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Space of Linear Differential Operators on the Real Line as a Module over the Lie Algebra of Vector Fields

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1 Introduction

The space of linear differential operators on a manifold M has various algebraic structures: the structure of associative algebra and of Lie algebra, and in the 1-dimensional case it can be considered as an infinite-dimensional Poisson space (with respect to the so-called Adler-Gelfand-Dickey bracket).

1.1 Diff(M)-module structures

One of the basic structures on the space of linear differential operators is a natural family of module structures over the group of diffeomorphisms Diff(M) (and of the Lie algebra of vector fields Vect(M)). These Diff(M)- (and Vect(M))-module structures are defined if one considers the arguments of differential operators as *tensor-densities* of degree λ on M.

In this paper we consider the space of differential operators on $\mathbf{R}$.[1] Denote by $\mathcal{D}^k$ the space of kth-order linear differential operators

$$A(\phi) = a_k(x)\frac{d^k\phi}{dx^k} + \cdots + a_0(x)\phi \tag{1}$$

where $a_i(x), \phi(x) \in C^\infty(\mathbf{R})$.

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[1] Particular cases of actions of Diff($\mathbf{R}$) and Vect($\mathbf{R}$) on this space were considered in classics (see [1], [14]). The well-known example is the Sturm-Liouville operator $d^2/dx^2 + a(x)$ acting on $-1/2$-densities (see, e.g., [1], [14], [13]). Already this simplest case leads to interesting geometric structures and is related to the so-called Bott-Virasoro group (cf. [7], [12]).

Define a 1-parameter family of Diff($\mathbf{R}$)-module structures on $C^\infty(\mathbf{R})$ by

$$g_\lambda^* \phi := \phi \circ g^{-1} \cdot \left(\frac{dg^{-1}}{dx}\right)^{-\lambda}$$

where $\lambda \in \mathbf{R}$ (or $\lambda \in \mathbf{C}$) is a parameter. Geometrically speaking, ϕ is a tensor-density of degree $-\lambda$:

$$\phi = \phi(x)(dx)^{-\lambda}.$$

A 1-parameter family of actions of Diff($\mathbf{R}$) on the space of differential operators (1) is defined by

$$g(A) = g_\lambda^* A (g_\lambda^*)^{-1}.$$

Denote by $\mathcal{D}_\lambda^k$ the space of operators (1) endowed with the defined Diff($\mathbf{R}$)-module structure. An infinitesimal version of this action defines a 1-parameter family of Vect($\mathbf{R}$)-module structures on $\mathcal{D}^k$ (see Section 3 for details).

1.2 The problem of isomorphism

Let M be a manifold, $\dim M \geq 2$. The problem of isomorphism of Diff(M)- (and Vect(M)-) module structures for different values of λ was stated in [4] and was saved in the case of second-order differential operators. In this case, different Diff(M)-module structures are isomorphic to each other for every λ except 3 critical values: $\lambda = 0, -1/2, -1$ (corresponding to differential operators on functions, 1/2-densities, and volume forms, respectively).

Geometric quantization gives an example of such a special Diff(M)-module: differential operators are considered as acting on 1/2-densities (see [8]).

Recently, P. B. A. Lecomte, P. Mathonet, and E. Tousset [9] showed that in the case of differential operators of order ≥ 3, Diff(M)-modules corresponding to λ and λ'-densities are isomorphic if and only if $\lambda + \lambda' = 1$. The unique isomorphism in this case is given by conjugation of differential operators.

These results solve the problem of isomorphism in the multidimensional case.

It was shown in [4], [9] that the case $\dim M = 1$ ($M = \mathbf{R}$ or S^1) is particular. It is richer in algebraic structures and therefore is of a special interest.

In this paper we solve the problem of isomorphism of Diff($\mathbf{R}$)-modules $\mathcal{D}_\lambda^k$ for any k. The result is as follows.

(a) The modules $\mathcal{D}_\lambda^3$ of *third-order* differential operators (1) are isomorphic to each other for all values of λ except 5 critical values:

$$\left\{0,\quad -1,\quad -\frac{1}{2},\quad -\frac{1}{2}+\frac{\sqrt{21}}{6},\quad -\frac{1}{2}-\frac{\sqrt{21}}{6}\right\}.$$

(This result was announced in [4].)

(b) The Diff($\mathbf{R}$)-modules $\mathcal{D}^k_\lambda$ and $\mathcal{D}^k_{\lambda'}$ on the space of differential operators (1) of order $k \geq 4$ are isomorphic if and only if $\lambda + \lambda' = -1$.

1.3 Intertwining operator

The most important result of the paper is a construction of the unique (up to a constant) equivariant linear operator on the space of third-order differential operators,

$$T\colon\ \mathcal{D}^3_\lambda \to \mathcal{D}^3_\mu, \tag{2}$$

for $\lambda, \mu \neq 0, -1, -1/2,\ -1/2 \pm \sqrt{21}/6$; see the explicit formulae (3), (7), and (8) below. It has nice geometric and algebraic properties and seems to be an interesting object to study.

The operator T is an analogue of the *second-order Lie derivative* from [4], intertwining different Diff(M)-actions on the space of second-order differential operators on a multidimensional manifold M.

1.4 Normal symbols

The main tool of this paper is the notion of a *normal symbol*, which we define in the case of fourth-order differential operators. We define an sl_2-equivariant way to associate a polynomial function of degree 4 on $T^*\mathbf{R}$ to a differential operator $A \in \mathcal{D}^4_\lambda$. In the case of second-order operators, the notion of a normal symbol was defined in [4]. This construction is related with the results of [3]. We discuss the geometric properties of the normal symbol and its relations to the intertwining operator (2).

2 Main results

We formulate here the main results of this paper. All the proofs will be given in Sections 3–7.

2.1 Classification of Diff(**R**)-modules

First, remark that for each value of k, there exists an isomorphism of Diff(**R**)-modules:

$$\mathcal{D}^k_\lambda \cong \mathcal{D}^k_{-1-\lambda}.$$

It is given by *conjugation* $A \mapsto A^*$:

$$A^* = \sum_{i=1}^{k} (-1)^i \frac{d^i}{dx^i} \circ a_i(x).$$

The following two theorems give a solution of the problem of isomorphism of Diff(**R**)-modules $\mathcal{D}^k_\lambda$ on the space $\mathcal{D}^k$. The first result was announced in [4].

Theorem 1. (i) All the Diff(**R**)-modules $\mathcal{D}^3_\lambda$ with $\lambda \neq 0, -1, -1/2, -1/2 + \sqrt{21}/6, -1/2 - \sqrt{21}/6$ are isomorphic to each other.

(ii) The modules $\mathcal{D}^3_0$, $\mathcal{D}^3_{-1/2}$, $\mathcal{D}^3_{-1/2+\sqrt{21}/6}$ are not isomorphic to $\mathcal{D}^3_\lambda$ for general λ. □

It follows from the general isomorphism $*$: $\mathcal{D}^k_\lambda \cong \mathcal{D}^k_{-1-\lambda}$ that

$$\mathcal{D}^3_0 \cong \mathcal{D}^3_{-1} \quad \text{and} \quad \mathcal{D}^3_{-1/2+\sqrt{21}/6} \cong \mathcal{D}^3_{-1/2-\sqrt{21}/6}.$$

Therefore, there exist 4 nonisomorphic Diff(**R**)-module structures on the space $\mathcal{D}^3$.

Theorem 2. For $k \geq 4$, the Diff(**R**)-modules $\mathcal{D}^k_\lambda$ and $\mathcal{D}^k_{\lambda'}$ are isomorphic if and only if $\lambda + \lambda' = -1$. □

This result shows that operators of order 3 play a special role in the 1-dimensional case (as operators of order 2 in the case of a manifold of dimension ≥ 2; cf. [4], [9]).

2.2 Intertwining operator T

Theorem 3. For $\lambda, \mu \neq 0, -1, -1/2, -1/2 \pm \sqrt{21}/6$ there exists a unique (up to a constant) isomorphism of Diff(**R**)-modules $\mathcal{D}^3_\lambda$ and $\mathcal{D}^3_\mu$. □

Let us give an explicit formula for the operator (2). Every differential operator of order 3 can be written (not in a canonical way) as a linear combination of 4 operators:

(a) zero-order operator of multiplication by a function: $\phi(x) \mapsto \phi(x) f(x)$,

(b) first-order operator of Lie derivative:
$$L^\lambda_X = X(x)\frac{d}{dx} - \lambda X'(x),$$
where $X' = dX/dx$,

(c) symmetric "anticommutator" of Lie derivatives:
$$[L^\lambda_X, L^\lambda_Y]_+ := L^\lambda_X \circ L^\lambda_Y + L^\lambda_Y \circ L^\lambda_X,$$

(d) symmetric third-order expression:

$$[L^\lambda_X, L^\lambda_Y, L^\lambda_Z]_+ := \mathrm{Sym}_{X,Y,Z}(L^\lambda_X \circ L^\lambda_Y \circ L^\lambda_Z)$$

for some vector fields $X(x)d/dx$, $Y(x)d/dx$, $Z(x)d/dx$.

Theorem 4. The formula

$$\begin{aligned}
T(f) &= \frac{\mu(\mu+1)(2\mu+1)}{\lambda(\lambda+1)(2\lambda+1)} f,\\
T(L^\lambda_X) &= \frac{3\mu^2+3\mu-1}{3\lambda^2+3\lambda-1} L^\mu_X,\\
T([L^\lambda_X, L^\lambda_Y]_+) &= \frac{2\mu+1}{2\lambda+1}[L^\mu_X, L^\mu_Y]_+,\\
T([L^\lambda_X, L^\lambda_Y, L^\lambda_Z]_+) &= [L^\mu_X, L^\mu_Y, L^\mu_Z]_+
\end{aligned} \tag{3}$$

defines an intertwining operator (2). □

A remarkable fact is that the formula (3) does not depend on the choice of X, Y, Z, and f representing the third-order operator. (Indeed, the formulae (7) and (8) below give the expression of T directly in terms of coefficients of differential operators.) Moreover, this property fixes the coefficients in (3) in a unique way (up to a constant).

Remarks. (a) In the case of a multidimensional manifold M, almost all $\mathrm{Diff}(M)$-module structures on the space of second-order differential operators are isomorphic to each other and the corresponding isomorphism is unique (up to a constant) [4]; there is no isomorphism between different $\mathrm{Diff}(M)$-module structures on the space of third-order operators, except the conjugation [9].

(b) The formula (3) gives an idea that it would be interesting to study the commutative algebra structure (defined by the anticommutator) on the Lie algebra of all differential operators.

3 Action of $\mathrm{Vect}(\mathbf{R})$ on the space $\mathcal{D}^4$

To prove Theorems 1–4, it is sufficient to consider only the $\mathrm{Vect}(\mathbf{R})$-action on $\mathcal{D}^k$. Indeed, since the $\mathrm{Diff}(\mathbf{R})$-action on the space of differential operators is local, therefore, the properties $\mathrm{Vect}(\mathbf{R})$- and $\mathrm{Diff}(\mathbf{R})$-equivariance are equivalent.

3.1 Definition of the family of $\mathrm{Vect}(\mathbf{R})$-actions

Let $\mathrm{Vect}(\mathbf{R})$ be the Lie algebra of smooth vector fields on $\mathbf{R}$

$$X = X(x)\frac{d}{dx}$$

with the commutator

$$\left[X(x)\frac{d}{dx}, Y(x)\frac{d}{dx}\right] = (X(x)Y'(x) - X'(x)Y(x))\frac{d}{dx},$$

where $X' = dX/dx$.

The action of $\mathrm{Vect}(\mathbf{R})$ on space $\mathcal{D}^k$ is defined by

$$\mathrm{ad}\, L_X^\lambda(A) := L_X^\lambda \circ A - A \circ L_X^\lambda$$

where

$$L_X^\lambda \phi = X(x)\phi'(x) - \lambda X'(x)\phi(x).$$

The last formula defines a 1-parameter family of $\mathrm{Vect}(\mathbf{R})$-actions on $C^\infty(\mathbf{R})$. One obtains a 1-parameter family of $\mathrm{Vect}(\mathbf{R})$-modules on $\mathcal{D}^k$.

Notation. (a) The operator L_X^λ is called the *operator of Lie derivative* of tensor-densities of degree $-\lambda$. Denote by $\mathcal{F}_\lambda$ the corresponding $\mathrm{Vect}(\mathbf{R})$-module structure on $C^\infty(\mathbf{R})$.

(b) As in the case of $\mathrm{Diff}(\mathbf{R})$-module structures, we denote by $\mathcal{D}_\lambda^k$ the space $\mathcal{D}^k$ as a $\mathrm{Vect}(\mathbf{R})$-module.

3.2 Explicit formula

Let us calculate explicitly the action of Lie algebra $\mathrm{Vect}(\mathbf{R})$ on the space $\mathcal{D}^4$. Given a differential operator $A \in \mathcal{D}^4$, let us use the following notation for the $\mathrm{Vect}(\mathbf{R})$-action $\mathrm{ad}\, L_X$:

$$\mathrm{ad}\, L_X(A) = a_4^X(x)\frac{d^4}{dx^4} + a_3^X(x)\frac{d^3}{dx^3} + a_2^X(x)\frac{d^2}{dx^2} + a_1^X(x)\frac{d}{dx} + a_0^X(x).$$

Lemma 3.1. The action $\mathrm{ad}\, L_X^\lambda$ of $\mathrm{Vect}(\mathbf{R})$ on space $\mathcal{D}^4$ is given by

$$\begin{aligned}
a_4^X &= L_X^4(a_4)\\
a_3^X &= L_X^3(a_3) + 2(2\lambda - 3)a_4X''\\
a_2^X &= L_X^2(a_2) + 3(\lambda - 1)a_3X'' + 2(3\lambda - 2)a_4X'''\\
a_1^X &= L_X^1(a_1) + (2\lambda - 1)a_2X'' + (3\lambda - 1)a_3X''' + (4\lambda - 1)a_4X^{IV}\\
a_0^X &= L_X^0(a_0) + \lambda(a_1X'' + a_2X''' + a_3X^{IV} + a_4X^{V}).
\end{aligned} \tag{4}$$

□

Proof. One gets easily the formula (4) from the definition

$$\operatorname{ad} L_X^\lambda(A) = [L_X^\lambda, A] = \left(X\frac{d}{dx} - \lambda X'\right)\left(a_4\frac{d^4}{dx^4} + \cdots + a_0\right) - \left(a_4\frac{d^4}{dx^4} + \cdots + a_0\right)\left(X\frac{d}{dx} - \lambda X'\right). \qquad \blacksquare$$

3.3 Remarks

It is convenient to interpret the action (4) as a *deformed* standard action of Vect($\mathbf{R}$) on the direct sum

$$\mathcal{F}_4 \oplus \mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0$$

(given by the first term of the right-hand side of each equality in the formula (4)). This interpretation is the motivation of the main construction of Section 4; it will be discussed in Section 7.2.

The main idea of the proof of Theorems 1 and 2 is to find some *normal form* (cf. [4]) of the coefficients $a_4(x), \ldots, a_0(x)$ for fourth-order differential operators on $\mathbf{R}$ which reduces the action (4) to a canonical form.

4 Normal form of a symbol

It is convenient to represent differential operators as polynomials on the cotangent bundle. The standard way to define a (total) symbol of an operator (1) is to associate to A the polynomial

$$P_A(x, \xi) = \sum_{i=0}^{k} \xi^i a_i(x)$$

on $T^*\mathbf{R} \cong \mathbf{R}^2$ (where ξ is a coordinate on the fiber). However, this formula depends on coordinates, and only the higher term $\xi^k a_k(x)$ of P_A (the principal symbol) has a geometric sense.

4.1 The main idea

The Lie algebra Vect($\mathbf{R}$) naturally acts on $C^\infty(T^*\mathbf{R})$ (it acts on the cotangent bundle). Consider a linear differential operator $A \in \mathcal{D}^4$. Let us look for a natural definition of a symbol of A in the form

$$\overline{P}_A(x, \xi) = \xi^4 \bar{a}_4(x) + \xi^3 \bar{a}_3(x) + \xi^2 \bar{a}_2(x) + \xi \bar{a}_1(x) + \bar{a}_0(x),$$

where the functions $\bar{a}_i(x)$ are linear expressions of the coefficients $a_i(x)$ and their derivatives.

Any symbol $\overline{P}(x,\xi)$ can be considered as a linear mapping

$$\mathcal{D}^4 \to \mathcal{F}_4 \oplus \mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0.$$

Indeed, the Lie algebra $\mathrm{Vect}(\mathbf{R})$ acts on each coefficient $\bar{a}_i(x)$ of the polynomial $\overline{P}_A(x,\xi)$ as on a tensor-density of degree $-i$:

$$L_X(\overline{P}_A) = \sum_{i=0}^{4} \xi^i L_X^i(\bar{a}_i).$$

However, there is no such mapping that is $\mathrm{Vect}(\mathbf{R})$-equivariant.

4.2 Definition

The *normal symbol* of $A \in \mathcal{D}_\lambda^4$ is a polynomial $\overline{P}_A(x,\xi)$ such that the linear mapping $A \mapsto \overline{P}_A$ is equivariant with respect to the subalgebra $sl_2 \subset \mathrm{Vect}(\mathbf{R})$ generated by the vector fields

$$\left\{\frac{d}{dx},\ \ x\frac{d}{dx},\ \ x^2\frac{d}{dx}\right\}.$$

Proposition. 4.1. (i) The following formula defines a normal symbol of a differential operator $A \in \mathcal{D}_\lambda^4$:

$$\begin{aligned}
\bar{a}_4 &= a_4\\
\bar{a}_3 &= a_3 + \frac{1}{2}(2\lambda-3)a_4'\\
\bar{a}_2 &= a_2 + (\lambda-1)a_3' + \frac{2}{7}(\lambda-1)(2\lambda-3)a_4''\\
\bar{a}_1 &= a_1 + \frac{1}{2}(2\lambda-1)a_2' + \frac{3}{10}(\lambda-1)(2\lambda-1)a_3''\\
&\quad + \frac{1}{15}(\lambda-1)(2\lambda-1)(2\lambda-3)a_4'''\\
\bar{a}_0 &= a_0 + \lambda a_1' + \frac{1}{3}\lambda(2\lambda-1)a_2'' + \frac{1}{6}\lambda(\lambda-1)(2\lambda-1)a_3'''\\
&\quad + \frac{1}{30}\lambda(\lambda-1)(2\lambda-1)(2\lambda-3)a_4^{(IV)}.
\end{aligned} \tag{5}$$

(ii) The normal symbol is defined uniquely (up to multiplication of each function $\bar{a}_i(x)$ by a constant). □

Proof. Direct calculation shows that the Vect($\mathbf{R}$)-action ad L^λ on $\mathcal{D}^4$ given by the formula (4) reads in terms of $\bar{a}_i$ as

$$
\begin{aligned}
\bar{a}_4^X &= L_X^4(\bar{a}_4)\\
\bar{a}_3^X &= L_X^3(\bar{a}_3)\\
\bar{a}_2^X &= L_X^2(\bar{a}_2) + \frac{2}{7}(6\lambda^2 + 6\lambda - 5)J_3(X, \bar{a}_4)\\
\bar{a}_1^X &= L_X^1(\bar{a}_1) + \frac{2}{5}(3\lambda^2 + 3\lambda - 1)J_3(X, \bar{a}_3)\\
&\quad + \frac{1}{6}\lambda(\lambda+1)(2\lambda+1)J_4(X, \bar{a}_4)\\
\bar{a}_0^X &= L_X^0(\bar{a}_0) + \frac{2}{3}\lambda(\lambda+1)J_3(X, \bar{a}_2)\\
&\quad + \frac{1}{6}\lambda(\lambda+1)(2\lambda+1)J_4(X, \bar{a}_3)\\
&\quad + \frac{1}{420}\lambda(\lambda+1)(12\lambda^2 + 12\lambda + 11)J_5(X, \bar{a}_4),
\end{aligned}
\tag{6}
$$

where $\bar{a}_i^X$ are coefficients of the normal symbol of the operator ad $L_X^\lambda(A)$, and the expressions J_m are

$$
\begin{aligned}
J_3(X, \bar{a}_s) &= X'''\bar{a}_s\\
J_4(X, \bar{a}_s) &= sX^{(IV)}\bar{a}_s + 2X'''\bar{a}_s'\\
J_5(X, \bar{a}_s) &= s(2s-1)X^{(V)}\bar{a}_s + 5(2s-1)X^{(IV)}\bar{a}_s' + 10X'''\bar{a}_s''.
\end{aligned}
$$

It follows that the mapping $\mathcal{D}^4 \to \mathcal{F}_4 \oplus \cdots \oplus \mathcal{F}_0$ defined by (5) is sl_2-equivariant. Indeed, for a vector field $X \in sl_2$ (which is a polynomial in x of degree ≤ 2) all the terms $J_m(X, \bar{a}_s)$ in (6) vanish.

Proposition 4.1 (i) is proven.

Let us prove the uniqueness. By definition, the functions $\bar{a}_i(x)$ are linear expressions in $a_s(x)$, and their derivatives

$$
\bar{a}_i(x) = \sum_{s,t} \alpha_t^s(x) a_s^{(t)}(x),
$$

where $a_s^{(t)} = d^t a_s/dx^t$, $\alpha_k^j(x)$ are some functions. The fact that the normal symbol $\overline{P}_A$ is sl_2-equivariant means that for a vector field $X \in sl_2$, $\bar{a}_i^X = L_X^i(a_i)$.

(a) Substitute $X = d/dx$ to obtain that the coefficients α_k^j do not depend on x.

(b) Substitute $X = x d/dx$ to obtain the condition $j - k = i$:

$$\bar{a}_i(x) = \sum_{j=4}^{i} \alpha^j a_j^{(j-i)}(x),$$

and $\alpha^i \neq 0$.

(c) Put $\alpha^i = 1$ and, finally, substitute $X = x^2 d/dx$ to obtain the coefficients from (5). Proposition 4.1 (ii) is proven. ■

The notion of normal symbol of a fourth-order differential operator plays a central role in this paper.

Remark: *the transvectants*. Operations $J_3(X, a_s), J_4(X, a_s), J_5(X, a_s)$ are particular cases of the following remarkable bilinear operations on tensor-densities. Consider the expressions

$$j_n(\phi, \psi) = \sum_{i+j=n} (-1)^i \binom{n}{i} \frac{(2\lambda - i)!(2\mu - j)!}{(2\lambda - n)!(2\mu - n)!} \phi^{(i)} \psi^{(j)}$$

where $\phi = \phi(x), \psi = \psi(x)$ are smooth functions.

This operation defines a unique (up to a constant) sl_2-equivariant mapping

$$\mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu-n}.$$

Operations $j_n(\phi, \psi)$ were discovered by Gordan [6]; they are also known as Rankin-Cohen brackets (see [11], [2]).

Note that the operations J_m from the formula (6) are proportional to j_n for $X \in \mathcal{F}_1$, $a_s \in \mathcal{F}_{-s}$.

5 Diagonalization of the operator T

We will obtain here an important property of the intertwining operator (2): in terms of normal symbol it has a diagonal form. We will also prove part (i) of Theorem 1 and Theorem 4.

Proof of Theorem 1, part (i). Let us define an isomorphism of modules $\mathcal{D}^3_\lambda$ and $\mathcal{D}^3_\mu$ for $\lambda \neq 0, -1, -1/2, -1/2 \pm \sqrt{21}/6$. Associate to $A \in \mathcal{D}^3_\lambda$ the operator $T(A) \in \mathcal{D}^3_\mu$:

$$T: \; a_3 \frac{d^3}{dx^3} + a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0 \;\longmapsto\; a_3^T \frac{d^3}{dx^3} + a_2^T \frac{d^2}{dx^2} + a_1^T \frac{d}{dx} + a_0^T$$

such that its standard symbol

$$\overline{P}_{T(A)} = \xi^3 \overline{a^T}_3(x) + \xi^2 \overline{a^T}_2(x) + \xi \overline{a^T}_1(x) + \overline{a^T}_0(x)$$

is given by

$$\overline{a^T}_3(x) = \bar{a}_3(x)$$
$$\overline{a^T}_2(x) = \frac{2\mu+1}{2\lambda+1}\bar{a}_2(x)$$
$$\overline{a^T}_1(x) = \frac{3\mu^2+3\mu-1}{3\lambda^2+3\lambda-1}\bar{a}_1(x)$$
$$\overline{a^T}_0(x) = \frac{\mu(\mu+1)(2\mu+1)}{\lambda(\lambda+1)(2\lambda+1)}\bar{a}_0(x). \tag{7}$$

It follows immediately from the formula (6) that this formula defines an isomorphism of Vect($\mathbf{R}$)-modules $T : \mathcal{D}^3_\lambda \cong \mathcal{D}^3_\mu$.

Theorem 1 (i) is proven. ∎

Proof of Theorem 4. Let us show that the operator (7) in terms of symmetric expressions of Lie derivatives is given by (3).

The first equality in (3) coincides with the last equality in (7).

(a) Consider a first-order operator of a Lie derivative

$$L^\lambda_X = X(x)\frac{d}{dx} - \lambda X'(x).$$

Its normal symbol defined by (5) is

$$\overline{P}_{L^\lambda_X} = \xi X(x).$$

One obtains the second equality of the formula (3).

(b) The anticommutator

$$[L^\lambda_X, L^\lambda_Y]_+ = 2XY\frac{d^2}{dx^2} + (1-2\lambda)(XY)'\frac{d}{dx} - \lambda(XY'' + X''Y) + 2\lambda^2 X'Y'$$

has the normal symbol

$$\overline{P}_{[L^\lambda_X, L^\lambda_Y]_+} = 2\xi^2 XY - \frac{2}{3}\lambda(\lambda+1)(XY'' + X''Y - X'Y'),$$

which also follows from (5). The third equality of (3) follows now from the second and the fourth ones of (7).

(c) The normal symbol of a third-order expression $[L^\lambda_X, L^\lambda_Y, L^\lambda_Z]_+ := \mathrm{Sym}_{X,Y,Z}(L^\lambda_X L^\lambda_Y L^\lambda_Z)$ can be also easily calculated from (5). The result is

$$\begin{aligned}\overline{P}_{[L^\lambda_X, L^\lambda_Y, L^\lambda_Z]_+} = {} & 6\xi^3 XYZ \\ & -(3\lambda^2+3\lambda-1)\xi\left(XYZ'' + XY''Z + X''YZ - \frac{1}{5}(XYZ)''\right) \\ & -\lambda(\lambda+1)(2\lambda+1)(XYZ''' + XY'''Z + X'''YZ).\end{aligned}$$

This formula implies the last equality of (3). ∎

Remarks. (a) The normal symbols of $[L^\lambda_X, L^\lambda_Y]_+$ and $[L^\lambda_X, L^\lambda_Y, L^\lambda_Z]_+$ are given by very simple and harmonic expressions (which implies the diagonal form (3) of the operator T). It would be interesting to understand better the geometric reason for this fact.

(b) Comparing the formulae (3) and (7), one finds a coincidence between coefficients. This fact shows that, in some sense, the symmetric expressions of Lie derivatives and the normal symbol represent the same thing in terms of differential operators and in terms of polynomial functions on $T^*\mathbf{R}$, respectively. We do not see any reason a priori for this remarkable coincidence.

6 Uniqueness of the operator T

In this section we prove that the isomorphism T defined by the formula (7) is unique (up to a constant). We also show that in the higher-order case $k \geq 4$ there is no analogue of this operator.

Proof of Theorem 3. The normal symbol of an operator $A \in \mathcal{D}^3_\lambda$ is at the same time a normal symbol of $T(A) \in \mathcal{D}^3_\mu$, since the operator T is equivariant. The normal symbol is unique up to normalization (cf. Proposition 4.1, part (ii)), and therefore the polynomial $\overline{P}_{T(A)}(x, \xi)$ defined by the formula (5) is of the form

$$\overline{P}_{T(A)}(x, \xi) = \alpha_3 \xi^3 \bar{a}_3(x) + \alpha_2 \xi^2 \bar{a}_2(x) + \alpha_1 \xi \bar{a}_1(x) + \alpha_0 \bar{a}_0(x),$$

where $\alpha_i \in \mathbf{R}$ are some constants depending on λ and μ. Choose $\alpha_3 = 1$. It follows immediately from the formula (6) (after substitution $a_4 \equiv 0$) that the formula (7) gives the unique choice of the constants $\alpha_2, \alpha_1, \alpha_0$ such that the operator T is equivariant.

Theorem 3 is proven. ■

Proof of Theorem 2. Suppose now that $\Phi : \mathcal{D}^4_\lambda \to \mathcal{D}^4_\mu$ is an isomorphism. In the same way, it follows that in terms of normal symbols, operator Φ is diagonal. More precisely, if $A \in \mathcal{D}^4_\lambda$, then the normal symbol of the operator $\Phi(A) \in \mathcal{D}^4_\mu$ is

$$\overline{P}_{\Phi(A)}(x, \xi) = \alpha_4 \xi^4 \bar{a}_4(x) + \alpha_3 \xi^3 \bar{a}_3(x) + \alpha_2 \xi^2 \bar{a}_2(x) + \alpha_1 \xi \bar{a}_1(x) + \alpha_0 \bar{a}_0(x),$$

where $\bar{a}_i$ are the components of the normal symbol of A, $\alpha_i \in \mathbf{R}$. The condition of equivariance implies

$$\begin{aligned}
\frac{\alpha_2}{\alpha_0} &= \frac{\lambda(\lambda+1)}{\mu(\mu+1)}, & \frac{\alpha_3}{\alpha_0} &= \frac{\lambda(\lambda+1)(2\lambda+1)}{\mu(\mu+1)(2\mu+1)} \\
\frac{\alpha_4}{\alpha_0} &= \frac{\lambda(\lambda+1)(12\lambda^2+12\lambda+11)}{\mu(\mu+1)(12\mu^2+12\mu+11)}, & \frac{\alpha_4}{\alpha_2} &= \frac{6\lambda^2+6\lambda-5}{6\mu^2+6\mu-5} \\
\frac{\alpha_3}{\alpha_1} &= \frac{3\lambda^2+3\lambda-1}{3\mu^2+3\mu-1}, & \frac{\alpha_4}{\alpha_1} &= \frac{\lambda(\lambda+1)(2\lambda+1)}{\mu(\mu+1)(2\mu+1)}.
\end{aligned}$$

This system of equations has solutions if and only if $\lambda = \mu$ or $\lambda + \mu = -1$. For $\lambda = \mu$ one has $\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$, and for $\lambda + \mu = -1$ one has $\alpha_0 = -\alpha_1 = \alpha_2 = -\alpha_3 = \alpha_4$. Theorem 2 is proven for $k = 4$.

Theorem 2 follows now from one of the results of [9]: given an isomorphism $\Phi : \mathcal{D}^k_\lambda \to \mathcal{D}^k_\mu$, then the restriction of Φ to $\mathcal{D}^4_\lambda$ is an isomorphism of Vect($\mathbf{R}$)-modules: $\mathcal{D}^4_\lambda \to \mathcal{D}^4_\mu$. (To prove this, it is sufficient to suppose equivariance of Φ with respect to the affine algebra with generators $d/dx, xd/dx$; see [9]).

This implies that $\lambda = \mu$ or $\lambda + \mu = -1$. Theorem 2 is proven. ■

7 Relation with the cohomology group $H^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$

The problem of isomorphism of Vect($\mathbf{R}$)-modules $\mathcal{D}^k_\lambda$ for different values of λ is related to the first cohomology group $H^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$. This cohomology group has already been calculated by B. L. Feigin and D. B. Fuchs (in the case of formal series) [5].

Nontrivial cocycles. The relation of Vect($\mathbf{R}$)-action on the space of differential operators and the cohomology groups $H^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$ is given by the following construction.

Let us associate to the bilinear mappings J_m, defined by the formula (6), a linear mapping $c_m : \mathrm{Vect}(\mathbf{R}) \to \mathrm{Hom}(\mathcal{F}_s, \mathcal{F}_{s+1-m})$:

$$c_m(X)(a) := J_m(X, a),$$

where $a \in \mathcal{F}_s$.

A remarkable property of transvectants J_3 and J_4 is the following lemma.

Lemma 7.1. For each value of s, the mappings c_3 and c_4 are nontrivial cocycles on Vect($\mathbf{R}$):

(i) $c_3 \in Z^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_s, \mathcal{F}_{s-2}))$,
(ii) $c_4 \in Z^1(\mathrm{Vect}(\mathbf{R}); \mathrm{Hom}(\mathcal{F}_s, \mathcal{F}_{s-3}))$. □

Proof. From the fact that the formula (6) defines a Vect($\mathbf{R}$) action, one checks that, for any $X, Y \in \mathrm{Vect}(\mathbf{R})$,

$$[L_X, c_m(Y)] - [L_Y, c_m(X)] = c_m([X, Y])$$

with $m = 3, 4$. This means that c_3 and c_4 are cocycles.

The cohomology classes $[c_3], [c_4] \neq 0$. Indeed, verify that c_3 and c_4 are cohomological to the nontrivial cocycles:

$$\widetilde{c}_3(X)(a) = X'''a + 2X''a',$$

$$\widetilde{c}_4(X)(a) = X'''a' + X''a''$$

from [5]. Lemma 7.1 is proven. ■

Proof of Theorem 1, part (ii). First, remark that for every $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$, the formula

$$\rho_X(a_3) = L_X^3(a_3)$$

$$\rho_X(a_2) = L_X^2(a_2)$$

$$\rho_X(a_1) = L_X^1(a_1) + \alpha_1 J_3(X, a_3)$$

$$\rho_X(a_0) = L_X^0(a_0) + \alpha_2 J_3(X, a_2) + \alpha_3 J_4(X, \bar{a}_3)$$

defines a Vect($\mathbf{R}$)-action. Indeed, this formula coincides with (6) in the case $a_4 \equiv 0$ and for the special values of $\alpha_1, \alpha_2, \alpha_3$; however, the constants $\alpha_1, \alpha_2, \alpha_3$ are independent.

The Vect($\mathbf{R}$)-action ρ is a *nontrivial 3-parameter deformation* of the standard action on the direct sum $\mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0$.

The fact that the cocycles c_3 and c_4 are nontrivial is *equivalent* to the fact that the defined Vect($\mathbf{R}$)-modules with

(a) $\alpha_1, \alpha_2, \alpha_3 \neq 0$,

(b) $\alpha_1 = 0$, $\alpha_2 \neq 0$, $\alpha_3 \neq 0$,

(c) $\alpha_1 \neq 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$,

(d) $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_3 = 0$,

are not isomorphic to each other.

The Vect($\mathbf{R}$)-modules $\mathcal{D}_\lambda^3$ (given by the formula (6) with $a_4 \equiv 0$) corresponds to the case (a) for general values of λ, to the case (b) for $\lambda = -1/2 \pm \sqrt{21}/6$, to the case (c) for $\lambda = -1/2$, and to the case (d) for $\lambda = 0, -1$. Therefore, one obtains 5 critical values of the degree for which Vect($\mathbf{R}$)-module structure on the space of third-order operators is special. Theorem 1 (ii) is proven. ■

Remark. For each value of λ, at least one of constants $\alpha_1, \alpha_2, \alpha_3 \neq 0$. This implies that the module $\mathcal{D}_\lambda^3$ is not isomorphic to the direct sum $\mathcal{F}_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0$.

8 Explicit formula for the intertwining operator

We give here the explicit formula for the operator (2) intertwining Vect($\mathbf{R}$)-actions on $\mathcal{D}^3$, which follows from the expression for the operator T in terms of the normal symbol (7).

For every $A \in \mathcal{D}^3_\lambda$, the operator $T(A) \in \mathcal{D}^3_\mu$

$$T(A) = a_3^T \frac{d^3}{dx^3} + a_2^T \frac{d^2}{dx^2} + a_1^T \frac{d}{dx} + a_0^T$$

is given by the following formula:

$$\begin{aligned}
a_3^T &= a_3, \\
a_2^T &= \frac{2\mu+1}{2\lambda+1} a_2 + \frac{3(\mu-\lambda)}{2\lambda+1} a_3', \\
a_1^T &= \frac{3\mu^2+3\mu-1}{3\lambda^2+3\lambda-1} a_1 \\
&\quad + \frac{(\lambda-\mu)(\mu(12\lambda-1)-\lambda+3)}{2(2\lambda+1)(3\lambda^2+3\lambda-1)} a_2' \\
&\quad + \frac{3}{2} \frac{\mu^2(5\lambda-1)-\mu(6\lambda^2+\lambda-1)+\lambda^3+2\lambda^2-\lambda}{(2\lambda+1)(3\lambda^2+3\lambda-1)} a_3'' \\
a_0^T &= \frac{\mu(\mu+1)(2\mu+1)}{\lambda(\lambda+1)(2\lambda+1)} a_0 \\
&\quad - \frac{\mu^3(3\lambda+5)-\mu^2(3\lambda^2-6)-\mu(5\lambda^2+6\lambda)}{(\lambda+1)(2\lambda+1)(3\lambda^2+3\lambda-1)} a_1' \\
&\quad + \frac{\mu^3(3-\lambda)-\mu^2(6\lambda^2+7\lambda-5)+\mu(7\lambda^3+4\lambda^2-5\lambda)}{2(\lambda+1)(2\lambda+1)(3\lambda^2+3\lambda-1)} a_2'' \\
&\quad - \frac{\mu^3(3\lambda^2+1)-3\mu^2(\lambda^2+2\lambda-1)-\mu(3\lambda^4-3\lambda^3-5\lambda^2+3\lambda)}{2(\lambda+1)(2\lambda+1)(3\lambda^2+3\lambda-1)} a_3'''.
\end{aligned} \tag{8}$$

We de not prove this formula since we do not use it in this paper.

Remarks. (a) If $\lambda = \mu$, then the operator T defined by this formula is the identity; if $\lambda + \mu = -1$, then T is the operator of conjugation.

(b) The fact that operator T is equivariant implies that *the formula (8) does not depend on the choice of the coordinate* x.

9 Discussion and final remarks

Let us give here a few examples and applications of the normal symbols (5).

Examples. The notion of normal symbol was introduced in [4] in the case of second-order differential operators. In this case, for $\lambda = 1/2$ (operators on $-1/2$-densities), the normal symbol (5) is just the standard total symbol: $\bar{a}_2 = a_2, \bar{a}_1 = a_1, \bar{a}_0 = a_0$. This corresponds to the classical example of second-order operators on $-1/2$-densities (cf. the footnote in the introduction).

In the same way, the semi-integer values $\lambda = 1, 3/2, 2, \ldots$ correspond to particularly simple expressions of the normal symbol for operators of order $k = 3, 4, 5, \ldots$.

Modules of second-order differential operators on $\mathbf{R}$. (a) The module of second-order operators $\mathcal{D}^2_\lambda$ with $\lambda = 0, -1$, is decomposed to a direct sum

$$\mathcal{D}^2_0 \cong \mathcal{D}^2_{-1} \cong \mathcal{F}_2 \oplus \mathcal{F}_1 \oplus \mathcal{F}_0.$$

Indeed, the coefficients $\bar{a}_2, \bar{a}_1, \bar{a}_0$ transform as tensor-densities (cf. formula (6)). This module is special: $\mathcal{D}^2_\lambda$ is not isomorphic to $\mathcal{D}^2_0$ for $\lambda \neq 0, -1$ (see [4]).

(b) For every $\lambda, \mu \neq 0, -1$, $\mathcal{D}^2_\lambda \cong \mathcal{D}^2_\mu$.

Operators on $1/2$*-densities.* For every $k \geq 3$, the module $\mathcal{D}^k_{-1/2}$ (corresponding to $1/2$-densities) is special. It is decomposed into a sum of submodules: of symmetric operators and of skew-symmetric operators.

Normal symbol and Weil symbol. The Weil quantization defines a 1-parameter family of mappings from the space of polynomials $C[\xi, x]$ to the space of differential operators on $\mathbf{R}$ with polynomial coefficients. One associates to a polynomial the symmetric expression in $\hbar(d/dx)$ and x: $F(\xi, x) \mapsto \mathrm{Sym}\, F(\hbar(d/dx), x)$. This 1-to-1 correspondence between differential operators and polynomials is sl_2-equivariant. However, in the Weil quantization the action of the Lie algebra sl_2 on differential operators is generated by $x^2, x(d/dx) + (d/dx)x, (d^2/dx^2)$, and, therefore, is completely different from the normal symbol.

Automorphic (pseudo-) differential operators. The notion of canonical symbol is related (and in some sense inverse) to the construction of the recent work by P. Cohen, Yu. I. Manin, and D. Zagier [3] of a PSL_2-equivariant (pseudo-) differential operator associated to a holomorphic tensor-density on the upper half-plane.

Exotic $\star$-product. Another way to understand this sl_2-equivariant correspondence between linear differential operators and polynomials in ξ, x leads to a $\star$-product on the algebra of Laurent polynomials on $T^*\mathbf{R}$. This $\star$-product is projectively invariant and nonequivalent to the standard Moyal-Weil $\star$-product (see [10]).

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