

# A RIEMANN-ROCH THEOREM FOR ONE-DIMENSIONAL COMPLEX GROUPOIDS

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## Abstract

We consider a smooth groupoid of the form  $\Sigma \rtimes \Gamma$  where  $\Sigma$  is a Riemann surface and  $\Gamma$  a discrete pseudogroup acting on  $\Sigma$  by local conformal diffeomorphisms. After defining a  $K$ -cycle on the crossed product  $C_0(\Sigma) \rtimes \Gamma$  generalising the classical Dolbeault complex, we compute its Chern character in cyclic cohomology, using the index theorem of Connes and Moscovici. This involves in particular a generalisation of the Euler class constructed from the modular automorphism group of the von Neumann algebra  $L^\infty(\Sigma) \rtimes \Gamma$ .

## I. Introduction

In a series of papers [4, 5], Connes and Moscovici proved a general index theorem for transversally (hypo)elliptic operators on foliations. After constructing  $K$ -cycles on the algebra crossed product  $C_0(M) \rtimes \Gamma$ , where  $\Gamma$  is a discrete pseudogroup acting on the manifold  $M$  by local diffeomorphisms [4], they developed a theory of characteristic classes for actions of Hopf algebras that generalise the usual Chern-Weil construction to the non-commutative case [5, 6]. The Chern character of the concerned  $K$ -cycles is then captured in the periodic cyclic cohomology of a particular Hopf algebra encoding the action of the diffeomorphisms on  $M$ . The nice thing is that this cyclic cohomology can be completely exhausted as Gelfand-Fuchs cohomology and renders the index computable. We shall illustrate these methods with a specific example, namely the crossed product of a Riemann surface  $\Sigma$  by a discrete pseudogroup  $\Gamma$  of local conformal mappings. We find that the relevant characteristic classes are the fundamental class  $[\Sigma]$  and a cyclic 2-cocycle on  $C_c^\infty(\Sigma) \rtimes \Gamma$  generalising the (Poincaré dual of the) usual Euler class. When applied to the  $K$ -cycle represented by the Dolbeault operator of  $\Sigma \rtimes \Gamma$ , this yields a non-commutative version of the Riemann-Roch theorem. Throughout the text we also stress the crucial role played by the modular automorphism group of the von Neumann algebra  $L^\infty(\Sigma) \rtimes \Gamma$ .

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## II. The Dolbeault $K$ -cycle

Let  $\Sigma$  be a Riemann surface without boundary and  $\Gamma$  a pseudogroup of local conformal mappings of  $\Sigma$  into itself. We want to define a  $K$ -cycle on the algebra  $C_0(\Sigma) \rtimes \Gamma$  generalising the classical Dolbeault complex. Following [4], the first step consists in lifting the action of  $\Gamma$  to the bundle  $P$  over  $\Sigma$ , whose fiber at point  $x$  is the set of Kähler metrics corresponding to the complex structure of  $\Sigma$  at  $x$ . By the obvious correspondence metric  $\leftrightarrow$  volume form,  $P$  is the  $\mathbb{R}_+^*$ -principal bundle of densities on  $\Sigma$ . The pseudogroup  $\Gamma$  acts canonically on  $P$  and we consider the crossed product  $C_0(P) \rtimes \Gamma$ .

Let  $\nu$  be a smooth volume form on  $\Sigma$ . As in [2], this gives a weight on the von Neumann algebra  $L^\infty(\Sigma) \rtimes \Gamma$  together with a representative  $\sigma$  of its modular automorphism group. Moreover  $\sigma$  leaves  $C_0(\Sigma) \rtimes \Gamma$  globally invariant and one has

$$C_0(P) \rtimes \Gamma = (C_0(\Sigma) \rtimes \Gamma) \rtimes_\sigma \mathbb{R} , \quad (1)$$

where the space  $P$  is identified with  $\Sigma \times \mathbb{R}$  thanks to the choice of the global section  $\nu$ . Therefore one has a Thom-Connes isomorphism [1]

$$K_i(C_0(\Sigma) \rtimes \Gamma) \rightarrow K_{i+1}(C_0(P) \rtimes \Gamma) , \quad i = 0, 1 , \quad (2)$$

and we shall obtain the desired  $K$ -homology class on  $C_0(P) \rtimes \Gamma$ . The reason for working on  $P$  rather than  $\Sigma$  is that  $P$  carries quasi  $\Gamma$ -invariant metric structures, allowing the construction of  $K$ -cycles represented by differential hypoelliptic operators [4].

More precisely, consider the product  $P \times \mathbb{R}$ , viewed as a bundle over  $\Sigma$  with 2-dimensional fiber. The action of  $\Gamma$  extends to  $P \times \mathbb{R}$  by making  $\mathbb{R}$  invariant. Up to another Thom isomorphism, the  $K$ -cycle may be defined on  $C_0(P \times \mathbb{R}) \rtimes \Gamma = (C_0(P) \rtimes \Gamma) \otimes C_0(\mathbb{R})$ . By a choice of horizontal subspaces on the bundle  $P \times \mathbb{R}$ , one can lift the Dolbeault operator  $\bar{\partial}$  of  $\Sigma$ . This yields the horizontal operator  $Q_H = \bar{\partial} + \bar{\partial}^*$ , where the adjoint  $\bar{\partial}^*$  is taken relative to the  $L^2$ -norm given by the canonical invariant measure on  $P \times \mathbb{R}$  (see [4] for details). Finally, consider the signature operator of the fibers,  $Q_V = d_V d_V^* - d_V^* d_V$ , where  $d_V$  is the vertical differential. Then the sum  $Q = Q_H + Q_V$  is a hypoelliptic operator representing our Dolbeault  $K$ -cycle.

This construction ensures that the principal symbol of  $Q$  is completely canonical, because related only to the fibration of  $P \times \mathbb{R}$  over  $\Sigma$ , and hence is invariant under  $\Gamma$ . Another choice of horizontal subspaces does not change the leading term of the symbol of  $Q$ . This is basically the reason why  $Q$  allows to construct a spectral triple (of even parity) for the algebra  $C_c^\infty(P \times \mathbb{R}) \rtimes \Gamma$ .

If  $\Gamma = \text{Id}$ , then  $C_0(P \times \mathbb{R}) \rtimes \Gamma = C_0(\Sigma) \otimes C_0(\mathbb{R}^2)$  and the addition of  $Q_V$  to  $Q_H$  is nothing else but a Thom isomorphism in  $K$ -homology

$$K^*(C_0(\Sigma)) \rightarrow K^*(C_0(P \times \mathbb{R})) \quad (3)$$

sending the classical Dolbeault elliptic operator  $\bar{\partial} + \bar{\partial}^*$  to  $Q$ .

Now we want to compute the Chern character of  $Q$  in the periodic cyclic cohomology  $H^*(C_c^\infty(P \times \mathbb{R}) \rtimes \Gamma)$  using the index theorem of [5]. We need first to construct an *odd* cycle by tensoring the Dolbeault complex with the spectral triple of the real line  $(C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), i\frac{\partial}{\partial x})$ . In this way we get a differential operator  $Q' = Q + i\frac{\partial}{\partial x}$  whose Chern character lives in the cyclic cohomology of  $(C_c^\infty(P) \rtimes \Gamma) \otimes C_c^\infty(\mathbb{R}^2)$ . By Bott periodicity it is just the cup product

$$\text{ch}_*(Q') = \varphi \# [\mathbb{R}^2] \quad (4)$$

of a cyclic cocycle  $\varphi \in HC^*(C_c^\infty(P) \rtimes \Gamma)$  by the fundamental class of  $\mathbb{R}^2$ . The main theorem of [5] states that  $\varphi$  can be computed from Gelfand-Fuchs cohomology, after transiting through the cyclic cohomology of a particular Hopf algebra. We perform the explicit computation in the remaining of the paper.

### III. The Hopf algebra and its cyclic cohomology

First we reduce to the case of a flat Riemann surface, since for any groupoid  $\Sigma \rtimes \Gamma$  one can find a flat surface  $\Sigma'$  and a pseudogroup  $\Gamma'$  acting by conformal transformations on  $\Sigma'$  such that  $C_0(\Sigma') \rtimes \Gamma'$  is Morita equivalent to  $C_0(\Sigma) \rtimes \Gamma$  (see [5] and section V below).

Let then  $\Sigma$  be a flat Riemann surface and  $(z, \bar{z})$  a complex coordinate system corresponding to the complex structure of  $\Sigma$ . Let  $F$  be the  $GL(1, \mathbb{C})$ -principal bundle over  $\Sigma$  of frames corresponding to the conformal structure.  $F$  is gifted with the coordinate system  $(z, \bar{z}, y, \bar{y})$ ,  $y, \bar{y} \in \mathbb{C}^*$ . A point of  $F$  is the frame

$$(y\partial_z, \bar{y}\partial_{\bar{z}}) \quad \text{at } (z, \bar{z}) . \quad (5)$$

The action of a discrete pseudogroup  $\Gamma$  of conformal transformations on  $\Sigma$  can be lifted to an action on  $F$  by pushforward on frames. More precisely, a holomorphic transformation  $\psi \in \Gamma$  acts on the coordinates by

$$z \rightarrow \psi(z) \quad \text{Dom}\psi \subset F \quad (6)$$

$$y \rightarrow \psi'(z)y, \quad \psi'(z) = \partial_z \psi(z) . \quad (7)$$

Let  $C_c^\infty(F)$  be the algebra of smooth complex-valued functions with compact support on  $F$ , and consider the crossed product  $\mathcal{A} = C_c^\infty(F) \rtimes \Gamma$ .  $\mathcal{A}$  is the associative algebra linearly generated by elements of the form  $fU_\psi^*$  with  $\psi \in \Gamma$ ,  $f \in C_c^\infty(F)$ ,  $\text{supp}f \subset \text{Dom}\psi$ . We adopt the notation  $U_\psi \equiv U_{\psi^{-1}}^*$  for the inverse of  $U_\psi^*$ . The multiplication rule

$$f_1 U_{\psi_1}^* f_2 U_{\psi_2}^* = f_1 (f_2 \circ \psi_1) U_{\psi_2 \psi_1}^* \quad (8)$$

makes good sense thanks to the condition  $\text{supp}f_i \subset \text{Dom}\psi_i$ . We introduce now the differential operators

$$X = y\partial_z \quad Y = y\partial_y \quad \bar{X} = \bar{y}\partial_{\bar{z}} \quad \bar{Y} = \bar{y}\partial_{\bar{y}} \quad (9)$$

forming a basis of the set of smooth vector fields viewed as a module over  $C^\infty(F)$ . These operators act on  $\mathcal{A}$  in a natural way:

$$X.(fU_\psi^*) = (X.f)U_\psi^* , \quad Y.(fU_\psi^*) = (Y.f)U_\psi^* \quad (10)$$

and similarly for  $\bar{X}, \bar{Y}$ . Remark that the system  $(z, \bar{z})$  determines a smooth volume form  $\frac{dz \wedge d\bar{z}}{2i}$  on  $\Sigma$ . This in turn gives a representative  $\sigma$  of the modular automorphism group of  $L^\infty(\Sigma) \rtimes \Gamma$ , whose action on  $C_c^\infty(\Sigma) \rtimes \Gamma$  reads (cf. [3] chap. III)

$$\sigma_t(fU_\psi^*) = |\psi'|^{2it} fU_\psi^* , \quad t \in \mathbb{R} . \quad (11)$$

We let  $D$  be the derivation corresponding to the infinitesimal action of  $\sigma$ :

$$D = -i \frac{d}{dt} \sigma_t|_{t=0} \quad D(fU_\psi^*) = \ln |\psi'|^2 fU_\psi^* . \quad (12)$$

The operators  $\delta_n, \bar{\delta}_n$ ,  $n \geq 1$  are defined recursively

$$\delta_n = \underbrace{[X, \dots [X, D] \dots]}_n \quad \bar{\delta}_n = \underbrace{[\bar{X}, \dots [\bar{X}, D] \dots]}_n . \quad (13)$$

Their action on  $\mathcal{A}$  are explicitly given by

$$\delta_n(fU_\psi^*) = y^n \partial_z^n (\ln \psi') fU_\psi^* , \quad \bar{\delta}_n(fU_\psi^*) = y^n \partial_{\bar{z}}^n (\ln \bar{\psi}') fU_\psi^* . \quad (14)$$

Thus  $\delta_n, \bar{\delta}_n$  represent in some sense the Taylor expansion of  $D$ . All these operators fulfill the commutation relations

$$\begin{aligned} [Y, X] &= X & [Y, \delta_n] &= n\delta_n \\ [X, \delta_n] &= \delta_{n+1} & [\delta_n, \delta_m] &= 0 \end{aligned} \quad (15)$$

and similarly for the conjugates  $\bar{X}, \bar{Y}, \bar{\delta}_n$ . Thus  $\{X, Y, \delta_n, \bar{X}, \bar{Y}, \bar{\delta}_n\}_{n \geq 1}$  form a basis of a (complex) Lie algebra. Let  $\mathcal{H}$  be its enveloping algebra. The remarkable fact is that  $\mathcal{H}$  is a Hopf algebra. First, the coproduct  $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$  is determined by the action of  $\mathcal{H}$  on  $\mathcal{A}$ :

$$\Delta h(a_1 \otimes a_2) = h(a_1 a_2) \quad \forall h \in \mathcal{H}, a_i \in \mathcal{A} . \quad (16)$$

One has

$$\begin{aligned} \Delta X &= 1 \otimes X + X \otimes 1 + \delta_1 \otimes Y \\ \Delta Y &= 1 \otimes Y + Y \otimes 1 \quad \Delta \delta_1 = 1 \otimes \delta_1 + \delta_1 \otimes 1 . \end{aligned} \quad (17)$$

$\Delta \delta_n$  for  $n > 1$  is obtained recursively from (13) using the fact that  $\Delta$  is an algebra homomorphism,  $\Delta(h_1 h_2) = \Delta h_1 \Delta h_2$ . Similarly for the conjugate elements. The counit  $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}$  satisfies simply  $\varepsilon(1) = 1$ ,  $\varepsilon(h) = 0 \quad \forall h \neq 1$ .

Finally,  $\mathcal{H}$  has an antipode  $S : \mathcal{H} \rightarrow \mathcal{H}$ , determined uniquely by the condition  $m \circ S \otimes \text{Id} \circ \Delta = m \circ \text{Id} \otimes S \circ \Delta = \eta \varepsilon$ , where  $m : \mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$  is the multiplication and  $\eta : \mathbb{C} \rightarrow \mathcal{H}$  the unit of  $\mathcal{H}$ . One finds

$$S(X) = -X + \delta_1 Y \quad S(Y) = -Y \quad S(\delta_1) = -\delta_1 . \quad (18)$$

Since  $S$  is an antiautomorphism:  $S(h_1 h_2) = S(h_2) S(h_1)$ , the values of  $S(\delta_n)$ ,  $n > 1$  follow.

We are interested now in the cyclic cohomology of  $\mathcal{H}$  [5, 6]. As a space, the cochain complex  $C^*(\mathcal{H})$  is the tensor algebra over  $\mathcal{H}$ :

$$C^*(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} . \quad (19)$$

The crucial step is the construction of a characteristic map

$$\gamma : \mathcal{H}^{\otimes n} \rightarrow C^n(\mathcal{A}, \mathcal{A}^*) \quad (20)$$

from the cochain complex of  $\mathcal{H}$  to the Hochschild complex of  $\mathcal{A}$  with coefficients in  $\mathcal{A}^*$  [3]. First  $F$  has a canonical  $\Gamma$ -invariant measure  $dv = dz d\bar{z} \frac{dy d\bar{y}}{(y\bar{y})^2}$ . This yields a trace  $\tau$  on  $\mathcal{A}$ :

$$\begin{aligned} \tau(f) &= \int_F f dv \quad f \in C_c^\infty(F) , \\ \tau(f U_\psi^*) &= 0 \quad \text{if } \psi \neq 1 . \end{aligned} \quad (21)$$

Then the characteristic map sends the  $n$ -cochain  $h_1 \otimes \dots \otimes h_n \in \mathcal{H}^{\otimes n}$  to the Hochschild cochain  $\gamma(h_1 \otimes \dots \otimes h_n) \in C^n(\mathcal{A}, \mathcal{A}^*)$  given by

$$\gamma(h_1 \otimes \dots \otimes h_n)(a_0, \dots, a_n) = \tau(a_0 h_1(a_1) \dots h_n(a_n)) , \quad a_i \in \mathcal{A} . \quad (22)$$

The cyclic cohomology of  $\mathcal{H}$  is defined such that  $\gamma$  is a morphism of cyclic complexes. One introduces the face operators  $\delta^i : \mathcal{H}^{\otimes(n-1)} \rightarrow \mathcal{H}^{\otimes n}$  for  $0 \leq i \leq n$ :

$$\begin{aligned} \delta^0(h_1 \otimes \dots \otimes h_{n-1}) &= 1 \otimes h_1 \otimes \dots \otimes h_{n-1} \\ \delta^i(h_1 \otimes \dots \otimes h_{n-1}) &= h_1 \otimes \dots \otimes \Delta h_i \otimes \dots \otimes h_{n-1} \quad 1 \leq i \leq n-1 \\ \delta^n(h_1 \otimes \dots \otimes h_{n-1}) &= h_1 \otimes \dots \otimes h_{n-1} \otimes 1 \end{aligned} \quad (23)$$

as well as the degeneracy operators  $\sigma_i : \mathcal{H}^{\otimes(n+1)} \rightarrow \mathcal{H}^{\otimes n}$

$$\sigma_i(h_1 \otimes \dots \otimes h_{n+1}) = h_1 \otimes \dots \otimes \varepsilon(h_{i+1}) \dots \otimes h_{n+1} \quad 0 \leq i \leq n . \quad (24)$$

Next, the cyclic structure is provided by the antipode  $S$  and the multiplication of  $\mathcal{H}$ . Consider the twisted antipode  $\tilde{S} = (\delta \otimes S) \circ \Delta$ , where  $\delta : \mathcal{H} \rightarrow \mathbb{C}$  is a character such that

$$\tau(h(a)b) = \tau(a \tilde{S}(h)(b)) \quad \forall a, b \in \mathcal{A} . \quad (25)$$

This last formula plays the role of ordinary integration by parts. One finds:

$$\begin{aligned} \delta(1) &= 1 , \quad \delta(Y) = \delta(\bar{Y}) = 1 \\ \delta(X) &= \delta(\bar{X}) = \delta(\delta_n) = \delta(\bar{\delta}_n) = 0 \quad \forall n \geq 1 . \end{aligned} \quad (26)$$

The definition implies  $\tilde{S}^2 = 1$ . Connes and Moscovici proved in [6] that the latter identity is sufficient to ensure the existence of a cyclicity operator  $\tau_n : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes n}$

$$\tau_n(h_1 \otimes \dots \otimes h_n) = (\Delta^{n-1} \tilde{S}(h_1)) \cdot h_2 \otimes \dots \otimes h_n \otimes 1, \quad (27)$$

with  $(\tau_n)^{n+1} = 1$ . Now  $C^*(\mathcal{H})$  endowed with  $\delta^i, \sigma_i, \tau_n$  defines a cyclic complex. The Hochschild coboundary operator  $b : \mathcal{H}^{\otimes n} \rightarrow \mathcal{H}^{\otimes(n+1)}$  is

$$b = \sum_{i=0}^{n+1} (-)^i \delta^i \quad (28)$$

and Connes' operator  $B : \mathcal{H}^{\otimes(n+1)} \rightarrow \mathcal{H}^{\otimes n}$  is

$$B = \sum_{i=0}^n (-)^{ni} (\tau_n)^i B_0 \quad B_0 = \sigma_n \tau_{n+1} + (-)^n \sigma_n. \quad (29)$$

They fulfill the usual relations  $B^2 = b^2 = bB + Bb = 0$ , so that  $C^*(\mathcal{H}, b, B)$  is a bicomplex. We define the cyclic cohomology  $HC^*(\mathcal{H})$  as the  $b$ -cohomology of the subcomplex of cyclic cochains. The corresponding *periodic* cyclic cohomology  $H^*(\mathcal{H})$  is isomorphic to the cohomology of the bicomplex  $C^*(\mathcal{H}, b, B)$  [3]. Furthermore, the definitions of  $\delta^i, \sigma_i, \tau_n$  imply that  $\gamma$  is a morphism of cyclic complexes. Consequently,  $\gamma$  passes to cyclic cohomology

$$\gamma : HC^*(\mathcal{H}) \rightarrow HC^*(\mathcal{A}), \quad (30)$$

as well as to periodic cyclic cohomology

$$\gamma : H^*(\mathcal{H}) \rightarrow H^*(\mathcal{A}). \quad (31)$$

In fact we are not interested in the frame bundle  $F$  but rather in the bundle of metrics  $P = F/SO(2)$ , where  $SO(2) \subset Gl(1, \mathbb{C})$  is the group of rotations of frames.  $P$  is gifted with the coordinate chart  $(z, \bar{z}, r)$  where the radial coordinate  $r$  is obtained from the decomposition

$$y = e^{-r+i\theta} \quad r \in \mathbb{R}, \theta \in [0, 2\pi). \quad (32)$$

The pseudogroup  $\Gamma$  still acts on  $P$  by

$$\begin{aligned} z &\rightarrow \psi(z) & \bar{z} &\rightarrow \overline{\psi(z)} \\ r &\rightarrow r - \frac{1}{2} \ln |\psi'(z)|^2. \end{aligned} \quad (33)$$

Define  $\mathcal{A}_1 = \mathcal{A}^{SO(2)} \subset \mathcal{A}$  the subalgebra of elements of  $\mathcal{A}$  invariant under the (right) action of  $SO(2)$  on  $F$ .  $\mathcal{A}_1$  is canonically isomorphic to the crossed product  $C_c^\infty(P) \rtimes \Gamma$ .  $P$  carries a  $\Gamma$ -invariant measure  $dv_1 = e^{2r} dz d\bar{z} dr$ , so that there is a trace on  $\mathcal{A}_1$ , namely

$$\begin{aligned} \tau_1(f) &= \int_P f dv_1 & f &\in C_c^\infty(P) \\ \tau_1(fU_\psi^*) &= 0 & \text{if } \psi &\neq 1. \end{aligned} \quad (34)$$

Thus passing to  $SO(2)$ -invariants yields an induced characteristic map

$$\gamma_1 : HC^*(\mathcal{H}, SO(2)) \rightarrow HC^*(\mathcal{A}_1) \quad (35)$$

from the relative cyclic cohomology of  $\mathcal{H}$ , with  $\gamma_1(h_1 \otimes \dots \otimes h_n)(a_0, \dots, a_n) = \tau_1(a_0 h_1(a_1) \dots h_n(a_n))$ ,  $a_i \in \mathcal{A}_1$ , where  $h_1 \otimes \dots \otimes h_n$  represents an element of  $HC^*(\mathcal{H}, SO(2))$ . The map  $\gamma_1$  generalises the classical Chern-Weil construction of characteristic classes from connexions and curvatures. In the crossed product case  $\Sigma \rtimes \Gamma$ , these classes are captured by the periodic cyclic cohomology of  $\mathcal{H}$ . The authors of [5] computed the latter as Gelfand-Fuchs cohomology. This is the subject of the next section.

#### IV. Gelfand-Fuchs cohomology

Let  $G$  be the group of complex analytic transformations of  $\mathbb{C}$ .  $G$  has a unique decomposition  $G = G_1 G_2$ , where  $G_1$  is the group of affine transformations

$$x \rightarrow ax + b, \quad x \in \mathbb{C}, \quad a, b \in \mathbb{C} \quad (36)$$

and  $G_2$  is the group of transformations of the form

$$x \rightarrow x + o(x). \quad (37)$$

Any element of  $G$  is then the composition  $k \circ \psi$  for  $k \in G_1$ ,  $\psi \in G_2$ . Since  $G_2$  is the left quotient of  $G$  by  $G_1$ ,  $G_1$  acts on  $G_2$  from the right: for  $k \in G_1$ ,  $\psi \in G_2$ , one has  $\psi \triangleleft k \in G_2$ . Similarly,  $G_2$  acts on  $G_1$  from the left:  $\psi \triangleright k \in G_1$ .

Remark that  $G_1$  is the crossed product  $\mathbb{C} \rtimes Gl(1, \mathbb{C})$ . The space  $\mathbb{C} \times Gl(1, \mathbb{C})$  is a prototype for the frame bundle  $F$  of a flat Riemann surface. This motivates the notation  $a = y$ ,  $b = z$  for the coordinates on  $G_1$ . Under this identification, the left action of  $G_2$  on  $G_1$  corresponds to the action of  $G_2$  on  $F$ : for a holomorphic transformation  $\psi \in G_2$ , one has

$$z \rightarrow \psi(z), \quad y \rightarrow \psi'(z)y, \quad (38)$$

with  $\psi(0) = 0$ ,  $\psi'(0) = 1$ . Furthermore, the vector fields  $X, \bar{X}, Y, \bar{Y}$  form a basis of invariant vector fields for the left action of  $G_1$  on itself, i.e. a basis of the (complexified) Lie algebra of  $G_1$ . Its dual basis is given by the left-invariant 1-forms (Maurer-Cartan form)

$$\begin{aligned} \omega_{-1} &= y^{-1} dz & \bar{\omega}_{-1} &= \bar{y}^{-1} d\bar{z} \\ \omega_0 &= y^{-1} dy & \bar{\omega}_0 &= \bar{y}^{-1} d\bar{y}. \end{aligned} \quad (39)$$

The left action  $G_2 \triangleright G_1$  implies a right action of  $G_2$  on forms by pullback. One has in particular, for  $\psi \in G_2$ ,

$$\omega_{-1} \circ \psi = \omega_{-1} \quad \omega_0 \circ \psi = \omega_0 + y \partial_z \ln \psi' \omega_{-1} \quad \text{and c.c.} \quad (40)$$

Consider now the discrete crossed product  $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$  where  $G_2$  acts on  $C_c^\infty(G_1)$  by pullback. As a coalgebra,  $\mathcal{H}$  is dual to the algebra  $\mathcal{H}_*$ . One has a natural action of  $\mathcal{H}$  on  $\mathcal{H}_*$ :

$$\begin{aligned} X.(fU_\psi^*) &= X.fU_\psi^* \quad f \in C_c^\infty(G_1), \psi \in G_2, \\ \delta_n(fU_\psi^*) &= y^n \partial_z^n \ln \psi' fU_\psi^*, \end{aligned} \quad (41)$$

and so on with  $Y, \bar{X} \dots$ . The operators  $\delta_n, \bar{\delta}_n$  have in fact an interpretation in terms of coordinates on the group  $G_2$ : for  $\psi \in G_2$ ,  $\delta_n(\psi)$  is by definition the value of the function  $\delta_n(U_\psi^*)U_\psi$  at  $1 \in G_1$ . For any  $k \in G_1$ , one has

$$[\delta_n(U_\psi^*)U_\psi](k) = \delta_n(\psi \triangleleft k). \quad (42)$$

Note that (40) rewrites

$$\omega_0 \circ \psi = \omega_0 + \delta_1(\psi \triangleleft k)\omega_{-1} \quad \text{at } k \in G_1. \quad (43)$$

The Hopf subalgebra of  $\mathcal{H}$  generated by  $\delta_n, \bar{\delta}_n$ ,  $n \geq 1$ , corresponds to the commutative Hopf algebra of functions on  $G_2$  which are *polynomial* in these coordinates.

Let  $A$  be the complexification of the formal Lie algebra of  $G$ . It coincides with the jets of holomorphic and antiholomorphic vector fields of any order on  $\mathbb{C}$ :

$$\begin{aligned} \partial_x &, \quad x\partial_x, \dots, x^n\partial_x, \dots & x \in \mathbb{C} \\ \partial_{\bar{x}} &, \quad \bar{x}\partial_{\bar{x}}, \dots, \bar{x}^n\partial_{\bar{x}}, \dots \end{aligned} \quad (44)$$

The Lie bracket between the elements of the above basis is thus

$$\begin{aligned} [x^n\partial_x, x^m\partial_x] &= (m-n)x^{n+m-1}\partial_x \quad \text{and c.c.} \\ [x^n\partial_x, \bar{x}^m\partial_{\bar{x}}] &= 0. \end{aligned} \quad (45)$$

Define the generator of dilatations  $H = x\partial_x + \bar{x}\partial_{\bar{x}}$  and of rotations  $J = x\partial_x - \bar{x}\partial_{\bar{x}}$ . They fulfill the properties

$$\begin{aligned} [H, x^n\partial_x] &= (n-1)x^n\partial_x & [H, \bar{x}^n\partial_{\bar{x}}] &= (n-1)\bar{x}^n\partial_{\bar{x}} \\ [J, x^n\partial_x] &= (n-1)x^n\partial_x & [J, \bar{x}^n\partial_{\bar{x}}] &= -(n-1)\bar{x}^n\partial_{\bar{x}}. \end{aligned} \quad (46)$$

We are interested in the Lie algebra cohomology of  $A$  (see [7]). The complex  $C^*(A)$  of cochains is the exterior algebra generated by the dual basis  $\{\omega^n, \bar{\omega}^n\}_{n \geq -1}$ :

$$\begin{aligned} \omega^n(x^m\partial_x) &= \delta_{n+1}^m & \omega^n(\bar{x}^m\partial_{\bar{x}}) &= 0 \\ \bar{\omega}^n(x^m\partial_x) &= 0 & \bar{\omega}^n(\bar{x}^m\partial_{\bar{x}}) &= \delta_{n+1}^m \quad \forall n \geq -1, m \geq 0, \end{aligned} \quad (47)$$

and the coboundary operator is uniquely defined by its action on 1-cochains

$$d\omega(X, Y) = -\omega([X, Y]) \quad \forall X, Y \in A. \quad (48)$$



From [5] we know that the *periodic* cyclic cohomology  $H^*(\mathcal{H}, SO(2))$  is isomorphic to the relative Lie algebra cohomology  $H^*(A, SO(2))$ , i.e. the cohomology of the basic subcomplex of cochains on  $A$  relative to the Cartan operation  $(L, i)$  of  $J$ :

$$L_J\omega = (i_Jd + di_J)\omega \quad \forall \omega \in C^*(A) . \quad (49)$$

We say that a cochain  $\omega \in C^*(A)$  is of weight  $r$  if  $L_H\omega = -r\omega$ . Remark that

$$L_H\omega^n = -n\omega^n , \quad L_H\bar{\omega}^n = -n\bar{\omega}^n \quad \forall n \geq -1 , \quad (50)$$

so that  $C^*(A)$  is the direct sum, for  $r \geq -2$ , of the spaces  $C_r^*(A)$  of weight  $r$ . Since  $[H, J] = 0$ ,  $C_r^*(A)$  is stable under the Cartan operation of  $J$  and we note  $C_r^*(A, SO(2))$  the complex of basic cochains of weight  $r$ . Then we have

$$C^*(A, SO(2)) = \bigoplus_{r=-2}^{\infty} C_r^*(A, SO(2)) . \quad (51)$$

For any cocycle  $\omega \in C_r^*(A, SO(2))$ ,

$$L_H\omega = di_H\omega = -r\omega \quad (52)$$

so that  $C_r^*(A, SO(2))$  is acyclic whenever  $r \neq 0$ . Hence  $H^*(A, SO(2))$  is equal to the cohomology of the finite-dimensional subcomplex  $C_0^*(A, SO(2))$ . The direct computation gives

$$\begin{aligned} H^0(A, SO(2)) &= \mathbb{C} && \text{with representative } 1 \\ H^2(A, SO(2)) &= \mathbb{C} && \omega^{-1}\omega^1 \\ H^3(A, SO(2)) &= \mathbb{C} && (\omega^{-1}\omega^1 - \bar{\omega}^{-1}\bar{\omega}^1)(\omega^0 + \bar{\omega}^0) \\ H^5(A, SO(2)) &= \mathbb{C} && \omega^1\omega^{-1}\bar{\omega}^1\bar{\omega}^{-1}(\omega^0 + \bar{\omega}^0) \end{aligned} \quad (53)$$

The other cohomology groups vanish.

Next we construct a map  $C$  from  $C^*(A)$  to the bicomplex  $(C^{n,m}, d_1, d_2)_{n,m \in \mathbb{Z}}$  of [3] chap. III.2.δ. Let  $\Omega^m(G_1)$  be the space  $m$ -forms on  $G_1$ .  $C^{n,m}$  is the space of totally antisymmetric maps  $\gamma : G_2^{n+1} \rightarrow \Omega^m(G_1)$  such that

$$\gamma(g_0g, \dots, g_ng) = \gamma(g_0, \dots, g_n) \circ g \quad g_i \in G_2, g \in G , \quad (54)$$

where  $g_i g$  is given by the right action of  $G$  on  $G_2$ , and  $G$  acts on  $\Omega^*(G_1)$  by pullback (left action of  $G$  on  $G_1$ ).

The first differential  $d_1 : C^{n,m} \rightarrow C^{n+1,m}$  is

$$(d_1\gamma)(g_0, \dots, g_{n+1}) = (-)^m \sum_{i=0}^{n+1} (-)^i \gamma(g_0, \dots, \check{g}_i, \dots, g_{n+1}) , \quad (55)$$

and  $d_2 : C^{n,m} \rightarrow C^{n,m+1}$  is just the de Rham coboundary on  $\Omega^*(G_1)$ :

$$(d_2\gamma)(g_0, \dots, g_n) = d(\gamma(g_0, \dots, g_n)) . \quad (56)$$

Of course  $d_1^2 = d_2^2 = d_1d_2 + d_2d_1 = 0$ . Remark that for  $\gamma \in C^{n,m}$ , the invariance property (54) implies

$$\gamma(g_0, \dots, g_n) \circ k = \gamma(g_0 \triangleleft k, \dots, g_n \triangleleft k) \quad \forall k \in G_1 , \quad (57)$$

in other words the value of  $\gamma(g_0, \dots, g_n) \in \Omega^m(G_1)$  at  $k$  is deduced from its value at 1.

Let us describe now the construction of  $C$ . As a vector space, the Lie algebra  $A$  is just the direct sum  $\mathbf{G}_1 \oplus \mathbf{G}_2$ ,  $\mathbf{G}_i$  being the (complexified) Lie algebra of  $G_i$ . The cochain complex  $C^*(A)$  is then the exterior product  $\Lambda A^* = \Lambda \mathbf{G}_1^* \otimes \Lambda \mathbf{G}_2^*$ . One identifies  $\mathbf{G}_1^*$  with the cotangent space  $T_1^*(G_1)$  of  $G_1$  at the identity. Since  $G_2$  fixes  $1 \in G_1$ , there is a right action of  $G_2$  on  $\Lambda \mathbf{G}_1^*$  by pullback. The basis  $\{\omega^{-1}, \omega^0, \bar{\omega}^{-1}, \bar{\omega}^0\}$  of  $\mathbf{G}_1^*$  is represented by left-invariant one-forms on  $G_1$  through the identification

$$\begin{aligned} \omega^{-1} &\rightarrow -\omega_{-1} = -y^{-1}dz & \bar{\omega}^{-1} &\rightarrow -\bar{\omega}_{-1} = -\bar{y}^{-1}d\bar{z} \\ \omega^0 &\rightarrow -\omega^0 = -y^{-1}dy & \bar{\omega}^0 &\rightarrow -\bar{\omega}^0 = -\bar{y}^{-1}d\bar{y}, \end{aligned} \quad (58)$$

and the right action of  $\psi \in G_2$  reads (cf. (40))

$$\omega^{-1} \cdot \psi = \omega^{-1}, \quad \omega^0 \cdot \psi = \omega^0 + \delta_1(\psi)\omega^{-1}. \quad (59)$$

Next, we view a cochain  $\omega \in C^*(A)$  as a cochain of the Lie algebra of  $G_2$  with coefficients in the right  $G_2$ -module  $\Lambda \mathbf{G}_1^*$ . It is represented by a  $\Lambda \mathbf{G}_1^*$ -valued right-invariant form  $\mu$  on  $G_2$ . Then  $C(\omega) \in C^{*,*}$  evaluated on  $(g_0, \dots, g_n) \in G_2^{n+1}$  is a differential form on  $G_1$  whose value at  $1 \in G_1$  is

$$C(\omega)(g_0, \dots, g_n) = \int_{\Delta(g_0, \dots, g_n)} \mu \in \Lambda T_1^*(G_1), \quad (60)$$

where  $\Delta(g_0, \dots, g_n)$  is the affine simplex in the coordinates  $\delta_i, \bar{\delta}_i$ , with vertices  $(g_0, \dots, g_n)$ . Let  $\{\rho_j\}$  be a basis of left-invariant forms on  $G_1$ . Then

$$C(\omega)(g_0, \dots, g_n) = \sum_j p_j(g_0, \dots, g_n) \rho_j \quad \text{at } 1 \in G_1, \quad (61)$$

where  $p_j(g_0, \dots, g_n)$  are polynomials in the coordinates  $\delta_i, \bar{\delta}_i$ . The invariance property (54) enables us to compute the value of  $C(\omega)(g_0, \dots, g_n)$  at any  $k \in G_1$ ,

$$C(\omega)(g_0, \dots, g_n)(k) = \sum_j p_j(g_0 \triangleleft k, \dots, g_n \triangleleft k) \rho_j \quad (62)$$

because  $\rho_j \circ k = \rho_j$ .

Connes and Moscovici showed in [5] that  $C$  is a morphism from  $C^*(A, d)$  to the bicomplex  $(C^{n,m}, d_1, d_2)_{n,m \in \mathbb{Z}}$ . In the relative case, it restricts to a morphism from  $C^*(A, SO(2), d)$  to the subcomplex  $(C_{bas.}^{n,m}, d_1, d_2)$  of antisymmetric cochains on  $G_2$  with values in the *basic* de Rham cohomology  $\Omega^*(P) = \Omega^*(G_1/SO(2))$ .

It remains to compute the image of  $H^*(A, SO(2))$  by  $C$ . We restrict ourselves to even cocycles, i.e. the unit  $1 \in H^0(A, SO(2))$  and the first Chern class  $c_1 \in H^2(A, SO(2))$ , defined as the class

$$c_1 = [2\omega^{-1}\omega^1]. \quad (63)$$

One has  $C(1) \in C_{bas.}^{0,0}$ . The immediate result is

$$C(1)(g_0) = 1, \quad g_0 \in G_2. \quad (64)$$

For the first Chern class, we must transform  $c_1$  into a right-invariant form on  $G_2$  with values in  $\Lambda T_1^*(G_1)$ . We already know that  $\omega^{-1}$  is represented by  $-\omega_{-1} = -y^{-1}dz$ , which satisfies  $\omega_{-1} \circ \psi = \omega_{-1}$ ,  $\forall \psi \in G_2$ . Next, the Taylor expansion of an element  $\psi \in G_2$  can be expressed in the coordinates  $\delta_n$  thanks to the obvious formula

$$\ln \psi'(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \delta_n(\psi) x^n, \quad \forall x \in \mathbb{C}. \quad (65)$$

One finds:

$$\psi(x) = x + \frac{1}{2} \delta_1(\psi) x^2 + \frac{1}{3!} (\delta_2(\psi) + \delta_1(\psi)^2) x^3 + O(x^4). \quad (66)$$

It shows that the cochain  $\omega^1 \in C^*(A)$  is represented by the right-invariant 1-form  $\frac{1}{2}d\delta_1$  on  $G_2$ . Thus at  $1 \in G_1$ ,  $C(c_1) \in C_{bas.}^{1,1}$  is given by

$$\begin{aligned} C(c_1)(g_0, g_1) &= \int_{\Delta(g_0, g_1)} -\omega_{-1} d\delta_1 \\ &= -\omega_{-1}(\delta_1(g_1) - \delta_1(g_0)) \quad g_i \in G_2, \end{aligned} \quad (67)$$

and at  $k \in G_1$ , the 1-form  $C(c_1)(g_0, g_1)$  is

$$C(c_1)(g_0, g_1) = -\omega_{-1}(\delta_1(g_1 \triangleleft k) - \delta_1(g_0 \triangleleft k)). \quad (68)$$

Since  $\omega_{-1} = y^{-1}dz$  and  $\delta_1(g \triangleleft k) = y \partial_z \ln g'(z)$ ,  $z$  and  $y$  being the coordinates of  $k$ , one has explicitly

$$C(c_1)(g_0, g_1) = -dz(\partial_z \ln g_1'(z) - \partial_z \ln g_0'(z)). \quad (69)$$

It is a basic form on  $G_1$  relative to  $SO(2)$ , then descends to a form on  $P = G_1/SO(2)$  as expected.

The last step is to use the map  $\Phi$  of [3] theorem 14 p.220 from  $(C^{n,m}, d_1, d_2)$  to the  $(b, B)$  bicomplex of the discrete crossed product  $C_c^\infty(P) \rtimes G_2$ . Define the algebra

$$\mathcal{B} = \Omega^*(P) \hat{\otimes} \Lambda \mathbb{C}(G_2'), \quad (70)$$

where  $\Lambda \mathbb{C}(G_2')$  is the exterior algebra generated by the elements  $\delta_\psi$ ,  $\psi \in G_2$ , with  $\delta_e = 0$  for the identity  $e$  of  $G_2$ . With the de Rham coboundary  $d$  of  $\Omega^*(P)$ ,  $\mathcal{B}$  is a differential algebra. Now form the crossed product  $\mathcal{B} \rtimes G_2$ , with multiplication rules

$$\begin{aligned} U_\psi^* \alpha U_\psi &= \alpha \circ \psi, \quad \alpha \in \Omega^*(P), \psi \in G_2 \\ U_{\psi_1}^* \delta_{\psi_2} U_{\psi_1} &= \delta_{\psi_2 \circ \psi_1} - \delta_{\psi_1}, \quad \psi_i \in G_2. \end{aligned} \quad (71)$$

Endow  $\mathcal{B} \rtimes G_2$  with the differential  $\tilde{d}$  acting on an element  $bU_\psi^*$  as

$$\tilde{d}(bU_\psi^*) = dbU_\psi^* - (-)^{\partial b} b \delta_\psi U_\psi^*, \quad (72)$$

where  $db$  comes from the de Rham coboundary of  $\Omega^*(P)$ . The map

$$\Phi : (C^{*,*}, d_1, d_2) \rightarrow (C_c^\infty(P) \times G_2, b, B) \quad (73)$$

is constructed as follows. Let  $\gamma \in C_{bas.}^{n,m}$ . It yields a linear form  $\tilde{\gamma}$  on  $\mathcal{B} \times G_2$ :

$$\begin{aligned} \tilde{\gamma}(\alpha \otimes \delta_{g_1} \dots \delta_{g_n}) &= \int_P \alpha \wedge \gamma(1, g_1, \dots, g_n), \quad \alpha \in \Omega^*(P), g_i \in G_2 \\ \tilde{\gamma}(bU_\psi^*) &= 0 \quad \text{if } \psi \neq 1. \end{aligned} \quad (74)$$

Then  $\Phi(\gamma)$  is the following  $l$ -cochain on  $C_c^\infty(P) \times G_2$ ,  $l = \dim P - m + n$

$$\begin{aligned} \Phi(\gamma)(x_0, \dots, x_l) &= \frac{n!}{(l+1)!} \sum_{j=0}^l (-)^{j(l-j)} \tilde{\gamma}(\tilde{d}x_{j+1} \dots \tilde{d}x_l x_0 \tilde{d}x_1 \dots \tilde{d}x_j), \\ x_i &\in C_c^\infty(P) \times G_2 \subset \mathcal{B} \times G_2. \end{aligned} \quad (75)$$

The essential tool is that  $\Phi$  is a morphism of bicomplexes:

$$\Phi(d_1\gamma) = b\Phi(\gamma), \quad \Phi(d_2\gamma) = B\Phi(\gamma). \quad (76)$$

Moreover, if  $d_1\gamma = d_2\gamma = 0$ ,  $\Phi(\gamma)$  is a cyclic cocycle. This happens in our case. Since  $P$  is a 3-dimensional manifold, the image of  $C(1)$  under  $\Phi$  is the cyclic 3-cocycle

$$\Phi(C(1))(x_0, \dots, x_3) = \int_P x_0 dx_1 \dots dx_3, \quad x_i \in C_c^\infty(P) \times G_2, \quad (77)$$

where  $d(fU_\psi^*) = dfU_\psi^*$  for  $f \in C_c^\infty(P)$ ,  $\psi \in G_2$ , and the integration is extended over  $\Omega^*(P) \times G_2$  by setting

$$\int_P \alpha U_\psi^* = 0 \quad \text{if } \psi \neq 1, \alpha \in \Omega^*(P). \quad (78)$$

The image of  $\gamma = C(c_1)$  is more complicated to compute. One has

$$\tilde{\gamma}(\alpha \otimes \delta_g) = - \int_P \alpha \wedge y^{-1} dz \delta_1(g \triangleleft k), \quad \alpha \in \Omega^2(P), g \in G_2 \quad (79)$$

where  $y^{-1} dz \delta_1(g \triangleleft k) = dz \partial_z \ln g'(z)$  is, of course, a 1-form on  $P$ .  $\Phi(\gamma)$  is the cyclic 3-cocycle

$$\begin{aligned} \Phi(\gamma)(f_0 U_{\psi_0}^*, \dots, f_3 U_{\psi_3}^*) &= -\tilde{\gamma}(f_0 U_{\psi_0}^* df_1 U_{\psi_1}^* df_2 U_{\psi_2}^* f_3 \delta_{\psi_3} U_{\psi_3}^* \\ &\quad + f_0 U_{\psi_0}^* df_1 U_{\psi_1}^* f_2 \delta_{\psi_2} U_{\psi_2}^* df_3 U_{\psi_3}^* \\ &\quad + f_0 U_{\psi_0}^* f_1 \delta_{\psi_1} U_{\psi_1}^* df_2 U_{\psi_2}^* df_3 U_{\psi_3}^*) \\ &= \tilde{\gamma}(f_0 (df_1 \circ \psi_0) (df_2 \circ \psi_1 \psi_0) (f_3 \circ \psi_2 \psi_1 \psi_0) \delta_{\psi_2 \psi_1 \psi_0} \\ &\quad + f_0 (df_1 \circ \psi_0) (f_2 \circ \psi_1 \psi_0) (df_3 \circ \psi_2 \psi_1 \psi_0) (\delta_{\psi_2 \psi_1 \psi_0} - \delta_{\psi_1 \psi_0}) \\ &\quad - f_0 (f_1 \circ \psi_0) (df_2 \circ \psi_1 \psi_0) (df_3 \circ \psi_2 \psi_1 \psi_0) (\delta_{\psi_1 \psi_0} - \delta_{\psi_0})), \end{aligned} \quad (80)$$

upon assuming that  $\psi_3\psi_2\psi_1\psi_0 = \text{Id}$ . Using the relation

$$\delta_1(\psi \triangleleft k) = [\delta_1(U_\psi^*)U_\psi](k) , \quad \forall k \in G_1, \psi \in G_2 \quad (81)$$

the computation gives

$$\Phi(\gamma)(x_0, \dots, x_3) = \int_P x_0(dx_1dx_2\delta_1(x_3) + dx_1\delta_1(x_2)dx_3 + \delta_1(x_1)dx_2dx_3)y^{-1}dz . \quad (82)$$

Now recall that  $P$  has an invariant volume form  $dv_1 = e^{2r} dz d\bar{z} dr$ . The differential  $df$  of a function on  $P$  makes use of the horizontal  $X = y\partial_z$ ,  $\bar{X} = \bar{y}\partial_{\bar{z}}$  and vertical  $Y + \bar{Y} = -\partial_r$  vector fields:

$$df = y^{-1}dzX.f + \bar{y}^{-1}d\bar{z}\bar{X}.f - dr(Y + \bar{Y}).f . \quad (83)$$

Then using the relations (40) one sees that  $\Phi(C(c_1))$  is a sum of terms involving the Hopf algebra

$$\Phi(C(c_1))(x_0, \dots, x_3) = \sum_i \int_P x_0 h_1^i(x_1) \dots h_3^i(x_3) dv_1 , \quad (84)$$

where the sum  $\sum_i h_1^i \otimes h_2^i \otimes h_3^i$  is a cyclic 3-cocycle of  $\mathcal{H}$  relative to  $SO(2)$ . This follows from the existence of the characteristic map (35)

$$HC^*(\mathcal{H}, SO(2)) \rightarrow HC^*(C_c^\infty(P) \rtimes G_2) \quad (85)$$

and the duality between  $\mathcal{H}$  and  $\mathcal{H}_* = C_c^\infty(G_1) \rtimes G_2$  (cf. [5]).

Returning to the initial situation, where  $F$  is the frame bundle of a flat Riemann surface  $\Sigma$ , and  $P = F/SO(2)$  the bundle of metrics, the above computation shows that the cyclic 3-cocycle on  $\mathcal{A}_1 = C_c^\infty(P) \rtimes \Gamma$

$$[c_1](a_0, \dots, a_3) = \sum_i \int_P a_0 h_1^i(a_1) \dots h_3^i(a_3) dv_1 , \quad a_i \in \mathcal{A}_1 , \quad (86)$$

is the image of  $C(c_1)$  by the characteristic map  $HC^*(\mathcal{H}, SO(2)) \rightarrow HC^*(\mathcal{A}_1)$ . Also the fundamental class

$$[P](a_0, \dots, a_3) = \int_P a_0 da_1 da_2 da_3 \quad (87)$$

is in the range of the characteristic map.

Since Connes and Moscovici showed that the Gelfand-Fuchs cohomology  $H^*(A, SO(2))$  is isomorphic to the periodic cyclic cohomology of  $\mathcal{H}$ , we have completely determined the odd part of the range of the characteristic map. We can summarize the result in the following

**Proposition 1** *Under the characteristic map*

$$H^*(A, SO(2)) \simeq H^*(\mathcal{H}, SO(2)) \rightarrow H^*(\mathcal{A}_1) , \quad (88)$$

the unit  $1 \in H^0(A, SO(2))$  maps to the fundamental class  $[P]$  represented by the cyclic 3-cocycle

$$[P](a_0, \dots, a_3) = \int_P a_0 da_1 da_2 da_3, \quad a_i \in \mathcal{A}_1, \quad (89)$$

and the first Chern class  $c_1 \in H^2(A, SO(2))$  gives the cocycle  $[c_1] \in HC^3(\mathcal{A}_1)$ :

$$[c_1](a_0, \dots, a_3) = \int_P a_0 (da_1 da_2 \delta_1(a_3) + da_1 \delta_1(a_2) da_3 + \delta_1(a_1) da_2 da_3) y^{-1} dz. \quad (90)$$

In section II we considered an odd  $K$ -cycle on  $C_0(P \times \mathbb{R}^2) \rtimes \Gamma$  represented by a differential operator  $Q'$ , which is equivalent, up to Bott periodicity, to an odd  $K$ -cycle on  $C_0(P) \rtimes \Gamma$ .  $Q'$  is a matrix-valued polynomial in the vector fields  $X, \bar{X}, Y + \bar{Y}$  and the partial derivatives along the two directions of  $\mathbb{R}^2$ . Its Chern character is the cup product

$$\text{ch}_*(Q') = \varphi \# [\mathbb{R}^2] \quad (91)$$

of a cyclic cocycle  $\varphi \in HC^{odd}(C_c^\infty(P) \rtimes \Gamma)$  by the fundamental class of  $\mathbb{R}^2$ . The index theorem of Connes and Moscovici states that  $\varphi$  is in the range of the characteristic map (we have to assume that the action of  $\Gamma$  on  $\Sigma$  has no fixed point). Hence it is a linear combination of the characteristic classes  $[P]$  and  $[c_1]$ . We shall determine the coefficients by using the classical Riemann-Roch theorem.

## V. A Riemann-Roch theorem for crossed products

We shall first use the Thom isomorphism in  $K$ -theory [1]

$$K_i(C_0(\Sigma) \rtimes \Gamma) \rightarrow K_{i+1}(C_0(P) \rtimes \Gamma) \quad (92)$$

to descend the characteristic classes  $[P]$  and  $[c_1]$  down to the cyclic cohomology of  $C_c^\infty(\Sigma) \rtimes \Gamma$ . Recall that  $C_0(P) \rtimes \Gamma$  is just the crossed product of  $C_0(\Sigma) \rtimes \Gamma$  by the modular automorphism group  $\sigma$  of the associated von Neumann algebra

$$C_0(P) \rtimes \Gamma = (C_0(\Sigma) \rtimes \Gamma) \rtimes_\sigma \mathbb{R}. \quad (93)$$

By homotopy we can deform  $\sigma$  continuously into the trivial action. For  $\lambda \in [0, 1]$ , let  $\sigma_t^\lambda = \sigma_{\lambda t}$ ,  $\forall t \in \mathbb{R}$ . Then  $\sigma^1 = \sigma$ ,  $\sigma^0 = \text{Id}$  and

$$(C_0(\Sigma) \rtimes \Gamma) \rtimes_{\text{Id}} \mathbb{R} = C_0(\Sigma) \rtimes \Gamma \otimes C_0(\mathbb{R}). \quad (94)$$

Next, the coordinate system  $(z, \bar{z})$  of  $\Sigma$  gives a smooth volume form  $\frac{dz \wedge d\bar{z}}{2i}$  together with a representative of  $\sigma$ , whose action on the subalgebra  $C_c^\infty(\Sigma) \rtimes \Gamma$  is

$$\sigma_t(fU_\psi^*) = f|\psi'|^{2it}U_\psi^*, \quad f \in C_c^\infty(\Sigma), \psi \in \Gamma, \quad (95)$$

and accordingly

$$\sigma_t^\lambda(fU_\psi^*) = f|\psi'|^{2i\lambda t}U_\psi^* . \quad (96)$$

Remark that the algebra  $(C_0(\Sigma) \rtimes \Gamma) \rtimes_{\sigma^\lambda} \mathbb{R}$  is equal to the crossed product  $C_0(P) \rtimes_\lambda \Gamma$  obtained from the following deformed action of  $\Gamma$  on  $P$ :

$$\begin{aligned} z &\rightarrow \psi(z) & \bar{z} &\rightarrow \overline{\psi(z)} \\ r &\rightarrow r - \frac{1}{2}\lambda \ln |\psi'(z)|^2 & \psi &\in \Gamma . \end{aligned} \quad (97)$$

Hence for any  $\lambda \in [0, 1]$ , one has a Thom isomorphism

$$\Phi^\lambda : K_0(C_0(\Sigma) \rtimes \Gamma) \rightarrow K_1(C_0(P) \rtimes_\lambda \Gamma) , \quad (98)$$

and  $\Phi^0$  is just the connecting map  $K_0(C_0(\Sigma) \rtimes \Gamma) \rightarrow K_1(S(C_0(\Sigma) \rtimes \Gamma))$ . We introduce also the family  $\{[P]^\lambda\}_{\lambda \in [0,1]}$  of cyclic cocycles

$$[P]^\lambda(a_0^\lambda, \dots, a_3^\lambda) = \int_P a_0^\lambda da_1^\lambda \dots da_3^\lambda , \quad \forall a_i^\lambda \in C_c^\infty(P) \rtimes_\lambda \Gamma . \quad (99)$$

One has  $[P]^1 = [P]$  and  $[P]^0 = [\Sigma] \# [\mathbb{R}] \in (C_c^\infty(\Sigma) \rtimes \Gamma) \otimes C_c^\infty(\mathbb{R})$ , where

$$[\Sigma](a_0, a_1, a_2) = \int_\Sigma a_0 da_1 da_2 \quad \forall a_i \in C_c^\infty(\Sigma) \rtimes \Gamma . \quad (100)$$

Moreover for any element  $[e] \in K_0(C_0(\Sigma) \rtimes \Gamma)$  such that  $\Phi^\lambda([e])$  is in the domain of definition of  $[P]^\lambda$ , the pairing

$$\langle \Phi^\lambda([e]), [P]^\lambda \rangle \quad (101)$$

depends continuously upon  $\lambda$ . Next for any  $\lambda \in (0, 1]$ , consider the vertical diffeomorphism of  $P$  whose action on the coordinates  $(z, \bar{z}, r)$  reads

$$\tilde{\lambda}(z) = z \quad \tilde{\lambda}(\bar{z}) = \bar{z} \quad \tilde{\lambda}(r) = \lambda r . \quad (102)$$

Thus for  $\lambda \neq 0$  one has an algebra isomorphism

$$\chi_\lambda : C_c^\infty(P) \rtimes_\lambda \Gamma \rightarrow C_c^\infty(P) \rtimes \Gamma \quad (103)$$

by setting

$$\chi_\lambda(fU_\psi^*) = f \circ \tilde{\lambda} U_\psi^* \quad \forall f \in C_c^\infty(P), \psi \in \Gamma . \quad (104)$$

For any  $\lambda \neq 0$ ,

$$(\chi_\lambda)_* \circ \Phi^\lambda = \Phi^1 , \quad (105)$$

$$(\chi_\lambda)^* [P]^1 = [P]^\lambda . \quad (106)$$

Eq.(105) comes from the unicity of the Thom map (cf. [1]), and (106) is obvious. Thus  $\langle \Phi^\lambda([e]), [P]^\lambda \rangle$  is constant for  $\lambda \neq 0$ , and by continuity at 0

$$\langle \Phi^1([e]), [P] \rangle = \langle [e], [\Sigma] \rangle . \quad (107)$$

This shows that the image of  $[P]$  by Thom isomorphism is the cyclic 2-cocycle  $[\Sigma]$  corresponding to the fundamental class of  $\Sigma$ . In exactly the same way we show that the image of  $[c_1]$  is the cyclic 2-cocycle  $\tau$  defined, for  $a_i = f_i U_{\psi_i}^* \in C_c^\infty(\Sigma) \rtimes \Gamma$ , by

$$\tau(a_0, a_1, a_2) = \int_{\Sigma} a_0 (da_1 \partial \ln \psi_2' a_2 + \partial \ln \psi_1' a_1 da_2) , \quad (108)$$

with  $\partial = dz \partial_z$ . Note that in the decomposition of the differential on  $\Sigma$ ,  $d = \partial + \bar{\partial}$ , both  $\partial$  and  $\bar{\partial}$  commute with the pullbacks by the conformal transformations  $\psi \in \Gamma$ .

So far we have considered a *flat* Riemann surface and the constructions we made were relative to a coordinate system  $(z, \bar{z})$ . We shall now remove this unpleasant feature by using the Morita equivalence [5]. In order to understand the general situation, let us first treat the particular case of the Riemann sphere  $S^2 = \mathbb{C} \cup \{\infty\}$ . We consider an open covering of the sphere by two planes:  $S^2 = U_1 \cup U_2$ ,  $U_1 = \mathbb{C}$ ,  $U_2 = \mathbb{C}$ , together with the glueing function  $g$ :

$$\begin{aligned} g : U_1 \setminus \{0\} &\rightarrow U_2 \setminus \{0\} \\ z &\mapsto \frac{1}{z} . \end{aligned} \quad (109)$$

The pseudogroup of conformal transformations  $\Gamma_0$  generated by  $\{U_g^*, U_g\}$  acts on the disjoint union  $\Sigma = U_1 \amalg U_2$ , which is flat. Then  $S^2$  is described by the groupoid  $\Sigma \rtimes \Gamma_0$ . If  $\Gamma$  is a pseudogroup of local transformations of  $S^2$ , there exists a pseudogroup  $\Gamma'$  containing  $\Gamma_0$ , acting on  $\Sigma$  and such that the crossed product  $C^\infty(S^2) \rtimes \Gamma$  is Morita equivalent to  $C_c^\infty(\Sigma) \rtimes \Gamma'$ . The latter splits into four parts: it is the direct sum, for  $i, j = 1, 2$ , of elements of the form  $f_{ij} U_{\psi_{ij}}^*$  with

$$\psi_{ij} : U_i \rightarrow U_j \quad \text{and} \quad \text{supp} f_{ij} \subset \text{Dom} \psi_{ij} . \quad (110)$$

For convenience, we adopt a matricial notation for any generic element  $b \in C_c^\infty(\Sigma) \rtimes \Gamma'$ :

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} , \quad b_{ij} = f_{ij} U_{\psi_{ij}}^* . \quad (111)$$

Now the Morita equivalence is explicitly realized through the following idempotent  $e \in C_c^\infty(\Sigma) \rtimes \Gamma'$ :

$$e = \begin{pmatrix} \rho_1^2 & \rho_1 \rho_2 U_g^* \\ U_g \rho_2 \rho_1 & U_g \rho_2^2 U_g^* \end{pmatrix} , \quad e^2 = e , \quad (112)$$

where  $\{\rho_i\}_{i=1,2}$  is a partition of unity relative to the covering  $\{U_i\}$ :

$$\rho_1 \in C_c^\infty(U_1) , \quad \rho_1^2 + \rho_2^2 = 1 \text{ on } S^2 = U_1 \cup \{\infty\} . \quad (113)$$

The reduction of  $C_c^\infty(\Sigma) \rtimes \Gamma'$  by  $e$  is the subalgebra

$$(C_c^\infty(\Sigma) \rtimes \Gamma')_e = \{b \in C_c^\infty(\Sigma) \rtimes \Gamma' / b = be = eb\} . \quad (114)$$



Its elements are of the form

$$ebe = \begin{pmatrix} \rho_1 c \rho_1 & \rho_1 c \rho_2 U_g^* \\ U_g \rho_2 c \rho_1 & U_g \rho_2 c \rho_2 U_g^* \end{pmatrix} \quad (115)$$

with  $c = \rho_1 b_{11} \rho_1 + \rho_2 U_g^* b_{21} \rho_1 + \rho_1 b_{12} U_g \rho_2 + \rho_2 U_g^* b_{22} U_g \rho_2$ . Then  $c$  can be considered as an element of  $C^\infty(S^2) \rtimes \Gamma$  under the identification  $S^2 = U_1 \cup \{\infty\}$ .  $(C_c^\infty(\Sigma) \rtimes \Gamma')_e$  and  $C^\infty(S^2) \rtimes \Gamma$  are isomorphic through the map

$$\begin{aligned} \theta : C^\infty(S^2) \rtimes \Gamma &\longrightarrow (C_c^\infty(\Sigma) \rtimes \Gamma')_e \\ a &\longmapsto \begin{pmatrix} \rho_1 a \rho_1 & \rho_1 a \rho_2 U_g^* \\ U_g \rho_2 a \rho_1 & U_g \rho_2 a \rho_2 U_g^* \end{pmatrix}. \end{aligned} \quad (116)$$

We are ready to compute the pullbacks of  $[\Sigma]$  and  $\tau \in HC^2(C_c^\infty(\Sigma) \rtimes \Gamma')$  by  $\theta$ . This yields the following cyclic 2-cocycles on  $C^\infty(S^2) \rtimes \Gamma$ :

$$\begin{aligned} \theta^*[\Sigma] &= [S^2], \\ (\theta^*\tau)(a_0, a_1, a_2) &= \int_{S^2} a_0 \left( da_1 (\partial \ln \psi'_2 a_2 + [a_2, \rho_2^2 \partial \ln g']) \right. \\ &\quad \left. + (\partial \ln \psi'_1 a_1 + [a_1, \rho_2^2 \partial \ln g']) \right) \\ &\quad - \int_{S^2} a_2 a_0 a_1 d(\rho_2^2) \partial \ln g', \end{aligned} \quad (117)$$

with  $a_i = f_i U_{\psi_i}^* \in C^\infty(S^2) \rtimes \Gamma$ . In formula (117),  $S^2 = U_1 \cup \{\infty\}$  is gifted with the coordinate chart  $(z, \bar{z})$  of  $U_1$ , which makes sense to  $\psi'_i(z) = \partial_z \psi_i(z)$  and  $g'(z) = \partial_z g(z) = -1/z^2$ , but gives singular expressions at 0 and  $\infty$ . We can overcome this difficulty by introducing a smooth volume form  $\nu = \rho(z, \bar{z}) \frac{dz \wedge d\bar{z}}{2i}$  on  $S^2$ . The associated modular automorphism group  $\sigma^\nu$  leaves  $C^\infty(S^2) \rtimes \Gamma$  globally invariant and is expressed in the coordinates  $(z, \bar{z})$  by

$$\sigma_t^\nu(fU_\psi^*) = \left( \frac{\nu \circ \psi}{\nu} \right)^{it} fU_\psi^* = \left( \frac{\rho \circ \psi}{\rho} |\partial_z \psi|^2 \right)^{it} fU_\psi^*, \quad \forall t \in \mathbb{R}. \quad (118)$$

Define the derivation  $\delta^\nu$  on  $C^\infty(S^2) \rtimes \Gamma$

$$\begin{aligned} \delta^\nu(fU_\psi^*) &\equiv -i \left[ \partial, \frac{d}{dt} \sigma_t^\nu \right] (fU_\psi^*)|_{t=0} \\ &= [\partial, \ln \left( \frac{\rho \circ \psi}{\rho} |\partial_z \psi|^2 \right)] (fU_\psi^*) \end{aligned} \quad (119)$$

$$= \partial \ln \psi' fU_\psi^* - [\partial \ln \rho, fU_\psi^*]. \quad (120)$$

One has

$$\partial \ln \psi' fU_\psi^* + [fU_\psi^*, \rho_2^2 \partial \ln g'] = \delta^\nu(fU_\psi^*) + [\partial \ln \rho - \rho_2^2 \partial \ln g', fU_\psi^*], \quad (121)$$

where the 1-form  $\omega = \partial \ln \rho - \rho_2^2 \partial \ln g'$  is globally defined, nowhere singular on  $S^2$ . Let  $R^\nu = \partial \bar{\partial} \ln \rho$  be the curvature 2-form associated to the Kähler metric  $\rho dz \otimes d\bar{z}$ . One has the commutation rule

$$(\bar{\partial} \delta^\nu + \delta^\nu \bar{\partial})a = [R^\nu, a] \quad \forall a \in C^\infty(S^2) \rtimes \Gamma. \quad (122)$$

Simple algebraic manipulations show that the following 2-cochain

$$\tau^\nu(a_0, a_1, a_2) = \int_{S^2} a_0(da_1\delta^\nu a_2 + \delta^\nu a_1 da_2) + \int_{S^2} a_2 a_0 a_1 R^\nu \quad (123)$$

is a cyclic cocycle. Moreover,  $\tau^\nu$  is cohomologous to  $\theta^*\tau$ . To see this, let  $\varphi$  be the cyclic 1-cochain

$$\varphi(a_0, a_1) = \int_{S^2} (a_0 da_1 - a_1 da_0) \omega . \quad (124)$$

Then for all  $a_i \in C^\infty(S^2) \rtimes \Gamma$ ,

$$\begin{aligned} (\tau^\nu - \theta^*\tau)(a_0, a_1, a_2) &= - \int_{S^2} (a_0 da_1 a_2 + a_2 da_0 a_1 + a_1 da_2 a_0) \omega \\ &= b\varphi(a_0, a_1, a_2) . \end{aligned} \quad (125)$$

It is clear now that the construction of characteristic classes for an arbitrary (non flat) Riemann surface  $\Sigma$  follows exactly the same steps as in the above example. Using an open cover with partition of unity, one gets the desired cyclic cocycles by pullback. Choose a smooth measure  $\nu$  on  $\Sigma$ , then the associated modular group is

$$\sigma_t^\nu(fU_\psi^*) = \left( \frac{\nu \circ \psi}{\nu} \right)^{it} fU_\psi^* \quad fU_\psi^* \in C_c^\infty(\Sigma) \rtimes \Gamma . \quad (126)$$

The corresponding derivation

$$D^\nu(fU_\psi^*) = \ln \left( \frac{\nu \circ \psi}{\nu} \right) fU_\psi^* \quad (127)$$

allows to define the noncommutative differential

$$\delta^\nu = [\partial, D^\nu] . \quad (128)$$

Then the characteristic classes of the groupoid  $\Sigma \rtimes \Gamma$  are given by  $[\Sigma]$  and  $[\tau^\nu] \in HC^2(C_c^\infty(\Sigma) \rtimes \Gamma)$ , where  $\tau^\nu$  is given by eq.(123) with  $S^2$  replaced by  $\Sigma$ .

In the case  $\Gamma = \text{Id}$ , the crossed product reduces to the commutative algebra  $C_c^\infty(\Sigma)$  for which ( $\delta^\nu = 0$ )

$$\tau^\nu(a_0, a_1, a_2) = \int_\Sigma a_0 a_1 a_2 R^\nu \quad (129)$$

is just the image of the cyclic 0-cocycle

$$\tau_0^\nu(a) = \int_\Sigma a R^\nu \quad (130)$$

by the suspension map in cyclic cohomology  $S : HC^*(C_c^\infty(\Sigma)) \rightarrow HC^{*+2}(C_c^\infty(\Sigma))$ . Thus the periodic cyclic cohomology class of  $\tau^\nu$  corresponds in de Rham homology to the cap product

$$\frac{1}{2\pi i} [\tau^\nu] = c_1(\kappa) \cap [\Sigma] \in H_0(\Sigma) \quad (131)$$

of the first Chern class of the holomorphic tangent bundle  $\kappa$  by the fundamental class. This motivates the following definition:

**Definition 2** Let  $\Sigma$  be a Riemann surface without boundary and  $\Gamma$  a discrete pseudogroup acting on  $\Sigma$  by local conformal transformations. Let  $\nu$  be a smooth volume form on  $\Sigma$ , and  $\sigma^\nu$  the associated modular automorphism group leaving  $C_c^\infty(\Sigma) \rtimes \Gamma$  globally invariant. Then the Euler class  $e(\Sigma \rtimes \Gamma)$  is the class of the following cyclic 2-cocycle on  $C_c^\infty(\Sigma) \rtimes \Gamma$

$$\frac{1}{2\pi i} \tau^\nu(a_0, a_1, a_2) = \frac{1}{2\pi i} \int_\Sigma (a_2 a_0 a_1 R^\nu + a_0 (da_1 \delta^\nu a_2 + \delta^\nu a_1 da_2)) , \quad (132)$$

where  $\delta^\nu$  is the derivation  $-i[\partial, \frac{d}{dt}\sigma_t^\nu|_{t=0}]$ , and  $R^\nu$  is the curvature of the Kähler metric determined by  $\nu$  and the complex structure of  $\Sigma$ . Moreover, this cohomology class is independent of  $\nu$ .

Now if  $\Gamma = \text{Id}$ , the operator  $Q$  of section II defines an element of the  $K$ -homology of  $\Sigma \times \mathbb{R}^2$ . It corresponds to the tensor product of the classical Dolbeault complex  $[\bar{\partial}]$  of  $\Sigma$  by the signature complex  $[\sigma]$  of the fiber  $\mathbb{R}^2$ , so that its Chern character in de Rham homology is the cup product

$$\begin{aligned} \text{ch}_*(Q) &= \text{ch}_*([\bar{\partial}]) \# \text{ch}_*([\sigma]) \\ &= ([\Sigma] + \frac{1}{2}c_1(\kappa) \cap [\Sigma]) \# 2[\mathbb{R}^2] \in H_*(\Sigma \times \mathbb{R}^2) \end{aligned} \quad (133)$$

which yields, by Thom isomorphism, the homology class on  $\Sigma$

$$2[\Sigma] + c_1(\kappa) \cap [\Sigma] \in H_*(\Sigma) . \quad (134)$$

Next for any  $\Gamma$ , we know from the last section that the Chern character of the Dolbeault  $K$ -cycle, expressed in the periodic cyclic cohomology of  $C_c^\infty(\Sigma) \rtimes \Gamma$ , is a linear combination of  $[\Sigma]$  and  $e(\Sigma \rtimes \Gamma)$ . Thus we deduce immediately the following generalisation of the Riemann-Roch theorem:

**Theorem 3** Let  $\Sigma$  be a Riemann surface without boundary and  $\Gamma$  a discrete pseudogroup acting on  $\Sigma$  by local conformal mappings without fixed point. The Chern character of the Dolbeault  $K$ -cycle is represented by the following cyclic 2-cocycle on  $C_c^\infty(\Sigma) \rtimes \Gamma$

$$\text{ch}_*(Q) = 2[\Sigma] + e(\Sigma \rtimes \Gamma) . \quad (135)$$

□

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