. Residues of a pair of functions.

The cocycle $c(F, G)$ can be represented as the "residue of the pair of functions" F and G. Recall that if $F \in \mathcal{F}_\lambda$, $G \in \mathcal{F}_\mu$, where $\lambda+\mu+m = n+1$, then $\operatorname{res}\{F, G\}_m = 0$. Assume that $\lambda+\mu+m \neq n+1$. The bilinear mapping

$$R_m(F,G) = \frac{1}{\lambda+\mu+m-n-1}\int_{S^{2n+1}}\{F,G\}_m\alpha\wedge\omega^n$$

is defined.

Assertion **4.2.** *The functional R_m extends to the space $\mathcal{F}_\lambda \otimes \mathcal{F}_{n-m+1-\lambda}$, and is given in this case by the expression*

$$R_m(F,G) = -\int_{S^{2n+1}} F_1\{G_1, \log r\}_m\alpha\wedge\omega^n.$$

4.4. Extensions of the associative algebra of functions.

Let $K = C^\infty(\mathbb{R}P^{2n+1})$ be the commutative associative algebra of smooth functions on $\mathbb{R}P^{2n+1}$. The construction of extensions of K as an associative algebra will be analogous to the construction of extensions of the Lie algebra of contact vector fields $\mathfrak{h} = \mathfrak{h}(\mathbb{R}P^{2n+1})$.

Let $\mathcal{F}$ be the space of smooth functions on $\mathbb{R}^{2n+2}\setminus\{0\}$. As an $\mathfrak{h}$-module the algebra K is isomorphic to the space $\mathcal{F}_0 \subset \mathcal{F}$ of all homogeneous functions of degree of homogeneity 0. We consider the space $\mathcal{F}[[t]]$ of formal series in the variable t with coefficients in $\mathcal{F}$. The Moyal product (2.11) defines an associative algebra structure on $\mathcal{F}[[t]]$. We consider the series of associative algebras

$$\mathcal{F}_m = \mathcal{F}[[t]] \,/\, (t^{m+1} = 0).$$

By definition the algebra K_m is a subalgebra of $\mathcal{F}_m$ of the form

$$K_m = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_m$$

(thus the extensions of the algebra of functions are defined by the Moyal product).

The construction of central extensions is also analogous to the case of Lie algebras. The explicit formula for a 2-cocycle $a : K_m \otimes K_m \to \mathbb{R}$ has the following form:

$$a(F,G) = \operatorname{res}(F \cdot G *_t \log r), \tag{4.6}$$

or, in more detail, $F = \sum F_i t^i$, $G = \sum G_j t^j$, $F_i \in \mathcal{F}_i$, $G_j \in \mathcal{F}_j$,

$$\alpha(F_i t^i, G_j t^j) = \sum_{i+j+2m=n+1} \int_{S^{2n+1}} F_i\{G_j, \log r\}_m \alpha \wedge \omega^n. \tag{4.6'}$$

***Theorem* 4.2.** *The cocycle α defines a non-trivial central extension of the associative algebra K_m for $m > n$.*

To avoid overloading the text with technical details, we shall not analyze the case of the associative algebras K_m in detail. All the computations here are analogous to the case of the Lie algebras $\mathfrak{h}_m$. The general theory of Hochschild cohomology of associative algebras is contained, for example, in [58].

§5. Examples

5.1. The Virasoro algebra.

Consider the plane $(\mathbb{R}^2, dp \wedge dq)$. The Lie algebra $\mathfrak{h}$ in this case is isomorphic to the Lie algebra of all vector fields on S^1 ($\cong \mathbb{R}P^1$):

$$\mathfrak{h} = \mathcal{F}_{-1} \cong \operatorname{Vect} S^1.$$

The space $\mathcal{F}_\lambda$ (of all functions on $\mathbb{R}^2 \setminus \{0\}$ that are homogeneous of degree -2λ) is isomorphic to the space of tensor densities on S^1 of degree λ. The isomorphism is given by an explicit formula. A homogeneous function $F \in \mathcal{F}_\lambda$ has the form $F(p, q) = r^{-2\lambda} f(\tau)$, where (r, τ) are polar coordinates. Then the isomorphism between $\mathcal{F}_\lambda$ and tensor densities is constructed in the following way:

$$F \longleftrightarrow f(\tau)(d\tau)^\lambda. \tag{5.1}$$

Via this isomorphism the Poisson bracket on $\mathbb{R}^2$ defines an invariant differential operator on tensor densities. Let $f = f(\tau)(d\tau)^\lambda$, $g = g(\tau)(d\tau)^\mu$; then

$$\{\mathfrak{h}, g\}_1 = \left(\mu f' g - \lambda f g'\right)(d\tau)^{\lambda+\mu+1}. \tag{5.2}$$

In particular, for $\lambda = \mu = -1$ the operation (5.2) becomes the usual commutator of vector fields on S^1:

$$\left[f(\tau)\frac{d}{d\tau}, g(\tau)\frac{d}{d\tau}\right] = (fg' - f'g)\frac{d}{d\tau}.$$

The Lie algebra Vect S^1 has a unique non-trivial central extension, defined by the cocycle

$$c\left(f(\tau)\frac{d}{d\tau}, g(\tau)\frac{d}{d\tau}\right) = \int_{S^1} f(\tau) g'''(\tau)\, d(\tau), \tag{5.3}$$

which is called the *Gel'fand–Fuks cocycle* (see [14]). The Lie algebra obtained as a result of this central extension is called the Virasoro algebra.

There is no difficulty in verifying that formula (5.3) coincides with (4.3).

5.2. Extensions of the Virasoro algebra.
The construction of the Lie algebras $\mathfrak{h}_m$ and their central extensions in the one-dimensional case gives a series of extensions of the Virasoro algebra. The first non-trivial extension is constructed in the following way. We consider the space $\mathcal{F}_5(S^1)$, consisting of tensor densities of degree 5 of the form

$$f = f(\tau)(d\tau)^5$$

(where τ is a parameter on S^1). It is known that there exists a non-trivial extension of the Lie algebra Vect $S^1 = \mathfrak{h}_1$ in the module $\mathcal{F}_5$ (see [12]). We shall show that this extension arises as a subalgebra of the Lie algebra $\mathfrak{h}_5(S^1)$.

***Lemma* 5.1.** *The seventh higher Poisson bracket is given by the* 2*-cocycle*

$$\{\,,\,\}_7 : \mathfrak{h}_1 \otimes \mathfrak{h}_1 \to \mathcal{F}_5.$$

Let $F, G \in \mathcal{F}_{-1}$ be functions of degree of homogeneity 2 on the plane. Then it is easy to check that $\{F, G\}_3 = \{F, G\}_5 \equiv 0$. Hence, using the Jacobi identity (2.10) it follows immediately that the mapping

$$\alpha(F, G) = \{F, G\}_7$$

is a 2-cocycle.

In the language of vector fields the explicit formula has the shape

$$\alpha\left(f(\tau)\frac{d}{d\tau}, g(\tau)\frac{d}{d\tau}\right) = \left(f^{III}g^{IV} - f^{IV}g^{III}\right)(d\tau)^5. \tag{5.4}$$

Thus, an extension of the Virasoro algebra is defined:

$$\left[\begin{pmatrix} f(\tau)\frac{d}{d\tau} \\ \varphi(\tau)d\tau^5 \\ \lambda \end{pmatrix}, \begin{pmatrix} g(\tau)\frac{d}{d\tau} \\ \psi(\tau)d\tau^5 \\ \mu \end{pmatrix}\right] = \begin{pmatrix} [f, g] \\ L_f\psi - L_g\varphi + \alpha(f, g) \\ c(f, g) \end{pmatrix}, \tag{5.5}$$

where $L_f\psi = (f\psi' + 5f'\psi)d\tau^5$ is the Lie derivative of a tensor density of degree 5 along a vector field.

We now carry out a detailed construction of the Lie algebras $\mathfrak{h}_m$ in the one-dimensional case.

5.3. One-dimensional case: the Lie algebras $\mathfrak{h}_m(S^1)$ and their central extensions.
We need to find an explicit formula for the higher Poisson brackets on tensor densities on S^1.

***Proposition* 5.1.** *The higher Poisson* (*Lagrange*) *brackets on the tensor densities* $f = f(\tau)(d\tau)^\lambda$, $g = g(\tau)(d\tau)^\mu$ *are defined by the following formula*:

$$\{f, g\}_m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\prod_{i=k}^{m-1}(2\lambda + i)\prod_{j=m-k}^{m-1}(2\mu + j)f^{(k)}(\tau)g^{(m-k)}(\tau)(d\tau)^{\lambda+\mu+m}. \tag{5.6}$$

Example. The infinitesimal term in the Moyal bracket is equal to

$$\{f,g\}_3 = 2\lambda(2\lambda+1)(2\lambda+2)fg''' - 3(2\lambda+1)(2\lambda+2)(2\mu+2)f'g'' + 3(2\lambda+2)(2\mu+1)(2\mu+2)f''g' - 2\mu(2\mu+1)(2\mu+2)f'''g.$$

Thus, if $f, g \in \mathcal{F}_{-1}$, then $\{f, g\}_3 \equiv 0$. In this case it is also easy to compute that $\{f, g\}_5 \equiv 0$. We note that if $f, g \in \mathcal{F}_0$, then $\{f, g\}_1 \equiv 0$. We make one more observation: if $f \in \mathcal{F}_{-1}$, $g \in \mathcal{F}_0$, then $\{f, g\}_1 = fg'+gf'$. Then

$$\int_{S^1} \{f,g\}_1 = 0.$$

Remark 5.1. The Gel'fand–Fuks cocycle (5.3) is the "regularized" residue of $\{f, g\}_3$. In fact, let $F_\varepsilon = r^\varepsilon F$, $G_\varepsilon = r^\varepsilon G$ be homogeneous functions of degree $2+\varepsilon$ on $\mathbb{R}^2\setminus\{0\}$ such that $F, G \in \mathcal{F}_{-1}$ respectively are mapped to f, g under the isomorphism (5.1). Then

$$c(f,g) = \operatorname{res}\left(\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{F_\varepsilon, G_\varepsilon\}_3\right).$$

Now we shall give several examples.

The central extension $\widehat{\mathfrak{h}_1(S^1)}$ of the Lie algebra $\mathfrak{h}_1(S^1)$ defines a Lie algebra structure on the space $\mathcal{F}_{-1} \oplus \mathcal{F}_1 \oplus \mathbb{R}$. In the old notations we have:

$$[U_f, U_g] = U_{fg'-f'g} + z\int_{S^1} fg'''\,d\tau,$$

$$[U_f, V_a] = V_{fa'+f'a} + z\int_{S^1} fa'\,d\tau,$$

$$[V_a, V_b] = 0$$

(where $f, g \in \mathcal{F}_{-1}$, $a, b \in \mathcal{F}_1$). The Lie algebra $\widehat{\mathfrak{h}_1(S^1)}$ contains the Virasoro algebra as a subalgebra, and the space $\mathcal{F}_1 \oplus \mathbb{R}$ is a module over it.

The Lie algebra $\widehat{\mathfrak{h}_2(S^1)}$ is constructed analogously: it is also the semidirect product of the Virasoro algebra by some module. The remaining Lie algebras of the series $\widehat{\mathfrak{h}_m(S^1)}$ are *non-trivial extensions of the Virasoro algebra.*

We allow ourselves to describe the Lie algebra $\widehat{\mathfrak{h}_3(S^1)}$ explicitly; this is the first non-trivial extension of the Virasoro algebra from our series. As a linear space

$$\widehat{\mathfrak{h}_3(S^1)} = \mathcal{F}_{-1} \oplus \mathcal{F}_1 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathbb{R}.$$

We again introduce a convenient notation: the element $(f, a, x, k, z) \in \widehat{\mathfrak{h}_3(S^1)}$ is denoted by $U_f + V_a + W_x + X_k + z$ (we recall that $z \in \mathbb{R}$ is an element of the centre, $x = x(\tau)(d\tau)^3$ is a cubic differential, and $k = k(\tau)(d\tau)^5$ is a differential of degree 5). The commutator in the Lie algebra $\widehat{\mathfrak{h}_3(S^1)}$ is defined by the following explicit formulae:

$$
\begin{aligned}
&[U_f, U_g] = U_{fg'-f'g} + \frac{2}{35} X_{f^{\mathrm{III}} g^{\mathrm{IV}} - g^{\mathrm{III}} f^{\mathrm{IV}}} + z \int_{S^1} f g^{\mathrm{III}} \, d\tau, \\
&[U_f, V_a] = V_{fa'+f'a} + 2W_{f^{\mathrm{III}} a} + X_{10 f^{\mathrm{III}} a'' - 15 f^{\mathrm{IV}} a^{\mathrm{II}} + 6 f^{\mathrm{V}} a} + z \int_{S^1} f a' \, d\tau, \\
(5.7) \qquad &[U_f, W_x] = W_{fx'+3f'x} + 28 X_{f^{\mathrm{III}} x}, \\
&[U_f, X_k] = X_{fk'+5f'k}, \\
&[V_a, V_b] = W_{a'b-b'a} + 2X_{b^{\mathrm{III}} a - 3b^{\mathrm{II}} a' + 3b' a'' - b a^{\mathrm{III}}}; \\
&[V_a, W_x] = X_{3a'x - ax'}, \\
&[V_a, X_k] = [W_x, W_y] = [W_x, X_k] = [X_k, X_l] = 0.
\end{aligned}
$$

The formulae (5.7) define a Lie algebra structure. We note that the Lie algebra (5.5) is a subalgebra of the Lie algebra $\widehat{\mathfrak{h}_3(S^1)}$. We shall return to this Lie algebra in Appendix 2 and we shall write down the analogue of the Korteweg-de Vries equation corresponding to it.

Comments. The series of Lie algebras $\widehat{\mathfrak{h}_m(S^1)}$ is the simplest of the series of Lie algebras considered in this paper. While this is the simplest case, others better demonstrate the connection of the Virasoro algebra with Weyl quantization: all the Lie algebras $\widehat{\mathfrak{h}_m(S^1)}$ are constructed using the Moyal bracket and they are all direct extensions of the Virasoro algebra.

5.4. The Lie algebra $\mathfrak{h}_1$.
We consider the standard symplectic space $\mathbb{R}^{2n+2}$ ($n \geqslant 1$). The Lie algebra $\mathfrak{h}_1$ is defined by the extension

$$0 \to \mathcal{F}_1 \to \mathfrak{h}_1 \to \mathfrak{h} \to 0$$

by the cocycle $\gamma_1(G, G) = \frac{1}{6}\{F, G\}_3$.

We shall describe the commutator in $\mathfrak{h}_1$ explicitly. As a linear space $\mathfrak{h}_1 = \mathfrak{h}$ ($\cong \mathcal{F}_{-1}$) $\oplus$ $\mathcal{F}_1$. Let $(F, A) \in \mathfrak{h}_1$ (where $F \in \mathcal{F}_{-1}$, $A \in \mathcal{F}_1$). We introduce a convenient notation, that is, we denote the element (F, A) by $U_F + V_A$. Then

$$
\begin{aligned}
&[U_F, U_G] = U_{\{F,G\}_1} + \frac{1}{6} V_{\{F,G\}_3}, \\
(5.8) \qquad &[U_F, V_B] = V_{\{F,B\}_1}, \\
&[V_A, V_B] = 0.
\end{aligned}
$$

(The construction of the Lie algebra $\mathfrak{h}_1$ is the standard construction of an extension by a cocycle with values in a module over a given Lie algebra.) As we shall see below, in the case $n = 0$ the restriction of the cocycle γ_1 to the algebra $\mathfrak{h}$ is identically equal to zero.

5.5. The Lie algebra $\mathfrak{h}_2$.
We again consider the case $n \geqslant 1$. The algebra $\mathfrak{h}_2$ is the result of the extension

$$0 \to \mathcal{F}_3 \to \mathfrak{h}_2 \to \mathfrak{h}_1 \to 0.$$

As a linear space $\mathfrak{h}_2 = \mathcal{F}_{-1} \oplus \mathcal{F}_1 \oplus \mathcal{F}_3$. We shall denote the element (F, A, X) by $U_F + V_A + W_X$. The commutator in $\mathfrak{h}_2$ has the form

$$\begin{aligned}
[U_F, U_G] &= U_{\{F,g\}_1} + \frac{1}{6}V_{\{F,G\}_3} + \frac{1}{5!}W_{\{F,G\}_5},\\
[U_F, V_B] &= V_{\{F,B\}_1} + \frac{1}{6}W_{\{F,B\}_3},\\
[U_F, W_Y] &= W_{\{F,Y\}_1},\\
[V_A, V_B] &= W_{\{A,B\}_1},\\
[V_A, W_X] &= [W_X, W_Y] = 0.
\end{aligned}$$

5.6. Contact Virasoro algebra on S^5.
The Lie algebra $\widehat{\mathfrak{h}_2(S^5)}$ has a non-trivial central extension, defined by the 2-cocycle

$$\begin{aligned}
c(U_F, U_G) &= \frac{1}{5!}\int_{S^5} F\{\log r, G\}_5\alpha \wedge \omega^2,\\
c(U_F, V_B) &= \frac{1}{3!}\int_{S^5} F\{\log r, B\}_3\alpha \wedge \omega^2,\\
c(V_A, V_B) &= \int_{S^5} A\{\log r, B\}_1\alpha \wedge \omega^2,\\
c(V_A, W_Y) &= c(W_X, W_X) = 0.
\end{aligned} \tag{5.9}$$

Thus, the cocycle (5.9) defines a Lie algebra structure on the space

$$\widehat{\mathfrak{h}_2(S^5)} = \mathcal{F}_{-1} \oplus \mathcal{F}_1 \oplus \mathcal{F}_3 \oplus \mathbb{R}.$$

This Lie algebra will be called the contact Virasoro algebra on S^5. As in the case of S^1 a series of extensions of this Lie algebra by the module of tensor fields on S^5 is defined: these are the algebras $\widehat{\mathfrak{h}_m(S^5)}$ $(m > 2)$.

5.7. Associative algebras.
The first of the associative algebras K_1 is an extension of the algebra of functions on S^{2n+1} by the module of contact 1-forms. The cocycle defining the extension is the usual Poisson bracket.

The algebra K_2 is the extension of K_1 by the module $\mathcal{F}_2$:

$$0 \to \mathcal{F}_2 \to K_2 \to K_1 \to 0.$$

Thus, as a linear space $K_2 = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$. We denote the element $(F, A, X) \in K_2$ by $\Phi_F + \Psi_A + \mathrm{X}_X$. The product in the algebra K_2 is defined by the formula

$$\begin{aligned}
\Phi_F * \Phi_G &= \Phi_{\{F,G\}_0} + \Psi_{\{F,G\}_1} + \frac{1}{2}\mathrm{X}_{\{F,G\}_2},\\
\Phi_F * \Psi_B &= \Psi_{\{F,B\}_0} + \mathrm{X}_{\{F,B\}_1},\\
\Phi_F * \mathrm{X}_Y &= \mathrm{X}_{\{F,Y\}_0},\\
\Psi_A * \Psi_B &= \mathrm{X}_{\{A,B\}_0},\\
\Psi_A * \mathrm{X}_X &= \mathrm{X}_X * \mathrm{X}_X = 0
\end{aligned}$$

(recall that $\{\cdot,\cdot\}_0$ coincides with ordinary multiplication).

We consider the case S^3 (the space $\mathcal{F}_2$ will then coincide with the space of differential forms of highest degree). The cocycle defining a central extension of the algebra $K_2(\mathbb{R}P^3)$ looks like

$$\begin{aligned}
c(\Phi_F, \Phi_G) &= \frac{1}{2}\int_{S^3} F\{\log r, G\}_2 \alpha \wedge \omega,\\
c(\Phi_F, \Psi_B) &= \int_{S^3} F\{\log r, B\}_1 \alpha \wedge \omega,\\
c(\Psi_A, \Psi_B) &= \int_{S^3} AB\alpha \wedge \omega.
\end{aligned}$$

We now consider two examples of the associative algebras $\widehat{K_m(S^1)}$. The first of them is a one-dimensional central extension of the algebra of functions $C^\infty(S^1)$. Let $f, g \in C^\infty(S^1)$, $\alpha, \beta \in \mathbb{R}$. We introduce an operation on the space $C^\infty(S^1) \oplus \mathbb{R}$ (where $\mathbb{R}$ is the centre):

$$(5.10) \qquad (f,\alpha) * (g,\beta) = \left(f \cdot g, \int_{S^1} fg'\, d\tau\right).$$

This elementary operation defines the structure of an associative algebra, but not that of a commutative algebra.

We shall also describe the algebra $\widehat{K_2(S^1)}$:

$$\begin{aligned}
\Phi_f * \Phi_g &= \Phi_{fg} - \frac{1}{2}\mathrm{X}_{f'g'} + \int_{S^1} fg'\, d\tau,\\
\Phi_f * \Psi_a &= \Psi_{fa} - 2\mathrm{X}_{f'a} + \int_{S^1} fa\, d\tau,\\
\Phi_f * \mathrm{X}_x &= \mathrm{X}_{fx},\\
\Psi_a * \Psi_b &= \Psi_{ab},\\
\Psi_a * \mathrm{X}_x &= \mathrm{X}_x * \mathrm{X}_x = 0.
\end{aligned}$$

CHAPTER II

DEFORMATIONS OF THE POISSON BRACKET AND *-PRODUCT ON AN ARBITRARY SYMPLECTIC MANIFOLD

This chapter is devoted to the generalization of the construction of the *-product and the Moyal bracket to the case of an arbitrary symplectic manifold. The existence theorem for the *-product in the general case was proved by De Wilde and Lecomte [56] (see also [57]). We shall show that the cohomological questions arising here inevitably lead to the idea of a graded Lie algebra. The approach that we propose supplies a simple proof of the De Wilde–Lecomte theorem.

Throughout this chapter we consider a manifold V of dimension $2n$ on which a differential 2-form $\omega \in \Omega^2(V)$ is fixed. The manifold is said to be symplectic if $d\omega = 0$, $\omega^n \neq 0$ at each point of V. We denote by N the space $C^\infty(V)$ of all smooth functions on V and by $\mathfrak{A}(V)$ the Lie algebra of all smooth vector fields on V. Each function f on V is associated with a *Hamiltonian vector field* $X_f \in \mathfrak{A}(V)$, defined by the condition

$$i(X_f)\omega = df.$$

The Poisson bracket on V is defined by the following formula. Let $f, g \in N$. Then

$$\{f, g\} = \omega(X_f, X_g).$$

§6. Formal deformations: definitions

This section is not a detailed presentation of the theory of deformations of algebras. A detailed survey of this topic can be found in the important paper [16].

6.1. Let A be an associative algebra over a field $\varkappa$ of characteristic zero (we are interested in the cases $\varkappa = \mathbb{R}, \mathbb{C}$). Consider the algebra $A[[t]] \otimes_\varkappa \varkappa[[t]]$ of formal power series in the variable t with coefficients in A.

A *formal deformation* of A is a bilinear mapping

$$\mu_t : A \times A \to A[[t]],$$

$$\mu_t(a, b) = ab + \sum_{p=1}^{\infty} \mu_p(a, b) t^p,$$

such that its formal extension, also denoted by μ_t, is defined:

$$\mu_t : A[[t]] \times A[[t]] \to A[[t]],$$

$$\mu_t\left(\sum_{i=0}^{\infty} a_i t^i, \sum_{j=0}^{\infty} b_j t^j\right) = \sum_{k=0}^{\infty}\left(\sum_{p+q+r=k} \mu_p(a_q b_r)\right) t^k$$

(where $\mu_0(a, b) = ab$).

A formal deformation is *associative* if

$$\mu_t(\mu_t(a,b),c) = \mu_t(a,\mu_t(b,c)) \tag{6.1}$$

for any a, b, $c \in A$.

Two formal deformations μ_t and μ_t' of A are said to be *equivalent* if there exists a mapping

$$\varphi_t : A \to A[[t]], \qquad \varphi_t(a) = a + \sum_{p=1}^{\infty} \varphi_p(a)t^p,$$

such that

$$\mu_t'(\varphi_t(a),\varphi_t(b)) = \varphi_t(\mu_t(a,b)). \tag{6.2}$$

If $\mu_p \equiv 0$ for $p > k$, we say that μ_t is a *polynomial deformation of degree* k.

On the other hand, we can define a formal deformation by replacing formal series $\varkappa[[t]]$ by the quotient $\varkappa[[t]]/(t^{k+1} = 0)$. In this case we shall say that a deformation *of order* k is defined; in particular, for $k = 1$ this is an *infinitesimal* deformation.

In an analogous way we define the concept of a formal deformation of a Lie algebra.

Let $\mathfrak{A}$ be a Lie algebra (over $\varkappa$) with commutator $[\,,\,]$. We set $\mathfrak{A}[[t]] = \mathfrak{A} \otimes_\varkappa \varkappa[[t]]$. A *formal deformation* of $\mathfrak{A}$ is a mapping

$$\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}[[t]],$$
$$X \times Y \mapsto [X,Y]_t,$$

that is bilinear, antisymmetric, has the form

$$[\,,\,]_t = [\,,\,] + \sum_{p=1}^{\infty} c_p t^p,$$

and has an extension to $\mathfrak{A}[[t]]$:

$$\mathfrak{A}[[t]] \times \mathfrak{A}[[t]] \to \mathfrak{A}[[t]],$$

$$\left[\sum_{i=0}^{\infty} X_i t^i, \sum_{j=0}^{\infty} Y_j t^j\right]_t = \sum_{p=0}^{\infty}\left(\sum_{i+j+k=p} c_k(X_i,Y_j)\right) t^p$$

(with the condition $c_0(X, Y) = [X, Y]$).

It is required that the operation $[\,,\,]_t$ satisfy the Jacobi identity:

$$[[X,Y]_t,Z]_t + [[Z,X]_t,Y]_t + [[Y,Z]_t,X]_t = 0. \tag{6.3}$$

Analogously we can define the concepts of *polynomial deformation of degree* k, *formal deformation of order* k, and *infinitesimal deformation*. Two deformations $[\,,\,]$ and $[\,,\,]_t'$ are said to be *equivalent* if there exists a linear map

$$\Phi_t : \mathfrak{A} \to \mathfrak{A}[[t]], \qquad \Phi_t(X) = X + \sum_{p=1}^{\infty} \Phi_p(X)t^p,$$

such that

$$[\Phi_t(X), \Phi_t(Y)]'_t = \Phi_t([X,Y]_t). \tag{6.4}$$

A Lie algebra is said to be *rigid* if any deformation of it is equivalent to the trivial deformation.

There exists a connection between the two types of formal deformations discussed above. To each associative algebra A there is a corresponding Lie algebra $L(A)$: as a linear space $L(A)$ coincides with A, and the commutator has the form $[a, b] = ab-ba$. If μ_t is a formal deformation of A, then its antisymmetrization $[a, b]_t = \mu_t(a, b)-\mu_t(b, a)$ is a formal deformation of $L(A)$.

6.2. Deformations of the Poisson bracket and *-product.
For a symplectic manifold (V, ω) one can immediately define two structures on the space $N = C^\infty(V)$, an associative algebra structure (defined by multiplication of functions) and a Lie algebra structure (defined by the Poisson bracket). We consider the formal deformations of both these structures simultaneously.

We define the space of multilinear mappings C from N into N, satisfying the following properties:

(1) they are *local*: for arbitrary functions $f_1, ..., f_p \in N$ we have

$$\operatorname{Supp} c\ (f_1,\dots,f_p) \subset \bigcap_{i=1}^{p} \operatorname{Supp}(f_i);$$

(2) they *vanish on constants*: if there exists an i for which $f_i = \text{const}$, then $c(f_1, ..., f_i, ..., f_p) = 0$.

A theorem of Jaak Peetre asserts that any such mapping is defined by a multilinear differential operator: for any $x \in V$ there exists a local chart $U \ni x$ and functions $a_{i_1 \dots i_p}$ on U such that, for all functions f_i with support in U,

$$c(f_1,\dots,f_p)(x) = \sum_{0 \le i_j \le M} a_{i_1\dots i_p} \partial_{i_1} f_1(x) \dots \partial_{i_p} f_p(x),$$

where ∂_{i_j} are differential operators on U of order $i_j > 0$ for arbitrary $j = 1, ..., p$. (This assertion holds for arbitrary manifolds, symplectic or not.)

Definition 6.1. A formal deformation of the associative algebra N is called a **-product* if it is defined by a mapping

$$N \times N \to N[[t]]$$

of the form

$$f \times g \mapsto f \underset{t}{*} g = fg + \sum_{p=1}^{\infty} \mu_p(f,g)t^p,$$

satisfying the conditions:

(1) the mappings μ_p are local and are equal to zero on constants;

(2) $\mu_p(f, g) = (-1)^p \mu_p(g, f)$;

(3) $\mu_1(f, g) = -\mu_1(g, f) = \{f, g\}$.

Conditions (2) and (3) immediately allow us to define a formal deformation of the Poisson bracket:

$$\{f,g\}_t = \frac{1}{2t}(f \underset{t}{*} g - g \underset{t}{*} f) = \{f,g\} + \sum_{p=1}^{\infty} \mu_{2p+1}(f,g)t^p.$$

The Jacobi identity for the operation $\{f, g\}_t$ is derived immediately from the associativity condition (4.2) for the operation $f \underset{t}{*} g$.

Remark 6.1. There exists a general algebraic definition of the concept of *-product. Let A be a Poisson algebra (that is, an associative algebra equipped with a Lie structure $a \otimes b \to \{a, b\}$ satisfying the Leibniz identity $\{a, bc\} = \{a, b\}c + b\{a, c\}$). Then a *-product is defined as a formal deformation of the associative multiplication in A which satisfies conditions (2) and (3). This definition is often called a quantization of the Poisson algebra A.

The Moyal product (2.10) is an example of a *-product, defined on $\mathbb{R}^{2n+2}$ with the standard symplectic structure. A significant fact if that *locally the Moyal product is a *-product on any symplectic manifold V* (up to the equivalence defined above). For any point $x \in V$ there exists a chart U with Darboux coordinates on it (the symplectic form

$$\omega\big|_U = \sum_{i=1}^{n} dp_i \wedge dq_i$$

in these coordinates) such that the *-product on U is equivalent to the Moyal product (see [33]). We see that this result can be used for the proof of an existence theorem for the *-product on any symplectic manifold.

The natural question—is a deformation of the Poisson bracket always connected with some *-product?—has a positive answer (at least, for an important special case).

Definition 6.2. The *Vey bracket* [54] is a formal deformation of the Poisson bracket, for which the operations C_p are defined by bilinear differential operators of order at most $2p+1$.

***Proposition* 6.1** (Lichnerowicz [33]). *For any Vey bracket on a symplectic manifold there exists a *-product such that the Vey bracket is its anti-symmetrization.*

Before we proceed to a systematic study of formal deformations of the Poisson bracket and *-product, we consider conditions (6.1) and (6.2) (existence conditions) as a first approximation. Isolating the terms of order 1 (the coefficients of t), we obtain

$$\mu_1(ab,c) + \mu_1(a,b)c = \mu_1(a,bc)$$
$$+ a\mu_1(b,c) \circlearrowleft C_1([X,Y],Z) + \circlearrowleft [C_1(X,Y),Z] = 0,$$

where the symbol $\circlearrowleft$ denotes the cyclic permutation of X, Y, Z. Exactly the same conditions for the equivalence of two formal deformations give the following equalities in the first order:

$$\mu_1(a,b) - \mu_1'(a,b) = a\varphi_1(b) + \varphi_1(a)b - \varphi_1(a,b),$$
$$C_1(X,Y) - C_1'(X,Y) = [X, \Phi_1(Y)] + [\Phi_1(X), Y] - \Phi([X,Y])$$

respectively.

From this one can immediately obtain cohomological conditions on the infinitesimal terms of formal deformations: *the operation* μ_1 *is a* 1*-cycle in the Hochschild complex of the algebra* A, and for equivalent deformations $\mu_1 - \mu_1'$ is a 1-*coboundary* (that is, the cocycles μ_1 and μ_1' define the same Hochschild cohomology class of A). Analogously, C_1 is a cocycle and $C_1 - C_1'$ is a coboundary in the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{A}$.

A convenient method of working with similar cohomological questions connected with formal deformations leads to graded Lie algebras, which will be the subject of the next section.

§7. Graded Lie algebras as a means of describing deformations

In this section we give a brief description of the technique proposed recently by De Wilde and Lecomte. An explicit presentation of the details can be found in [51] and [57].

7.1. Graded Lie algebras and associated structures.

Definition 7.1. A *graded Lie algebra* is a space equipped with a grading $L^* = \oplus_{i\in\mathbb{Z}} L^i$ and a bilinear operation

$$[\cdot,\cdot] : L^* \otimes L^* \to L^*,$$

satisfying the following conditions:

$$(1) \qquad A \in L^a, B \in L^b \Rightarrow [A,B] \in L^{a+b},$$
$$(2) \qquad A \in L^a, B \in L^b \Rightarrow [A,B] = (-1)^{ab+1}[B,A]$$

(antisymmetry);

$$(3) \qquad A \in L^a, B \in L^b, C \in L^c \Rightarrow \sum_{(a,b,c)} (-1)^{ac}[[A,B],C] = 0$$

is the graded Jacobi identity.

Remark 7.1. If $\mathbb{Z}$ is replaced by $\mathbb{Z}_2$ in the definition, then we obtain the definition of a Lie superalgebra.

Definition 7.2. An *associated structure* on a graded Lie algebra L^* is an element $c \in L^1$ that satisfies the condition $[c, c] = 0$.

We define the cohomology related to this object. We consider the operator

$$\partial_c : L^* \to L^*, \qquad \partial_c(X) = [c, X].$$

It is easy to check that $\partial_c \cdot \partial_c = 0$ (which follows from the Jacobi identity). This allows us to consider the cohomology space $H_c^*(L)$, defined in the usual way:

$$H_c^p(L) = \operatorname{Ker} \partial_c|_{L^p} / \operatorname{Im} \partial_c|_{L^{p-1}}.$$

Moreover, the commutator in the algebra L^* can be restricted to cocycles, which turns the space $H_c^*(L)$ into a *graded Lie algebra.*

This formalism turns out to be suitable for the study of deformations: we consider a deformation of $c \in L^1$, which is a family $c_t \in L^1$, depending on a parameter t, with the conditions $c_0 = c$, $[c_t, c_t] = 0$. It is possible to obtain different variants of deformations by imposing different conditions on the dependence of c_t on the parameter. In this paper we shall consider *formal deformations* $c_t \in L^1[[t]] = L^1 \otimes_\varkappa \varkappa[[t]]$ and understand the condition

$$[c_t, c_t] = 0 \tag{7.1}$$

as an identity in the algebra $L^*[[t]] = L^* \otimes_\varkappa \varkappa[[t]]$. It is not complicated to obtain a concept of a formal deformation of order k, replacing $L^*[[t]]$ by $L^*[[t]]/(t^{k+1} = 0)$ (a deformation of order $k = 1$ is termed "infinitesimal"). In exactly the same way we can consider polynomial deformations and analytic deformations, as was done above.

It is natural to assume a deformation c_t to be *trivial* if it is obtained by a "change of coordinates" in L^1. More precisely, we consider a family $\Phi_t \in L^0$, $\Phi_0 = Id$, and the deformation

$$c_t = [c, \Phi_t]. \tag{7.2}$$

Condition (7.1) gives a chain of identities

$$\begin{gathered}[c, c_1] = 0,\\ 2[c, c_2] + [c_1, c_1] = 0,\\ \dots\dots\dots\dots\dots\dots\end{gathered} \tag{7.1'}$$

while condition (7.2) gives

$$\begin{gathered}c_1 = [c, \Phi_1],\\ c_2 = [c, \Phi_2],\\ \dots\dots\dots\end{gathered} \tag{7.2'}$$

The first identity of (7.1′) means that c_1 is a 1-cocycle relative to ∂_c. The first identity of (7.2′) means that this cocycle is trivial.

Thus, infinitesimal deformations correspond to the cohomology classes c_1 in $H_c^1(L)$. The second equality of (7.1′) can be written in the form $\partial_c(c_2) = \frac{1}{2}[c_1, c_1]$. Thus, the cohomology class of $[c_1, c_1]$ in $H^2(L)$ is an obstruction to the extension of an infinitesimal deformation to a deformation of order 2. The succeeding identities of (7.1′) can be written in the form

$$\partial_n(c) = P_n(c_1, \dots, c_{n-1}),$$

where $P_n(c_1, ..., c_{n-1})$ is an element of L^2. For example, $P_3(c_1, c_2) = \frac{1}{2}[c_1, c_2] + \frac{1}{2}[c_2, c_1]$, and so on. It is not hard to see that this element is a cocycle. Thus we shall successively obtain a series of obstructions to the extension of an infinitesimal deformation to a formal deformation.

Thus, if the space $H_c^1(L)$ classifies infinitesimal deformations, then the computation of obstructions to extension consists in studying the commutator on cohomology:

$$[\,,\,] : H_c^1(L) \otimes H_c^1(L) \to H_c^2(L).$$

In particular, if $H_c^1(L) = 0$, then all deformations are equivalent to a trivial deformation; if $H_c^2(L) = 0$, then any infinitesimal deformation extends to a formal deformation.

7.2. Graded cohomology.
There are cases when a formal deformation exists for any associated structure $c \in L^1$ and we can speak about a universal deformation. We need the idea of graded cohomology of the algebra L^*.

Let $C_{gr}^q(L^*, L^*)$ be the space of q-linear forms of cochains on L^* with values in L^*, satisfying the following condition (of skew-symmetry):

$$c(\dots, A, \dots, B, \dots) = (-1)^{ab+1} c(\dots, B, \dots, A, \dots).$$

We define a grading on this space: the number γ is the degree of the cochain $c \in C_{gr}^q(L^*, L^*)$ if

$$\deg c(A_1, \dots, A_q) = \gamma + \sum_{i=1}^{q} \deg(A_i).$$

The definition of the differential $d_q : C_{gr}^q(L^*, L^*) \to C_{gr}^{q+1}(L^*, L^*)$ is analogous to the case of Lie algebras:

$$\begin{aligned} dc(A_1, \dots A_{q+1}) = & \sum_{1 \le r \le q+1} (-1)^{r + \sum_{i>r} a_i a_r} A_r c(A_1, \dots, \widehat{A}_r, \dots, A_{q+1}) \\ & + \sum_{1 \le r < s \le q+1} (-1)^{r+s-1+a_{rs}} c([A_r, A_s], A_1, \dots, \widehat{A}_r, \dots, \widehat{A}_s, \dots, A_{q+1}), \end{aligned}$$

where

$$a_{rs} = \sum_{i<r} a_i a_r + \sum_{j<s, j \neq r} a_j a_s.$$

The only modification is a rule according to which the coefficient $(-1)^{ab+1}$ arises in a permutation A of degree a and B of degree b. This cohomology was proposed by Braconnier and Leites [32], who only considered the case of Lie superalgebras. A second grading on the cohomology space $H_{gr}^q(L^*)$ is defined by the grading in the algebra L^*. The corresponding bigraded cohomology space is denoted by $H_{gr}^q(L^*)_r$.

For every associated structure c on L^* we define a mapping

$$\bigoplus_{p\in\mathbb{Z}} H^p_{gr}(L^*)_{r-p} \xrightarrow{\gamma^r_c} H^r_c(L),$$

associating

$$\gamma^r_c(\omega) = \omega(c,\dots,c)$$

with each cocycle $\omega \in H^p_{gr}(L^*)_{r-p}$. A direct check shows that this mapping acts on cohomology (recall that $c \in L$ and $[c, c] = 0$).

In particular, for $r = 1$ we obtain

$$\bigoplus_{p\in\mathbb{Z}} H^p_{gr}(L^*)_{1-p} \xrightarrow{\gamma^1_c} H^1_c(L).$$

The main result of this construction that we need is the following.

***Assertion* 7.1** (see [50], [51], [56], [57]). *The cohomology classes of the image of γ^1_c induce infinitesimal deformations of associated structures on L^*, which always extend to formal deformations.*

In this case we can speak about universal deformations. This result also proves the fact that the image of γ^r_c is contained in the centre of the Lie algebra $H^*_c(L)$ (see [30] for the details). An example of this construction can be found in [50].

We give two examples of graded Lie algebras, which will be used in what follows (see [51] for subsequences and a set of analogous examples).

7.3. The algebra $M^*(E)$ (see [57]).
Let E be a linear space, and $M^p(E)$ the space of $(p+1)$-linear mappings from $E^{\otimes p}$ into E. In this case,

$$M^*(E) = \bigoplus_{p=-1}^{\infty} M^p(E)$$

(by convention $M^{-1}(E) = E$). Let $A \in M^a(E)$, $B \in M^b(E)$. We define $i(A)B \in M^{a+b}(E)$ by the formula

$$i(A)B(x_1,\dots,x_{a+b}) = \sum_{k=0}^{b}(-1)^{ka}B(x_0,\dots,x_{k-1},A(x_k,\dots,x_{k+a}),\dots,x_{a+b}).$$

We set $A\Delta B = i(A)B+(-1)^{ab+1}i(B)A$. We obtain a graded Lie algebra structure on $M^*(E)$.

Let $c \in M^1(E)$ be a bilinear mapping from $E \otimes E$ into E. Then

$$c \,\Delta\, c(x_0,x_1,x_2) = 2\,[c(c(x_0,x_1),x_2) - c(x_0,c(x_1,x_2))]\,.$$

Therefore $c\Delta c = 0$ if and only if c defines an associative multiplication on E.

Thus, the associated structures on $M^*(E)$ are the associative algebra structures on the space E. The formalism given above, applied to the algebra $M^*(E)$, gives a deformation theory of associative algebra structures on E.

We denote by A the Lie algebra defined by an associative multiplication c on E. The cohomology of $M^*(E)$ associated with c is identical to the Hochschild cohomology of A with coefficients in the adjoint representation. As a result we obtain (after re-indexing) the following isomorphism:

$$H_c^p(M^*(E)) = HH^{p+1}(A, A).$$

The role of Hochschild cohomology in the deformation theory of associative algebras is well known thanks to the work of Gerstenhaber (see [15], [16]). A detailed study of Hochschild homology and cohomology can be found in the book of Loday [35] concerning cyclic cohomology.

7.4. The Richardson–Nijenhuis algebras $A^*(E)$.
Again let E be a linear space. We denote by $A^p(E)$ the space of skew-symmetric $(p+1)$-linear mappings from $E^{\otimes(p+1)}$ into E. We consider the space

$$A^*(E) = \bigotimes_{p=-1}^{\infty} A^p(E),$$

where $A^{-1}(E) = E$. A Lie algebra structure on $A^*(E)$ can be defined via the Lie structure on $M^*(E)$. Let $\alpha : M^*(E) \to A^*(E)$ be the antisymmetrization. If $A \in A^a(E)$ and $B \in A^b(E)$, we set

$$[A, B] = \frac{(a+b+1)!}{(a+1)!\,(b+1)!}\alpha(A \vartriangle B).$$

All the properties of a graded Lie algebra are easily verified. In particular, if $c \in A^1(E)$, then $[c, c] \in A^2(E)$ is defined by the formula

$$[c, c](X, Y, Z) = 2\circlearrowleft C(c(X, Y), Z).$$

The equality $[c, c] = 0$ means in this case that the bilinear antisymmetric operation c satisfies the Jacobi identity. The associated structures on $A^*(E)$ therefore coincide with the Lie algebra structures on E. We now see that the cohomology of a Lie algebra with values in the coadjoint representation is well adapted to the study of its deformations. Indeed, let $\mathfrak{A}$ be a Lie algebra defined on the space E via a structure c satisfying $[c, c] = 0$. We will obtain an isomorphism

$$H_c^p(A^*(E)) = H^{p+1}(\mathfrak{A}, \mathfrak{A}).$$

The connection between deformations of a Lie algebra and its cohomology was discovered in the fundamental papers of Richardson and Nijenhuis (see, for example, [42]). (Some authors denote the operation $X \to [X, X]$ from $H_c^1(A^*(E))$ into $H_c^2(A^*(E))$ by S_q.)

In cases when E is infinite-dimensional we usually need to consider some topology on the spaces of mappings. We are interested in the case when $E = N$, where N is the space of smooth functions on some manifold V. We require that the multilinear mappings from E to E satisfy the locality property that was defined above. The graded Lie algebras defined on these spaces are denoted respectively by $A^*_{\mathrm{loc}}(N)$ and $M^*_{\mathrm{loc}}(N)$. (It is easy to check that the locality property is preserved under commutation in the graded Lie algebras.) For technical reasons we shall also consider operations from $N^{\otimes p}$ into N, vanishing on constants (see above) and corresponding to multilinear differential operators. The corresponding graded Lie algebras are denoted by $A^*_{\mathrm{loc},nc}(N)$ and $M^*_{\mathrm{loc},nc}(N)$. The spaces $A^0_{\mathrm{loc}}(N) = M^0_{\mathrm{loc}}(N)$ are identified with algebras of differential operators on the manifold V. (We note that some authors have considered other structures on these spaces, in particular, Hopf algebra structures.)

§8. Cohomology computations and their consequences

Here we give some results relating to the computation of cohomology, which will be used to study deformations of the product and Poisson bracket on the space of functions on a symplectic manifold.

8.1. Hochschild cohomology of the algebra of functions.
We return to the study of the algebra of functions $N = C^\infty(V)$ on a manifold V. Let c denote the product structure in N. The study of local deformations of c is connected with the computation of the spaces

$$H^p_c(M^*_{\mathrm{loc}}(N)) = HH^{p+1}_{\mathrm{loc}}(N,N) \quad \text{for} \quad p=1,2.$$

(The Hochschild cohomology considered here is also local: it is constructed from cochains satisfying the locality condition.) We let $\Omega_P(V)$ denote the space of covariant tensor fields of degree p on V.

***Theorem* 8.1.** $HH^p_{\mathrm{loc}}(N, N) = \Omega_p(V)$.

This theorem was proved independently by Hochschild, Kostant and Rosenberg (see [35]). It is valid for an arbitrary manifold V. We note that these authors formulated the theorem in a different form. In the form given above the theorem was rediscovered independently by Cohen, Gutt and De Wilde.

Let Λ be a skew-symmetric tensor of degree p on V (a p-vector field). We define a Hochschild p-cochain on N by the formula

$$\tilde{\Lambda}(f_1,\dots,f_p) = \langle \Lambda, df_1 \wedge \cdots \wedge df_p \rangle .$$

This defines a mapping $\Omega_p(V) \to CH^p_{\mathrm{loc}}(N, N)$. It is easy to check that $\tilde{\Lambda}$ is a cocycle. The theorem means that the corresponding cohomology classes contain all the Hochschild cohomology. On the other hand, this means that

$$H^*_c(M^*_{\mathrm{loc}}(N)) = H^*_c(M^*_{\mathrm{loc},nc}(N)).$$

Remark 8.1. The graded Lie algebra structure on $M^*(N)$ induces a graded Lie algebra structure on $\Omega_*(V)$, which coincides with the Schouten–Nijenhuis bracket on $\Omega_*(V)$.

In the case when V is a Poisson manifold, we can naturally take Λ to be a bivector defining the Poisson bracket: $\{f, g\} = \langle\Lambda, df \wedge dg\rangle$. Then $\Lambda \in \Omega_2(V)$ (and has degree 1 relative to the grading in $M^*(N)$). The resulting cohomology, associated with Λ, is called Λ-cohomology (it was proposed independently by Brylinski and Lichnerowicz). In the case when Λ has maximal rank, which corresponds to the Poisson bracket on a symplectic manifold, Λ-cohomology reduces to ordinary cohomology of the manifold V.

8.2. Cohomology of the Poisson Lie algebra on a symplectic manifold. Let (V, ω) be a symplectic manifold, and $c : f \otimes g \to \{f, g\}$ the Poisson bracket on it. We regard c as an element of the graded Lie algebra $A^*_{\mathrm{loc},nc}(N) : c \in A^1_{\mathrm{loc},nc}(N)$, where $[c, c] = 0$. The associated cohomology coincides with the cohomology of the Lie algebra N with values in N (see [12]):

$$H^p_c(A^*_{\mathrm{loc},nc}(N)) = H^{p+1}_{\mathrm{Lie}}(N, N).$$

This cohomology in turn coincides with the cohomology of the Lie algebras of Hamiltonian vector fields on V, having a unique Hamiltonian (exact Hamiltonian fields). In fact, the Lie algebra N splits into a direct sum: $N = \mathrm{Ham}(V) \oplus \mathbb{R}$, where $\mathrm{Ham}(V)$ is the Lie algebra of all exact Hamiltonian fields on V and $\mathbb{R}$ is the space of constant functions. (For the basics of the cohomology theory of algebras of vector fields see [12].) The problem of computing the cohomology of the Lie algebra $\mathrm{Ham}(V)$ in the general case is as yet unsolved. Here we give computations of $H^i(N, N)$ for $i \leqslant 3$ and a geometric interpretation of all the cohomology classes that we find.

We note that $H^0(N, N) = \mathbb{R}$ corresponds to the space of constant functions.

The space $H^1(N, N)$ is also easy to compute. The derivations of the Lie algebra N are the vector fields that commute with Λ. They satisfy the condition $L_X\omega = 0$ and thus derivations of N correspond to Hamiltonian vector fields on V. The exact sequence

$$0 \to \mathrm{Ham}(V) \to \mathrm{Vect}(V, \omega) \to H^1_{\mathrm{DR}}(V) \to 0$$

(where $\mathrm{Vect}(V, \omega)$ is the Lie algebra of Hamiltonian fields on V; the third mapping has the form $X \mapsto [i_X\omega]$) immediately gives the answer:(1)

$$H^1(N, N) = H^1_{\mathrm{DR}}(V).$$

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(1) The authors thank the referee for pointing out the paper "On Poisson manifolds and the Schouten bracket" (Funktsional. Anal. i Prilozhen. **22**:1 (1988), 1–11. MR **89k**:58011. = Functional Anal. Appl. **22** (1988), 1–9) by Yu.M. Vorob'ev and M.V. Karasev, and also the book "Non-linear Poisson brackets. Geometry and quantization" (in Russian) Nauka, Moscow 1990, by Karasev and V.P. Maslov, in which analogous results were obtained.

This equality can be generalized. We consider the inclusion of the space of differential p-forms on V

$$\Omega^p(V) \hookrightarrow A^p_{\mathrm{loc},nc}(N),$$

defined as follows: with each p-form σ one associates a cochain $\widetilde{\sigma}(f_1,\dots,f_p) = \sigma(X_{f_1},\dots,X_{f_p})$. A well-known property of the Lie algebra of Hamiltonian vector fields is the formula

$$d\widetilde{\sigma} = [c,\widetilde{\sigma}]$$

which allows us to obtain an inclusion $H^p_{\mathrm{DR}}(V) \hookrightarrow H^p(N, N)$.

***Proposition* 8.1** (Gutt, Lecomte, and De Wilde).

$$H^2_{\mathrm{Lie}}(N,N) = H^2_{\mathrm{DR}}(V) \oplus \mathbb{R}.$$

The one-dimensional space complementing the de Rham cohomology is generated by an additional class, which by tradition is denoted by S^3_Γ. We describe it in detail. We denote by $P(V) \xrightarrow{\pi} V$ the frame bundle over the manifold V. We fix an arbitrary linear connection θ in this bundle. Let X, Y be vector fields on V, and $\widetilde{X}$, $\widetilde{Y}$ horizontal vector fields on $P(V)$ (which are obtained by lifting X, Y). The differential 2-form on $P(V)$ of the form $T_r(L_{\widetilde{X}}\theta \wedge L_{\widetilde{Y}}\theta)$ is projected onto the base (this fact is easy to check). The mapping

$$(X,Y) \mapsto \Phi_\Gamma(X,Y) = T_r(L_{\widetilde{X}}\theta \wedge L_{\widetilde{Y}}\theta) \in \Omega^2(V)$$

defines a 2-cocycle on the Lie algebra $\mathrm{Vect}(V)$ of all vector fields on V with values in the space $\Omega^2(V)$ (which is a $\mathrm{Vect}(V)$-module relative to the operation of Lie derivative). The cohomology class $[\Phi_\Gamma]$ in $H^2(\mathrm{Vect}(V), \Omega^2(V))$ does not depend on the choice of the connection θ.

Now if V is a manifold on which a Poisson bracket is defined, then we can define a 2-cocycle on the Lie algebra N with values in N:

$$S^3_\Gamma(f,g) = \langle \Phi_\Gamma(X_f, X_g), \Lambda \rangle . \tag{8.1}$$

The corresponding cohomology class of $H^2_{\mathrm{Lie}}(N, N)$ is also denoted by S^3_Γ. This cohomology class is defined by any Poisson manifold V without any assumption of the non-degeneracy of the Poisson structure. It is an amusing exercise to check that

$$S^3_\Gamma(f,g) = \{f,g\}_3,$$

in the case when V is the standard symplectic space, that is, S^3_Γ coincides with the cocycle defining the Moyal deformation $(\cdot)$ on the infinitesimal level.

This class admits a generalization of arbitrarily high degree. If $T : \mathrm{gl}(n, \mathbb{R})^{\otimes p} \to \mathbb{R}$ is an invariant skew-symmetric mapping, we set

$$(X_1,\dots,X_p) \mapsto T(L_{\widetilde{X}_1}\theta \wedge \cdots \wedge L_{\widetilde{X}_p}\theta).$$

Thus, we define a cohomology class in

$$H^p(\mathrm{Vect}(V), \Omega^p(V)).$$

This cohomology class (in a somewhat different form) was proposed by Gel'fand in his 1970 International Congress talk at Nice [13].

The associative multiplication on N defines a product on cohomology:

$$H^p(N,N) \otimes H^q(N,N) \to H^{p+q}(N,N). \tag{8.2}$$

In particular, multiplication by $S_\Gamma^3 \in H^2(N, N)$ gives a mapping of $H^1(N, N) = H^1_{\mathrm{DR}}(V)$ into $H^3(N, N)$.

The complete answer for the space $H^3(N, N)$ depends on the value of the first Pontryagin class of the manifold V, $p_1(V) \in H^4_{\mathrm{DR}}(V)$.

***Proposition* 8.2.**

$$H^3(N,N) = H^1_{\mathrm{DR}}(V) \oplus H^3_{\mathrm{DR}}(V) \qquad \text{if } p_1(V) \neq 0,$$
$$H^3(N,N) = H^1_{\mathrm{DR}}(V) \oplus H^3_{\mathrm{DR}}(V) \oplus \mathbb{R} \quad \text{if } p_1(V) = 0.$$

The cohomology class corresponding to the one-dimensional space in the case when $p_1(V) = 0$ is obtained by transgression (see [57] for a detailed proof and construction).

Analogous questions in the case of an arbitrary Poisson manifold have been considered by Maslov and his students (see the footnote above and also [50]). The computations for the case of a linear Poisson structure on a dual space to a Lie algebra were carried out in the dissertation of Melotte [40].

We again consider the Richardson–Nijenhuis bracket:

$$\begin{gathered} H^2(N,N) \otimes H^2(N,N) \to H^3(N,N), \\ c_1' \otimes c_2 \mapsto [[c_1, c_2]]. \end{gathered} \tag{8.3}$$

It is easy to verify that if $c_i = [\widetilde{\omega}_i]$ for $\omega_i \in \Omega^2(V)$, then $[[c_1, c_2]] = [\{\widetilde{\omega}_1, \widetilde{\omega}_2\}]$, where $\{\omega_1, \omega_2\}$ is the graded Lie algebra bracket on differential forms, defined for all Poisson manifolds. Thus we shall show that the image of the mapping (8.3) lies in the component $H^3_{\mathrm{DR}}(V) \hookrightarrow H^3(N, N)$. This proves a result of Vey [54].

***Proposition* 8.3.** *If $H^3_{\mathrm{DR}}(V) = 0$, then any infinitesimal deformation of the Poisson bracket can be extended to a formal deformation.*

In the case of the standard symplectic space $\mathbb{R}^{2n+2}$, $H^2(N, N) = \mathbb{R}$, $H^3(N, N) = 0$, which immediately proves Theorem 2.2 on the uniqueness of the Moyal deformation. The same property remains valid for the Moyal product.

***Proposition* 8.4** ("Quantum Darboux theorem"). *Any *-product on the standard symplectic space* $\mathbb{R}^{2n+2}$ *is equivalent to the Moyal product.*

We now proceed to a detailed study of deformations of the associative product of functions on a symplectic manifold.

§9. Existence of a *-product

We give two approaches to the solution of this problem.

9.1. Construction via a covering.
The idea is very natural: cover a symplectic manifold by Darboux charts, take the Moyal product on each chart and, using the preceding proposition, glue each pair of Moyal products on each intersection.

Let $(U_\alpha, \varphi_\alpha)$ be a covering of V by Darboux charts such that all the intersections $U_{\alpha_1 \dots \alpha_k} = U_{\alpha_1} \cap \dots \cap U_{\alpha_k}$ are contractible. We denote by $N_t(U_\alpha)$ the algebra of formal series in the variable t with coefficients in the algebra $N(U_\alpha)$ of smooth functions on U_α. Let M_α denote the Moyal product on $N_t(U_\alpha)$ and P_α a formal deformation of the Poisson bracket that is compatible with M_α.

There is an isomorphism

$$T_{\alpha\beta} : (N_t(U_{\alpha\beta}, P_\alpha) \to (N_t(U_{\alpha\beta}), P_\beta),$$

and it can be chosen so that $T_{\alpha\beta} = Id + tT'_{\alpha\beta}$, where $T'_{\alpha\beta} \in A'_{loc,nc}(N(U_{\alpha\beta})[[t]])$. In order to define a globally defined deformation, it is necessary that $T_{\alpha\beta}$ satisfy the condition $T_{\alpha\beta}T_{\beta\gamma}T_{\gamma\alpha} = Id$ (Čech cocycle). A direct attempt to satisfy this condition a priori encounters an obstruction in the third (Čech) cohomology group of the manifold V. (This obstruction was discovered by Vey; see Proposition 8.3 above.) A method of overcoming these difficulties, based on an idea of Omori et al, was proposed by Lecomte and De Wilde. The idea consists in the study of automorphisms of the Lie algebra $(N_t(U_\alpha), P_\alpha)$ and its derivations (as before we restrict ourselves to derivations satisfying the locality condition).

Definition 9.1. A mapping $T : N_t(U_\alpha) \to N_t(U_\alpha)$ is said to be *formal* if it is obtained by formal extension of a mapping of $N(U_\alpha)$ into itself:

$$T\left(\sum_{k=0}^{\infty} f_k t^k\right) = \sum_{k=0}^{\infty} T(f_k) t^k.$$

***Proposition* 9.1.** *All the formal local derivations of the algebra* $N_t(U_\alpha)$ *are inner: they are all given in the form*

$$T(u) = ad_{f_t}(u) = P_\alpha(f_t, u),$$

where $f_t \in N_t(U_\alpha)$.

This property is no longer true if we remove the assumption of "formality".

***Theorem* 9.1** [57]. *There exists a derivation of the Moyal bracket which is not inner and is such that all the local derivations of $N_t(U_\alpha)$ are represented in the form $T = \lambda\theta + \mathrm{ad}_{f_t}$.*

This derivation is defined in the following way: for any point $x_0 \in U_\alpha$ we consider the Euler field

$$\xi_{x_0} = \sum_{i=1}^{2n} (x - x_0)_i \frac{\partial}{\partial x_i}$$

and denote the corresponding Lie derivative by $L_{\xi_{x_0}}$. Then

$$\theta = -2\left(Id + t\frac{\partial}{\partial t}\right) + L_{\xi_{x_0}}. \tag{9.1}$$

The fact that θ is a derivation of the Moyal bracket and the independence from the choice of the point x_0 are both easily verified.

Thus, we can consider the extension of the Lie algebra $(N_t(U_\alpha), P_\alpha)$ (of the Moyal bracket defined locally in the domain U_α) via the outer derivation θ.

Remark 9.1. Recall that if $\mathfrak{G}$ is a Lie algebra satisfying the condition $[\mathfrak{G}, \mathfrak{G}] = \mathfrak{G}$ and $H^1(\mathfrak{G}, \mathfrak{G}) := \mathrm{Ext}(\mathfrak{G}, \mathbb{R}) = \mathbb{R}$, then there is a well-defined extension

$$0 \to \mathfrak{G} \to \widetilde{\mathfrak{G}} \to \mathbb{R} \to 0$$

with commutator

$$[X + \lambda\theta, Y + \mu\theta] = [X, Y] + \lambda\theta(Y) - \mu\theta(X),$$

where θ is an outer derivation (representing a non-trivial cohomology class in $H^1(\mathfrak{G}, \mathfrak{G})$). It is easy to show that $H^1(\widetilde{\mathfrak{G}}, \widetilde{\mathfrak{G}}) = 0$, that is, all the derivations of the Lie algebra $\widetilde{\mathfrak{G}}$ are inner.

In the case under consideration the resulting extension of the Moyal bracket has the following form:

$$0 \to N_t(U_\alpha) \to A_t(U_\alpha) \to \mathbb{R} \to 0.$$

Definition 9.2. *A regular automorphism of order k* is an automorphism T of the Lie algebra $(N_t(U_\alpha), P_\alpha)$ of the form $T = Id + t^k T'$, where $T' : N_t(U_\alpha) \to N_t(U_\alpha)$ is a local mapping (having the form of a formal series in t), each term of which satisfies condition (1) of §6.2.

***Lemma* 9.1.** *A regular automorphism of the Moyal bracket can be written in the form*

$$T = \exp(t^k \,\mathrm{ad}_f), \tag{9.2}$$

In other words,

$$T(g) = \sum_{p=0}^{\infty} \frac{t^{kp}}{p!} \mathrm{ad}_f^p(g)$$

where $\mathrm{ad}_f^p(g) = \{f, g\}_t$ is the Moyal bracket of the functions f and g.

A regular automorphism of order k of the Moyal algebra $(N_t(U_\alpha), P_\alpha)$ extends in a unique way to an automorphism of the Lie algebra $A_t(U_\alpha)$ of the form $T = \exp(t^k \operatorname{ad}_{Af})$, where ad_{Af} is the adjoint action of f in $A_t(U_\alpha)$.

***Assertion* 9.1.** *All the automorphisms of the Lie algebra $A_t(U_\alpha)$ are inner, and an arbitrary regular automorphism of $A_t(U_\alpha)$ of order k can be represented in the form* (9.1).

This fact is a simple consequence of the triviality of the first cohomology space of $A_t(U_\alpha)$.

9.2. Global construction.
For each pair of domains U_α, $U_\beta \subset V$ there exists an isomorphism of Lie algebras

$$T_{\alpha\beta} : (N_t(U_\alpha), P_\alpha) \xrightarrow{\cong} (N_t(U_\beta), P_\beta)$$

(by Proposition 8.4), having the form $T_{\alpha\beta} = Id + tT'_{\alpha\beta}$. The restriction $T_{\alpha\beta \mid U_\alpha \cap U_\beta}$ can be extended to an automorphism

$$Q_{\alpha\beta} : A_t(U_{\alpha\beta}) \to A_t(U_{\alpha\beta}),$$

which is regular and of order 1. The composition

$$Q_{\alpha\beta\gamma} = Q_{\alpha\beta} Q_{\beta\gamma} Q_{\gamma\alpha}$$

is a regular automorphism of order 1 of the Lie algebra $A(U_{\alpha\beta\gamma})$. All the automorphisms of $A(U_{\alpha\beta\gamma})$ are inner. This means that there exists a $q_{\alpha\beta\gamma} \in N_t(U_{\alpha\beta\gamma})$ for which

$$Q_{\alpha\beta\gamma} = \exp(\operatorname{ad}_A q_{\alpha\beta\gamma}).$$

The set $q_{\alpha\beta\gamma}$ defines a Čech cocycle corresponding to the covering (U_α). The cohomology class that it defines is trivial (which follows from Assertion 9.1). In other words, the Čech cocycle $q_{\alpha\beta\gamma}$ that arises is a coboundary. This fact means that the automorphisms $Q_{\alpha\beta}$ can be replaced by automorphisms $\overline{Q}_{\alpha\beta}$ such that $\overline{Q}_{\alpha\beta\gamma} = \overline{Q}_{\alpha\beta}\overline{Q}_{\beta\gamma}\overline{Q}_{\gamma\alpha}$ will be a regular automorphism of order 2. Continuing these arguments, we eventually arrive at a series of isomorphisms $Q'_{\alpha\beta} : A_t(U_{\alpha\beta}) \to A_t(U_{\alpha\beta})$ satisfying the condition $Q'_{\alpha\beta}Q'_{\beta\gamma}Q'_{\gamma\alpha} = Id$, which also remains valid under restriction to $N_t(U_{\alpha\beta\gamma})$. Thus, we shall obtain a series of isomorphisms $T'_{\alpha\beta} : N_t(U_{\alpha\beta}) \to N_t(U_{\alpha\beta})$ with the condition $T'_{\alpha\beta}T'_{\beta\gamma}T'_{\gamma\alpha} = Id$.

This argument proves the existence of a globally defined deformation of the Poisson bracket on a symplectic manifold. Locally in each Darboux chart U_α this deformation coincides with the Moyal bracket. Analogous arguments are applicable for the proof of the existence of a *-product defined on the whole symplectic manifold V, which coincides locally with the Moyal product.

Remark 9.2. A geometric interpretation of the construction of the set of extended algebras $A_t(U)$ can be given in terms of the *Weyl fibration*.

We consider the fibration of symplectic frames over the symplectic manifold V. The resulting algebra coincides with the corresponding Weyl algebra. It can also be interpreted as the universal enveloping algebra of the Heisenberg algebra.

This construction can be considered from the point of view of the concept of a "quantum manifold" (in the terminology of Cartier) associated with an arbitrary symplectic manifold. It is likely that quantum groups should arise in this context in a natural way.

9.3. Construction via graded cohomology classes.
This construction was used in the original proof of De Wilde and Lecomte [56]. Its first step consists in the construction of graded cohomology classes of the algebra $A^*_{\text{loc},nc}(N)$ (recall that this denotes the graded Lie algebra of multidifferential operators on V; see §8). Here we follow the presentation of [57].

Definition 9.3. Every closed 2-form $\sigma \in \Omega^2(V)$ corresponds to a 2-cocycle of weight -1 : $\Theta^2_\sigma \in C^2_{-1}(A^*_{\text{loc},nc}(N))$. This cocycle is defined locally: on every neighbourhood $U \subset V$ there are a 1-form α_U such that $\sigma|_U = d\alpha|_U$ and a vector field X_U on U such that $\sigma = d(i_{X_U}\omega)$. We consider a mapping defined on the space of cochains on the Lie algebra $\text{Vect}(V)$ with values in N:

$$\mu^* : A^*_{\text{loc}}(\text{Vect}(V), N) \to A^*_{\text{loc},nc}(N),$$

such that $\mu^*(c)(f_1, \ldots, f_p) = c(X_{f_1}, \ldots, X_{f_p})$. We note that μ^* commutes with the differential and there is a

$$\tau : A^*_{\text{loc},nc}(N) \to A^*_{\text{loc}}(\text{Vect}(V), N),$$

that inverts μ^* : $\mu^*\tau = Id_{A^*_{\text{loc},nc}(N)}$.

Let $A \in A^a_{\text{loc},nc}(N)$ and $B \in A^b_{\text{loc},nc}(N)$. Then

$$\Theta^2_\sigma(A, B)\big|_U = \mu^*[i(X_U)\tau[A, B]] - (-1)^b[\mu^* iX_U\tau(A), B] - [A, \mu^* iX_U\tau(B)].$$

It is not hard to verify that in this way we obtain a 2-cocycle on the algebra $A^*_{\text{loc},nc}(N)$. The mapping $[\sigma] \mapsto [\Theta^2_\sigma]$ gives a well-defined linear mapping

$$H^2_{\text{DR}}(V) \to H^2_{\text{gr}}(A^*_{\text{loc},nc}(N))_{-1}.$$

We shall use this 2-cocycle Θ^2_σ to construct formal deformations of the Poisson bracket (analogous to what was done above). Thus we will construct a so-called universal deformation.

The Poisson bracket on V can be understood as an element of the algebra $A^*_{\text{loc},nc}(N)$. We denote this element by

$$P \in A^0_{\text{loc},nc}(N).$$

We associate the cocycle Θ^r_σ with the operator

$$D^r : A^*_{\text{loc},nc}(N) \to A^*_{\text{loc},nc}(N)$$

by the formula

$$D^r(A) := \Theta^r_\sigma(A, P).$$

We shall indicate a method of obtaining a formal deformation of P (that is, of the Poisson bracket on V) as a solution of some formal differential equation.

We consider a formal series $P_t \in A^0_{\mathrm{loc},nc}(N)[[t]]$ such that $P_0 = P$ and the following (formal) first-order differential equation holds:

$$\left(t\frac{\partial}{\partial t} + 1\right) + \frac{1}{2}\Theta^r_\sigma(P_t, P_t) = 0. \tag{9.3}$$

For the first three terms we obtain

$$P + \frac{1}{2}\Theta^r_\sigma(P, P) = 0, \tag{9.3'}$$

$$2P_1 + \Theta^r_\sigma(P, P_1) = 0, \tag{9.3''}$$

$$3P_2 + \Theta^r_\sigma(P, P_2) + \Theta^r_\sigma(P_1, P_1) = 0. \tag{9.3'''}$$

The first relation holds automatically. The second can be written in the form $(D^r+2)P_1 = 0$. For (9.3″) to hold it suffices to take $P_1 = \mu^*(\Omega)$, where Ω is an arbitrary closed form (see [57], p. 929), which is easy to verify. Relation (9.3‴) has the following form: $(D^r + 3)P + \frac{1}{2}\Theta^r_\sigma(P_1, P_1) = 0$. A formal verification easily allows us to see that

$$P_2 = -\frac{1}{2}(D^r + 1)\Theta^r_\sigma(P_1, P_1)$$

is a solution of (9.3‴). Since $(D^r+3)(S^3_\Gamma) = 0$, the general solution of equation (9.3‴) is

$$P_2 = \lambda S^3_\Gamma - \frac{1}{2}(D^r + 1)\Theta^r_\sigma(P_1, P_1). \tag{9.4}$$

***Assertion* 9.2.** *The operator* (D^r+k+1) *is invertible for* $k > 2$.

The proof is an easy check.

Corollary. *Equation* (9.3) *has a unique solution for any fixed pair*

$$P_1 = \mu^*(\Omega), \qquad P_2 = -\frac{1}{2}(D^r + 1)\Theta^r_\sigma(\mu^*(\Omega), \mu^*(\Omega)).$$

Remark 9.3. Formula (9.4) classifies the deformations of the Poisson bracket on the infinitesimal level. The corollary stated above is actually a theorem on the extension of an infinitesimal deformation to a formal deformation.

We shall now show that a solution P_t of equation (9.3) is in fact a formal deformation of the Poisson bracket. This means that the following relation (in the Lie algebra $A^*_{\mathrm{loc},nc}(N)$) holds:

$$[P_t, P_t] = 0.$$

We prove this by induction on k. We assume that it holds on level $k-1$. Since

$$\begin{aligned}\left(t\frac{\partial}{\partial t}+2\right)([P_t,P_t]) &= 2\left[\left(t\frac{\partial}{\partial t}+1\right)P_t,P_t\right]\\ &= [\Theta^r_\sigma(P_t,P_t),P_t]\\ &= \Theta^r_\sigma([P_t,P_t],P_t),\end{aligned}$$

and since Θ^r_σ satisfies a cocycle condition, on level k we obtain

$$(k+2)[P_t,P_t]_k = -\Theta^r_\sigma([P_t,P_t]_k,P);$$

thus,

$$(D^r+k+2)([P_t,P_t]_k) = 0.$$

Now we obtain $[P_t, P_t]_k = 0$ from Assertion 9.2, as required.

Remark 9.4. Deformations with the condition $\lambda = 0$ in the form (9.4) were studied in detail in [31]. In exactly the same way we can consider deformations with the condition $\Omega = 0$. In this case all the odd-numbered terms P_{2k+1} vanish; on the infinitesimal level the deformation is defined by the cocycle S^3_Γ. Thus, we are led to the universal deformation that we have already considered in §9.1.

The proof of the existence of a *-product that is compatible with a given deformation of the Poisson bracket can easily be obtained in an analogous fashion. We limit ourselves to a reference to [33].

Remark 9.5. A symplectic manifold V is said to be *homogeneous* if there exists a vector field X on it such that $[X, P] = -P$. It turns out that we can obtain the homogeneity of every term defining the deformation of the Poisson bracket (see [34]). This property is decisive for the generalization of the results of Chapter I to the case of an arbitrary contact manifold.

CHAPTER III

EXTENSIONS OF THE LIE ALGEBRA OF CONTACT VECTOR FIELDS ON AN ARBITRARY CONTACT MANIFOLD

In this chapter we generalize the results of Chapter I.

We recall that an odd-dimensional manifold M^{2n+1} is called a contact manifold if it has a fixed distribution of hyperplanes in the tangent space that is non-degenerate at every point. These hyperplanes are called contact planes.

Locally a contact distribution can be defined using a 1-form α on M that vanishes on the contact planes. The form α can be chosen so that the $(2n+1)$-form $\alpha\wedge(d\alpha)^n$, defined locally, is non-degenerate. The form α is defined up to multiplication by a non-zero function at every point. It is called the contact form.

All contact manifolds are locally diffeomorphic to each other. There exist local coordinates on M (contact Darboux coordinates) in which the contact structure is defined by the 1-form

$$\alpha = \sum_{i=1}^{n} \frac{x_i\, dy_i - y_i\, dx_i}{2} - dz.$$

§10. Lagrange bracket

10.1. Symplectization.
There is a canonical procedure for constructing a symplectic manifold associated with a given contact manifold M^{2n+1}. It was proposed by Arnol'd [1] (see also [2]) and is called symplectization.

The contact distribution in the tangent bundle TM defines a dual line subbundle of the cotangent bundle:

$$\begin{array}{ccc} S & \subset & T^*M \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & M & \end{array}$$

(the fibre S_m over a point $m \in M$ consists of the covectors in $T_m^* M$ that vanish on a contact hyperplane in $T_m M$).

A section of the bundle $\pi : S \to M$ is a 1-form on M that vanishes on the contact distribution (not necessarily non-degenerate). We shall call the space of all sections $\Gamma(S)$ the space of *contact forms on M*.

In the total space of the bundle S a canonical symplectic structure is defined, the restriction of the standard symplectic form on T^*M. Thus, S is a symplectic manifold. It is called the *symplectization* of M. The symplectic form on S is exact: it is a differential 1-form σ on S, defined as follows. The value of the form σ on a tangent vector $v \in T_s S$, applied at a point $s \in S$, is equal to the value of the vector s on the projection $\pi_* v$:

$$\sigma_s(v) = s(\pi_* v).$$

(The form σ is nothing but the restriction to S of the standard Liouville 1-form on T^*M.)

10.2. Contact tensor fields.
We consider the line bundle $\Lambda^{2n+1} T^*M \to M$. There is an isomorphism between this bundle and a tensor power of S.

***Proposition* 10.1.** *The bundle $\Lambda^{2n+1}T^*M$ is isomorphic to the bundle $S^{\otimes(n+1)}$.*

We give an equivalent formulation.

We denote by $\mathfrak{h}(M)$ the Lie algebra of all vector fields on M that preserve the contact structure (contact vector fields).

Note first that the group of all diffeomorphisms of M that preserve the contact structure acts on S. (Indeed, every diffeomorphism of M lifts to a diffeomorphism of the cotangent bundle T^*M, and the contact diffeomorphisms obviously preserve the subbundle $S \subset T^*M$.) Thus, the space of sections of S (and hence those of $S^{\otimes n}$) is a module over the group of contact diffeomorphisms of M (and hence, over the Lie algebra of contact vector fields on M).

Secondly, note that multiplication by scalars is defined on the bundle space of S. The action of the group of contact diffeomorphisms obviously commutes with multiplication by scalars. Thus, a space of homogeneous functions on S is defined. We denote the space of functions that are homogeneous of degree $-\lambda$ by $\mathcal{F}_\lambda(M)$ (compare §3). The space $\mathcal{F}_\lambda(M)$ is an $\mathfrak{h}(M)$-module.

***Proposition* 10.1′.** *The following three spaces are isomorphic as $\mathfrak{h}(M)$-modules:*

a) *the space of contact* 1*-forms;*

b) $\mathcal{F}_1(M)$ *(the space of homogeneous functions on S of degree of homogeneity* -1);

c) *the space of tensor densities of M of degree* $1/(n+1)$.

Proof. Let $G : M \to M$ be a contact diffeomorphism and α a contact 1-form. Then $G^*(\alpha)$ is also a contact 1-form, that is, it differs from α by multiplication by some function (say m_G):

$$G^*\alpha = m_G\alpha.$$

The forms of degree $2n+1$ on M are proportional to the volume form $\Omega = \alpha \wedge (d\alpha)^n$. Then $G^*\Omega = m_G\alpha \wedge (dm_G\alpha)^n = (m_G)^{n+1}\Omega$. Therefore a differential form on M of highest degree transforms like the $(n+1)$-st power of the contact form α (that is, like a section of the bundle $S^{\otimes(n+1)}$). From this we get the isomorphism between the spaces a) and c). The isomorphism between the spaces a) and b) is obvious.

***Corollary* 1.** *The space $\mathcal{F}_\lambda(M)$ is isomorphic to the space of tensor densities on M of degree* $\lambda/(n+1)$.

***Corollary* 2.** *There exists a natural isomorphism*

$$\mathcal{F}_\lambda(M) \cong \mathcal{F}_{(n+1)-\lambda}(M). \tag{10.1}$$

Definition. The Poisson bracket on the symplectic manifold S defines an operation

$$\{\,,\}: \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+1} \tag{10.2}$$

on the space of tensor densities on M. This operation is called the *Lagrange bracket*. In Darboux coordinates on M it is defined by formula (3.5).

Remark 10.1. a) The space $\mathcal{F}_{-1}(M)$ is a Lie algebra relative to the bracket (10.2). This Lie algebra is isomorphic to $\mathfrak{h}(M)$. Indeed, the space of homogeneous functions of degree 1 on M forms a Lie algebra isomorphic to the Lie algebra of all Hamiltonian vector fields on M that commute with multiplication by a scalar, and a homogeneous Hamiltonian function corresponds to a homogeneous vector field.

b) For $\mu = -1$ the operation (10.2) defines the Lie derivative along a contact vector field of a tensor density of degree $\lambda/(n+1)$.

Thus, the Lagrange bracket on a contact manifold is defined on the space of tensor densities (see also [1]). It satisfies the Jacobi and Leibniz identities. The tensor densities of degree $-1/(n+1)$ form a Lie algebra which is isomorphic to the Lie algebra $\mathfrak{h}(M)$ of all contact vector fields. They are called *contact Hamiltonians.*

§11. Extensions and modules of tensor fields

11.1. Extension of the Lie algebra $\mathfrak{h}(M)$ by the modules $\mathcal{F}_\lambda$.
We construct extensions of the Lie algebra of all contact vector fields on M by the modules of tensor fields. These extensions are connected with the deformation of the Poisson bracket on the symplectization S, which was called the universal deformation in Chapter II. The scheme of constructing these extensions repeats the scheme of §3.

***Theorem* 11.1.** *There exists a formal deformation of the Poisson bracket on the symplectic manifold S*

$$\{F,G\}_t = \{F,G\} + \sum_{k=1}^{\infty} t^k P_k(F,G) \tag{11.1}$$

such that every operation P_k, restricted to homogeneous functions, defines a mapping

$$P_k : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+2k+1}. \tag{11.2}$$

Proof. We start with the first term. It is necessary to show that there exists a cocycle P_1 on the Poisson algebra $N(S)$ such that $P_1 : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+3}$.

***Lemma* 11.1.** *The cocycle S^3_Γ* (defined in §8.2) *satisfies the condition*

$$S^3_\Gamma : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+3}.$$

This lemma was first proved by Lichnerowicz in [34] using the choice of a suitable linear connection on S. It is also not hard to verify it in local coordinates (locally the cocycle S^3_Γ coincides with the operation $\{\,,\}_3$ in the Moyal bracket).

Corollary of the lemma. *The cocycle S^3_Γ defines a non-trivial extension of the Lie algebra of contact vector fields on M by the space of contact* 1*-forms*:

$$0 \to \mathcal{F}_1(M) \to \mathfrak{h}(M) \oplus \mathcal{F}_1(M) \to \mathfrak{h}(M) \to 0.$$

We return to the proof of the theorem. We shall prove it by induction on k. We use the Jacobi identity for the brackets (11.1). At level k we obtain the relation

$$dP_k(F,G,H) = \circlearrowleft \left[\sum_{m=3}^{k-1} P_m(F, P_{k-m}(G,H))\right], \tag{11.3}$$

which coincides with the identities (2.10), which we called higher Jacobi identities. It immediately follows from this that the differential dP_k is an operation of "weight" $2k+1$. More precisely,

$$dP_k : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \otimes \mathcal{F}_\nu \to \mathcal{F}_{\lambda+\mu+\nu+2k+1}.$$

The differential preserves the homogeneity condition. On the space of homogeneous 3-cochains of degree of homogeneity greater than 3 the operator d^{-1} (on the image of d) is defined in the natural way (compare [57], p. 932).

This follows from the classification of the three-dimensional cohomology (Proposition 8.2). Therefore there exists a unique operator P_k with the condition (11.2), as was required.

Now the construction of §3 can be applied. The result of this is a series of extensions $\mathfrak{h}_m(M)(\cdot)$ of the Lie algebra $\mathfrak{h}(M)$. This proves Theorem 1.1. Analogous arguments prove Theorem 1.3.

Remark 11.1. In [47] the Lie algebras $\mathfrak{h}_m(M)$ were constructed under the condition that the contact structure on the manifold M is compatible with some projective structure.

11.2. Central extensions.

Assume that the contact manifold M has dimension $4k+1$. We construct non-trivial central extensions of the Lie algebra $\mathfrak{h}_m(M)$ for $m > k$ under the assumption that the symplectization bundle $S \to M$ has a section γ which vanishes nowhere. Our construction will again repeat the construction of Chapter I (see §4).

Definition 11.1. The space $\mathcal{F}_k(M)$ is isomorphic to the space of differential forms of degree $4k+1$ on M. Therefore an $\mathfrak{h}(M)$-invariant functional

$$\int : \mathcal{F}_k \to \mathbb{R}$$

is defined (in Chapter I for the case of the sphere this functional was called the residue).

We also introduce a radius-function on S. We set $r|_\gamma \equiv 1$ and define r on the whole manifold S via the action of the group $\mathbb{R}^*$ (multiplication by scalars) as a homogeneous function of degree 1.

***Assertion* 11.1.** *The formulae* (4.3) *define a non-trivial scalar-valued* 2*-cocycle on the Lie algebra* $\mathfrak{h}_m(M)$ *for* $m > k$.

The proof is carried out like that of Theorem 1.2.

APPENDIX 1

EXTENSIONS OF THE LIE ALGEBRA OF DIFFERENTIAL OPERATORS

The Lie algebra of differential operators $DO(M^n)$ on a compact manifold M^n has a non-trivial central extension [6], [49]. In the one-dimensional case a central extension of the Lie algebra $DO(S^1)$ was first proposed in a paper by Kac and Peterson [26] (see also [45]). This non-trivial extension is unique. A central extension of the Lie algebra $PDO(S^1)$ of pseudodifferential operators on S^1 was constructed by Kravchenko and Khesin [27] (see also [28]). The connection of these Lie algebras with the Gel'fand–Dikii (Dickey) algebras and so-called W-algebras was given by Radul [49].

It turns out that central extensions of the Lie algebras $DO(S^1)$ and $PDO(S^1)$ are connected with the Moyal bracket on $\mathbb{R}^2$. Here we propose an invariant description of these Lie algebras and of central extensions of them.

We recall that $\mathcal{F}_\lambda = \mathcal{F}_\lambda(S^1)$ is the space of tensor fields of degree $-\lambda$ on S^1. This space is isomorphic to the space of homogeneous functions on $\mathbb{R}^2$ of degree of homogeneity 2λ. The isomorphism can be defined by the formula

$$\varphi(\tau)(d\tau)^{-\lambda} \mapsto r^{2\lambda}\varphi(\tau), \tag{1}$$

where r is the radius and τ is an angle. Formula (1) is written in coordinates, but the isomorphism between tensor fields and homogeneous functions is invariantly defined. Here the Lie algebra Vect $S^1 \cong \mathcal{F}_1$ is isomorphic to the Lie algebra of functions on $\mathbb{R}^2$, homogeneous of degree 2, and the isomorphism (1) is an isomorphism of Vect S^1-modules.

We consider the space

$$\mathcal{F} = \dots\mathcal{F}_{-2} \oplus \mathcal{F}_{-1} \oplus \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \dots.$$

Definition 1. We introduce a Lie algebra structure on the space $\mathcal{F}$. An element $F \in \mathcal{F}$ can be conveniently represented as a formal series:

$$F = \sum_{k=-\infty}^{N} t^k F_k,$$

where $F_k \in \mathcal{F}_k$ and t is a formal variable. We define the commutator on $\mathcal{F}$ by the formula

$$\{F, G\}_t := \sum_{i,j,k} \frac{t^{i+j-2k-1}}{(2k+1)!}\{F_i, G_j\}_{2k+1}. \tag{2}$$

***Assertion* 1.** *The Lie algebra $\mathcal{F}$ is isomorphic to the Lie algebra of pseudodifferential operators of the form*

$$L = \sum_{k=-\infty}^{N} u_k \left(\frac{\partial}{\partial\tau}\right)^k$$

on the circle.

Remark 1. The Lie algebra $\mathcal{F}$ is actually nothing but the Moyal algebra on $\mathbb{R}^2\setminus\{0\}$ (that is, the deformation of the Poisson bracket on $\mathbb{R}^2\setminus\{0\}$, restricted to a special class of functions). In fact, the space $\mathcal{F}$ consists of functions that are homogeneous on $\mathbb{R}^2\setminus\{0\}$ of degree of homogeneity an integer multiple of two. The operation (2) is the usual Moyal bracket.

The residue is defined as usual, a linear functional

$$\mathrm{res} : \mathcal{F} \to \mathbb{R}$$

such that

$$\mathrm{res}(F) = \int_{S^1} F_{-1}.$$

We consider the function $\log r$ on $\mathbb{R}^2\setminus\{0\}$.

***Assertion* 2.** a) *The mapping* $\alpha : \mathcal{F} \to \mathcal{F}$

$$\alpha(F) = \{\log r, F\}_t \tag{3}$$

defines a non-trivial 1*-cocycle on* $\mathcal{F}$.

b) *The mapping* $c : \mathcal{F} \otimes \mathcal{F} \to \mathbb{R}$

$$c(F, G) = \mathrm{res}\,(F \cdot \alpha(G)) \tag{4}$$

defines a non-trivial 2*-cocycle on* $\mathcal{F}$.

Thus we have defined a central extension of the Lie algebra $\mathcal{F}$ (that is, a central extension of the Lie algebra of pseudodifferential operators on S^1).

Remark 2. The central extension we have constructed coincides with the central extension from [27]. Formulae (3) and (4) are another invariant form of writing the beautiful formula discovered by Kravchenko and Khesin. In [27] (see also [28]) the cocycle α can be represented in the form

$$\alpha(f_m\partial^m) = [\log\partial, f_m\partial^m]$$

in the language of pseudodifferential symbols. One can verify that the cocycle c in the components of pseudodifferential operators has the form

$$c\,(f_m\partial^m, g_n\partial^n) = \sum_{k+l=m+n+1} (-1)^{m+n} \sum_{r=0}^{n} \frac{(-1)^r n!}{r!\,(n-r)\,!(m+n+1-r)} \int_{S^1} g_n f_m^{(m+n+1)}.$$

APPENDIX 2

EXAMPLES OF EQUATIONS OF KORTEWEG-DE VRIES TYPE

There is a profound and beautiful connection between the Virasoro algebra and the Korteweg-de Vries (KdV) equation

$$\dot{u} = 3u'u + cu'''$$

(where $u = u(t, r)$, $\dot{u} = \partial u/\partial t$, $u' = \partial u/\partial r$, and c is a scalar). This connection is that the KdV equation is realized as a vector field on the dual space to the Virasoro algebra, and this vector field is Hamiltonian simultaneously with respect to two natural Poisson structures. This realization gives a simple method of interpreting the KdV equation. This approach is called "the method of Poisson pairs", the "Lenart scheme", the "Adler scheme", and others. For the historical details we refer the reader to [37], [38].

Later on this method proved its uniqueness. It has been successfully applied to other Lie algebras (generalizing the Virasoro algebra). In this way new integrable systems have been found which generalize the Korteweg-de Vries equation (see [8]).

This appendix states the problem. We write down equations which are obtained by a formal application of the method to the Lie algebra (5.5). It is obvious that these equations are Hamiltonian and that they have several first integrals. We pose the question of their complete integrability. The proof of this fact could be obtained if we could find the corresponding analogues of the Miura transformation [37] (see [8]).

A. Method.

Let $\mathfrak{A}$ be a finite-dimensional Lie algebra, and $\mathfrak{A}^*$ the space of linear functionals on $\mathfrak{A}$. There is a natural Poisson structure on $\mathfrak{A}^*$, the *Kirillov bracket*. Let F and G be functions on $\mathfrak{A}^*$. Then

$$\{F, G\}(u) = \langle [dF(u), dG(u)], u \rangle ,$$

where u is a point of $\mathfrak{A}^*$; $dF(u)$ and $dG(u)$ are the differentials of F and G at the point u and are elements of the space $(\mathfrak{A}^*)^* \cong \mathfrak{A}$. Thus, a Poisson bracket is defined on the space $\mathfrak{A}^*$.

Every function H on $\mathfrak{A}^*$ corresponds to a Hamiltonian vector field

$$\dot{u} = \mathrm{ad}^*_{dH(u)}\, u,$$

where $\mathrm{ad}^*_{dH(u)}$ is the operator of the coadjoint action of the element $dH(u) \in \mathfrak{A}$ on the space $\mathfrak{A}$. We denote this field by $c(H)$.

We consider a new Poisson bracket on $\mathfrak{A}^*$. We fix an arbitrary element $u_0 \in \mathfrak{A}^*$ and set

$$\{F, G\}_0(u) = \langle [dF(u), dG(u)], u_0 \rangle .$$

From the fact that this is a Poisson bracket it follows that the bilinear functional $x \otimes y \to \langle [x, y], u_0 \rangle$ is a cocycle on $\mathfrak{A}$. (This cocycle is a coboundary.)

A Hamiltonian vector field with Hamiltonian in this structure has the form

$$\dot{u} = \mathrm{ad}^*_{dH(u)}\, u_0 .$$

We denote this field by $c_0(H)$.

***Assertion* A.1.** *The Poisson brackets* $\{\,,\}$ *and* $\{\,,\}_0$ *satisfy the condition that any linear combination*

$$\alpha\{\,,\,\}+\beta\{\,,\,\}_0 \tag{1}$$

is again a Poisson bracket (satisfies the Jacobi identity).

This fact is equivalent to the fact that the mapping $x\otimes y\mapsto\langle[x,y],u_0\rangle$ is a 2-cocycle on $\mathfrak{A}$.

Remark A.1. Assertion A.1 has already been used in [1]. Two Poisson brackets that satisfy the condition (1) are called a *Poisson pair.* Not infrequently the bracket $\{\,,\}_0$ is called the first Poisson structure and the bracket $\{\,,\}$ is called the second Poisson structure.

Let $x_0\in \mathrm{st}_{u_0}$ be an arbitrary element of the stabilizer of $u_0\in\mathfrak{A}^*$. Then the linear function

$$I(u)=\langle x_0,u\rangle$$

is in involution with any function relative to the bracket $\{\,,\}_0$ (that is, all the Hamiltonian fields are tangent to its level curves).

The essence of the method is as follows. We write the Hamiltonian vector field $c(I)$:

$$\dot u=\mathrm{ad}^*_{x_0}u. \tag{2}$$

An important result is the following.

***Assertion* A.2.** *The field* $c(I)$ *is Hamiltonian relative to the Poisson bracket* $\{\,,\}_0$.

In fact, we check that it preserves the bracket $\{\,,\}_0$. We let v_0 denote the field (2). We need to verify that

$$v_0\{F,G\}_0=\{v_0F,G\}_0+\{F,v_0G\}_0.$$

This follows from the condition $x\in\mathrm{st}_{u_0}$.

Let $I_1(u)$ be the Hamiltonian of the vector field (2) relative to the first Hamiltonian structure. We consider the Hamiltonian vector field $c(I_1)$. (It turns out that this field is again Hamiltonian relative to the structure $\{\,,\}_0$ with Hamiltonian $I_2(u)$.) Iterating this procedure, we obtain a chain of functions I, I_1, ..., I_k, ... In order to prove that at each stage the Hamiltonian field with Hamiltonian I_k relative to the structure $\{\,,\}$ is also Hamiltonian relative to $\{\,,\}_0$, we consider the Poisson bracket

$$\{\,,\,\}_0+\varepsilon\{\,,\,\}. \tag{3}$$

***Assertion* A.3.** *An invariant of the bracket* (3) *has the form*

$$I_\varepsilon=I+\varepsilon I_1+\cdots+\varepsilon^k I_k+\cdots.$$

Indeed, a Hamiltonian field with Hamiltonian I_ε relative to the bracket (3) has the form

$$\sum \varepsilon^k \left(c_0(I_k) + c(I_{k-1})\right).$$

By hypothesis this field is equal to zero (since I_ε is an invariant of the bracket (3)). Therefore we obtain the condition $c_0(I_k) = c(I_{k-1})$.

***Assertion* A.4.** *All the functions I_k are in involution with each other*:

$$\{I_k, I_n\} = \{I_k, I_n\}_0 = 0.$$

In fact, suppose that $k < n$. Then

$$\{I_k, I_n\} = \{I_{k+1}, I_n\}_0 = \{I_{k+1}, I_{n-1}\} = \dots .$$

Continuing the chain of equalities, we obtain $\{I_l, I_l\} = 0$ or $\{I_l, I_l\}_0 = 0$.

Thus the method of Poisson pairs allows us to construct a chain of functions I_k in involution. This means that all the functions are first integrals of each of the vector fields $c(I_k)$. One can hope that the integrals I_k turn out to be sufficient to prove their integrability. There are no general theorems on this subject, and the integrability must be verified separately for each Lie algebra.

B. An equation of Korteweg-de Vries type for the extended Virasoro algebra (5.5). We give here the result of the computations for the Lie algebra (5.5):

$$\begin{aligned} \dot{u} &= 3(uv)' - \lambda(v^2)' = 2v^{(7)}v - \tfrac{5}{2}v^{(6)}v' - 2(v^{\mathrm{IV}}v - v^{\mathrm{III}}v')^{\mathrm{III}} + cv^{\mathrm{III}}, \\ \dot{v} &= 3v'v, \end{aligned}$$

where λ, c are parameters and $u(x)$ and $v(x)$ are functions.

This equation is obtained by applying the formal scheme. The corresponding Miura transformation (and hence whether it is completely integrable) is unknown.

It is also interesting to consider the analogous equation in the case of the contact Virasoro algebra.

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