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Conformally equivariant quantum Hamiltonians

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Abstract. Let (M, g) be a pseudo-Riemannian manifold and $\mathcal{F}_\lambda(M)$ the space of densities of degree λ on M. We study the space $\mathcal{D}^2_{\lambda,\mu}(M)$ of second-order differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. If (M, g) is conformally flat with signature $p - q$, then $\mathcal{D}^2_{\lambda,\mu}(M)$ is viewed as a module over the group of conformal transformations of M. It turns out that, for almost all values of $\mu - \lambda$, the $\mathrm{O}(p + 1, q + 1)$-modules $\mathcal{D}^2_{\lambda,\mu}(M)$ and the space of symbols (i.e., of second-order polynomials on T^*M) are canonically isomorphic. This yields a conformally equivariant quantization for quadratic Hamiltonians. We furthermore show that this quantization map extends to arbitrary pseudo-Riemannian manifolds and depends only on the conformal class $[\mathrm{g}]$ of the metric. As an example, the quantization of the geodesic flow yields a novel conformally equivariant Laplace operator on half-densities, as well as the well-known Yamabe Laplacian. We also recover in this framework the multi-dimensional Schwarzian derivative of conformal transformations.

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1. Introduction

The aim of this article is to investigate the relationship between differential operators on a smooth pseudo-Riemannian manifold (M, g) of signature $p - q$, and the polynomial functions on its cotangent bundle T^*M.

We will consider the space $\mathcal{D}(M)$ of differential operators on C^∞-function of M viewed as a module for the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M. We are, in fact, interested in a two-parameter family of modules which can be understood as follows. Considering that the arguments of these differential operators are, indeed, tensor densities of, say weight λ, and their values tensor densities of weight μ, we will, hence, deal with a new $\mathrm{Diff}(M)$-module structure denoted by $\mathcal{D}_{\lambda,\mu}(M)$.

The natural $\mathrm{Diff}(M)$-module of symbols associated with $\mathcal{D}_{\lambda,\mu}(M)$ is the space of fiberwise polynomials on T^*M with values in the $(\mu - \lambda)$-densities over M. Therefore, we have a one-parameter family of $\mathrm{Diff}(M)$-modules, $\mathcal{S}_\delta(M)$, where $\delta = \mu - \lambda$.

The modules $\mathcal{D}_{\lambda,\mu}(M)$ have already been considered in the classic literature on differential operators and, more recently, in a series of papers [8], [9], [10], [19], [13], [20], [12], [22]. The general problem of classification of these Diff(M)-modules has been solved in these articles.

We will be considering the modules of second-order operators, $\mathcal{D}^2_{\lambda,\mu}(M)$, and symbols, $\mathcal{S}^2_\delta(M)$.

The main purpose of this article is to define a canonical isomorphism

$$\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^2_{\mu-\lambda}(M) \xrightarrow{\cong} \mathcal{D}^2_{\lambda,\mu}(M) \tag{1.1}$$

that satisfies the following properties:

1. It is conformally *invariant*, i.e., it depends only on the conformal class [g] of the metric.
2. In the conformally flat case, it is *equivariant* with respect to $\mathrm{O}(p+1,q+1)$, the group of conformal diffeomorphisms.

We will show that the isomorphism (1.1) exists for generic λ and μ; in the most interesting case $\lambda = \mu = \frac{1}{2}$, it provides a natural quantization of the cotangent bundle of a pseudo-Riemannian manifold.

In the conformally flat case, the problem has been solved in [9]: the isomorphism (1.1) is characterized by the second property of $\mathrm{O}(p+1,q+1)$-equivariance and is essentially unique (up to a natural normalization).

This article constitutes the final stage of work started with the preprint [9] where the point of view of conformally equivariant quantization was first espoused.

1.1. Conformally flat case

Let us assume that the manifold M is endowed with a flat conformal structure which enables us to look for a conformally equivariant quantization with respect to the group $\mathrm{O}(p+1,q+1)$ (or its Lie algebra $\mathrm{o}(p+1,q+1)$) where $\dim(M) = p+q$ acting (locally) on M. The starting point of the present article consists in the following two results which first appeared in [9].

Theorem 1.1. *Given a conformally flat pseudo-Riemannian manifold M of dimension $n = p+q \geq 2$,*

(i) *there exists an isomorphism* (1.1) *of* $\mathrm{o}(p+1,q+1)$*-modules provided*

$$\mu - \lambda \notin \left\{ \frac{2}{n}, \frac{n+2}{2n}, 1, \frac{n+1}{n}, \frac{n+2}{n} \right\}. \tag{1.2}$$

(ii) *For every λ and μ as in* (1.2)*, this isomorphism is unique under the condition that the principal symbol be preserved at each order.*

Theorem 1.1 has recently been generalized in [10] to the case of higher-order polynomials. The purpose of this paper is to provide explicit expressions for the isomorphism (1.1) that are out of reach in the higher-order case. Such a study is motivated by the special relevance of second-order Hamiltonians in mathematical physics.

The singular values (1.2) of the shift $\delta = \mu - \lambda$ are called *resonances* and lead to special and interesting modules. It worth noticing that for any λ and μ there is no isomorphism (1.1) equivariant with respect to the full group $\mathrm{Diff}(M)$.

Theorem 1.2. *For each resonant value of δ, there exist particular pairs (λ, μ) of weights such that the $\mathrm{o}(p+1, q+1)$-modules $\mathcal{S}^2_\delta(M)$ and $\mathcal{D}^2_{\lambda,\mu}(M)$ are isomorphic, namely*

δ	$\frac{2}{n}$	$\frac{n+2}{2n}$	1	$\frac{n+1}{n}$	$\frac{n+2}{n}$
λ	$\frac{n-2}{2n}$	$0, \frac{n-2}{2n}$	0	$0, -\frac{1}{n}$	$-\frac{1}{n}$
μ	$\frac{n+2}{2n}$	$\frac{n+2}{2n}, 1$	1	$\frac{n+1}{n}, 1$	$\frac{n+1}{n}$

(1.3)

The isomorphism (1.1) is, in fact, not unique; there exists a one-parameter family of such isomorphisms in each resonant case.

Remark 1.3. This point of view on equivariant quantization was adopted in [20] where a projectively equivariant symbol calculus and quantization was introduced if M is endowed with a flat projective structure. In this case the group of (local) symmetries is $G = \mathrm{SL}(n+1, \mathbb{R})$ with $n = \dim(M)$. See also [18] for a cohomological treatment of this subject. Bearing in mind that the best-known geometries associated with a local and maximal symmetry group are the projective and conformal geometries (cf. [20], [3]), we have been led to look, in the same spirit, for a conformally equivariant quantization.

Remark 1.4. In the particular case $n = 1$, the projective and conformal symmetries coincide; our results are in full accordance with those obtained in [13], [12], [7] and the resonances are simply $\{1, \frac{3}{2}, 2\}$.

1.2. Generic pseudo-Riemannian case

It turns out that the isomorphism $\mathcal{Q}_{\lambda,\mu}$ makes sense for an arbitrary pseudo-Riemannian manifold (not necessarily conformally flat). We will prove here the following fundamental property of this isomorphism: $\mathcal{Q}_{\lambda,\mu}$ depends only on the conformal class [g] of the metric (i.e., it is conformally invariant) — see Theorem 4.7.

We will show that the condition of conformal invariance uniquely determines the isomorphism $\mathcal{Q}_{\lambda,\mu}$ in some natural class of differential linear maps. This enables us to introduce a conformally invariant quantization on the cotangent bundle of a pseudo-Riemannian manifold.

Note that we understand the term "quantization" in a somewhat generalized sense as λ and μ remain essentially arbitrary. In the case $\lambda = \mu = \frac{1}{2}$, we recover the usual terminology using the Hilbert space of half-densities considered in the framework of geometric quantization. However, our approach yields a new form for the quantized Hamiltonian. For example, our quantization of the geodesic flow brings a novel coefficient in front of the scalar curvature.

As an illustration of our general results, we consider a number of examples. The celebrated Yamabe-Laplace operator is, among others, naturally included into our considerations. It appears as the quantized geodesic flow in one of the resonant cases (1.3); the corresponding symbol is itself conformally invariant. This explains why the Yamabe-Laplace operator is the unique conformally invariant Laplace-Beltrami operator. It should be stressed, however, that it is unjustified to consider the Yamabe-Laplace operator as quantum Hamiltonian for the geodesic flow.

The paper is organized as follows.

In Section 2 we recall the basic definitions concerning the space of differential operators on tensor densities, as well as the space of symbols. We put emphasis on their $\mathrm{Diff}(M)$- and $\mathrm{Vect}(M)$-module structures.

We present, in Section 3, an explicit intrinsic formula for the isomorphism $\mathcal{Q}_{\lambda,\mu}$ which defines our conformally equivariant quantization map $\mathcal{Q}_{\lambda,\mu;\hbar}$ for an arbitrary pseudo-Riemannian manifold. We also prove the conformal invariance of $\mathcal{Q}_{\lambda,\mu}$.

Section 5 provides specific examples, namely the quantization of the geodesic flow, and of the magnetic minimal coupling prescription. We also give examples of quantized Hamiltonians pertaining to the resonant cases.

We develop in Section 6 the algebraic theory of Euclidean invariants which we use in the proofs of the uniqueness theorem for the conformally equivariant quantization map.

In Sections 7 and 8 we give the technical proofs of the main theorems.

Following the concluding Section 9, an Appendix presents in a somewhat detailed fashion the covariant calculus for density-valued symbols entering the technical calculations.

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2. Differential operators and symbols

2.1. Differential operators on tensor densities

Let us start with the definition of the Diff(M)-module $\mathcal{D}_{\lambda,\mu}(M)$ (or $\mathcal{D}_{\lambda,\mu}$ for short) of differential operators on a smooth manifold M with $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

Consider the determinant bundle $\Lambda^n T^*M \to M$. Let us recall that a tensor density of degree λ on M is a smooth section, ϕ, of the line bundle $|\Lambda^n T^*M|^{\otimes\lambda}$. The space of tensor densities of degree λ is naturally a Diff(M)-module which we call $\mathcal{F}_\lambda$.

It is evident that $\mathcal{F}_0 = C^\infty(M)$; if M is oriented, the space $\mathcal{F}_1$ coincides with the space of differential n-forms: $\mathcal{F}_1 = \Omega^n(M)$.

Definition 2.1. An operator $A : \mathcal{F}_\lambda \to \mathcal{F}_\mu$ is called a local operator on M if, for all $\phi \in \mathcal{F}_\lambda$, one has $\mathrm{Supp}(A(\phi)) \subset \mathrm{Supp}(\phi)$.

It is a classical result (see [24]) that such operators are in fact locally given by differential operators. The space $\mathcal{D}_{\lambda,\mu}$ of differential operators from λ-densities to μ-densities on M is naturally a Diff(M)-module.

There is a filtration $\mathcal{D}^0_{\lambda,\mu} \subset \mathcal{D}^1_{\lambda,\mu} \subset \cdots \subset \mathcal{D}^k_{\lambda,\mu} \subset \cdots$, where the module of zero-order operators $\mathcal{D}^0_{\lambda,\mu} \cong \mathcal{F}_{\mu-\lambda}$ consists of multiplication by $(\mu - \lambda)$-densities. The higher-order modules are defined by induction: $A \in \mathcal{D}^k_{\lambda,\mu}$ if $[A, f] \in \mathcal{D}^{k-1}_{\lambda,\mu}$ for every $f \in C^\infty(M)$.

To our knowledge, the whole family of modules of differential operators viewed as a deformation were first studied in [8] in the case $\lambda = \mu$; see also [19], [20], [13], [12], [22].

2.2. Classical examples

(a) The best known example is the Sturm-Liouville operator $L = (d/dx)^2 + u(x)$ in the one-dimensional case, $M = S^1$. It should, indeed, be considered as an element $L \in \mathcal{D}^2_{-\frac{1}{2},\frac{3}{2}}$ as $\lambda = -1/2$ and $\mu = 3/2$ are the only degrees for which its form is preserved by the action of Diff(S^1).

(b) Again, in the one-dimensional case, the study of the modules $\mathcal{D}^k_{\frac{1-k}{2},\frac{1+k}{2}}$ goes back to the pioneering work of Wilczynski [27].

(c) Yet another remarkable example is provided by the Yamabe-Laplace operator $A = \Delta - (n-2)/(4(n-1))\,R$, where Δ is the usual Laplace-Beltrami operator and R the scalar curvature on a (pseudo-)Riemannian manifold (M, g) of dimension $n \geq 2$. (See, e.g. [1].) This operator has been extensively used in the mathematical and physical literature because of its characteristic property of being invariant under conformal changes of metrics. It is well known that $A \in \mathcal{D}^2_{\frac{n-2}{2n},\frac{n+2}{2n}}$.

Observe that, for $n = 1$, the latter module of differential operators precisely coincides with the Sturm-Liouville module. We will see that this is by no means accidental and will prove below (Section 5.6) that the suitably regularized Yamabe operator equals $\Delta - S(\varphi)/(2\mathrm{g})$, where S is the Schwarzian derivative and φ the diffeomorphism which defines the metric $\mathrm{g} = \varphi^*(dx^2)$.

(d) The special module $\mathcal{D}_{\frac{1}{2},\frac{1}{2}}$ has been introduced in the context of geometric quantization by Blattner [2] and Kostant [17]. This module will also naturally arise in our quantization procedure.

2.3. The modules $\mathcal{F}_\lambda$ and $\mathcal{D}_{\lambda,\mu}$

If M is orientable, which we will assume throughout the paper, then $\mathcal{F}_\lambda$ can be identified with $C^\infty(M)$ as a vector space. Given a volume form, vol, on M, one can write any λ-tensor density as $\phi = f\,|\mathrm{vol}|^\lambda$ with $f \in C^\infty(M)$, and define the $\mathrm{Diff}(M)$-module structure of $\mathcal{F}_\lambda$ via the action of $\varphi \in \mathrm{Diff}(M)$:

$$\varphi_\lambda(f) = \varphi_*(f)\left|\frac{\varphi_*\,\mathrm{vol}}{\mathrm{vol}}\right|^\lambda. \tag{2.1}$$

With this identification, the module $\mathcal{D}_{\lambda,\mu}$ can be viewed as a two-parameter family of the standard module $\mathcal{D}_{0,0}$ of differential operators on smooth functions $\mathcal{F}_0$. The natural $\mathrm{Diff}(M)$-action on $\mathcal{D}_{\lambda,\mu}$ then reads

$$\varphi_{\lambda,\mu}(A) = \varphi_\mu \circ A \circ \varphi_\lambda^{-1}. \tag{2.2}$$

The expression of a differential operator $A \in \mathcal{D}^k_{\lambda,\mu}$ in a local coordinate system (x^i) is then

$$A = A_k^{i_1\ldots i_k}\partial_{i_1}\ldots\partial_{i_k} + \cdots + A_1^i\partial_i + A_0 \tag{2.3}$$

where $\partial_i = \partial/\partial x^i$ and $A_\ell^{i_1\ldots i_\ell} \in C^\infty(M)$ with $\ell = 0, 1, \ldots, k$. (From now on we suppose a summation over repeated indices.)

The infinitesimal version of the action (2.2) is

$$L_X^{\lambda,\mu}(A) = L_X^\mu\, A - A\, L_X^\lambda \tag{2.4}$$

where $X \in \mathrm{Vect}(M)$, while the infinitesimal version of the action (2.1) is given by the Lie derivative on $\mathcal{F}_\lambda$, namely

$$L_X^\lambda(f) = X(f) + \lambda\;\mathrm{Div}(X)\,f. \tag{2.5}$$

2.4. The module of symbols $\mathcal{S}_\delta$

Consider the space $\mathcal{S} = \Gamma(S(TM))$ of contravariant symmetric tensor fields on M which is naturally a $\mathrm{Diff}(M)$-module. We can locally identify $\mathcal{S}$ with the space of polynomials

$$P(\xi) = \sum_{\ell=0}^{k} P_\ell^{i_1 \dots i_\ell} \xi_{i_1} \cdots \xi_{i_\ell}, \tag{2.6}$$

with $P_\ell^{i_1 \dots i_\ell} \in C^\infty(M)$, on the cotangent bundle of M.

Definition 2.2. The one-parameter family of $\mathrm{Diff}(M)$-actions on $\mathcal{S}$:

$$\varphi_\delta(P) = \varphi_*(P) \left| \frac{\varphi_* \, \mathrm{vol}}{\mathrm{vol}} \right|^\delta \tag{2.7}$$

identifies the space $\mathcal{S}$ with the $\mathrm{Diff}(M)$-module $\mathcal{S} \otimes \mathcal{F}_\delta$. We denote this module by $\mathcal{S}_\delta$.

We will need in the sequel the infinitesimal version of the $\mathrm{Diff}(M)$-action on $\mathcal{S}_\delta$. The action of $\mathrm{Vect}(M)$ on $\mathcal{S}_\delta$ deduced from (2.7) reads as

$$L_X^\delta(P) = L_X(P) + \delta \, \mathrm{Div}(X) \, P \tag{2.8}$$

where

$$L_X = X^i \frac{\partial}{\partial x^i} - \xi_j \partial_i X^j \frac{\partial}{\partial \xi_i} \tag{2.9}$$

is the cotangent lift of $X \in \mathrm{Vect}(M)$.

Again, there is a filtration $\mathcal{S}_\delta^0 \subset \mathcal{S}_\delta^1 \subset \cdots \subset \mathcal{S}_\delta^k \subset \cdots$, where $\mathcal{S}_\delta^k$ denotes the space of symbols of degree less or equal to k. In contrast to the filtration on the space $\mathcal{D}_{\lambda,\mu}$ of differential operators, the above filtration on the space of symbols actually leads to a $\mathrm{Diff}(M)$-invariant graduation

$$\mathcal{S}_\delta = \bigoplus_{k=0}^{\infty} \mathcal{S}_{k,\delta} \tag{2.10}$$

where $\mathcal{S}_{k,\delta}$ denotes the space of homogeneous polynomials (isomorphic to $\mathcal{S}_\delta^k / \mathcal{S}_\delta^{k-1}$).

3. Quantization map in the conformally flat case

There is no fully $\mathrm{Diff}(M)$-equivariant quantization since the modules $\mathcal{D}_{\lambda,\mu}$ are not isomorphic to the module $\mathcal{S}_{\mu-\lambda}$ of symbols. One is thus led to impose some extra geometric structure on M and to look for a symbol calculus, equivariant with respect to the automorphisms of this structure.

In this article, we will assume — unless otherwise stated — that the manifold M is endowed with a flat conformal structure.

3.1. A compendium on conformally flat structures

A conformal structure on a manifold M is given by a smooth field $[\,g\,]$ of directions of metrics. This structure is called flat if M can be locally identified with $\mathbb{R}^n$ endowed with the canonical action of the conformal Lie algebra $\mathrm{o}(p+1,q+1)$, where $n=p+q$.

The Lie algebra $\mathrm{o}(p+1,q+1)\subset \mathrm{Vect}(\mathbb{R}^n)$ is generated by the vector fields:

$$\begin{aligned} X_i &= \frac{\partial}{\partial x^i},\\ X_{ij} &= x_i\frac{\partial}{\partial x^j}-x_j\frac{\partial}{\partial x^i},\\ X_0 &= x^i\frac{\partial}{\partial x^i},\\ \overline{X}_i &= x_jx^j\frac{\partial}{\partial x^i}-2x_ix^j\frac{\partial}{\partial x^j} \end{aligned} \tag{3.1}$$

with $i,j=1,\ldots,n$; we have used the notation $x_i=g_{ij}x^j$ where the flat metric $g=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ has trace $p-q$.

The subalgebra generated by the vector fields X_i and X_{ij} is the Euclidean Lie algebra $\mathrm{e}(p,q)=\mathrm{o}(p,q)\ltimes\mathbb{R}^n$. The operator X_0 is the generator of homotheties while the vector fields $\overline{X}_i$ generate inversions.

Remark 3.1.

(a) It is well known that the conformal flatness of a n-dimensional pseudo-Riemannian manifold is equivalent to the vanishing of the Weyl curvature tensor if $n\geq 4$, and to that of the Weyl-Schouten curvature tensor if $n=3$ [1]. All two-dimensional pseudo-Riemannian manifolds are conformally flat.

(b) In the one-dimensional case the conformal Lie algebra is isomorphic to the projective Lie algebra since $\mathrm{o}(2,1)\cong\mathrm{sl}(2,\mathbb{R})$.

3.2. Conformal equivariance

Let M be endowed with a flat conformal structure: there exists a local action of the group $\mathrm{O}(p+1,q+1)$ on M, which enables us to restrict the $\mathrm{Diff}(M)$-modules $\mathcal{D}_{\lambda,\mu}$ to the conformal group. Our problem amounts then to the determination of intertwining differentiable linear maps $\mathcal{Q}^k_{\lambda,\mu}$ between the $\mathrm{o}(p+1,q+1)$-modules $\mathcal{S}^k_{\lambda-\mu}$ and $\mathcal{D}^k_{\mu,\lambda}$.

Here, we give the solution for the case $k=2$ which is the most relevant one for applications. Indeed, the existence and uniqueness of a conformally equivariant quantization map for any order k has recently been established in [10]; however, no explicit formula is available.

3.3. Expression in adapted coordinates

If we fix the local coordinate system on M for which the generators of $\mathrm{o}(p+1,q+1)$ retain the form (3.1), we have the following

Theorem 3.2. *For any dimension, n, and any δ as in* (1.2) *the unique conformally equivariant isomorphism $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^2_\delta \to \mathcal{D}^2_{\lambda,\mu}$, viz*

$$P \longmapsto \mathcal{Q}_{\lambda,\mu}(P) = A_2^{ij}\partial_i\partial_j + A_1^i\partial_i + A_0$$

that preserves the principal symbol is as follows

$$\begin{cases} A_2^{ij} = P_2^{ij} \\ A_1^i = P_1^i + \beta_1\,\partial_j P_2^{ij} + \beta_2\, g^{ij} g_{k\ell}\,\partial_j\, P_2^{k\ell} \\ A_0 = P_0 + \alpha\,\partial_i P_1^i + \beta_3\,\partial_{ij} P_2^{ij} + \beta_4\, g^{ij} g_{k\ell}\,\partial_{ij} P_2^{k\ell} \end{cases} \tag{3.2}$$

where $P(\xi) = P_2^{ij}\xi_i\xi_j + P_1^i\xi_i + P_0 \in \mathcal{S}^2_\delta$; the numerical coefficients are given by

$$\alpha = \frac{\lambda}{1-\delta} \tag{3.3}$$

and

$$\begin{aligned} \beta_1 &= \frac{2(n\lambda+1)}{2+n(1-\delta)} \\ \beta_2 &= \frac{n(\lambda+\mu-1)}{(2+n(1-\delta))(2-n\delta)} \\ \beta_3 &= \frac{n\lambda(n\lambda+1)}{(1+n(1-\delta))(2+n(1-\delta))} \\ \beta_4 &= \frac{n\lambda(n^2\mu(2-\lambda-\mu) + 2(n\lambda+1)^2 - n(n+1))}{(1+n(1-\delta))(2+n(1-\delta))(2+n(1-2\delta))(2-n\delta)}. \end{aligned} \tag{3.4}$$

We will prove this theorem in Section 8.3.

In the one-dimensional case, $n = 1$, formula (3.2) can be written as

$$\begin{cases} A_2 = P_2 \\ A_1 = P_1 + \dfrac{2\lambda+1}{2-\delta}\, P_2' \\ A_0 = P_0 + \dfrac{\lambda}{1-\delta}\, P_1' + \dfrac{\lambda(2\lambda+1)}{(3-2\delta)(2-\delta)}\, P_2''. \end{cases} \tag{3.5}$$

Remark 3.3. The projectively equivariant isomorphism $\mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ has been constructed in [20] in the special case $\lambda = \mu$ in any dimension. (See also [18] for arbitrary λ and μ, and [7], [12] for the one-dimensional case.)

4. Quantization map in the generic case

In this section we show that the isomorphism (3.2) can be generalized to an arbitrary pseudo-Riemannian manifold. This kind of problem is natural and often arises in conformal geometry. We refer to [4], [11], [14], [15] for the study of conformally invariant differential operators on tensor fields. Let us nevertheless emphasize that our modules $\mathcal{D}_{\lambda,\mu}$ of differential operators are totally different, and non-isomorphic to the latter spaces, cf. [8].

4.1. The covariant derivative of densities

Given a conformally flat manifold M of signature $p-q$, one can choose, locally, a pseudo-Riemannian metric g which represents the conformal class of the manifold. We will denote by ∇ the Levi-Civita connection. Let us now recall the definition of the covariant derivative of densities. If $\phi \in \mathcal{F}_\lambda$, then $\nabla\phi \in \Omega^1(M) \otimes \mathcal{F}_\lambda$ is defined by $\nabla\phi = df \otimes |\mathrm{vol}|^\lambda$, using the local representation $\phi = f\,|\mathrm{vol}|^\lambda$ with $f \in C^\infty(M)$.

Choose an arbitrary coordinate system (x^i) on M (with associated coordinate system (ξ_i, x^i) on T^*M); one has, for every $\phi \in \mathcal{F}_\lambda$, the local expression

$$\nabla_i\phi = \partial_i\phi - \lambda\Gamma_i\phi \tag{4.1}$$

with $\Gamma_i = \Gamma_{ij}^j$.

4.2. Case of first-order polynomials

Let us start with the simplest case, namely that of first-order symbols $\mathcal{S}_\delta^1$ on the cotangent bundle of any pseudo-Riemannian manifold (M, g).

Definition 4.1. For any $\delta \neq 1$, we call quantization map the linear map $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}_\delta^1 \to \mathcal{D}_{\lambda,\mu}^1$ defined by

$$\mathcal{Q}_{\lambda,\mu}(P) = P_1^i\nabla_i + \alpha\nabla_i(P_1^i) + P_0 \tag{4.2}$$

where $P(\xi) = P_1^i\xi_i + P_0$ and α is as in (3.3).

Note that the map (4.2) preserves the principal symbol, and coincides with (3.2) in the flat case.

It can be verified that $\mathcal{Q}_{\lambda,\mu}$ in (4.2) is, actually, equivariant with respect to the full Lie algebra $\mathrm{Vect}(M)$. This formula holds in any dimension.

Remark 4.2. In the resonant case, $\delta = 1$, the modules are still isomorphic if and only if $(\lambda,\mu) = (0,1)$ as given by Theorem 1.2. The isomorphism is not unique and given by the formula (4.2) with arbitrary α.

4.3. Case of quadratic polynomials in higher dimension

Let us now give our main definition of the quantization map for homogeneous second-order symbols $\mathcal{S}_{2,\delta}$ in the case $\dim(M) \geq 3$.

Definition 4.3. If $n \geq 3$, for any δ as in (1.2) we call quantization map the linear map $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}_{2,\delta} \to \mathcal{D}^2_{\lambda,\mu}$ which preserves the principal symbol and is defined by

$$\begin{aligned}\mathcal{Q}_{\lambda,\mu}(P) =& P^{ij}\nabla_i\nabla_j \\ &+ \left(\beta_1\nabla_i P^{ij} + \beta_2\, \mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i P^{k\ell}\right)\nabla_j \\ &+ \beta_3\nabla_i\nabla_j(P^{ij}) + \beta_4\, \mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i\nabla_j(P^{k\ell}) + \beta_5 R_{ij}P^{ij} + \beta_6 R\, \mathrm{g}_{ij}P^{ij}\end{aligned} \tag{4.3}$$

where $P(\xi) = P^{ij}\xi_i\xi_j$; the coefficients $\beta_1, \dots, \beta_4$ are given by (3.4) and

$$\begin{aligned}\beta_5 &= \frac{n^2\lambda(\mu-1)}{(n-2)(1+n(1-\delta))} \\ \beta_6 &= \frac{n^2\lambda(\mu-1)(n\delta-2)}{(n-1)(n-2)(1+n(1-\delta))(2+n(1-2\delta))}\end{aligned} \tag{4.4}$$

and R_{ij} (resp. R) denote the Ricci tensor components (resp. the scalar curvature) of the metric g.

This definition introduces the main object of our study; the crucial property of the map (4.3) is its conformal invariance (see Theorem 4.7). Again, the map (4.3) coincides with (3.2) in the flat case.

Remark 4.4. Another quantization formula for second-order polynomials has been proposed in [21] using a (pseudo-)Riemannian metric on M and the local identification of T^*M with $\mathbb{R}^{2n}$ endowed with its standard $\mathrm{sp}(2n,\mathbb{R})$ action.

The general formula (4.3) for the quantization map is obviously non applicable in the cases $n = 1$ and $n = 2$. We must therefore consider each of these cases separately.

4.4. One-dimensional case and the Schwarzian derivative

Let us consider the one-dimensional case for which all metrics are equivalent. In this case, say $M = S^1$, the metric retains the form $\mathrm{g} = \varphi^*(dx^2)$ for some $\varphi \in \mathrm{Diff}(S^1)$ with x an arbitrary coordinate.

Definition 4.5. If $n = 1$, and $\delta \neq \frac{3}{2}, 2$, we introduce the quantization map as

$$\begin{aligned}\mathcal{Q}_{\lambda,\mu}(P) =& P\,\nabla^2 + \frac{2\lambda+1}{2-\delta}\,(\nabla P)\nabla \\ &+ \frac{\lambda(2\lambda+1)}{(3-2\delta)(2-\delta)}\,(\nabla^2 P) - \frac{2\lambda(\mu-1)}{3-2\delta}\,\frac{S(\varphi)}{\mathrm{g}}P\end{aligned} \tag{4.5}$$

where $P(\xi) = P\xi^2$ and

$$S(\varphi) = \frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 \tag{4.6}$$

is the Schwarzian derivative of φ.

Note that this expression is precisely (3.5) if one fixes an affine coordinate. Comparison with the expression (4.3) strengthens the saying according to which the Schwarzian derivative is nothing but "curvature".

4.5. Two-dimensional case and the conformal Schwarzian

The two-dimensional case, $n = 2$, is especially interesting since all surfaces (M, g) are conformally flat. The Riemann uniformization theorem can be invoked to express the metric (locally) as

$$\mathrm{g} = F^{-1}\varphi^*\mathrm{g}_0 \tag{4.7}$$

where φ is a conformal diffeomorphism of M, and $F \in C^\infty(M, \mathbb{R}^*_+)$, and g_0 is a metric of constant curvature. Let us emphasize that this weaker form of the uniformization theorem still holds in the Lorentz case (see, e.g., [25]).

There exists in the recent literature an interesting generalization of the Schwarzian derivative for conformal diffeomorphisms in the multi-dimensional case. In the situation (4.7) with $F = e^{2f}$, the Schwarzian derivative of φ is defined [23], [6] as the symmetric twice-covariant tensor $S(\varphi)$ such that

$$S(\varphi)(X,Y) = X(Yf) - (\nabla_X Y)f - (Xf)(Yf) + \frac{1}{2}\|df\|^2_{\mathrm{g}}\,\mathrm{g}(X,Y) \tag{4.8}$$

for any $X, Y \in \mathrm{Vect}(M)$.

In our notation, it reads

$$S(\varphi) = \frac{1}{2F}\nabla dF - \frac{3}{4F^2}\,dF \otimes dF + \frac{1}{8F^2}\,\mathrm{g}^{-1}(dF, dF)\,\mathrm{g}. \tag{4.9}$$

This new object will enter naturally the expression of the conformally equivariant map (1.1) for surfaces.

Note that the definition (4.9) yields the classical Schwarzian derivative in the one-dimensional case.

Definition 4.6. If $n = 2$, for any δ as in (1.2) we put for the quantization map:

$$\begin{aligned} \mathcal{Q}_{\lambda,\mu}(P) = & P^{ij}\nabla_i\nabla_j \\ & + \left(\beta_1\nabla_i P^{ij} + \beta_2\,\mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i P^{k\ell}\right)\nabla_j \\ & + \beta_3\nabla_i\nabla_j(P^{ij}) + \beta_4\,\mathrm{g}^{ij}\mathrm{g}_{k\ell}\nabla_i\nabla_j(P^{k\ell}) \\ & + \frac{4\lambda(\mu-1)}{2\delta-3}\Big(S(\varphi)_{ij}P^{ij} + \frac{1}{8(\delta-1)}R\,\mathrm{g}_{ij}P^{ij}\Big) \end{aligned} \tag{4.10}$$

where $P(\xi) = P^{ij}\xi_i\xi_j$ and $S(\varphi)$ is as in (4.9) while R denotes the scalar curvature of g; the coefficients $\beta_1, \ldots, \beta_4$ are given by (3.4).

Notice that the scalar curvature of g is related to the trace of the Schwarzian derivative by

$$R = -2\mathrm{g}^{ij} S(\varphi)_{ij} \tag{4.11}$$

provided (4.7) holds.

4.6. Conformal invariance

The preceding definitions for the quantization map were actually prompted by a fundamental requirement, namely that of the conformal invariance of $\mathcal{Q}_{\lambda,\mu}$ with respect to a rescaling of the metric.

Theorem 4.7. *The map* $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^2_{\mu-\lambda} \longrightarrow \mathcal{D}^2_{\lambda,\mu}$ *defined by* (4.2), (4.3), (4.5) *and* (4.10) *is conformally invariant, i.e., it depends only on the conformal class of the metric.*

Proof. Let us choose another metric $\widehat{\mathrm{g}} = F\mathrm{g}$ with F a strictly positive valued function. In the special case of conformally flat manifolds, the map $\mathcal{Q}_{\lambda,\mu}$ is given, in an adapted coordinate system, by Theorem 3.2. Now, the adapted coordinate systems for g and $\widehat{\mathrm{g}}$ are the same. This proves the theorem in the conformally flat case, in particular for $n = 1$ and $n = 2$ in full generality.

The case of an arbitrary pseudo-Riemannian manifold needs a separate proof which goes as follows. We have

$$\widehat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \frac{1}{2F}\left(F_i\delta^k_j + F_j\delta^k_i - F^k\mathrm{g}_{ij}\right) \tag{4.12}$$

where we have used the notation $F_i = \partial_i F$ and $F^k = \mathrm{g}^{jk}F_j$.

Let us start with the proof for first-order symbols. With the help of

$$\widehat{\nabla}_i\phi = \nabla_i\phi - \frac{n\lambda}{2}\frac{F_i}{F}\phi \qquad \text{and} \qquad \widehat{\nabla}_iP^i_1 = \nabla_iP^i_1 + \frac{n(1-\delta)}{2}\frac{F_iP^i_1}{F}$$

and, using (4.1), for every $P \in \mathcal{S}^1_\delta$ we find that

$$\widehat{\mathcal{Q}}_{\lambda,\mu}(P) = \mathcal{Q}_{\lambda,\mu}(P) + \frac{n}{2}\left(\alpha(1-\delta) - \lambda\right)\frac{F_iP^i}{F}.$$

The equality $\widehat{\mathcal{Q}}_{\lambda,\mu}(P) = \mathcal{Q}_{\lambda,\mu}(P)$ is now equivalent to (3.3).

As for the second-order symbols, $P \in \mathcal{S}_{2,\delta}$, the proof involves the calculation of $\widehat{\nabla}_i\widehat{\nabla}_j\phi$ and $\widehat{\nabla}_iP^{jk}$ together with $\widehat{\nabla}_i\widehat{\nabla}_jP^{k\ell}$, which is straightforward. It also relies on the well-known transformation law [1] of the Ricci tensor, $\mathrm{Ric} = R_{ij}\, dx^i \otimes dx^j$, under a conformal rescaling, $\widehat{\mathrm{g}} = F\mathrm{g}$, namely

$$\begin{aligned}\widehat{\mathrm{Ric}} = \mathrm{Ric} - \frac{(n-2)}{2}\left(\frac{\nabla dF}{F} - \frac{3}{2}\,\frac{dF\otimes dF}{F^2}\right)\\ -\frac{1}{2}\left(\frac{\Delta F}{F} - \frac{(n-4)}{2}\,\frac{\|dF\|^2}{F^2}\right)\mathrm{g}\end{aligned} \tag{4.13}$$

where $\Delta F = \mathrm{g}^{ij}\nabla_i\partial_j F$ and $\|dF\|^2 = \mathrm{g}^{ij}\partial_i F\partial_j F$. The scalar curvature transforms accordingly as

$$\widehat{R} = \frac{R}{F} - (n-1)\left(\frac{\Delta F}{F^2} + \frac{(n-6)}{4}\,\frac{\|dF\|^2}{F^3}\right). \tag{4.14}$$

Using the formula (4.3) as an Ansatz with undetermined coefficients $\beta_1, \ldots, \beta_6$, a tedious calculation then shows that the condition $\widehat{\mathcal{Q}}_{\lambda,\mu}(P) = \mathcal{Q}_{\lambda,\mu}(P)$ is equivalent to an overdetermined linear system of 9 equations for these coefficients. For generic values of δ, the solution turns out to be unique and given by (3.4) and (4.4). $\square$

5. Applications

We apply these results to the quantization of the geodesic flow on a conformally flat manifold (M, g), where, locally, $\mathrm{g}_{ij} = F\, g_{ij}$ for some smooth strictly positive function F, i.e., to the quantization of the quadratic polynomial $H = \mathrm{g}^{ij}\xi_i\xi_j$ on T^*M. We will furthermore quantize the Hamiltonian $H = \mathrm{g}^{ij}(\xi_i - A_i)(\xi_j - A_j)$, where $A = A_i\, dx^i$ is a $U(1)$-connection, describing the motion of a charged particle on a conformally flat manifold, minimally coupled to an electromagnetic field. We will also pay special attention to the resonant cases corresponding to the table (1.3).

5.1. Introducing $i\hbar$

Let us now introduce, as usual, a real parameter $\hbar$ and replace the momenta ξ_j by their quantum substitutes $i\hbar\xi_j$. More specifically, let us consider a new operator on symbols $\mathcal{I}_\hbar : \mathcal{S}_\delta \to (\mathcal{S}_\delta)^{\mathbb{C}}$ by

$$\mathcal{I}_\hbar(P)(\xi) = P(i\hbar\,\xi). \tag{5.1}$$

We will then define a one-parameter family of conformally equivariant quantization maps $\mathcal{Q}_{\lambda,\mu;\hbar} : \mathcal{S}^2_\delta \to (\mathcal{D}^2_{\lambda,\mu})^{\mathbb{C}}$ by

$$\mathcal{Q}_{\lambda,\mu;\hbar} = \mathcal{Q}_{\lambda,\mu} \circ \mathcal{I}_\hbar \tag{5.2}$$

where $\mathcal{Q}_{\lambda,\mu}$ is as in (4.2), (4.3) and $\mathcal{I}_\hbar$ given by (5.1).

Let us recall that if

$$\lambda + \mu = 1, \tag{5.3}$$

there exists a $\mathrm{Vect}(M)$-invariant pairing $(\mathcal{F}_\lambda)^{\mathbb{C}} \otimes (\mathcal{F}_\mu)^{\mathbb{C}} \to \mathbb{C}$ defined by

$$\varphi \otimes \psi \mapsto \int_M \overline{\varphi}\, \psi \tag{5.4}$$

for compactly-supported densities. The quantization map (5.2) enjoys the following crucial property.

Proposition 5.1. *The differential operators $\mathcal{Q}_{\lambda,\mu;\hbar}(P)$ defined by* (5.2) *are symmetric (i.e., formally self-adjoint) for the pairing* (5.4) *provided* (5.3) *holds.*

Proof. A more general version of this proposition has been proved in [10]. In our case, it can be proved directly. Easy calculation already gives, in any coordinate system, the (formal) adjoints $(P^{jk}\partial_j\partial_k)^* = \partial_j\partial_k \circ P^{jk}$, and $(P^j\partial_j)^* = -\partial_j \circ P^j$. Using the expression (3.2) in an adapted coordinate system, we find that $\mathcal{Q}_{\lambda,\mu;\hbar}(P)$ is symmetric for any $P \in \mathcal{S}^2_\delta$ if and only if $\alpha = \frac{1}{2}$, $\beta_1 = 1$ and $\beta_2 = 0$. Returning to the values (3.3) and (3.4) of the numeric coefficients, these conditions are satisfied if $\lambda + \mu = 1$. □

5.2. Conformally equivariant Laplacian in the generic case

Consider the quadratic polynomial $H \in \mathcal{S}_{2,\delta}$ given in local coordinates on T^*M by

$$H = \mathrm{g}^{ij}\xi_i\xi_j \tag{5.5}$$

where $\mathrm{g} = \mathrm{g}_{ij}\, dx^i \otimes dx^j$ is a pseudo-Riemannian metric of signature $p - q$ on M.

Proposition 5.2. *In the case $n \geq 2$, and for λ, μ fulfilling the condition* (1.2)*, the quantization map* (5.2) *yields the following expression:*

$$\mathcal{Q}_{\lambda,\mu;\hbar}(H) = -\hbar^2 \left(\Delta + C_{\lambda,\mu}\, R\right) \tag{5.6}$$

with

$$C_{\lambda,\mu} = \frac{n^2\lambda(\mu - 1)}{(n-1)(n+2-2n\delta)} \tag{5.7}$$

where Δ is the Laplace operator and R the scalar curvature of (M, g).

Proof. If $n \geq 3$, let us substitute the symbol H given by (5.5) into the formula (4.3). The second-order term of $\mathcal{Q}_{\lambda,\mu}(H)$ is nothing but the Laplace operator Δ. Since all covariant derivatives $\nabla_i \mathrm{g}^{jk}$ vanish, we are left with the scalar term $(\beta_5 + n\beta_6)R$. The result follows from (4.4); note that the coefficient $\hbar^2$ comes from (5.2) applied to the quadratic-homogeneous polynomial H.

In the case $n = 2$, the result follows from (4.10) and (4.11). □

5.3. The quantum Hamiltonian

In the special instance where $H \in \mathcal{S}_0 \cong \mathrm{Pol}(T^*M)$, the Hamiltonian flow of H projects onto the geodesics of (M, g). Furthermore, in the most interesting case

$$\lambda = \mu = \frac{1}{2} \tag{5.8}$$

naturally associated with geometric quantization, the operator (5.6) takes the form

$$\mathcal{Q}_{\frac{1}{2},\frac{1}{2};\hbar}(H) = -\hbar^2 \left(\Delta - \frac{n^2}{4(n-1)(n+2)} R \right). \tag{5.9}$$

The self-adjoint operator (5.9) on the Hilbert space $\overline{\mathcal{F}_{\frac{1}{2}}}$ (the completion of the compactly supported half-densities) is a natural new candidate for the quantized Hamiltonian of the geodesic flow on a (pseudo-)Riemannian manifold. None of the expressions obtained in the literature by different methods of quantization (see, e.g., [8] for relevant references) corresponds to this one; all these expressions therefore lack the conformal equivariance property (in the conformally flat case).

5.4. Minimal coupling and quantization

One can, as well, incorporate into the Hamiltonian (5.5) additional terms needed to describe electromagnetic interaction. This is usually performed via the so-called "minimal coupling" prescription to a $U(1)$-connection, locally given by $A = A_i \, dx^i$. This procedure leads to a Hamiltonian $H \in \mathcal{S}^2_\delta$ of the form

$$H = \mathrm{g}^{jk}(\xi_j - A_j)(\xi_k - A_k) \tag{5.10}$$

on any pseudo-Riemannian manifold (M, g).

Proposition 5.3. *In the case $n \geq 2$, and for λ, μ as in* (1.2)*, the quantization map* (5.2) *yields*

$$\begin{aligned} \mathcal{Q}_{\lambda,\mu;\hbar}(H) = & -\hbar^2 \mathrm{g}^{jk} \Big(\nabla_j + \frac{i}{\hbar} A_j \Big) \Big(\nabla_k + \frac{i}{\hbar} A_k \Big) - \hbar^2 C_{\lambda,\mu} R \\ & + i\hbar \, \frac{(1 - \lambda - \mu)}{(1 - \delta)} \, \mathrm{g}^{jk} \nabla_j A_k \end{aligned} \tag{5.11}$$

where $C_{\lambda,\mu}$ is given by (5.7)*.*

The proof of the above proposition is completely analogous to that of Proposition 5.2 and will be omitted.

Notice that the first line in (5.11) corresponds to what is called quantum minimal coupling in the physics literature. Thus, our conformally equivariant quantization $\mathcal{Q}_{\lambda,\mu;\hbar}$ intertwines minimal coupling if and only if condition (5.3) holds.

5.5. The resonant cases: the Yamabe operator and its analogs

According to Theorem 1.2, there exist pairs (λ,μ) for which the modules $\mathcal{D}^2_{\lambda,\mu}$ and $\mathcal{S}^2_{\mu-\lambda}$ are isomorphic. However, we mentioned that this isomorphism is not unique. But, imposing the condition (5.8) for the module $\mathcal{D}^2_{\lambda,\mu}$ enables us to look for the operators $\mathcal{Q}_{\lambda,\mu;\hbar}(H)$ which are symmetric (formally self-adjoint).

Proposition 5.4. *In each of the following resonant cases, there exists a unique isomorphism $\mathcal{Q}_{\lambda,\mu;\hbar}$ for which the operator $\mathcal{Q}_{\lambda,\mu;\hbar}(H)$ is symmetric:*

$$\mathcal{Q}_{\frac{n-2}{2n},\frac{n+2}{2n};\hbar}(H) = -\hbar^2\Big(\Delta - \frac{n-2}{4(n-1)}\,R\Big), \tag{5.12}$$

$$\mathcal{Q}_{0,1;\hbar}(H) = -\hbar^2\Delta, \tag{5.13}$$

$$\mathcal{Q}_{-\frac{1}{n},\frac{n+1}{n};\hbar}(H) = -\hbar^2\Big(\Delta + \frac{1}{(n-1)(n+2)}\,R\Big). \tag{5.14}$$

The proof of the preceding proposition will be given in Section 8.6.

We notice that the constraint (5.8) selects only three (out of five) resonances in (1.3).

We recognize in (5.12) the so-called “Yamabe” operator and in (5.13) the ordinary Laplace operator on functions. At last, the operator (5.14) is a new $\mathrm{o}(p+1,q+1)$-equivariant Laplacian which should be put quite on the same footing as the other two.

Remarks.

(a) It is well known that the Yamabe operator (5.12) is the unique Laplace operator which is invariant under conformal changes of metrics: $\mathrm{g}\mapsto F\,\mathrm{g}$. In this framework, the symbol $H\in\mathcal{S}_{2,\delta}$, given by (5.5) with $\delta=\frac{2}{n}$, is also invariant under conformal changes of metrics.

(b) Note that, in contradistinction with the operator (5.9), the conformal Laplacians (5.12)–(5.14) cannot serve as self-adjoint quantum-mechanical operators on a Hilbert space since $\lambda\neq\mu$.

(c) It is worth mentioning that the numerical coefficients in front of the scalar curvature in (5.12)–(5.14) actually correspond to the expression (5.7) that holds in the generic case.

5.6. The Sturm-Liouville operator

The operator $\mathcal{Q}_{\lambda,\mu;\hbar}(H)$ for the symbol (5.5) can be computed in the case $n=1$ by (4.5); it appears to be still defined in the resonant case, $\delta=2$. In general, it does not yield an $\mathrm{sl}(2,\mathbb{R})$-equivariant quantization map $\mathcal{Q}_{\lambda,\lambda+2;\hbar}$ unless $\lambda=-\frac{1}{2}$

and $\mu = \frac{3}{2}$ (the "Yamabe" weights in (5.12)). In this case one obtains a special instance of Sturm-Liouville operator

$$\mathcal{Q}_{-\frac{1}{2},\frac{3}{2};\hbar}(H) = -\hbar^2\Big(\Delta - \frac{S(\varphi)}{2\mathrm{g}}\Big) \tag{5.15}$$

which can be interpreted as the Yamabe operator in the one-dimensional case.

We now start the more technical part of our work in which we will provide the proofs of the main theorems. We will derive the formulæ for the conformally equivariant isomorphism (1.1) by means of the algebraic theory of invariants.

6. Euclidean invariant theory

In this section we will recall the results of [9] (see also [10]); we will introduce a Lie algebra of differential operators acting on the space of symbols $\mathcal{S}_{\mu-\lambda}$ and commuting with the canonical action of the Euclidean algebra. The associated universal enveloping algebra will provide us with the ingredients needed to construct the conformally equivariant map $\mathcal{Q}_{\lambda,\mu}$ (see [10] for the abstract theory of conformally equivariant quantization) on a conformally flat n-dimensional manifold. Throughout this section we will assume $n \geq 2$.

6.1. The Weyl-Brauer Theorem

Consider first the space of polynomials $\mathbb{C}[\xi_1, \dots, \xi_n]$ with the canonical action of the orthogonal Lie algebra $\mathrm{o}(p,q)$ with $p+q=n$, generated by $X_{ij} = \xi_i\partial/\partial\xi^j - \xi_j\partial/\partial\xi^i$ (cf. (3.1)). A classical theorem [26], [5] states that the commutant $\mathrm{o}(p,q)^!$ in the space $\mathrm{End}(\mathbb{C}[\xi_1, \dots, \xi_n])$ is the associative algebra generated by

$$\mathrm{R} = \xi^i\xi_i, \qquad \mathrm{E} = \xi_i\frac{\partial}{\partial\xi_i} + \frac{n}{2}, \qquad \mathrm{T} = \frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\xi_i} \tag{6.1}$$

whose commutation relations are those of $\mathrm{sl}(2,\mathbb{R})$. We will find it useful to deal with the Euler operator

$$\mathcal{E} = \mathrm{E} - \frac{n}{2}. \tag{6.2}$$

The explicit formulæ in coordinates are as follows:

$$\begin{aligned} \mathrm{R}(P_k)^{i_1\dots i_k ij} &= P_k^{(i_1\dots i_k}g^{ij)}, \\ \mathcal{E}(P_k)^{i_1\dots i_k} &= k\,P_k^{i_1\dots i_k}, \\ \mathrm{T}(P_k)^{i_1\dots i_{k-2}} &= k(k-1)g_{ij}P_k^{iji_1\dots i_{k-2}}, \end{aligned} \tag{6.3}$$

where round brackets denote symmetrization.

Remark. The associative algebra $\mathrm{o}(p,q)^!$ is isomorphic to the universal enveloping algebra $U(\mathrm{sl}(2,\mathbb{R}))$. It is worth noticing that the converse property holds, namely $\mathrm{sl}(2,\mathbb{R})^! = U(\mathrm{o}(p,q))$ showing that $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{o}(p,q)$ form a so-called dual pair of Lie algebras.

6.2. The Lie algebra of Euclidean invariants

Consider then the space of polynomials $\mathbb{C}[x^1,\ldots,x^n,\xi_1,\ldots,\xi_n]$ with the canonical action of the Euclidean Lie algebra $\mathrm{e}(p,q) = \mathrm{o}(p,q) \ltimes \mathbb{R}^n$ generated by the canonical lifts to $T^*\mathbb{R}^n$ of the vector fields X_{ij} and X_i given by (3.1). We are thus looking for the commutant $\mathrm{e}(p,q)^!$ in $\mathrm{End}(\mathbb{C}[x^1,\ldots,x^n,\xi_1,\ldots,\xi_n])$. The following propositions originally proved in [9] extend the Weyl-Brauer theorem.

Proposition 6.1.

(i) *The* $\mathrm{sl}(2,\mathbb{R})$*-module structure on* $\mathbb{C}[x^1,\ldots,x^n,\xi_1,\ldots,\xi_n]$ *extends to a module structure for the semi-direct product* $\mathrm{sl}(2,\mathbb{R})\ltimes\mathrm{h}_1$*, where* h_1 *is the Heisenberg Lie algebra generated by*

$$\mathrm{G} = \xi^i \frac{\partial}{\partial x^i}, \qquad \mathrm{D} = \frac{\partial}{\partial \xi_i}\frac{\partial}{\partial x^i}, \qquad \mathrm{L} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x_i}. \tag{6.4}$$

(ii) *The commutant* $\mathrm{e}(p,q)^!$ *is the associative algebra generated by the operators given in* (6.1) *and* (6.4).

Proof. Consider the commutant $\mathrm{o}(p,q)^!$ in the space $\mathrm{End}(\mathbb{C}[x^1,\ldots,x^n,\xi_1,\ldots,\xi_n])$. As in the proof of the Weyl-Brauer theorem we identify these endomorphisms with polynomials $\mathbb{C}[x^1,\ldots,x^n,p_1,\ldots,p_n,\xi_1,\ldots,\xi_n,y^1,\ldots,y^n]$, where the p_i and y^i are in duality with x^i and ξ_i, respectively. According to [26] the $\mathrm{o}(p,q)$-invariant polynomials are generated by the ten (scalar) products: $x_ix^i, p_ix^i,\ldots,y_iy^i$. These second-order polynomials form a Poisson algebra isomorphic to $\mathrm{sp}(4,\mathbb{R})$ therefore $\mathrm{o}(p,q)^!$ is isomorphic to (some quotient of) $U(\mathrm{sp}(4,\mathbb{R}))$.

The commutant $\mathrm{e}(p,q)^!$ is the subalgebra of $\mathrm{o}(p,q)^!$ which is invariant under translations generated by $\partial/\partial x^i$. This subalgebra is clearly generated by $\xi^i\xi_i$, ξ_iy^i, y_iy^i, ξ^ip_i, y^ip_i, p^ip_i, in other words by the operators (6.1) and (6.4). □

Again, one easily finds that

$$\begin{aligned} \mathrm{G}(P_k)^{i_1\ldots i_k i} &= \partial_j P_k^{(i_1\ldots i_k} g^{i)j}, \\ \mathrm{D}(P_k)^{i_1\ldots i_{k-1}} &= k\,\partial_i P_k^{ii_1\ldots i_{k-1}}, \\ \mathrm{L}(P_k)^{i_1\ldots i_k} &= g^{ij}\partial_i\partial_j P_k^{i_1\ldots i_k}. \end{aligned} \tag{6.5}$$

Remark. If $n \geq 3$, one has $\mathrm{o}(p,q)^! = U(\mathrm{sp}(4,\mathbb{R}))$ and $\mathrm{sp}(4,\mathbb{R})^! = U(\mathrm{o}(p,q))$. This is also a well-known instance of duality between the orthogonal and symplectic algebras.

We furthermore prove the following:

Theorem 6.2. *The commutant* $\mathrm{e}(p,q)^!$ *is isomorphic to quotient algebra* $U(\mathrm{sl}(2,\mathbb{R})\ltimes \mathrm{h}_1)/\mathcal{I}$ *where the ideal* $\mathcal{I}$ *is as follows:*

(i) *If* $n=2$*, the ideal* $\mathcal{I}$ *is generated by*

$$Z=(\mathcal{C}+\tfrac{3}{2})\,\mathrm{L}+\tfrac{1}{4}\big(\mathrm{D}\,[\mathrm{G},\mathcal{C}]+[\mathrm{G},\mathcal{C}]\,\mathrm{D}-\mathrm{G}\,[\mathrm{D},\mathcal{C}]-[\mathrm{D},\mathcal{C}]\,\mathrm{G}\big), \tag{6.6}$$

where $\mathcal{C}=\mathrm{E}^2-\frac{1}{2}(\mathrm{R}\,\mathrm{T}+\mathrm{T}\,\mathrm{R})$ *is the Casimir of* $\mathrm{sl}(2,\mathbb{R})$*;*

(ii) *if* $n\geq 3$*, one has*

$$\mathcal{I}=\{0\}. \tag{6.7}$$

Proof. Again, we identify the generators (6.1), (6.4) with the six quadratic polynomials given in the preceding proof.

If $n\geq 3$, one finds that these polynomials are functionally, hence algebraically independent. Indeed, $d(\xi^i\xi_i)\wedge d(\xi_j y^j)\wedge\cdots\wedge d(p^k p_k)\neq 0$.

In the case $n=2$, any five distinct polynomials from the previous set of quadratic polynomials turn out to be independent. One then checks that the operator given by Z in (6.6) vanishes identically. Moreover, $Z\in U(\mathrm{sl}(2,\mathbb{R})\ltimes \mathrm{h}_1)$ is of minimal degree (three). Working, as above, in terms of polynomials (principal symbols), one immediately gets, by using the implicit functions theorem, that any other polynomial in this ideal is a multiple of the symbol of Z. □

We do not know whether the converse to Theorem 6.2 is true: our conjecture is that $(\mathrm{sl}(2,\mathbb{R})\ltimes \mathrm{h}_1)^!=U(\mathrm{e}(p,q))$ for $n\geq 3$, in other words, $U(\mathrm{e}(p,q))^{!!}=U(\mathrm{e}(p,q))$. Similar problems have recently been investigated by A.A. Kirillov [16].

7. Equation characterizing conformal equivariance

7.1. Equivariance with respect to the affine subalgebra

We first consider, for the sake of completeness, the case of the whole affine Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$.

Lemma 7.1. *The actions* (2.4) *and* (2.8)*,* (2.9) *of the affine Lie algebra* $\mathrm{gl}(n,\mathbb{R})\ltimes\mathbb{R}^n$ *on the modules* $\mathcal{D}_{\lambda,\mu}$ *and* $\mathcal{S}_{\mu-\lambda}$ *for the local expressions* (2.3) *and* (2.6) *coincide identically.*

Proof. The $\mathrm{Vect}(M)$-action (2.4) has the following form in local coordinates:

$$L_X^{\lambda,\mu}(A)_\ell=L_X^{\mu-\lambda}(A_\ell)+(\text{higher order derivatives of }X) \tag{7.1}$$

for $X\in\mathrm{Vect}(M)$. The affine Lie algebra being characterized by the property that all second derivatives $\partial_i\partial_j X^k$ vanish, (7.1) implies that each coefficient of the operator A transforms as a symbol of degree ℓ. □

From now on, we identify locally the operators and the symbols by using the formula (7.1).

7.2. Action of the inversions on $\mathcal{D}^k_{\lambda,\mu}$

At this stage, we need an explicit formula for the action (7.1) of the inversions, generated by $\overline{X}_i$ (see (3.1)), on the space of differential operators.

In order to make calculations more systematic, let us introduce the following useful notation

$$L_{\overline{X}} = \xi_i \otimes L_{\overline{X}_i} \tag{7.2}$$

which captures the entire structure of the abelian subalgebra of inversions. Experience proved that this operator is compatible with all algebraic structures introduced so far.

Lemma 7.2. *The action of the inversions on $\mathcal{D}^k_{\lambda,\mu}$ takes, with the convention* (7.2)*, the following form:*

$$L^{\lambda,\mu}_{\overline{X}}(A)_\ell = L^{\mu-\lambda}_{\overline{X}}(A_\ell) + (\ell+1)\left(-\frac{1}{2}\ell \;\mathrm{R}\;\mathrm{T} + 2(\ell+n\lambda)\right) A_{\ell+1} \tag{7.3}$$

for $\ell = 0, 1, \ldots, k$.

Proof. Standard calculation leads to the general expression:

$$\begin{aligned} L^{\lambda,\mu}_X(A)^{i_1\ldots i_\ell}_\ell =& L^{\mu-\lambda}_X(A_\ell)^{i_1\ldots i_\ell} \\ & - \frac{\ell+1}{2}\sum_{s=1}^{\ell} A^{iji_1\ldots\widehat{i_s}\ldots i_\ell}_{\ell+1}\partial_i\partial_j X^{i_s} - (\ell+1)\lambda A^{ii_1\ldots i_\ell}_{\ell+1}\partial_i\partial_j X^j \\ & + \text{ (higher order derivatives of } X) \end{aligned}$$

for any $X \in \mathrm{Vect}(M)$. In the case of inversions, namely, if $X = \overline{X}_r$, one has

$$\partial_i\partial_j\overline{X}^s_r = 2\left(g_{ij}\delta^s_r - \delta^s_i g_{jr} - \delta^s_j g_{ir}\right), \tag{7.4}$$

where g_{ij} are the components of the flat metric on $\mathbb{R}^n$ given in Section 3.1. The previous formula, therefore, becomes

$$\begin{aligned} L^{\lambda,\mu}_{\overline{X}_r}(A)^{i_1\ldots i_\ell}_\ell =& L^{\mu-\lambda}_{\overline{X}_r}(A_\ell)^{i_1\ldots i_\ell} \\ & - (\ell+1)\sum_{s=1}^{\ell} g_{ij}\, A^{iji_1\ldots\widehat{i_s}\ldots i_\ell}_{\ell+1}\delta^r_{i_s} \\ & + 2(\ell+1)(\ell+n\lambda)A^{ri_1\ldots i_\ell}_{\ell+1}. \end{aligned}$$

Then, using (6.3), one finds that the second term in the sum $\xi_r L^{\lambda,\mu}_{\overline{X}_r}(A)_\ell$ is equal to $-\frac{1}{2}\ell(\ell+1)\,\mathrm{R}\;\mathrm{T}(A_{\ell+1})$. The third term in the same expression is plainly proportional to the identity. □

7.3. Equivariance equation

It is now possible to derive the main equation that guarantees the equivariance of the symbol map and the quantization map with respect to the inversions.

Proposition 7.3. *A linear map $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^k_{\mu-\lambda} \to \mathcal{D}^k_{\lambda,\mu}$ intertwines the action of the inversions if and only if the following equation holds:*

$$[\mathcal{Q}_{\lambda,\mu}, L^{\mu-\lambda}_{\overline{X}}] = \big(-\tfrac{1}{2}\,\mathrm{R}\;\mathrm{T}(\mathcal{E}-1) + 2\mathcal{E} + 2(n\lambda - 1)\big)\,\mathcal{E} \circ \mathcal{Q}_{\lambda,\mu}. \tag{7.5}$$

Proof. The equivariance condition writes that $\mathcal{Q}_{\lambda,\mu} \circ L^{\mu-\lambda}_{\overline{X}} = L^{\lambda,\mu}_{\overline{X}} \circ \mathcal{Q}_{\lambda,\mu}$. Applying then equation (7.3) to this condition readily yields the result. □

8. Proofs of the main results in the conformally flat case

8.1. Locality of the $\mathrm{o}(p+1, q+1)$-equivariant maps

It should be emphasized that the isomorphism (1.1) is necessarily given by a differential map, namely (3.2). This fact is already guaranteed by the equivariance with respect to the subalgebra $\mathbb{R} \ltimes \mathbb{R}^n$ generated by homotheties and translations (which is a common subalgebra of $\mathrm{o}(p+1, q+1)$ and $\mathrm{sl}(n+1, \mathbb{R})$), i.e., by the

Proposition 8.1 [20]**.** *If $k \geq \ell$, any $\mathbb{R} \ltimes \mathbb{R}^n$-equivariant map $\mathcal{S}^k_\delta \to \mathcal{S}^\ell_\delta$ is local.*

By Peetre's theorem [24] such maps are locally given by differential operators.

8.2. The Ansatz

We will use our previous results on the universal enveloping algebra $U(\mathrm{sl}(2,\mathbb{R}) \ltimes \mathrm{h}_1)$ to determine an adequate Ansatz for the quantization map $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^k_{\mu-\lambda} \to \mathcal{D}^k_{\lambda,\mu}$, which turns out to be more convenient in our framework. But, an identical general Ansatz would apply just as well to the symbol map.

Proposition 8.1, together with the generalized Weyl-Brauer Theorem 6.1, leads to the general form for an $\mathrm{e}(p,q)$-equivariant quantization map $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^k_{\mu-\lambda} \to \mathcal{D}^k_{\lambda,\mu}$ given by differential operators $\mathcal{Q}_{\lambda,\mu} = C_{r,e,g,d,\ell,t}\; \mathrm{R}^r\; \mathrm{E}^e\; \mathrm{G}^g\; \mathrm{D}^d\; \mathrm{L}^\ell\; \mathrm{T}^t$, where $C_{r,e,g,d,\ell,t}$ are constant coefficients.

Imposing, furthermore, the equivariance of $\mathcal{Q}_{\lambda,\mu}$ with respect to homotheties generated by X_0 from (3.1), one readily finds that $t = r + g + \ell$ and obtains that any $\mathrm{o}(p+1, q+1)$-equivariant map $\mathcal{Q}_{\lambda,\mu} : \mathcal{S}^k_{\mu-\lambda} \to \mathcal{D}^k_{\lambda,\mu}$ is of the form

$$\mathcal{Q}_{\lambda,\mu} = C_{r,e,g,d,\ell}\; \mathrm{R}^r_0\; \mathcal{E}^e\; \mathrm{G}^g_0\; \mathrm{D}^d\; \mathrm{L}^\ell_0, \tag{8.1}$$

where we have put

$$\mathrm{R}_0 = \mathrm{R}\ \mathrm{T}, \qquad \mathrm{G}_0 = \mathrm{G}\ \mathrm{T}, \qquad \mathrm{L}_0 = \mathrm{L}\ \mathrm{T}\,. \tag{8.2}$$

We will also impose the natural normalization condition which demands that the principal symbol be preserved:

$$C_{r,e,0,0,0} = \begin{cases} 1 & \text{if } (r,e) = (0,0) \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

8.3. Solving the equivariance equation: proof of Theorem 3.2

In the case of second-order differential operators, which is the one this article is devoted to, our Ansatz (8.1) implies that $\mathrm{e}(p,q)$-equivariant maps:

(a) $\mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta$ are linear combinations of D and G_0 for $k = 1, 2$;

(b) $\mathcal{S}^k_\delta \to \mathcal{S}^{k-2}_\delta$ are linear combinations of D^2 and L_0 for $k = 2$ (note that in this special case the other operators taken from (8.1), namely G_0^2 and $\mathrm{G}_0\,\mathrm{D}$ are expressible in terms of the latter).

Furthermore, the monomials in R_0 vanish because of the normalization condition (8.3); the terms $\mathrm{R}_0\ \mathrm{D}, \mathrm{R}_0\ \mathrm{G}_0, \ldots$ are identically zero for $k \le 2$.

Proposition 8.2. *There exists a unique linear map*

$$\mathcal{Q}_{\lambda,\mu} = \mathrm{Id} + \gamma_1\,\mathrm{G}_0 + \gamma_2\,\mathrm{D} + \gamma_3 \mathcal{E}\,\mathrm{D} + \gamma_4\,\mathrm{L}_0 + \gamma_5\,\mathrm{D}^2 \tag{8.4}$$

satisfying the equivariance equation (7.5) *provided condition* (1.2) *holds; it is given by*

$$\begin{aligned}
\gamma_1 &= \frac{n(\lambda+\mu-1)}{2(n\delta-2)(n(\delta-1)-2)},\\
\gamma_2 &= \frac{\lambda}{1-\delta},\\
\gamma_3 &= \frac{1-\lambda-\mu}{(\delta-1)(n(\delta-1)-2)},\\
\gamma_4 &= \frac{n\lambda\Big(2+(4\lambda-1)n+(2\lambda^2-\lambda\mu-\mu^2+2\mu-1)n^2\Big)}{2(n(\delta-1)-1)(n(2\delta-1)-2)(n\delta-2)(n(\delta-1)-2)},\\
\gamma_5 &= \frac{n\lambda(n\lambda+1)}{2(n(\delta-1)-1)(n(\delta-1)-2)}.
\end{aligned} \tag{8.5}$$

Proof. Let us compute the left hand side of the equation (7.5) where the quantization map given by our Ansatz (8.4). We need the commutators of the differential operators entering (8.4) with the Lie derivative $L^\delta_{\overline{X}_i}$ with respect to the generators $\overline{X}_i$ given by (3.1). Using the notation (7.2) we first prove the

Lemma 8.3. *The following commutation relations hold:*

$$\begin{aligned}
[\mathrm{R}_0, L^\delta_X] &= 0,\\
[\mathcal{E}, L^\delta_X] &= 0,\\
[\mathrm{G}_0, L^\delta_X] &= 2\,\mathrm{R}_0(\mathcal{E} - n\delta),\\
[\mathrm{D}, L^\delta_X] &= -2\,\mathrm{R}_0 + 4\mathcal{E}^2 - 2(n(\delta-1)+2)\mathcal{E},\\
[\mathrm{L}_0, L^\delta_X] &= -4\,\mathrm{R}_0\,\mathrm{D} + 8\mathcal{E}\,\mathrm{G}_0 + 2(n(1-2\delta)-2)\,\mathrm{G}_0,\\
[\mathrm{D}^2, L^\delta_X] &= -4\,\mathrm{R}_0\,\mathrm{D} - 2\,\mathrm{G}_0 + 8\mathcal{E}^2\,\mathrm{D} + 4(n(1-\delta)-1)\mathcal{E}\,\mathrm{D}\,.
\end{aligned} \tag{8.6}$$

Proof. One finds, using (2.8), (2.9) and (3.1), $[\mathrm{D}, L^\delta_{X_i}] = -2\xi_i\,\mathrm{T} + 4\mathcal{E}\partial_{\xi_i} - 2n(\delta-1)\partial_{\xi_i}$. Then, the final expression for $[\mathrm{D}, L^\delta_X]$ follows from the definition of the operators R_0 and $\mathcal{E}$ given by (6.1), (8.2) and (6.2). The other commutators in (8.6) are derived in the same fashion with the help of the commutation relations of the operators (6.1) and (6.4). □

Using the commutation relations (8.6), we find

$$\begin{aligned}
[\mathcal{Q}_{\lambda,\mu}, L^{\mu-\lambda}_X] = & 2\gamma_1(\mathrm{R}_0\,\mathcal{E} - n\delta\,\mathrm{R}_0)\\
& + 2\gamma_2(-\mathrm{R}_0 + 2(\mathcal{E}^2 - \mathcal{E}) - n(\delta-1)\mathcal{E})\\
& + 2\gamma_3(-\mathrm{R}_0(\mathcal{E}-1) + (2 - n(\delta-1))(\mathcal{E}^2 - \mathcal{E}))\\
& + 2\gamma_4(-2\,\mathrm{R}_0\,\mathrm{D} + 4\mathcal{E}\,\mathrm{G}_0 + (n(1-2\delta)-2)\,\mathrm{G}_0)\\
& + 2\gamma_5(-\mathrm{G}_0 - 2\,\mathrm{R}_0\,\mathrm{D} + 4\mathcal{E}^2\,\mathrm{D} + 2(n(1-\delta)-1)\mathcal{E}\,\mathrm{D})
\end{aligned}$$

while the right hand side of (7.5) is given by

$$\begin{aligned}
\Big(-\frac{1}{2}\,\mathrm{R}_0(\mathcal{E}-1) + 2\mathcal{E} + 2(n\lambda - 1)\Big)\,\mathcal{E}\circ\mathcal{Q}_{\lambda,\mu} = &\\
& \Big(-\frac{1}{2}\,\mathrm{R}_0(\mathcal{E}-1) + 2(\mathcal{E} + n\lambda - 1)\Big)\,\mathcal{E}\\
& + 2(\mathcal{E} + n\lambda - 1)\mathcal{E}(\gamma_1\,\mathrm{G}_0 + \gamma_2\,\mathrm{D} + \gamma_3\mathcal{E}\,\mathrm{D})
\end{aligned}$$

since the extra terms, namely $(\mathcal{E}-1)\mathcal{E}(\gamma_1\,\mathrm{G}_0 + \gamma_2\,\mathrm{D} + \gamma_3\mathcal{E}\,\mathrm{D} + \gamma_4\,\mathrm{L}_0 + \gamma_5\,\mathrm{D}^2)$ and $\mathcal{E}(\gamma_4\,\mathrm{L}_0 + \gamma_5\,\mathrm{D}^2)$ obviously vanish on the space of second-order symbols.

Now, the equivariance condition (7.5) amounts to equating the two previous expressions. Identifying the coefficients of $\mathrm{R}_0, \mathrm{G}_0, \mathrm{D}$ and the scalar terms (of order one and two), respectively, one gets the following system of linear equations:

$$\begin{cases} (2-n\delta)\gamma_1 - (\gamma_2+\gamma_3) = -\dfrac{1}{2}, \\ (n(1-2\delta)+2)\gamma_4 - \gamma_5 = n\lambda\gamma_1, \\ 2(n(1-\delta)+1)\gamma_5 = n\lambda(\gamma_2+\gamma_3), \\ (1-\delta)\gamma_2 = \lambda, \\ (2+n(1-\delta))(\gamma_2+\gamma_3) = n\lambda + 1. \end{cases} \tag{8.7}$$

The solution of this system is unique and given by (8.5). □

Example. Proposition (8.2) yields, in particular, the following half-density quantization map:

$$\mathcal{Q}_{\frac{1}{2},\frac{1}{2}} = \mathrm{Id} + \frac{1}{2}\,\mathrm{D} + \frac{n}{8(n+1)(n+2)}\,\mathrm{L}_0 + \frac{n}{8(n+1)}\,\mathrm{D}^2\,. \tag{8.8}$$

Let us notice that the map $\mathcal{Q}_{\lambda,\mu}$ defined by (8.4) and (8.5) coincides with the quantization map (3.2) in an adapted coordinate system. Proposition 8.2 is nothing but a rephrasing of Theorem 3.2 whose proof is now complete.

8.4. Proof of Theorem 1.1

The $\mathrm{o}(p+1,q+1)$-equivariant quantization map (8.4) precisely coincides with the expression (3.2), since, taking into account the formulæ (6.3) and (6.5), one easily establishes the correspondence between the coefficients (8.5) and (3.3), (3.4).

We have thus proved the existence of an isomorphism (1.1) provided the coefficients (8.5) are well defined, i.e., condition (1.2) holds. This proves part (i) of Theorem 1.1.

Then the formula (8.1) and the normalization condition (8.3) ensure that, up to a multiplicative constant, every $\mathrm{o}(p+1,q+1)$-equivariant quantization map (1.1) is, indeed, of the form (8.4). The uniqueness of the quantization map (part (ii) of Theorem 1.1) immediately follows from Proposition 8.2.

8.5. Proof of Theorem 1.2

The system (8.7) determines all $\mathrm{o}(p+1,q+1)$-equivariant linear maps from $\mathcal{S}^2_{\mu-\lambda}$ to $\mathcal{D}^2_{\lambda\mu}$. In the resonant cases, this system has, in general, no solution. However, solving it for $\gamma_1,\ldots,\gamma_5$ and λ as an extra indeterminate, one immediately obtains the values of λ and μ displayed in (1.3).

In doing so, one finds that the coefficient γ_3 remains undetermined for the third resonance, and γ_4 for the rest.

8.6. Proof of Proposition 5.4

Returning to the basic system (8.7) in the presence of resonances, we easily find that the free parameter γ_3 (resp. γ_4) is uniquely determined in each resonant case where $\lambda + \mu = 1$ if we require that the operators $\mathcal{Q}_{\lambda,\mu;\hbar}(P)$ be symmetric for all $P \in \mathcal{S}^2_\delta$. In such cases, the explicit expressions (5.12)–(5.14) are obtained in the same manner as in the proof of Proposition 5.2.

9. Conclusion and outlook

In this work, we have taken a first step towards a conformally invariant quantization, i.e., depending only on the conformal class of a pseudo-Riemannian metric. This program is now achieved for the case of second-order symbols and differential operators. The general case still remains to be tackled, however computations seem much more intricate.

Our original idea was to relate geometric quantization and deformation quantization in a somewhat novel fashion, namely by using, from the start, equivariance with respect to some structural symmetry group (e.g. the conformal group). In the conformally flat case, it has been proved [10] that there exists, for any order, a canonical quantization map equivariant with respect to the action of the conformal group. This conformally flat case is particular and we have been able to extend the second-order quantization map to the case of an arbitrary pseudo-Riemannian manifold.

As a by-product, we have obtained a new quantization of the geodesic flow (5.9) on the Hilbert space of half-densities. We have also related the Yamabe operator to other conformally equivariant Laplacians on resonant modules of densities, and derived the quantum version of minimal coupling in the same framework.

We have also chosen to put aside the cohomological content of many aspects of the problem. It should be stressed that Lie algebra cohomology proved useful in earlier work [8], [20], [12], [18] on the modules of differential operators. The resonances appearing in (1.2) should thus certainly hide nontrivial $\mathrm{o}(p+1, q+1)$-cohomology classes.

Let us finish by mentioning a crucial property of the conformal algebra which was of central importance in our work. The Lie algebra $\mathrm{o}(p+1, q+1)$ is a maximal Lie subalgebra of $\mathrm{Vect}(\mathbb{R}^n)$ in the sense that any larger subalgebra is infinite-dimensional (see [3]). This property implied the uniqueness of the isomorphisms of the modules of differential operators and symbols under study. Recall that the same is true for the projective Lie algebra $\mathrm{sl}(n+1, \mathbb{R})$.

10. Appendix

Let us show how the formulæ (4.2), (4.3), (4.5) and (4.10) for the quantization map for a general curved pseudo-Riemannian metric stem from formula (3.2) in the conformally flat case. We therefore develop in this Appendix the covariant calculus for density-valued symbols which is needed for this purpose.

10.1. First-order symbols

Let us consider a homogeneous first-order polynomial $P \in \mathcal{S}_{1,\delta}$. From the expression (4.1) of the covariant derivative of a tensor density, one gets

$$\nabla_i P^i = \partial_i P^i + (1-\delta)\Gamma_i P^i.$$

We then deduce from the formula (3.2) in an adapted coordinate system that

$$\begin{aligned}\mathcal{Q}_{\lambda,\mu}(P) &= P^i \partial_i + \frac{\lambda}{1-\delta}\Big(\nabla_i P^i - (1-\delta)\Gamma_i P^i\Big)\\ &= P^i \partial_i - \lambda\, \Gamma_i P^i + \frac{\lambda}{1-\delta}\nabla_i P^i\\ &= P^i \nabla_i + \alpha\, \nabla_i P^i\end{aligned}$$

thanks to (3.3) and (4.1), hence, the formula (4.2).

Note that the quantization map (3.2) is already intrinsic when restricted to first-order polynomials.

10.2. Second-order symbols

Consider now a homogeneous second-order polynomial $P \in \mathcal{S}_{2,\delta}$ and let us, again, use the formula (3.2) in adapted coordinates. Our goal is to rewrite this expression in a general coordinate system, but in the case of a conformally flat metric.

The highest order term in the operator $\mathcal{Q}_{\lambda,\mu}(P)$ retains the following form:

$$\begin{aligned}P^{ij}\partial_i\partial_j =& P^{ij}\nabla_i\nabla_j + \left(P^{jk}\Gamma^i_{jk} + 2\lambda P^{ij}\Gamma_j\right)\nabla_i\\ &+ P^{ij}\left(\lambda^2\Gamma_i\Gamma_j + \lambda\partial_i\Gamma_j\right)\end{aligned}$$

which can be deduced from (4.1). Let us notice that in order to obtain such a seemingly standard expression, we actually need to differentiate λ-densities and tensor fields with values in $\mathcal{F}_\lambda$. Neither the latter formula nor the following ones are common in differential geometry.

The two first-order terms in $\mathcal{Q}_{\lambda,\mu}(P)$ read

$$\begin{aligned}(\partial_j P^{ij})\partial_i =& (\nabla_j P^{ij})\nabla_i - \left(P^{jk}\Gamma^i_{jk} + (1-\delta)P^{ij}\Gamma_j\right)\nabla_i\\ &+ \lambda(\nabla_i P^{ij})\Gamma_j - \lambda P^{ij}\left(\Gamma^k_{ij}\Gamma_k + (1-\delta)\Gamma_i\Gamma_j\right)\end{aligned}$$

and

$$\mathrm{g}^{ij}\mathrm{g}_{k\ell}(\partial_j P^{k\ell})\partial_i = \mathrm{g}^{ij}\mathrm{g}_{k\ell}(\nabla_j P^{k\ell})\nabla_i - \mathrm{g}^{ij}\mathrm{g}_{k\ell}\left(2P^{m\ell}\Gamma^k_{jm} - \delta P^{k\ell}\Gamma_j\right)\nabla_i + \lambda\Gamma_i\,\mathrm{g}^{ij}\mathrm{g}_{k\ell}\left(\nabla_j P^{k\ell} - 2P^{m\ell}\Gamma^k_{jm} + \delta P^{k\ell}\Gamma_j\right).$$

At last, the two zero-order terms in $\mathcal{Q}_{\lambda,\mu}(P)$ are as follows:

$$\partial_i\partial_j P^{ij} = \nabla_i\nabla_j P^{ij} - 2(1-\delta)(\nabla_i P^{ij})\Gamma_j - 2(\nabla_i P^{jk})\Gamma^i_{jk} - P^{ij}\left(\partial_k\Gamma^k_{ij} + (1-\delta)\partial_i\Gamma_j - 2\Gamma^\ell_{ik}\Gamma^k_{j\ell} - (1-2\delta)\Gamma^k_{ij}\Gamma_k - (1-\delta)^2\Gamma_i\Gamma_j\right)$$

and

$$\mathrm{g}^{ij}\mathrm{g}_{k\ell}\partial_i\partial_j P^{k\ell} = \mathrm{g}^{ij}\mathrm{g}_{k\ell}\Big(\nabla_i\nabla_j P^{k\ell} - 4(\nabla_j P^{\ell m})\Gamma^k_{im} + 2\delta(\nabla_j P^{k\ell})\Gamma_i + (\nabla_m P^{k\ell})\Gamma^m_{ij} - 2P^{\ell m}\partial_i\Gamma^k_{jm} + \delta P^{k\ell}\partial_i\Gamma_j + 2P^{\ell m}(\Gamma^r_{im}\Gamma^k_{jr} - 2\delta\Gamma^k_{im}\Gamma_j) + 2P^{mr}\Gamma^k_{im}\Gamma^\ell_{jr} + \delta^2 P^{k\ell}\Gamma_i\Gamma_j\Big).$$

To obtain the final formula for the quantization map, let us collect the above terms within the expression (3.2) where the coefficients $\beta_1,\ldots,\beta_4$ are considered undetermined. We also need to use the Christoffel symbols of the conformally flat metric $\mathrm{g} = Fg$, namely

$$\Gamma^k_{ij} = \frac{1}{2F}\left(F_i\delta^k_j + F_j\delta^k_i - F^k g_{ij}\right) \tag{10.1}$$

where g is some flat metric and $F^k = g^{jk}F_j$ (see (4.12).

The second-order term we get is plainly $P^{ij}\nabla_i\nabla_j$. Then, the first-order term in $\mathcal{Q}_{\lambda,\mu}(P)$ is just given by the second line of (4.3) if we impose the following conditions

$$\Big(1 - \beta_1 + \frac{n}{2}\left(2\lambda - \beta_1(1-\delta)\right)\Big)\,P^{ij}F_j = 0$$

and

$$\Big(-\frac{1}{2}(1-\beta_1) + \beta_2\Big(\frac{n\delta}{2} - 1\Big)\Big)\,g^{ij}g_{k\ell}\,P^{k\ell}F_j = 0$$

for the extra non-intrinsic terms; these conditions are satisfied if and only if β_1 and β_2 are as in (3.4). As for the zero-order terms, we, again, have to rule out two non-intrinsic terms, i.e.,

$$\Big(\lambda\beta_1 - 2\beta_3\Big(1 - \delta + \frac{1}{n}\Big)\Big)\,(\nabla_i P^{ij})F_j = 0$$

and

$$\Big(\lambda\beta_2 + \frac{\beta_3}{n} + \beta_4\Big(2\delta - 1 - \frac{2}{n}\Big)\Big)\,g^{ij}g_{k\ell}(\nabla_i P^{k\ell})F_j = 0.$$

These conditions determine β_3 and β_4 in accordance with (3.4).

We finally check that the remaining zero-order terms in $\mathcal{Q}_{\lambda,\mu}(P)$ are as follows:

$$\begin{aligned}\frac{n^2\lambda(1-\mu)}{2(1+n(1-\delta))}\Bigg[\frac{P^{ij}F_{ij}}{F}-\frac{3}{2}\frac{P^{ij}F_iF_j}{F^2}+\frac{1}{2+n(1-2\delta)}g^{ij}g_{k\ell}\,\cdot \\ \cdot\,\left(\frac{P^{k\ell}F_{ij}}{F}-\frac{1}{2}(2+n(\delta-1))\frac{P^{k\ell}F_iF_j}{F^2}\right)\Bigg]\end{aligned} \tag{10.2}$$

where $F_{ij}=\partial_i\partial_jF$. At this stage, some more ingredients are needed, namely the Ricci tensor and the scalar curvature for the the conformally flat metric $\mathrm{g}=Fg$ with Christoffel symbols (10.1). The corresponding expressions can be easily deduced from (4.13) and (4.14).

One checks that, in the case $n\geq 3$, the expression (10.2) organizes as the combination $\beta_5P^{ij}R_{ij}+\beta_6P^{ij}\mathrm{g}_{ij}R$ where β_5 and β_6 are rigidly fixed and coincide with (4.4). This computation therefore justifies our main Definition 4.3.

In the lower dimensional cases, $n=1$ and $n=2$, the calculation is similar to that of the higher dimensional case $n\geq 3$.

In order to give an intrinsic interpretation of (10.2), we resort to the definition (4.9) of the Schwarzian derivative in the case $n=2$. The final formula (4.10) for $\mathcal{Q}_{\lambda,\mu}(P)$ then readily follows from the expression (4.14) of the scalar curvature for the conformally flat metric $\mathrm{g}=Fg$.

In the case $n=1$, the formula (4.5) is obtained exactly in the same way as (4.10).

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