

## Lorentzian Worldlines and the Schwarzian Derivative

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The aim of this note is to relate the classical Schwarzian derivative and the geometry of Lorentz surfaces of constant curvature.

1. The starting point of our investigations lies in the following result (established together with L. Guieu). Consider a curve  $y = f(x)$  in the Lorentz plane with the metric  $g = dx dy$ . If  $f'(x) > 0$ , then the Lorentz curvature of this curve can be calculated,  $\varrho(x) = f''(x)(f'(x))^{-3/2}$ , and it satisfies the following remarkable property:

$$\sqrt{f'(x)} \varrho'(x) = S(f)(x), \tag{1}$$

where

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2 \tag{2}$$

stands for the Schwarzian derivative of  $f$ . (It is well known that the Schwarzian derivative defines a quadratic differential, and we write  $S(f) = S(f)(x) dx^2$ .)

2. It is now natural to look for all Lorentz metrics admitting this specific property. We first consider an orientation-preserving diffeomorphism  $f: \mathbb{R}P^1 \rightarrow \mathbb{R}P^1$ . Its graph is a time-like curve in the torus  $\mathbb{R}P^1 \times \mathbb{R}P^1$  endowed with the Lorentz metric  $g = g(x, y) dx dy$ , where  $g(x, y)$  is a positive function. Denoting by  $t$  the Lorentz arc length (the so-called proper time), we have the following assertion.

**Theorem.** For the equation

$$d\varrho dt = S(f) \tag{3}$$

to hold for any orientation-preserving diffeomorphism  $f$  of  $\mathbb{R}P^1$ , it is necessary and sufficient that

$$g = \frac{dx dy}{(axy + bx + cy + d)^2}, \tag{4}$$

where  $a, b, c$ , and  $d$  are arbitrary real constants.

Note that equation (3) is the intrinsic form of (1). The metric (4) is actually defined (and nonsingular) on the complement  $\Sigma \subset \mathbb{R}P^1 \times \mathbb{R}P^1$  of the graph of the linear-fractional transformation  $y = -(bx + d)/(ax + c)$  associated with the singular set of the metric. Clearly,  $\Sigma$  has the topology of a cylinder  $\mathbb{R} \times \mathbb{T}$ .

3. The scalar curvature of the metric (4) is constant,  $R = 8(ad - bc)$ . It is well known [9, 6] that any Lorentz metric  $g$  of constant curvature can locally be reduced to one of the following forms:

$$g = dx dy \quad \text{for } R = 0, \tag{5}$$

$$g = \frac{8}{R} \frac{dx dy}{(x - y)^2} \quad \text{for } R \neq 0. \tag{6}$$

In our case (4), this reduction is global and can be obtained by the action of  $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$  on  $\mathbb{R}P^1 \times \mathbb{R}P^1$ .

4. **Proof of the theorem.** Recall that the curvature of a curve on a (pseudo-)Riemannian surface  $(\Sigma, g)$  is given by  $\varrho = \omega(v, a) g(v, v)^{-3/2}$ , where  $v$  stands for the velocity and  $a = \nabla_v v$  for the acceleration

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vector ( $\omega$  is the surface 2-form associated with  $g$ , and  $\nabla$  the Levi-Civita connection). For a time-like curve  $\tau \mapsto (x(\tau), y(\tau))$  in  $\mathbb{R}P^1 \times \mathbb{R}P^1$ , one has

$$\varrho = \frac{x'y'' - x''y'}{g^{1/2}(x'y')^{3/2}} - \frac{x'\partial_x g - y'\partial_y g}{g^{3/2}(x'y')^{1/2}}$$

as a function of the parameter  $\tau$ . We can readily see that

$$\begin{aligned} \sqrt{g(v, v)} \varrho' = & -\frac{x'''}{x'} + \frac{3}{2} \left(\frac{x''}{x'}\right)^2 + \frac{y'''}{y'} - \frac{3}{2} \left(\frac{y''}{y'}\right)^2 \\ & - x'^2 \left[ \frac{\partial_x^2 g}{g} - \frac{3}{2} \left(\frac{\partial_x g}{g}\right)^2 \right] + y'^2 \left[ \frac{\partial_y^2 g}{g} - \frac{3}{2} \left(\frac{\partial_y g}{g}\right)^2 \right]. \end{aligned} \quad (7)$$

In the right-hand side of equation (7) we recognize the difference  $S(y)(\tau) - S(x)(\tau)$  of the Schwarzian derivatives.

Let us show that the extra terms vanish simultaneously if and only if the metric is given by (4). Indeed, if  $g(x, y) = \partial_x \varphi(x, y) = \partial_y \tilde{\varphi}(x, y)$  with  $S(\varphi)(x) = S(\tilde{\varphi})(y) = 0$ , then  $\varphi(x, y) = (\alpha(y)x + \beta(y))/(\gamma(y)x + \delta(y))$  and  $\tilde{\varphi}(x, y) = (\tilde{\alpha}(x)y + \tilde{\beta}(x))/(\tilde{\gamma}(x)y + \tilde{\delta}(x))$ , where the unimodularity condition holds:  $\alpha\delta - \beta\gamma = \tilde{\alpha}\tilde{\delta} - \tilde{\beta}\tilde{\gamma} = 1$ . Since  $\partial_x \varphi(x, y) = 1/(\gamma(y)x + \delta(y))^2 = \partial_y \tilde{\varphi}(x, y) = 1/(\tilde{\gamma}(x)y + \tilde{\delta}(x))^2$ , it follows that the functions  $\gamma, \delta, \tilde{\gamma}$ , and  $\tilde{\delta}$  are affine, which immediately yields (4).

Substituting  $y = f(x)$  and  $\tau = x$ , and using the definition of the arc length,  $g(v, v) = f'(x)(dx/dt)^2 = 1$ , we readily obtain (3) from (7).  $\square$

5. Amazingly, our standpoint allows us to recover the definition [4] of the relative Schwarzian derivative of two mappings of  $\mathbb{R}P^1$ .

**Corollary.** *Let  $\tau \mapsto (x(\tau), y(\tau))$  be a curve in  $\mathbb{R}P^1 \times \mathbb{R}P^1$ , as in the theorem above. Then equation (3) becomes*

$$d\varrho dt = S(x, y), \quad (8)$$

where  $S(x, y)$  stands for the relative Schwarzian derivative of the variables  $x$  and  $y$ .

6. Recall that, in the classical Liouville theory associated with Riemannian surfaces of constant curvature, the Schwarzian derivative naturally enters the transformation law of the Kähler metric under conformal transformations (see, e.g., [1, p. 118]).

An analogous phenomenon occurs in the case of (real) Lorentz surfaces, namely, the Schwarzian derivative (2) of a conformal diffeomorphism  $f \in \text{Conf}(\Sigma, g) \cong \text{Diff}(\mathbb{R}P^1)$  for the metric (6) is interpreted in [5] as an obstruction for  $f$  to be an isometry. The conformal classes of the metrics (5) and (6) are studied in [2], where these classes are shown to be symplectomorphic to coadjoint orbits of the Virasoro group. In these papers, the Schwarzian derivative arises as a 1-cocycle on the group  $\text{Conf}(\Sigma, g)$  and encodes the behavior of the metric near the conformal boundary.

However, the results of this note have no direct relationship to those cited above.

7. Recently, E. Ghys proved that, for any diffeomorphism  $f$  of  $\mathbb{R}P^1$ , the Schwarzian derivative  $S(f)$  has at least four geometrically distinct zeroes [3] (see also [7, 8]). This theorem was discovered by Ghys as an analog of the classical four-vertex theorem: *the curvature of any smooth closed plane curve has at least four geometrically distinct critical points*. We have thus proved that the Ghys theorem is precisely the four-vertex theorem for time-like closed curves in  $\Sigma \subset \mathbb{R}P^1 \times \mathbb{R}P^1$  endowed with the Lorentzian metric of constant curvature (4).

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