A cohomology free description of eigencones in type A, B and C

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Abstract

Let K be a compact connected Lie group. The triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint K-orbits such that $\mathcal{O}_1+\mathcal{O}_2+\mathcal{O}_3$ contains 0 are parametrized by a closed convex polyhedral cone, called the eigencone of K. For K simple of type A, B or C we give an inductive cohomology free parametrization of the minimal set of linear inequalities which characterizes the eigencone of K.

1 Introduction

1.1 — We first explain the Horn conjecture which answers the following elementary question:

What can be said about the eigenvalues of a sum of two Hermitian matrices, in terms of the eigenvalues of the summands?

If A is a Hermitian n by n matrix, we will denote by $\lambda(A) = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{R}^n$ its spectrum. Consider the following set:

$$\mathrm{Horn}_{\mathbb{R}}(n) = \{(\lambda(A), \lambda(B), \lambda(C)) \in \mathbb{R}^{3n} : \begin{array}{l} A, B, C \text{ are 3 Hermitian matrices} \\ \text{s.t. } A+B+C=0\}. \end{array}$$

It turns out that $\mathrm{Horn}_{\mathbb{R}}(n)$ is a closed convex polyhedral cone in $\mathbb{R}^{3n}$. We now want to explain the Horn conjecture which describes inductively a list of linear inequalities which characterizes this cone. Let $\mathcal{P}(r,n)$ denote the set of parts of $\{1, \cdots, n\}$ with r elements. Let $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r,n)$. We set: $\lambda^I = (i_r - r, i_{r-1} - (r-1), \cdots, i_2 - 2, i_1 - 1)$. We will denote by 1^r the vector $(1, \cdots, 1)$ in $\mathbb{R}^r$.

Theorem 1 *Let (λ, μ, ν) be a triple of non-increasing sequences of n real numbers. Then, $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \tag{1}$$

and for any $r = 1, \cdots, n-1$, for any $(I, J, K) \in \mathcal{P}(r,n)^3$ such that

$$(\lambda^I, \lambda^J, \lambda^K - 2(n-r)1^r) \in \mathrm{Horn}_{\mathbb{R}}(r), \tag{2}$$

we have:

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0. \tag{3}$$

Note that if starting with a point in $\mathrm{Horn}_{\mathbb{R}}(r)$, one adds 1^r to one factor add -1^r to another one, one stays in $\mathrm{Horn}_{\mathbb{R}}(r)$. This remark implies that condition (2) is symmetric in I, J and K.

In 1962, Horn [Hor62] conjectured Theorem 1. This conjecture was proved by combining works by Klyachko [Kly98] and Knutson-Tao [KT99] (see also [Ful00] for a survey). Despite the proof, the statement of Theorem 1 is as elementary as the Horn problem is. Note that I, J and K are sets of indexes in inequality (3) whereas λ^I, λ^J and λ^K are eigenvalues of Hermitian matrices in condition (2). This very curious remark certainly contributed to the success of the Horn conjecture.

As pointed out by C. Woodward, Theorem 1 has a weakness. Indeed, it gives redundant inequalities. To describe a minimal set of inequalities, we need to introduce some notation. Let $\mathbb{G}(r,n)$ be the Grassmann variety of r-dimensional subspaces of $\mathbb{C}^n$. Consider its cohomology ring $\mathrm{H}^*(\mathbb{G}(r,n), \mathbb{Z})$. To any $I \in \mathcal{P}(r,n)$ is associated a Schubert class $\sigma_I \in \mathrm{H}^*(\mathbb{G}(r,n), \mathbb{Z})$. There are two usual ways, obtained one from each other composing by Poincaré duality, to assign a Schubert class σ_I to I. Our choice is detailed in Paragraph 2.1.1. Let $[\mathrm{pt}] \in \mathrm{H}^{2r(n-r)}(\mathbb{G}(r,n), \mathbb{Z})$ denote the Poincaré dual class of the point. Belkale proved in [Bel01] the following:

Theorem 2 *Let (λ, μ, ν) be a triple of non-increasing sequences of n real numbers. Then, $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$ if and only if*

$$\sum_i \lambda_i + \sum_j \mu_j + \sum_k \nu_k = 0 \tag{4}$$

and for any $r = 1, \cdots, n-1$, *for any* $(I, J, K) \in \mathcal{P}(r,n)^3$ *such that*

$$\sigma_I.\sigma_J.\sigma_K = [\mathrm{pt}] \in \mathrm{H}^*(\mathbb{G}(r,n), \mathbb{Z}), \tag{5}$$

we have:

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j + \sum_{k \in K} \nu_k \leq 0. \tag{6}$$

The statement of Theorem 2 is not elementary, but as proved by Knutson-Tao-Woodward in [KTW04] it is optimal:

Theorem 3 *In Theorem 2, no inequality can be omitted.*

1.2 — Klyachko obtained in [Kly98] a variation of Theorem 2:

Theorem 4 *Theorem 2 holds replacing condition (5) by*

$$\sigma_I.\sigma_J.\sigma_K = c[\mathrm{pt}], \text{ for a nonzero integer c.} \tag{7}$$

If λ, μ and ν are three partitions then we denote by $c_{\lambda\,\mu}^{\nu} \in \mathbb{Z}^+$ the associated Littlewood-Richardson coefficient. These integers control the cup product in the cohomology of the Grassmanians. For example, condition (7) is equivalent to the nonzeroness of some Littlewood-Richardson coefficient.

If $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, we set $\nu^\vee = (-\nu_n \geq \cdots \geq -\nu_1)$. A consequence of Knutson-Tao's saturation theorem is that

Theorem 5 *The coefficient* $c_{\lambda\,\mu}^{\nu}$ *is nonzero if and only if* $(\lambda, \mu, \nu^\vee) \in \mathrm{Horn}_{\mathbb{R}}(n)$.

Compare Theorems 2 and 4. The advantage of Theorem 2 is obvious: it gives exactly the minimal set of linear inequalities needed to characterized $\mathrm{Horn}_{\mathbb{R}}(n)$. The advantage of Theorems 4 is that by Theorem 5, condition (7) can be reinterpreted in terms of Horn cones. This is the key point to explain the inductive nature of Theorem 1. Finally, Theorem 1 gives an inductive algorithm to decide if a given point belongs to $\mathrm{Horn}_{\mathbb{R}}(n)$. Equivalently, it gives an inductive algorithm to decide if a given Littlewood-Richardson is zero or positive.

1.3 — In this work, we give an **inductive algorithm to decide if a given Littlewood-Richardson coefficient equals to one or not**. More precisely, our algorithm proceeds inductively and decides if a given

Littlewood-Richardson coefficient is equal to zero or if it is equal to one or it is greater than one. In particular, our algorithm decides if condition (5) is fulfilled. The combination of this algorithm with Theorems 2 and 3 gives an inductive description of the minimal set of inequalities of $\mathrm{Horn}_{\mathbb{R}}(n)$. Note that our algorithm uses Derksen-Weyman's one (see [DW02]) as a procedure.

Our algorithm will be detailed in Section 4.1. We now explain the main point of the algorithm. First, Horn's conjecture allows to decide if a given Littlewood-Richardson coefficient is zero or not. So, the remaining question is to decide if a given Littlewood-Richardson coefficient is less or equal to one or not.

First, we have a geometrical construction to obtain such coefficients. Indeed, consider a product $Y = G/P \times G/Q \times G/R$ of three compact $G = \mathrm{GL}_n(\mathbb{C})$-homogeneous spaces which contains a dense G-orbit. Let $\mathcal{L}$ be a G-linearized line bundle on Y. Then, the dimension of the space $H^0(Y, \mathcal{L})^G$ of G-invariant sections of $\mathcal{L}$ is a Littlewood-Richardson coefficient c. The fact that Y contains a dense G-orbit implies that

$$c \leq 1. \tag{8}$$

On the other hand, a Derksen-Weyman's theorem (see [DW10, Rot10, Res10c]) shows that some Littlewood-Richardson coefficients c are equal to a product of two other ones c_1 and c_2 (associated to smaller partitions); $c = c_1.c_2$. The obvious remark is that

$$c = 1 \iff c_1 = c_2 = 1. \tag{9}$$

Roughly speaking, our algorithm is based on the fact that *any Littlewood-Richardson coefficient* c *equal to one, can be obtained applying finitely many times assertions (8) and (9).* In particular, the LR-coefficients associated to regular partitions equal to one can be obtained using the fact that $SL_2(\mathbb{C})$ has a dense orbit in $(\mathbb{P}^1)^3$ and Derksen-Weyman's theorem.

1.4 — We now want to explain a generalization of the Horn problem. Let G (e.g. $G = \mathrm{GL}_n(\mathbb{C})$) be a reductive complex group and U (e.g. $U = U_n(\mathbb{C})$) be a maximal compact subgroup. Let $\mathfrak{u}$ denote its Lie algebra. We are interested in the following problem: what are the triples $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ of adjoint orbits such that $\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3$ contains 0.

Let T be a maximal torus of G such that $T \cap U$ is a Cartan subgroup of U. Let $\mathfrak{t}$ denote its Lie algebras and $\mathfrak{t}^+$ be a fixed Weyl chamber of $\mathfrak{t}$. It turns out that the triples of orbits as above are parametrized by a closed convex polyhedral cone contained in $(\mathfrak{t}^+)^3$ (see Section 7.2 for details). We will denote by $\Gamma(U)$ this cone. Using the Cartan-Killing form one can identify $\Gamma(U(n))$ with $\mathrm{Horn}_{\mathbb{R}}(n)$.

We now introduce notation to describe a minimal set of inequalities for $\Gamma(U)$.

Let α be a simple root of G and let ω_α be the corresponding fundamental weight. We consider the standard maximal parabolic subgroup P_α associated to α. Let W denote the Weyl group of G. The Weyl group W_α of P_α is also the stabilizer of ω_α.

Consider now the cohomology group $\mathrm{H}^*(G/P_\alpha, \mathbb{Z})$: it is freely generated by the Schubert classes σ_w parametrized by the cosets $w \in W/W_\alpha$. In [BK06], Belkale-Kumar defined a new product denoted $\odot_0$ on $\mathrm{H}^*(G/P_\alpha, \mathbb{Z})$. We can now state the main result of [BK06] which generalizes Theorem 2:

Theorem 6 *We assume that U is semisimple. Let $(\xi, \zeta, \eta) \in (\mathfrak{t}^+)^3$. Then, (ξ, ζ, η) belongs to $\Gamma(U)$ if and only if for any simple root α and any triple of Schubert classes σ_u, σ_v and σ_w in $\mathrm{H}^*(G/P_\alpha, \mathbb{Z})$ such that*

$$\sigma_u \odot_0 \sigma_v \odot_0 \sigma_w = [\mathrm{pt}], \qquad (10)$$

we have:

$$\omega_\alpha(u^{-1}\xi) + \omega_\alpha(v^{-1}\zeta) + \omega_\alpha(w^{-1}\eta) \leq 0. \qquad (11)$$

In [Res10a], the following generalization of Theorem 3 is obtained:

Theorem 7 *In Theorem 6, no inequality can be omitted.*

1.5 — For U simple of type B or C, in Theorems 19 and 20 below, we prove that each condition (10) is equivalent to the fact that two Littlewood-Richardson coefficients (for ordinary grassmanian !) are equal to one. The combination of Algorithm 4.1 and these results gives a **cohomology free description of the minimal set of inequalities for** $\Gamma(U)$. Note that in [BK10], Belkale-Kumar gave a redundant cohomology free description of $\Gamma(U)$.

1.6 — The paper is organized as follows. In Section 2, we introduce basic material about the Littlewood-Richardson coefficients and the Horn cone. In Section 3, we recall some useful results about quiver representations. In Section 4, we state and prove our inductive algorithm to decide if a given

Littlewood-Richardson coefficient equals to one or not. In Section 5, we introduce a parametrization of the Schubert classes of any complete rational homogeneous space and give some examples. In Section 6, we recall from [BK06] the notion of Levi-movability. In Section 7, we recall some results about eigencones. In Sections 8 and 9, we prove our results about the cohomology of isotropic and odd orthogonal Grassmannians.

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2 The Horn cone

2.1 The Littlewood-Richardson coefficients

2.1.1 — Schubert Calculus. Let $\mathbb{G}(r,n)$ be the Grassmann variety of r-dimensional subspaces L of a fixed n-dimensional vector space V. Let $F_\bullet$: $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V$ be a complete flag of V.

If $a \leq b$, we will denote by $[a;b]$ the set of integers between a and b. Let $\mathcal{P}(r,n)$ denote the set of subsets of $[1;n]$ with r elements. For any $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r,n)$, the Schubert variety $\Omega_I(F_\bullet)$ in $\mathbb{G}(r,n)$ is defined by

$$\Omega_I(F_\bullet) = \{L \in \mathbb{G}(r,n)\ :\ \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The Poincaré dual of the homology class of $\Omega_I(F_\bullet)$ does not depend on $F_\bullet$; it is denoted by σ_I. The σ_I's form a $\mathbb{Z}$-basis for the cohomology ring of $\mathbb{G}(r,n)$. The class associated to $[1;r]$ is the class of the point; it will be denoted by [pt]. It follows that for any subsets I, $J \in \mathcal{P}(r,n)$, there is a unique expression

$$\sigma_I.\sigma_J = \sum_{K\in\mathcal{P}(r,n)} c_{IJ}^K \sigma_K,$$

for integers c_{IJ}^K. We define $K^\vee$ by: $i \in K^\vee$ if and only if $n+1-i \in K$. Then, σ_K and $\sigma_{K^\vee}$ are Poincaré dual. So, if the sum of the codimensions of $\Omega_I(F_\bullet)$, $\Omega_J(F_\bullet)$ and $\Omega_K(F_\bullet)$ equals the dimension of $\mathbb{G}(r,n)$, we have

$$\sigma_I.\sigma_J.\sigma_K = c_{IJ}^{K^\vee}[\mathrm{pt}].$$

We set

$$c_{IJK} := c_{IJ}^{K^\vee}.$$

Note that $c_{IJK} = c_{JIK} = c_{IKJ} = \dots$

2.1.2 — Recall that the irreducible representations of $G = \mathrm{GL}_n(\mathbb{C})$ are indexed by sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$. Let us denote by Λ_n^+ the set of such sequences. We set $|\lambda| = \lambda_1 + \cdots + \lambda_n$. Denote the representation corresponding to λ by V_λ. For example, the representation V_{1^n} is the determinant representation of $\mathrm{GL}_n(\mathbb{C})$. Define the Littlewood-Richardson coefficients $c_{\lambda\,\mu}^{\nu} \in \mathbb{N}$ by:

$$V_\lambda \otimes V_\mu = \sum_{\nu \in \Lambda_n^+} c_{\lambda\,\mu}^{\nu} V_\nu.$$

For $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, we set: $\nu^\vee = (-\nu_n \geq \cdots \geq -\nu_1)$. Then, $V_{\nu^\vee}$ is the dual of V_ν. Finally, we set

$$c_{\lambda\mu\nu}^n = c_{\lambda\mu}^{\nu^\vee}.$$

Note that $c_{\lambda\mu\nu}^n$ is the dimension of the subspace $(V_\lambda \otimes V_\mu \otimes V_\nu)^G$ of G-invariant vectors in $V_\lambda \otimes V_\mu \otimes V_\nu$. Consider:

$$\mathrm{Horn}(n) := \{(\lambda, \mu, \nu) \in (\Lambda_n^+)^3 \ : \ c_{\lambda\mu\nu}^n \neq 0\}.$$

2.1.3 — We will use the standard correspondence between elements $I = \{i_1 < \cdots < i_r\}$ of $\mathcal{P}(r,n)$ and partitions $\lambda^I \in \Lambda_r^+$ such that $\lambda_1 \leq n-r$ and $\lambda_r \geq 0$. This correspondence is obtained by defining

$$\lambda^I = (i_r - r, i_{r-1} - (r-1), \cdots, i_2 - 2, i_1 - 1).$$

Note that the dimension of Ω_I equals $|\lambda^I|$ and that the degree of σ_I is $2(r(n-r) - |\lambda^I|)$. Authors use the base $\theta_I = \sigma_I^\vee$ for the cohomology group of the Grassmanian. In this case, the degree of θ_I is $2|\lambda^I|$. For I, J and K in $\mathcal{P}(r,n)$, Lesieur showed in 1947 (see [Les47]) that:

$$\theta_I.\theta_J.\theta_{K^\vee} = c_{\lambda^I\lambda^J}^{\lambda^K}[\mathrm{pt}].$$

Applying Poincaré duality, we deduce that

$$c_{IJK} = c_{\lambda^I\lambda^J\lambda^K - 2(n-r)1^r}^r.$$

2.2 The Horn cone

Let $I = \{i_1 < \cdots < i_r\} \in \mathcal{P}(r,n)$ and $\lambda \in \Lambda_n^+$. We set

$$\lambda_I = (\lambda_{i_1} \geq \cdots \geq \lambda_{i_r}).$$

In particular, $|\lambda_I| = \sum_{i\in I} \lambda_i$. Let $I, J, K \in \mathcal{P}(r, n)$. We define the "linear form" φ_{IJK} on $(\Lambda_n^+)^3$ by:

$$\varphi_{IJK}(\lambda, \mu, \nu) = |\lambda_I| + |\mu_J| + |\nu_K|.$$

Combining [Bel01] and [KT99], we obtain the following description of $\mathrm{Horn}(n)$:

Theorem 8 *Let $(\lambda, \mu, \nu) \in (\Lambda_n^+)^3$. The point (λ, μ, ν) belongs to* $\mathrm{Horn}(n)$ *if and only if*

$$|\lambda| + |\mu| + |\nu| = 0,$$

and for any $r \in [1; n-1]$, for any $(I, J, K) \in \mathcal{P}(r, n)$ such that $\sigma_I \sigma_J \sigma_K =$ [pt]*, we have:*

$$\varphi_{IJK}(\lambda, \mu, \nu) \leq 0.$$

3 Quivers

In this section, we explain how a Derksen-Weyman's algorithm (see [DW02]) on quiver representations can be applied to decide if a product of three flag manifolds contains a dense orbit or not. Let us first introduce standard notation on quivers.

3.1 Definitions

Let Q be a quiver (that is, a finite oriented graph) with vertexes Q_0 and arrows Q_1. We assume that Q has no oriented cycle. An arrow $a \in Q_1$ has initial vertex ia and terminal one ta. A representation R of Q is a family $(V(s))_{s\in Q_0}$ of finite dimensional vector spaces and a family of linear maps $u(a) \in \mathrm{Hom}(V(ia), V(ta))$ indexed by $a \in Q_1$. The dimension vector of R is the family $(\dim(V(s)))_{s\in Q_0} \in \mathbb{N}^{Q_0}$.

Let us fix $\alpha \in \mathbb{N}^{Q_0}$ and a vector space $V(s)$ of dimension $\alpha(s)$ for each $s \in Q_0$. Set

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a\in Q_1} \mathrm{Hom}(V(ia), V(ta)).$$

The group $\mathrm{GL}(\alpha) = \prod_{s\in Q_0} \mathrm{GL}(V(s))$ acts naturally on $\mathrm{Rep}(Q, \alpha)$.

3.2 Our question in terms of quivers

Let $\{a_1 < \cdots < a_p\}$ be a part of $[1; n-1]$. We will consider the following flag variety:

$$\mathcal{F}l_n(a_1, \cdots, a_p) := \{(V_1 \subset \cdots \subset V_p)\} \subset \mathbb{G}(a_1, n) \times \cdots \times \mathbb{G}(a_p, n).$$

Let also $\{b_1 < \cdots < b_q\}$ and $\{c_1 < \cdots < c_r\}$ be two other parts of $[1; n]$. Recall that we want to decide if $\mathcal{F}l_n(a_1, \cdots, a_p) \times \mathcal{F}l_n(b_1, \cdots, b_q) \times \mathcal{F}l_n(c_1, \cdots, c_r)$ contains a dense GL_n-orbit.

Consider the following quiver T_{pqr} with $p+q+r+1$ vertexes and $p+q+r$ arrows:

$$\begin{array}{ccccccccccc}
 & & x_1 & \longrightarrow & x_2 & \longrightarrow & x_3 & \cdots\cdots & x_p & & \\
 & & & & & & & & & \searrow & \\
y_1 & \longrightarrow & y_2 & \longrightarrow & y_3 & \cdots\cdots & \cdots\cdots & \cdots\cdots & y_q & \longrightarrow & x_{p+1} = y_{q+1} = z_{r+1}. \\
 & & & & & & & & & \nearrow & \\
 & & & & z_1 & \longrightarrow & z_2 & \cdots\cdots & z_r & &
\end{array}$$

Consider the following vector dimension α of T_{pqr}:

$$\alpha = \left\{ \begin{array}{ccccccccccc}
 & & a_1 & \longrightarrow & a_2 & \longrightarrow & a_3 & \cdots\cdots & a_p & & \\
 & & & & & & & & & \searrow & \\
b_1 & \longrightarrow & b_2 & \longrightarrow & b_3 & \cdots\cdots & \cdots\cdots & \cdots\cdots & b_q & \longrightarrow & n. \\
 & & & & & & & & & \nearrow & \\
 & & & & c_1 & \longrightarrow & c_2 & \cdots\cdots & c_r & &
\end{array} \right.$$

We have the well known

Lemma 1 *We recall that α is increasing on each harm. Then, the following are equivalent:*

(i) $\mathcal{F}l_n(a_1, \cdots, a_p) \times \mathcal{F}l_n(b_1, \cdots, b_q) \times \mathcal{F}l_n(c_1, \cdots, c_r)$ *contains a dense* GL_n*-orbit;*

(ii) $\mathrm{Rep}(T_{pqr}, \alpha)$ *contains a dense* $\mathrm{GL}(\alpha)$*-orbit.*

Proof. Let R be a general representation of T_{pqr} of dimension α. If s is a vertex of T_{pqr}, $V(s)$ denotes the vector space of R at s and $u(s)$ the linear map (if there exists) associated to the arrow a in T_{pqr} such that $ia = s$. Since α is increasing on each harm, for any arrow a, the linear map $u(a)$ is injective. In particular, the flag:

$$\xi_x = \Big(V(x_{p+1}) \supset u(x_p)(V(x_p)) \supset (u(x_p) \circ u(x_{p-1}))(V(x_{p-1})) \supset \cdots \Big)$$

has dimension $n > a_p > a_{p-1} \cdots$. So, we obtain a point (ξ_x, ξ_y, ξ_z) in $\mathcal{F}l_n(a_1, \cdots, a_p) \times \mathcal{F}l_n(b_1, \cdots, b_q) \times \mathcal{F}l_n(c_1, \cdots, c_r)$. It is easy to see that $\mathrm{GL}(\alpha).R$ is dense in $\mathrm{Rep}(T_{pqr}, \alpha)$ if and only if $\mathrm{GL}_n.(\xi_x, \xi_y, \xi_z)$ is dense in $\mathcal{F}l_n(a_1, \cdots, a_p) \times \mathcal{F}l_n(b_1, \cdots, b_q) \times \mathcal{F}l_n(c_1, \cdots, c_r)$. □

3.3 A Kac theorem

We follow notation of Section 3.1. We now recall a Kac's theorem which gives a criterion to decide if $\mathrm{Rep}(Q, \alpha)$ contains a dense $\mathrm{GL}(\alpha)$-orbit.

We call $\alpha = \alpha_1 + \cdots + \alpha_s$ the *canonical decomposition of* α if a general representation of dimension α is a direct sum of indecomposable representations of dimensions α_1, $\alpha_2, \cdots, \alpha_s$.

For $\alpha, \beta \in \mathbb{N}^{Q_0}$, the Ringle form is defined by:

$$\langle \alpha, \beta \rangle = \sum_{s \in Q_0} \alpha(s)\beta(s) - \sum_{a \in Q_1} \alpha(ia)\beta(ta).$$

Theorem 9 *(see [Kac82, Proposition 4]) Let $\alpha = \alpha_1 + \cdots + \alpha_s$ be the canonical decomposition of α. Then $\mathrm{Rep}(Q, \alpha)$ contains a dense $\mathrm{GL}(\alpha)$-orbit if and only if for any $i = 1, \cdots, s$ we have $\langle \alpha_i, \alpha_i \rangle = 1$.*

In [DW02], Derksen-Weyman describe an efficient algorithm to compute the canonical decomposition of a vector dimension. With Theorem 9, this gives **an algorithm to decide if** $\mathrm{Rep}(Q, \alpha)$ **contains a dense** $\mathrm{GL}(\alpha)$**-orbit.** Combining this remark with Lemma 1, we obtain

Proposition 1 *Derksen-Weyman's algorithm allows to decide if the variety*

$$\mathcal{F}l_n(a_1, \cdots, a_p) \times \mathcal{F}l_n(b_1, \cdots, b_q) \times \mathcal{F}l_n(c_1, \cdots, c_r)$$

contains a dense GL_n-orbit.

Remark 1 *It would be interesting to have a classification of the triples of parabolic subgroups (P, Q, R) of $G = \mathrm{GL}_n$ such that $G/P \times G/Q \times G/R$ contains an open G-orbit; instead an algorithm to decide if it is. In [MWZ99], Magyar-Weyman-Zelevinsky gives a classification of such triples such that $G/P \times G/Q \times G/R$ contains finitely many orbits. If one of P, Q, R is a Borel subgroup these two conditions are actually equivalent. Indeed, if $G/B \times G/Q \times G/R$ contains an open G-orbit, $G/Q \times G/R$ is a spherical G-variety and contains by [Bri86] finitely many B-orbits. The case when $P = Q = R$ is maximal was obtained in [Pop07].*

4 An algorithm

4.1 Description of the algorithm

In this section, we give an inductive (on n) algorithm to decide if a given element of $(\Lambda_n^+)^3$ belong to the following subset of Horn(n):

$$\mathrm{Horn}^1(n) := \{(\lambda, \mu, \nu) \in (\Lambda_n^+)^3 \;:\; c^n_{\lambda\mu\nu} = 1\}.$$

If $\lambda \in \Lambda_n^+$, we define the type of λ by

$$\mathrm{type}(\lambda) := \{j = 1, \cdots, n-1 \mid \lambda_j \neq \lambda_{j+1}\}.$$

The following procedure `IsLR01` takes an integer $n \geq 1$ and three elements λ, μ and ν in Λ_n^+ as arguments. It answers either $c = c^n_{\lambda\mu\nu} = 0$ or $c = 1$ or $c \geq 2$.

`IsLR01` (n, λ, μ, ν)

(i) `If` $n = 1$ `then`
(`if` $\lambda_1 + \mu_1 + \nu_1 = 0$ `then decide` $c = 1$ `and stop`
`else decide` $c = 0$ `and stop`)

(ii) `For every` $r = 1, \cdots, n-1$ `and` $(I,\, J,\, K)$ `in` $\mathcal{P}(r, n)$ `s.t.`
`IsLR01`$(r, \lambda^I, \lambda^J, \lambda^K - 2(n-r)1^r) = 1$ `do`

(a) `Compute` $\phi = \varphi_{IJK}(\lambda, \mu, \nu)$.

(b) `If` $\phi > 0$ `then decide` $c = 0$ `and stop.`

```
(c) If  φ = 0 then
        Using recursively IsLR01 with r
           decide if c_1 = c^r_{λ_I,μ_J,ν_K} = 0,1 or is ≥ 2.
        Using recursively IsLR01 with n − r
           decide if c_2 = c^{n−r}_{λ_{I^c},μ_{J^c},ν_{K^c}} = 0,1 or is ≥ 2.
         If c_1 = 0 or c_2 = 0 then    decide c = 0 and stop.
         If c_1 = 1 and c_2 = 1 then   decide c = 1 and stop.
         Otherwise                     decide c ≥ 2 and stop.
```

(iii) If step (ii) was inconclusive, then check if

$$\mathcal{F}l_n(\mathrm{type}(\lambda)) \times \mathcal{F}l_n(\mathrm{type}(\mu)) \times \mathcal{F}l_n(\mathrm{type}(\nu))$$

contains a dense GL_n-orbit (using the algorithm of Section 3).
If it does then decide $c = 1$ and stop
else decide $c \geq 2$ and stop.

The proof of the algorithm need some preparation.

4.2 Modularity and GIT

4.2.1 — Non-standard GIT. Let G be a reductive group acting on an irreducible projective variety X. Let $\mathrm{Pic^G(X)}$ denote the group of G-linearized line bundles on X. For $\mathcal{L} \in \mathrm{Pic^G(X)}$, we denote by $\mathrm{H}^0(X, \mathcal{L})$ the G-module of regular sections of $\mathcal{L}$ and by $\mathrm{H}^0(X, \mathcal{L})^G$ the subspace of G-invariant sections. For any $\mathcal{L} \in \mathrm{Pic^G(X)}$, we set

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X \,:\, \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ s.t. } \sigma(x) \neq 0\}.$$

Note that this definition of $X^{\mathrm{ss}}(\mathcal{L})$ is as in [MFK94] if $\mathcal{L}$ is ample but not in general. We consider the following projective variety:

$$X^{\mathrm{ss}}(\mathcal{L})/\!/G := \mathrm{Proj} \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G,$$

and the natural G-invariant morphism

$$\pi \,:\, X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G.$$

If $\mathcal{L}$ is ample π is a good quotient.

4.2.2 — We denote by $\mathrm{mod}(X, G)$ the minimal codimension of G-orbits in X. Recall that X is projective, but note that the notation $\mathrm{mod}(X, G)$ will be used for any irreducible G-variety X.

Proposition 2 *We assume that X is smooth. The maximal of the dimensions of the varieties $X^{\rm ss}(\mathcal{L})/\!/G$ for $\mathcal{L} \in {\rm Pic^G(X)}$ is equal to* ${\rm mod}(X, G)$.

Proof. Let $\mathcal{L} \in {\rm Pic^G(X)}$. Since π is G-invariant, we have:

$$\dim(X^{\rm ss}(\mathcal{L})/\!/G) \leq {\rm mod}(X^{\rm ss}(\mathcal{L}), G) = {\rm mod}(X, G).$$

Conversely, set $m = {\rm mod}(X, G)$. It remains to construct $\mathcal{L}$ such that $\dim(X^{\rm ss}(\mathcal{L})/\!/G) \geq m$. It is well known that m is the transcendence degree of the field $\mathbb{C}(X)^G$ of G-invariant rational functions on X. Let $f_1, \cdots, f_m$ be algebraically independent elements of $\mathbb{C}(X)^G$. For each $i = 1, \cdots, m$, consider the two effective divisors D_i^0 and D_i^∞ such that ${\rm div}(f_i) = D_i^0 - D_i^\infty$. Consider the line bundle $\mathcal{L}_i = \mathcal{O}(D_i^0) = \mathcal{O}(D_i^\infty)$. Let σ_i^0 be a regular section of $\mathcal{L}_i$ such that ${\rm div}(\sigma_i^0) = D_i^0$. Since D_i^0 is G-stable, there exists a unique G-linearization of $\mathcal{L}_i$ such that σ_i^0 is G-invariant; we now consider $\mathcal{L}_i$ endowed with this linearization. There exists a unique section σ_i^∞ of $\mathcal{L}_i$ such that $f_i = \sigma_i^0/\sigma_i^\infty$; since f_i and σ_i^0 are G-invariant, so is σ_i^∞.

Set $\mathcal{L} = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_m$. Consider the following G-invariant sections of $\mathcal{L}$:

$$\begin{array}{l} \tau_i = \sigma_1^\infty \otimes \cdots \otimes \sigma_{i-1}^\infty \otimes \sigma_i^0 \otimes \sigma_{i+1}^\infty \otimes \cdots \otimes \sigma_m^\infty \quad \forall i = 1, \cdots, m, \\ \tau_0 = \sigma_1^\infty \otimes \cdots \otimes \sigma_m^\infty. \end{array}$$

Consider now the rational map

$$\begin{array}{cccc} \theta\ : & X^{\rm ss}(\mathcal{L}) & \longrightarrow & \mathbb{P}^m \\ & x & \longmapsto & [\tau_0(x) : \cdots : \tau_m(x)]. \end{array}$$

Since $f_1, \cdots, f_m$ are algebraically independent, θ is dominant. Since θ is defined by G-invariant sections of $\mathcal{L}$, it factors by $\pi\,:\, X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})/\!/G$. It follows that $\dim(X^{\rm ss}(\mathcal{L})/\!/G) \geq m$. □

4.2.3 — We assume here that ${\rm Pic^G(X)}$ has finite rank and consider the rational vector space ${\rm Pic^G(X)}_{\mathbb{Q}} := {\rm Pic^G(X)} \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $X^{\rm ss}(\mathcal{L}) = X^{\rm ss}(\mathcal{L}^{\otimes n})$ for any positive integer n, one can define $X^{\rm ss}(\mathcal{L})$ for any element in ${\rm Pic^G(X)}_{\mathbb{Q}}$. The set of ample line bundles in ${\rm Pic^G(X)}$ generates an open convex cone ${\rm Pic^G(X)}_{\mathbb{Q}}^+$ in ${\rm Pic^G(X)}_{\mathbb{Q}}$. The following cone was defined in [DH98] and will be called the *ample GIT-cone*:

$$\mathcal{AC}^G(X) := \{\mathcal{L} \in {\rm Pic^G(X)}_{\mathbb{Q}}^+\ :\ {\rm X^{ss}}(\mathcal{L}) \neq \emptyset\}.$$

Indeed, since the product of two nonzero G-invariant sections of two line bundles is a nonzero G-invariant section of the tensor product of the two line bundles, $\mathcal{AC}^G(X)$ is convex. The following result is certainly well-known and can be deduced from [Res10b, Proposition 1.1]:

Proposition 3 *The dimension of $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ does not depend on $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$.*

4.2.4 — We now consider the case when X is a product of flag manifolds:

Lemma 2 *We assume that X is a product of flag manifolds for G and that $\mathcal{AC}^G(X)$ is nonempty. For any $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$, the dimension of $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ equals* $\mathrm{mod}(X, G)$.

Proof. The lemma is a consequence of Propositions 3 and 2. The only difficulty is that Proposition 3 deals with ample line bundles and Proposition 2 concerns any line bundle.

Let $\mathcal{M} \in \mathrm{Pic^G(X)}$ such that $X^{\mathrm{ss}}(\mathcal{M})$ is not empty. By [Res10a, Proposition 10], $\mathcal{M}$ belongs to the closure of $\mathcal{AC}^G(X)$. By [Res10a, Lemma 7], there exists $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$ such that $X^{\mathrm{ss}}(\mathcal{L}) \subset X^{\mathrm{ss}}(\mathcal{M})$. Corresponding to this inclusion we have a dominant (and so surjective) morphism $X^{\mathrm{ss}}(\mathcal{L})/\!/G \longrightarrow X^{\mathrm{ss}}(\mathcal{M})/\!/G$. In particular, we have:

$$\dim(X^{\mathrm{ss}}(\mathcal{L})/\!/G) \geq \dim(X^{\mathrm{ss}}(\mathcal{M})/\!/G).$$

With Proposition 3, this implies that for any $\mathcal{L}$ in the relative interior of $\mathcal{AC}^G(X)$, the dimension of $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ equals the maximal dimension of the varieties $X^{\mathrm{ss}}(\mathcal{M})/\!/G$ for $\mathcal{M} \in \mathrm{Pic^G(X)}$. With Proposition 2, this implies the lemma. □

4.3 Properties of the LR-coefficients

4.3.1 — **Saturation.** Let $(\lambda, \mu, \nu) \in \Lambda_n^+$ and m be a positive integer. Knutson-Tao proved in [KT99]:

Theorem 10 *If $c^n_{m\lambda\, m\mu\, m\nu} \neq 0$ then $c^n_{\lambda\mu\nu} \neq 0$.*

A geometric proof is given in [Bel06]. Note that this statement is a corollary (or a part) of Theorem 8 and was already stated in the introduction.

4.3.2 — **The Fulton conjecture.** Let $(\lambda, \mu, \nu) \in \Lambda_n^+$ and m be a positive integer. Knutson-Tao proved in [KT99] the following Fulton conjecture:

Theorem 11 *If $c^n_{\lambda\mu\nu} = 1$ then $c^n_{m\lambda\, m\mu\, m\nu} = 1$ for any positive integer m.*

Geometric proofs of this result are given in [Bel07, Res08b, BKR10].

4.3.3 — LR-coefficients on the boundary of Horn(n)**.** The following theorem has been proved independently in [KTT09] and [DW10]. Alternative proofs can be found in [Rot10, Res10c].

Theorem 12 *Let* $(\lambda, \mu, \nu) \in (\Lambda_n^+)^3$. *Let* $(I, J, K) \in \mathcal{P}(r, n)$ *such that*

$$\sigma_I.\sigma_J.\sigma_K = [\mathrm{pt}].$$

If $\varphi_{IJK}(\lambda, \mu, \nu) = 0$, *then*

$$c^n_{\lambda\,\mu\,\nu} = c^r_{\lambda_I\,\mu_J\,\nu_K} \cdot c^{n-r}_{\lambda_{I^c}\,\mu_{J^c}\,\nu_{K^c}}.$$

4.4 Proof of the algorithm

Theorem 13 *The algorithm described in Section 4.1 decides if* $c^n_{\lambda\mu\nu} = 0$, 1 *or if* $c^n_{\lambda\mu\nu} \geq 2$.

Proof. The case $n = 1$ is obvious. Moreover, the procedure is used recursively three times with strictly smaller n. So, the procedure finishes.

If algorithm stop in case (ii)b, we have

$$\varphi_{IJK}(\lambda, \mu, \nu) > 0$$

for some (I, J, K) appearing in Theorem 2. This implies that $c^n_{\lambda\mu\nu} = 0$.

If the algorithm stop in case (ii)c, ϕ is equal to 0. Then Theorem 12 shows that $c = c_1.c_2$; and the algorithm works in this case.

We now consider case (iii). In this case, for any $r = 1, \cdots, n-1$ and for any $(I, J, K) \in \mathcal{P}(r, n)$ such that $\sigma_I.\sigma_J.\sigma_K = [\mathrm{pt}]$, we have:

$$\varphi_{IJK}(\lambda, \mu, \nu) < 0.$$

So, Theorem 2 shows that $(\lambda, \mu, \nu) \in \mathrm{Horn}_{\mathbb{R}}(n)$.

Let T and B be the usual maximal torus and Borel subgroup of GL_n. Then, λ corresponds to a character of T or B. The group B fixes a unique point in $\mathcal{F}l_n(\mathrm{type}(\lambda))$ whose the stabilizer in G will be denoted by P. Moreover, λ extends to unique character of P. Similarly, we can think about μ and ν as characters of parabolic subgroups Q and R. Consider the $G = \mathrm{GL}_n^3$-variety $X = \mathcal{F}l_n(\mathrm{type}(\lambda)) \times \mathcal{F}l_n(\mathrm{type}(\mu)) \times \mathcal{F}l_n(\mathrm{type}(\nu)) = G/P \times G/Q \times G/R$. Let $\mathcal{L}$ be the GL_n^3-linearized line bundle on X associated to (λ, μ, ν) (see Paragraph 7.1.1 below for details). It is well known that $\mathcal{L}$

is ample and that $H^0(X, \mathcal{L}^{\otimes m}) = V_{m\lambda}^* \otimes V_{m\mu}^* \otimes V_{m\nu}^*$, for any positive integer m.

Let $\bar{\mathcal{L}}$ be the GL_n-linearized line bundle on X obtained by restriction the action of GL_n^3 to the diagonal. Since each $\varphi_{IJK}(\lambda, \mu, \nu) < 0$, Theorem 2 implies that $\bar{\mathcal{L}}$ belongs to the relative interior of $\mathcal{AC}^G(X)$. Now, Lemma 2 implies that the dimension of $X^{\mathrm{ss}}(\bar{\mathcal{L}})/\!/G$ is $\mathrm{mod}(X, G)$.

Assume now that Derksen-Weyman's algorithm decides that X does not contain an open G-orbit; that is $\mathrm{mod}(X, G) > 0$. Since the dimension of $X^{\mathrm{ss}}(\bar{\mathcal{L}})/\!/G$ is positive, there exists a positive integer such that $c^n_{m\lambda\, m\mu\, m\nu} \geq 2$. Now, Fulton's conjecture implies that $c^n_{\lambda\mu\nu} \neq 1$. But, Theorem 10 implies that $c^n_{\lambda\mu\nu} \neq 0$. Finally, $c^n_{\lambda\mu\nu} \geq 2$.

Assume finally that $\mathrm{mod}(X, G) = 0$. Since $X^{\mathrm{ss}}(\bar{\mathcal{L}})/\!/G$ is a point, $c^n_{\lambda\mu\nu} \leq 1$. But, Theorem 10 implies that $c^n_{\lambda\mu\nu} \neq 0$. Finally, $c^n_{\lambda\mu\nu} = 1$. □

5 A parametrization of Schubert varieties

In this section, we recall some properties about the inversion sets introduced by Kostant in [Kos61].

5.1 The general case

5.1.1 — Let G be a complex reductive group. Let $T \subset B$ be a maximal torus and a Borel subgroup of G.

Let Φ (resp. Φ^+) denote the set of roots (resp. positive roots) of G. Set $\Phi^- = -\Phi^+$. Let Δ denote the set of simple roots. Let us consider the set $X(T)^+$ of dominant characters of T. Let W denote its Weyl group.

5.1.2 — Let P be a standard (ie which contains B) parabolic subgroup of G and L denote its Levi subgroup containing T. Let W_L denote the Weyl group of L and Φ_L denote the set of roots of L. We consider the homogeneous space G/P. Its base point is denoted by P.

For $w \in W/W_L$, we consider the associated Schubert variety $\Omega(w)$ which is the closure of BwP/P.

If G/P is a Grassmannian, the Schubert varieties are classically parametrized by partitions (see Paragraphs 2.1.1 and 2.1.3). We are going to generalize this parametrization. The set of weights of T acting on the tangent space $T_P G/P$ is $-(\Phi^+ \backslash \Phi_L)$. Set

$$\Lambda(G/P) = -(\Phi^+ \backslash \Phi_L).$$

Let W^P denote the set of minimal length representatives of elements in W/W_L. Let $w \in W^P$. Consider $w^{-1}\Omega(w)$: it is a closed T-stable subvariety of G/P containing P and smooth at P. The tangent space $T_P w^{-1}\Omega(w)$ is called *the centered tangent space of* $\Omega(w)$. We set:

$$\Lambda_w = \{\alpha \in \Lambda(G/P) \,:\, \alpha \text{ is not a weight of } T \text{ in } T_P w^{-1}\Omega(w)\}.$$

Let $\mathcal{P}(\Lambda(G/P))$ denote the set of parts of $\Lambda(G/P)$. We have the following easy lemma (see [Bou02]).

Lemma 3 *We have* $\Lambda_w = \{\alpha \in \Lambda(G/P) \,:\, -w\alpha \in \Phi^+\}$*, and the map* $W^P \longrightarrow \mathcal{P}(\Lambda(G/P))$*,* $w \mapsto \Lambda_w$ *is injective. Moreover, the codimension of* $\Omega(w)$ *is the cardinality of* Λ_w*.*

5.1.3 — We write $\alpha \prec \beta$ if $\beta - \alpha$ is a non-negative combination of positive roots.

If λ is a one parameter subgroup of G then the set of $g \in G$ such that $\lim_{t\to 0} \lambda(t)g\lambda(t^{-1})$ exists in G is a parabolic subgroup $P(\lambda)$ of G. Moreover, any parabolic subgroup of G can be obtained in such a way. Let us fix a one parameter subgroup λ of T such that $P = P(\lambda)$. Let $\langle \cdot\,,\,\cdot \rangle$ denote the natural paring between one parameter subgroups and characters of T.

Lemma 4 *Let* $\alpha \in \Lambda_w$ *and* $\beta \in \Lambda(G/P)$*. We assume that* $\langle \lambda, \alpha \rangle = \langle \lambda, \beta \rangle$ *and* $\beta \prec \alpha$*.*

Then, $\beta \in \Lambda_w$*.*

Proof. We have to prove that $w\beta \in \Phi^-$. But $w\beta = w\alpha + w(\beta - \alpha)$. Since $\langle \lambda, \beta - \alpha \rangle = 0$, $\beta - \alpha$ belongs to the root lattice of L. But, $\beta \prec \alpha$; so, $\beta - \alpha$ is a non-negative combination of negative roots of L. Since $w \in W^P$, $w\Phi_L^- \subset \Phi^-$. Finally, $w(\beta - \alpha)$ is a non-negative combination of negative roots. If follows that $w\beta \prec w\alpha$ and $w\beta \in \Phi^-$. □

Lemma 4 implies that Λ_w is an order ideal on each strata given by λ. More precisely, Kostant characterized [Kos61, Proposition 5.10] the parts of $\Lambda(G/P)$ equal to Λ_w for some $w \in W^P$.

5.2 The case SL_n

5.2.1 — Let V be a n-dimensional vector space and set $G = \mathrm{SL}(V)$. Let $\mathcal{B} = (e_1, \cdots, e_n)$ be a basis of V. Let T be the maximal torus of G consisting of diagonal matrices in $\mathcal{B}$ and B the Borel subgroup of G consisting of

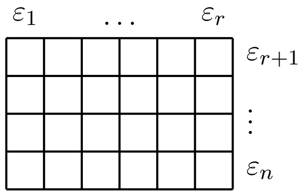

Figure 1: $\Lambda(\mathbb{G}(r,n))$

upper triangular matrices. Let ε_i denote the character of T which maps $\operatorname{diag}(t_1,\cdots,t_n)$ to t_i; we have $X(T)=\oplus_i\mathbb{Z}\varepsilon_i/\mathbb{Z}\sum_i\varepsilon_i$. Here, we have:

$$\begin{array}{l}\Phi^+=\{\varepsilon_i-\varepsilon_j\,:\,i<j\},\\ \Delta=\{\alpha_r=\varepsilon_r-\varepsilon_{r+1}\,:\,r=1,\cdots,n-1\}.\end{array}$$

The Weyl group W of G is the symmetric group S_n acting on n letters. We will denote by $F(r)$ the span of $e_1,\cdots,e_r$.

5.2.2 — Let α_r be a simple root, P_r be the corresponding maximal standard parabolic subgroup of G and L_r be its Levi subgroup containing T. The homogeneous space G/P_r with base point P_r is the Grassmannian $\mathbb{G}(r,n)$ of r-dimensional subspaces of V with base point $F(r)$. The tangent space $T_{F(r)}\mathbb{G}(r,n)$ identifies with $\operatorname{Hom}(F(r),V/F(r))$. The natural action of L_r which is isomorphic to $\mathrm{S}(\mathrm{GL}(F(r))\times\mathrm{GL}(V/F(r)))$ makes this identification equivariant.

Consider $\Lambda(\mathbb{G}(r,n))=\Phi^-\backslash\Phi_{L_r}$ as in Paragraph 5.1.2:

$$\Lambda(\mathbb{G}(r,n))=\{\varepsilon_i-\varepsilon_j\,:\,1\leq j\leq r<i\leq n\}.$$

We now represent $\Lambda(\mathbb{G}(r,n))$ by a rectangle with $r\times(n-r)$ boxes: the box at row i and the column j represents the root $\varepsilon_{r+i}-\varepsilon_j$ (see Figure 1).

Note that Lemma 4 asserts in this case that the Λ_w's are Young diagrams (oriented as on Figure 2).

5.2.3 — If $I\in\mathcal{P}(r,n)$, we set $F(I)=\operatorname{Span}(e_i\,:\,i\in I)$. Let $I=\{i_1<\cdots<i_r\}$ and $\Omega(I)$ the corresponding Schubert variety, that is the closure of $B.F(I)$. Set $\{i_{r+1}<\cdots<i_n\}=I^c$. Set $w_I=(i_1,\cdots,i_n)\in S_n=W$; then, $w_I\in W^{P_r}$ and represents $\Omega(I)$. Set $\Lambda_I=\Lambda_{w_I}$; we have:

$$\Lambda_I=\{\varepsilon_i-\varepsilon_j\,:\,w_I(j)<w_I(i)\text{ and }j\leq r<i\}.$$

To obtain Λ_I on Figure 2, one can proceeds as follows. Index the columns (resp. rows) of Figure 1 by I (resp. I^c). Now, a given box belongs to Λ_I if

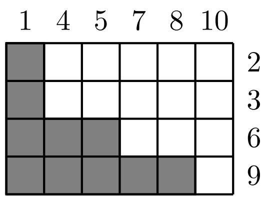

Figure 2: An example of Λ_I

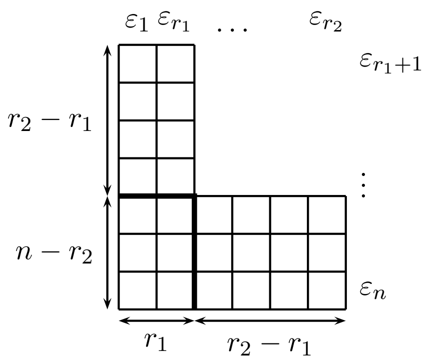

Figure 3: $\Lambda(\mathcal{F}l_n(r_1, r_2))$

and only if the index of its column is less that those of its row. For example, if $I = \{1, 4, 5, 7, 8, 10\} \in \mathcal{P}(6, 10)$, Λ_I is the set of black boxes on Figure 2.

Note that Λ_I is the **complement of the transpose of** the Young diagram of λ^I as defined in Paragraph 2.1.3.

5.2.4 — We now consider the case of a two step flag manifold $\mathcal{F}l_n(r_1, r_2)$. Here, $\Lambda(\mathcal{F}l_n(r_1, r_2))$ is the union of three rectangles of size $r_1 \times (r_2 - r_1)$, $(r_2 - r_1) \times (n - r_2)$ and $r_1 \times (n - r_2)$ (see Figure 3). These three rectangles are denoted by R_0, R_1 and R_2 respectively.

The Schubert varieties are naturally parametrized by the set $\mathcal{S}(\mathcal{F}l_n(r_1, r_2))$ of the pairs $(I^1, I^2) \in \mathcal{P}(r_1, n) \times \mathcal{P}(r_2, n)$ such that $I^1 \subset I^2$. Let $(I^1, I^2) \in \mathcal{S}(\mathcal{F}l_n(r_1, r_2))$. To obtain Λ_p on Figure 3, one can proceed as follows. Index the r_1 first columns (resp. $r_2 - r_1$ first rows) of Figure 3 by I^1 (resp. $I^2 - I^1$). Index the following $r_2 - r_1$ columns (resp. $n - r_2$ rows) of Figure 3 by $I^2 - I^1$ (resp. $[1, n] - I^2$). Now, a given box belongs to $\Lambda_{(I^1, I^2)}$ if and only if the index of its column is less that those of its row. For example, if $n = 9$, $I^1 = \{3, 7\}$ and $I^2 = I^1 \cup \{1, 5, 6, 8\}$, one obtains $\Lambda_{(I^1, I^2)}$ on Figure 4.

Remark 2 *Lemma 4 means that $\Lambda_{(I^1, I^2)}$ is the union of three Young dia-*

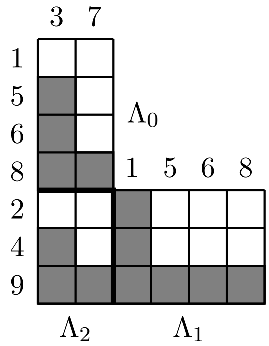

Figure 4: An example of $\Lambda_{(I^1,I^2)}$ for $(I^1, I^2) \in \mathcal{S}(\mathcal{F}l_n(r_1, r_2))$

grams as on Figure 4.

5.2.5 — We now consider the following characteristic function:

$$\begin{array}{cccl} \chi_{(I^1,I^2)} : & [1;n] & \longrightarrow & \{0,1,2\} \\ & i & \longmapsto & \left\{ \begin{array}{lll} 1 & \text{if} & i \in I^1, \\ 2 & \text{if} & i \in I^2 - I^1, \\ 0 & \text{if} & i \notin I^2. \end{array} \right. \end{array}$$

We think about $\chi_{(I^1,I^2)}$ as a word of length n with letters in $\{0, 1, 2\}$. If one cancels the letters 2 in this word, one obtains the characteristic function of a part I_2 of $[1; n-(r_2-r_1)]$ with r_1 elements. If one cancels the letters 1 in this word and then replaces 2 by 1, one obtains the characteristic function of a part I_1 of $[1; n-r_1]$ with r_2-r_1 elements. If one cancels the letters 0 in this word and then replaces 2 by 0, one obtains the characteristic function of a part I_0 of $[1; r_2]$ with r_1 elements. We just defined a map:

$$\begin{array}{ccl} \mathcal{S}(\mathcal{F}l_n(r_1, r_2)) & \longrightarrow & \mathcal{P}(r_1, n+r_1-r_2) \times \mathcal{P}(r_2-r_1, n-r_1) \times \mathcal{P}(r_1, r_2) \\ (I^1, I^2) & \longmapsto & (I_2, I_1, I_0). \end{array} \tag{12}$$

Let $(I^1, I^2) \in \mathcal{S}(\mathcal{F}l_n(r_1, r_2))$. Set $\Lambda_i = \Lambda_{(I^1,I^2)} \cap R_i$.

Proposition 4 *With above notation, Λ_i is the partition associated to the part I_i, for $i = 0$, 1 and 2.*

Proof. The proof is direct with the description of $\Lambda_{(I^1,I^2)}$ made in Paragraph 5.2.4. □

5.2.6 — We now consider the particular case when $n - r_2 = r_1$. So consider $\mathcal{F}l_n(r, n-r)$. In this case $\Lambda(G/P)$ is symmetric under the diagonal dashed line on Figure 5 below. Let τ denote this symmetry.

For $i \in [1; n]$, we set $\overline{i} = n + 1 - i$. The symmetry τ corresponds to the involution $\overline{\square}$. More precisely, we have:

Lemma 5 *Let $(I^1, I^2) \in \mathcal{S}(\mathcal{F}l_n(r, n-r))$. Set $J^1 = I^1$, $J^2 = I^2 - I^1$ and $J^3 = [1; n] - J^2$.*

Consider $(\overline{J^3}, \overline{J^2}, \overline{J^1})$; and $(\overline{J^3}, \overline{J^2} \cup \overline{J^3}) \in \mathcal{S}(\mathcal{F}l_n(r, n-r))$.

Then, $\tau(\Lambda_{(I^1,I^2)}) = \Lambda_{(\overline{J^3}, \overline{J^2} \cup \overline{J^3})}$.

Proof. The proof is direct with the description of $\Lambda_{(I^1,I^2)}$ made in Paragraph 5.2.4. $\square$

5.3 The case Sp_{2n}

5.3.1 — Root system.

Let V be a $2n$-dimensional vector space and $\mathcal{B} = (e_1, \cdots, e_{2n})$ be a basis of V. Consider the following $n \times n$ matrix J_n:

$$J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}; \tag{13}$$

and the bilinear symplectic form ω on V with matrix

$$\omega = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

Let G be the associated symplectic group. Set $T = \{\mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}) : t_i \in \mathbb{C}^*\}$. Let B be the Borel subgroup of G consisting of upper triangular matrices of G. For $i \in [1, n]$, let ε_i denote the character of T which maps $\mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1})$ to t_i; we have $X(T) = \oplus_i \mathbb{Z}\varepsilon_i$. Here, we have:

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \,:\, 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \,:\, 1 \leq i \leq n\},$$
$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\, \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots,\, \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\, \alpha_n = 2\varepsilon_n\}.$$

If $i \in [1; 2n]$, we set $\overline{i} = 2n + 1 - i$. The Weyl group W of G is a subgroup of the Weyl group S_{2n} of $\mathrm{SL}(V)$:

$$W = \{w \in S_{2n} \,:\, w(\overline{i}) = \overline{w(i)} \;\; \forall i \in [1; 2n]\}.$$

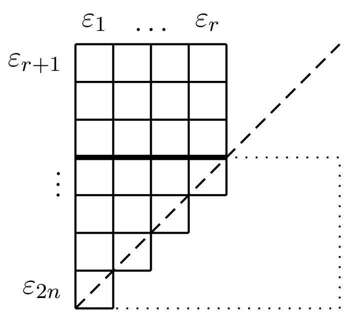

Figure 5: Roots of $T_{F(r)}\mathbb{G}_\omega(r, 2n)$

We will denote by $F(r)$ (resp. $\overline{F}(r)$) the span of $e_1, \cdots, e_r$ (resp. $e_{\overline{1}}, \cdots, e_{\overline{r}}$). We will denote by $V(r)$ the span of $e_{r+1}, \cdots, e_{\overline{r+1}}$.

5.3.2 — Tangent space of isotropic Grassmanians. Let α_r be a simple root, P_r be the corresponding maximal standard parabolic subgroup of G and L_r be its Levi subgroup containing T. The homogeneous space G/P_r with base point P_r is the isotropic Grassmannian $\mathbb{G}_\omega(r, 2n)$ of r-dimensional subspaces M of V such that $\omega(M, M) = 0$ with base point $F(r)$.

Note that $V = F(r) \oplus V(r) \oplus \overline{F}(r)$. Moreover, $F(r)^{\perp\omega} = F(r) \oplus V(r)$, and ω identifies $\overline{F}(r)$ with the dual of $F(r)$. The tangent space $T_{F(r)}\mathbb{G}_\omega(r, 2n)$ identifies with $\mathrm{Hom}(F(r), V(r)) \oplus S^2F(r)^*$. The natural action of L_r which is isomorphic to $\mathrm{GL}(F(r)) \times \mathrm{Sp}(V(r))$ makes this identification equivariant.

For convenience we set for $i = 1, \cdots, n$, $\varepsilon_{\overline{i}} := -\varepsilon_i$. Then,

$$\Phi^- = \{\varepsilon_i - \varepsilon_j \ : \ 1 \leq j \qquad < i \leq \overline{j} \leq 2n\}, \text{ and}$$
$$\Lambda(\mathbb{G}_\omega(r, 2n)) = \{\varepsilon_i - \varepsilon_j \ : \ 1 \leq j \leq r < i \leq \overline{j} \leq 2n\}.$$

We now represent each element of $\Lambda(\mathbb{G}_\omega(r, 2n))$ by a box on Figure 5. The box at row i and column j corresponds to $\varepsilon_{r+i} - \varepsilon_j$.

The boxes corresponding to roots of $S^2F(r)^*$ (resp. $\mathrm{Hom}(F(r), V(r))$) are in the triangular (resp. rectangular) part of Figure 5.

5.3.3 — Schubert varieties of isotropic Grassmanians. If $I \in \mathcal{P}(r, 2n)$ then we set $\overline{I} = \{\overline{i} \ : \ i \in I\}$ and

$$\mathcal{S}(\mathbb{G}_\omega(r, 2n)) := \{I \in \mathcal{P}(r, 2n) \ : \ I \cap \overline{I} = \emptyset\}.$$

The subspace $F(I)$ belongs to $\mathbb{G}_\omega(r, 2n)$ if and only if $I \in \mathcal{S}(\mathbb{G}_\omega(r, 2n))$; so, the Schubert varieties $\Psi(I)$ of $\mathbb{G}_\omega(r, 2n)$ are indexed by $I \in \mathcal{S}(\mathbb{G}_\omega(r, 2n))$.

If $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r, 2n)$, we set $i_{\overline{k}} = \overline{i_k}$ and write $(I \cup \overline{I})^c = \{i_{r+1} < \cdots < i_{\overline{r+1}}\}$. Then, the element of W^{P_r} which corresponds to $\Psi(I)$ is $w_I = (i_1, \cdots, i_{2n})$.

5.3.4 — We now want to describe $\Lambda_I = \Lambda_{w_I}$. Consider $(I \subset \overline{I}^c) \in \mathcal{S}(\mathcal{F}l_{2n}(r, 2n-r))$. We draw $\Lambda_{(I,\overline{I}^c)}$ on Figure 5 including the dotted part.

Proposition 5 *(i) The part $\Lambda_{(I,\overline{I}^c)}$ is symmetric relatively to the dashed line.*

(ii) The part Λ_I is the intersection of $\Lambda(\mathbb{G}_\omega(r, 2n))$ and $\Lambda_{(I,\overline{I}^c)}$.

Proof. The first assertion is a direct consequence of Lemma 5. Consider W as a subgroup of S_{2n} as in Paragraph 5.3.1. Then, w_I is the element of S_{2n} corresponding to the Schubert class $(I, \overline{I}^c)$ in $\mathcal{S}(\mathcal{F}l_{2n}(r, 2n-r))$ as in Paragraph 5.1.2. The second assertion follows. □

5.4 The case SO_{2n+1}

5.4.1 — **Root system.**

Let V be a $2n+1$-dimensional vector space and $\mathcal{B} = (e_1, \cdots, e_{2n+1})$ be a basis of V. We denote by $(x_1, \cdots, x_{2n+1})$ the dual basis. If $i \in [1; 2n+1]$, we set $\overline{i} = 2n+2-i$. Let G be the special orthogonal group associated to the quadratic form

$$Q = x_{n+1}^2 + \sum_{i=1}^{n} x_i x_{\overline{i}}.$$

Set $T = \{\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1}) \ : \ t_i \in \mathbb{C}^*\}$. Let B the Borel subgroup of G consisting of upper triangular matrices of G. Let ε_i denote the character of T which maps $\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1})$ to t_i; we have $X(T) = \oplus_{i=1}^n \mathbb{Z}\varepsilon_i$. Here, we have:

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \ : \ 1 \le i < j \le n\} \cup \{\varepsilon_i \ : \ 1 \le i \le n\},$$
$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3, \cdots,\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = \varepsilon_n\}.$$

The Weyl group W of G is a subgroup of the Weyl group S_{2n+1} of $\mathrm{SL}(V)$:

$$W = \{w \in S_{2n+1} \ : \ w(\overline{i}) = \overline{w(i)} \ \ \forall i \in [1; 2n+1]\}.$$

We will denote by $F(r)$ (resp. $\overline{F}(r)$) the span of $e_1, \cdots, e_r$ (resp. $e_{\overline{1}}, \cdots, e_{\overline{r}}$). We will denote by $V(r)$ the span of $e_{r+1}, \cdots, e_{\overline{r+1}}$.

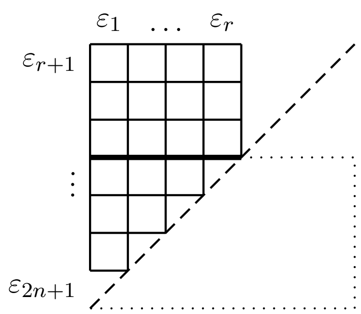

Figure 6: Roots of $T_{F(r)}\mathbb{G}_Q(r, 2n+1)$

5.4.2 — Tangent space of orthogonal Grassmanians. Let α_r be a simple root, P_r be the corresponding maximal parabolic subgroup of G and L_r be its Levi subgroup containing T. For $r \leq n$, we denote by $\mathbb{G}_Q(r, 2n+1)$ the orthogonal Grassmannian of r-dimensional subspaces M of V such that $Q_{|M} = 0$. The homogeneous space G/P_r with base point P_r is $\mathbb{G}_Q(r, 2n+1)$ with base point $F(r)$.

Note that $V = F(r) \oplus V(r) \oplus \overline{F}(r)$. Moreover, $F(r)^{\perp_Q} = F(r) \oplus V(r)$, and Q identifies $\overline{F}(r)$ with the dual of $F(r)$. The tangent space $T_{F(r)}\mathbb{G}_Q(r, 2n+1)$ identifies with $\mathrm{Hom}(F(r), V(r)) \oplus \bigwedge^2 F(r)^*$. The natural action of L_r which is isomorphic to $S(\mathrm{GL}(F(r)) \times \mathrm{O}(V(r)))$ makes this identification equivariant.

We set for $i \in [1, n]$, $\varepsilon_{\overline{i}} := -\varepsilon_i$, and $\varepsilon_{n+1} = 0$. Then, we have:

$$\Phi^- = \{\varepsilon_i - \varepsilon_j \; : \; j \qquad < i < \overline{j}\}, \text{ and}$$
$$\Lambda(\mathbb{G}_Q(r, 2n+1)) = \{\varepsilon_i - \varepsilon_j \; : \; j \leq r < i < \overline{j}\}.$$

We now represent each element of $\Lambda(\mathbb{G}_Q(r, 2n+1))$ by a box on Figure 6.

The boxes corresponding to roots of $\bigwedge^2 F(r)^*$ (resp. $\mathrm{Hom}(F(r), V(r))$) are in the triangular (resp. rectangular) part of Figure 6.

5.4.3 — Schubert varieties of orthogonal Grassmanians. If $I \in \mathcal{P}(r, 2n+1)$ then we set $\overline{I} = \{\overline{i} \; : \; i \in I\}$ and

$$\mathcal{S}(\mathbb{G}_Q(r, 2n+1)) := \{I \in \mathcal{P}(r, 2n+1) \; : \; I \cap \overline{I} = \emptyset\}.$$

The subspace $F(I)$ belongs to $\mathbb{G}_Q(r, 2n+1)$ if and only if $I \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$; so, the Schubert varieties $\Psi(I)$ of $\mathbb{G}_Q(r, 2n+1)$ are indexed by $I \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$. If $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r, 2n+1)$, we set $i_{\overline{k}} = \overline{i_k}$ and write $(I \cup \overline{I})^c = \{i_{r+1} < \cdots < i_{\overline{r+1}}\}$. Then, the element of W^{P_r} which corresponds to $\Psi(I)$ is $w_I = (i_1, \cdots, i_{2n+1})$.

5.4.4 — We now want to describe $\Lambda_I = \Lambda_{w_I}$. Consider $(I \subset \overline{I}^c) \in \mathcal{S}(\mathcal{F}l_{2n+1}(r, 2n+1-r))$. We draw $\Lambda_{(I,\overline{I}^c)}$ on Figure 6 including the dotted part. Then, we obtain easily:

Proposition 6 *(i) The part $\Lambda_{(I,\overline{I}^c)}$ is symmetric relatively to the dashed line.*

(ii) The part Λ_I is the intersection of $\Lambda(\mathbb{G}_Q(r, 2n+1))$ and $\Lambda_{(I,\overline{I}^c)}$.

6 Levi-movability

In this section, we recall the Belkale-Kumar notion of Levi-movability (see [BK06]). We follow notation of Section 5.1.

6.1 Cohomology of G/P

6.1.1 — Let σ_w denote the Poincaré dual of the homology class of $\Omega(w)$. We have:

$$\mathrm{H}^*(G/P, \mathbb{Z}) = \oplus_{w \in W^P} \mathbb{Z}\sigma_w.$$

The dual of the class σ_w is denoted by $\sigma_w^\vee$. Note that σ_e is the class of the point. Let σ_u, σ_v, σ_w be three Schubert classes (with $u, v, w \in W^P$). If there exists an integer d such that $\sigma_u.\sigma_v.\sigma_w = d\sigma_e$, we set $c_{uvw} = d$ and we set $c_{uvw} = 0$ otherwise. These coefficients are the (symmetrized) structure coefficients of the cup product on $\mathrm{H}^*(G/P, \mathbb{Z})$ in the Schubert basis in the following sense:

$$\sigma_u.\sigma_v = \sum_{w \in W^P} c_{uvw}\sigma_w^\vee;$$

and $c_{uvw} = c_{vuw} = c_{uwv}$.

6.1.2 — Let u, v and w in W^P. Let us consider the tangent space T_u of the $u{-}1BuP/P$'s at the point P; and, similarly T_v and T_w. Using the transversality theorem of Kleiman, Belkale-Kumar showed in [BK06, Proposition 2] the following important lemma:

Lemma 6 *The coefficient c_{uvw} is nonzero if and only if there exist $p_u, p_v, p_w \in P$ such that the natural map*

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{p_uT_u} \oplus \frac{T_P(G/P)}{p_vT_v} \oplus \frac{T_P(G/P)}{p_wT_w},$$

is an isomorphism.

Then, Belkale-Kumar defined Levi-movability:

Definition 1 *The triple $(\sigma_u,\, \sigma_v,\, \sigma_w)$ is said to be Levi-movable if there exist $l_u, l_v, l_w \in L$ such that the natural map*

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{l_uT_u} \oplus \frac{T_P(G/P)}{l_vT_v} \oplus \frac{T_P(G/P)}{l_wT_w},$$

is an isomorphism.

We set:

$$c_{uvw}^{\odot_0} = \begin{cases} c_{uvw} & \text{if } (\sigma_u,\, \sigma_v,\, \sigma_w) \text{ is Levi} - \text{movable;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that in [RR09], an equivalent characterization of Levi-movability is given. We define on the group $\mathrm{H}^*(G/P,\mathbb{Z})$ a bilinear product $\odot_0$ by the formula:

$$\sigma_u \odot_0 \sigma_v = \sum_{w\in W^P} c_{uvw}^{\odot_0} \sigma_w^\vee.$$

By [BK06, Definition 18], we have:

Theorem 14 *The product $\odot_0$ is commutative, associative and satisfies Poincaré duality.*

Remark that if G/P is cominuscule, P and L-orbits in T_PG/P are equal. In particular, in this case the product $\odot_0$ is the usual cup product.

7 Cones associated to groups

7.1 The tensor product cone

In this section, we will define a generalization of the Horn cone for any semisimple group G. We will also recall some results about these cones. We follow notation of Section 6.

7.1.1 — The Borel-Weil theorem. Let ν be a character of B. Let $\mathbb{C}_\nu$ denote the field $\mathbb{C}$ endowed with the action of B defined by $b.\tau = \nu(b)\tau$ for all $\tau \in \mathbb{C}_\nu$ and $b \in B$. The fiber product $G \times_B \mathbb{C}_{-\nu}$ is a G-linearized line bundle on G/B, denoted by $\mathcal{L}_\nu$. In fact, the map $X(B) = X(T) \longrightarrow \mathrm{Pic^G(G/B)}$, $\nu \longmapsto \mathcal{L}_\nu$ is an isomorphism. Moreover, $\mathcal{L}_\nu$ is generated by its sections if and only if it has nonzero sections if and only if ν is dominant; and, $\mathrm{H}^0(G/B, \mathcal{L}_\nu)$ is isomorphic to the dual V_ν^* of the irreducible G-module V_ν of highest weight ν.

7.1.2 — We set: $X(T)_\mathbb{Q} = X(T) \otimes \mathbb{Q}$. The set of triples $(\lambda, \mu, \nu) \in (X(T)^+)^3$ such that $V_\lambda \otimes V_\mu \otimes V_\nu$ contains nonzero G-invariant vectors is a finitely generated semigroup. We will denote by $\mathcal{LR}(G)$ the convex hull in $X(T)^3_\mathbb{Q}$ of this semigroup: it is a closed convex rational polyhedral cone.

Set $X = (G/B)^3$. Identifying $X(T^3)$ with $X(T)^3$, for any $(\lambda, \mu, \nu) \in X(T)^3$, we obtain a G^3-linearized line bundle $\mathcal{L}_{\lambda,\mu,\nu}$ on X. Applying the Borel-Weil theorem, we obtain

$$\mathcal{LR}(G) = \{(\lambda\ , \mu, \nu) \in X(T)^3 \otimes \mathbb{Q}\ :\ \exists n > 0\ \ \mathrm{H}^0(X, \mathcal{L}_{\lambda,\mu,\nu}^{\otimes n})^G \neq \{0\}\}.$$

Since G is assumed to be semisimple, we have isomorphisms $X(T^3)_\mathbb{Q} \simeq \mathrm{Pic}^{\mathrm{G}^3}(\mathrm{X})_\mathbb{Q} \simeq \mathrm{Pic}^{\mathrm{G}}(\mathrm{X})_\mathbb{Q}$. With these identifications, $\mathcal{LR}(G)$ is the closure of $\mathcal{AC}^G((G/B)^3)$ (see for example [Res10a, Proposition 10]).

7.1.3 — Let α be a simple root of G, P_α denote the associated maximal standard parabolic subgroup and L_α denote its Levi subgroup containing T. Set $W_\alpha = W_{L_\alpha}$. Consider the one parameter subgroup $\omega_{\alpha^\vee}$ (with usual notation) of the center of L_α. We now state the main result of [BK06]:

Theorem 15 *Here G is assumed to be semisimple. Let $(\lambda, \mu, \nu) \in X(T)^3_\mathbb{Q}$ dominant. Then, $(\lambda, \mu, \nu) \in \mathcal{LR}(G)$ if and only if*

$$\langle u\omega_{\alpha^\vee}, \lambda\rangle + \langle v\omega_{\alpha^\vee}, \mu\rangle + \langle w\omega_{\alpha^\vee}, \nu\rangle \leq 0, \tag{14}$$

for all simple root α and all triple $(u, v, w) \in W/W_\alpha$ with $c_{uvw}^{\odot_0} = 1$.

Let α and $(u, v, w) \in W/W_\alpha$ be as in the theorem. The set of $(\lambda, \mu, \nu) \in \mathcal{LR}(G)$ for which inequality (14) becomes an equality is a face of $\mathcal{LR}(G)$ denoted by $\mathcal{F}(\alpha, u, v, w)$. The following statement, proved in [Res10a], shows that no inequality (14) can be omitted in Theorem 15.

Theorem 16 *Let α and $(u, v, w) \in W/W_\alpha$ be as in Theorem 15. Then, $\mathcal{F}(\alpha, u, v, w)$ is a codimension one face of $\mathcal{LR}(G)$ intersecting the strictly dominant chamber.*

7.1.4 — We now want to understand better the faces $\mathcal{F}(\alpha, u, v, w)$. Consider the fixed point set $X^{\omega_{\alpha^\vee}}$ of $\omega_{\alpha^\vee}$ acting on X. Then,

$$C(u, v, w) = L_\alpha u^{-1}B \times L_\alpha v^{-1}B \times L_\alpha w^{-1}B$$

is an irreducible component of $X^{\omega_{\alpha^\vee}}$. Note that $B_L = B \cap L_\alpha$ is a Borel subgroup of L_α. If each w_i belongs to W^P, we fix an isomorphism between $(L_\alpha/B_L)^3$ and $C(u, v, w)$ by

$$(l_u B_L, l_v B_L, l_w B_L) \longmapsto l_u u^{-1}B \times l_v v^{-1}B \times l_w w^{-1}B;$$

it is well-defined since $u,\ v,\ w \in W^P$. In particular, the group $\mathrm{Pic}^{\mathrm{L}_\alpha^3}(\mathrm{C(u, v, w)})$ is isomorphic to $\mathrm{Pic}^{\mathrm{L}_\alpha^3}((\mathrm{L}_\alpha/\mathrm{B}_\mathrm{L})^3)$; that is, to $X(T)^3$. With these identifications the restriction morphism $\mathrm{Pic}^{\mathrm{G}^3}(\mathrm{X}) \longrightarrow \mathrm{Pic}^{\mathrm{L}_\alpha^3}(\mathrm{C(u, v, w)})$ is

$$\begin{array}{cccc} \rho_{uvw}: & X(T)^3 & \longrightarrow & X(T)^3 \\ & (\lambda, \mu, \nu) & \longmapsto & (u^{-1}\lambda, v^{-1}\mu, w^{-1}\nu). \end{array}$$

The following statement is [Res09, Lemma 1]:

Theorem 17 *Let α and $(u, v, w) \in W/W_\alpha$ be such that $\sigma_u.\sigma_v.\sigma_w \neq 0$. Then, for any $(\lambda, \mu, \nu) \in \mathcal{LR}(G)$,*

$$\langle u\omega_{\alpha^\vee}, \lambda\rangle + \langle v\omega_{\alpha^\vee}, \mu\rangle + \langle w\omega_{\alpha^\vee}, \nu\rangle \leq 0,$$

holds. Let $\mathcal{F}(\alpha, u, v, w)$ denote the corresponding face of $\mathcal{LR}(G)$. If $(\lambda, \mu, \nu) \in X(T)^3 \otimes \mathbb{Q}$ is dominant then $(\lambda, \mu, \nu) \in \mathcal{F}(\alpha, u, v, w)$ if and only if $\rho_{uvw}(\lambda, \mu, \nu) \in \mathcal{LR}(L_\alpha)$.

The following criterion to decide if $c^{\odot_0}_{uvw} = 1$ or not will play a central role in the sequence of this article.

Corollary 1 *Let α and $(u, v, w) \in W/W_\alpha$ be as in Theorem 17. Then, $c^{\odot_0}_{uvw} = 1$ if and only if $\mathcal{F}(\alpha, u, v, w)$ intersects the interior of the dominant chamber of $X(T^3)_\mathbb{Q}$.*

Proof. The direct implication is a consequence of Theorem 16. Conversely, the cone $\mathcal{LR}(L_\alpha)$ has codimension one (the rank of the center of L_α) in $X(T)^3_\mathbb{Q}$. So, since $\mathcal{F}(\alpha, u, v, w)$ intersects the interior of the dominant chamber of $X(T^3)_\mathbb{Q}$, Theorem 17 implies that $\mathcal{F}(\alpha, u, v, w)$ has codimension one. So, the corresponding inequality has to appear in Theorem 15. This implies that $c^{\odot_0}_{uvw} = 1$. □

7.2 The eigencone

Let us fix a maximal compact subgroup U of G in such a way that $T \cap U$ is a Cartan subgroup of U. Let $\mathfrak{u}$ and $\mathfrak{t}$ denote the Lie algebras of U and T. Let $\mathfrak{t}^+$ be the Weyl chamber of $\mathfrak{t}$ corresponding to B. Let $\sqrt{-1}$ denote the usual complex number. It is well known that $\sqrt{-1}\mathfrak{t}^+$ is contained in $\mathfrak{u}$ and that the map:

$$\begin{array}{ccc} \mathfrak{t}^+ & \longrightarrow & \mathfrak{u}/U \\ \xi & \longmapsto & U.(\sqrt{-1}\xi) \end{array}$$

is an homeomorphism. Consider the set

$$\Gamma(U) := \{(\xi, \zeta, \eta) \in (\mathfrak{h}^+)^3 \; : \; U.(\sqrt{-1}\xi) + U.(\sqrt{-1}\zeta) + U.(\sqrt{-1}\eta) \ni 0\}.$$

Let $\mathfrak{u}^*$ (resp. $\mathfrak{t}^*$) denote the dual (resp. complex dual) of $\mathfrak{u}$ (resp. $\mathfrak{t}$). Let $\mathfrak{t}^{*+}$ denote the dominant chamber of $\mathfrak{t}^*$ corresponding to B. By taking the tangent map at the identity, one can embed $X(T)^+$ in $\mathfrak{t}^{*+}$. Note that, this embedding induces a rational structure on the complex vector space $\mathfrak{t}^*$. In particular, we can embed $\mathcal{L}R(G)$ in $(\mathfrak{t}^{*+})^3$: let $\tilde{\mathcal{L}R}(G)$ denote the so obtained part of $\mathfrak{t}^{*+}$.

Now, using the Cartan-Killing form, we identify $\mathfrak{t}^+$ and $\mathfrak{t}^{*+}$. In particular, we can embed $\Gamma(U)$ in $(\mathfrak{t}^{*+})^3$; the so obtained cone is denoted by $\tilde{\Gamma}(U)$.

Theorem 18 *The set $\Gamma(U)$ is a closed convex polyhedral cone. Moreover, $\tilde{\mathcal{L}R}(G)$ is the set of the rational points of $\tilde{\Gamma}(U)$.*

8 About the cohomology of $\mathbb{G}_\omega(r, 2n)$

8.1 — This section is concerned by coefficient structures of the cohomology of ordinary and isotropic Grassmannians. To avoid any confusion, those concerning ordinary and isotropic Grassmanians will be denoted by c and d respectively. Note that, since ordinary Grassmannian is cominuscule, $c^{\odot_0} = c$.

In Paragraph 5.3.4, we defined combinatorially an injective map

$$\begin{array}{ccc} \mathcal{S}(\mathbb{G}_\omega(r, 2n)) & \hookrightarrow & \mathcal{S}(\mathcal{F}l_{2n}(r, 2n - r)). \\ I & \longmapsto & (I, \bar{I}^c) \end{array} \qquad (15)$$

Set $\mathcal{S}(\mathbb{G}(r,n)) = \mathcal{P}(r,n)$. In Paragraph 5.2.5, we defined combinatorially an injective map

$$\begin{array}{ccc} \mathcal{S}(\mathcal{F}l_{2n}(r,2n-r)) & \hookrightarrow & \mathcal{S}(\mathbb{G}(r,2n-r)) \times \mathcal{S}(\mathbb{G}(2(n-r),2n-r)) \times \mathcal{S}(\mathbb{G}(r,2r)). \\ (I^1, I^2) & \longmapsto & (I_0, I_1, I_2) \end{array} \tag{16}$$

By composing these two injective maps and then forgetting I_1, we obtain an injective map

$$\begin{array}{ccc} \mathcal{S}(\mathbb{G}_\omega(r,2n)) & \hookrightarrow & \mathcal{S}(\mathbb{G}(r,2n-r)) \times \mathcal{S}(\mathbb{G}(r,2r)). \\ I & \longmapsto & (I_0, I_2) \end{array} \tag{17}$$

The aim of this section is to prove that this immersion is relevant relatively to the Belkale-Kumar product. We will also use the following particular case of the construction in Paragraph 5.3.4:

$$\begin{array}{ccc} \mathcal{S}(\mathbb{G}_\omega(r,2r)) & \hookrightarrow & \mathcal{S}(\mathbb{G}(r,2r)). \\ I & \longmapsto & I = I_2 \end{array} \tag{18}$$

8.2 — The following result is due to Belkale-Kumar:

Proposition 7 *Let I, J, $K \in \mathcal{S}(\mathbb{G}_\omega(r,2n))$ such that $|\Lambda_I| + |\Lambda_J| + |\Lambda_K| = \dim \mathbb{G}_\omega(r,2n)$. Notations I_0, J_0, K_0, I_2, J_2 and K_2 refer to the map (23). Note also that I_2, J_2 and K_2 belong to $\mathcal{S}(\mathbb{G}_\omega(r,2r))$.*

The following are equivalent:

- *(i)* $d^{\odot_0}_{IJK} \neq 0$;
- *(ii)* $|\Lambda_{I_0}| + |\Lambda_{J_0}| + |\Lambda_{K_0}| = 2r(n-r)$ *and* $d_{IJK} \neq 0$;
- *(iii)* $d_{I_2 J_2 K_2} \neq 0$ *and* $c_{I_0\, J_0\, K_0} \neq 0$.

Remark 3 *The first assertion concerns a structure coefficient of $(H^*(\mathbb{G}_\omega(r,2n),\odot_0)$, the second one concerns $H^*(\mathbb{G}_\omega(r,2n)$ and the last one concerns structure coefficients of $H^*(\mathbb{G}_\omega(r,2r)$ and $H^*(\mathbb{G}(r,2n-r)$.*

Proof. This is essentially [BK10, Theorem 30]. We include a brief discussion for completeness.

The equivalence between the two first assertions is [RR09, Proposition 2.4]. We use notation of Section 5.3 for Sp_{2n}. Consider the decomposition of $T_{P_r}\mathbb{G}_\omega(r,2n)$ as sum of irreducible L_r-modules. The centered tangent space of $\Omega_I(\mathbb{G}_\omega(r,2n))$ decomposes as the sum of those of $\Omega_{I_0}(\mathbb{G}(r,2n-r))$ and those of $\Omega_{I_2}(\mathbb{G}_\omega(r,2r))$. Since (I,J,K) is Levi-movable, one immediately deduces that (I_2,J_2,K_2) and (I_0,J_0,K_0) are. In particular, Lemma 6 implies that $d_{I_2J_2K_2}\neq 0$ and $c_{I_0\,J_0\,K_0}\neq 0$.

The fact that the last assertion implies the second one is the difficult part of [BK10, Theorem 30]. □

8.3 — Here comes our main result about cohomology of $\mathbb{G}_\omega(r,2n)$; it allows to characterize the condition $d^{\odot_0}_{IJK}=1$ in terms of the Littlewood-Richardson coefficients.

Theorem 19 *Let I, J, $K\in\mathcal{S}(\mathbb{G}_\omega(r,2n))$ such that $|\Lambda_I|+|\Lambda_J|+|\Lambda_K|=\dim\mathbb{G}_\omega(r,2n)$. Notations I_0,J_0,K_0,I_2,J_2 and K_2 refer to the map (23). The following are equivalent:*

- *(i) $d^{\odot_0}_{IJK}=1$;*
- *(ii) $d_{I_2J_2K_2}=1$ and $c_{I_0\,J_0\,K_0}=1$;*
- *(iii) $c_{I_2J_2K_2}=1$ and $c_{I_0\,J_0\,K_0}=1$.*

Proof. We first prove that assertion (i) implies assertion (ii). Proposition 7 implies that $d_{I_2J_2K_2}\neq 0$ and $c_{I_0\,J_0\,K_0}\neq 0$. Now, by Corollary 1, it is sufficient to prove that the two faces $\mathcal{F}_2$ and $\mathcal{F}_0$ of $\mathcal{LR}(\mathrm{Sp}_{2r})$ and $\mathcal{LR}(\mathrm{GL}_{2n-r})$ corresponding to these coefficients intersect the strictly dominant chambers. We are going to prove this by constructing explicitly matrices whose the corresponding spectrum yield points of $\mathcal{F}_2$ and $\mathcal{F}_0$. The starting point is that the assumption $d^{\odot_0}_{IJK}=1$ yields matrices whose spectrum have certain properties.

We first make more explicit the description of the face $\mathcal{F}(r,I,J,K)$ of $\mathcal{LR}(\mathrm{Sp}_{2n})$ associated to $d^{\odot_0}_{IJK}=1$ as in Theorem 16. By Theorem 16, there exists a point (λ,μ,ν) in $\mathcal{F}(r,I,J,K)$ such that λ, μ and ν are strictly dominant. Let us use notation of Section 5.3 for the data associated to the group Sp_{2n}. We write $\lambda=\sum\lambda_i\varepsilon_i\in X(T)$, we recall that $\bar{i}=2n+1-i$ and we set $\lambda_{\bar{i}}=-\lambda_i$ for $i\in[1,n]$. We use similar notation for μ and ν. A direct computation shows the linear equation of $\mathcal{F}(r,I,J,K)$ is

$$\sum_{i\in I}\lambda_i+\sum_{j\in J}\mu_j+\sum_{k\in K}\nu_k=0. \tag{19}$$

Let us consider the isomorphism ρ_{IJK} of $X(T)^3$ defined in Paragraph 7.1.4. By Theorem 17, $\rho_{IJK}(\mathcal{F}(r,I,J,K))\subset\mathcal{LR}(L_r)$. Using Theorem 18, we now identify $\mathcal{LR}(L_r)$ with $\Gamma(L_r \cap U_{2n}(\mathbb{C}))$. We now want to describe a point in $\Gamma(L_r \cap U_{2n}(\mathbb{C}))$ corresponding to $\rho_{IJK}(\lambda, \mu, \nu)$.

The elements of $Lie(L_r)$ have the following form:

$$A = \left(\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & A_2 & 0 \\ \hline 0 & 0 & -(J_r\, {}^t A_1\, J_r) \end{array} \right), \tag{20}$$

where $A_1 \in Lie(\mathrm{GL}_r)$ and $A_2 \in Lie(\mathrm{Sp}_{2(n-r)})$ and J_r is defined by formula (13) (with r in place of n). Let $\mathfrak{t}_r^+$ be the dominant (relatively to $B \cap L_r$) chamber of the Cartan subalgebra $\mathfrak{t}$ of L_r. Note that $\mathfrak{t}_r^+$ is the set of diagonal real matrices $\mathrm{diag}(\alpha_1, \cdots, \alpha_n, -\alpha_n, \cdots, -\alpha_1)$ such that $\alpha_1 \geq \cdots \geq \alpha_r$ and $\alpha_{r+1} \geq \cdots \geq \alpha_n$. Recall that we have an homeomorphism $\pi : \mathfrak{t}_r^+ \longrightarrow Lie(L_r \cap U_{2n}(\mathbb{C}))/(L_r \cap U_{2n}(\mathbb{C}))$. Let $A \in Lie(L_r \cap U_{2n})$ as in formula (20). By juxtaposition of the spectrums of $\sqrt{-1}A_1$, $\sqrt{-1}A_2$ and $-\sqrt{-1}(J_r\, {}^t A_1\, J_r)$ (each one in non-increasing order), we obtain a point $\xi(A)$ in $\mathfrak{t}_r^+ \subset \mathbb{R}^{2n}$. Note that $\pi(\xi(A)) = (L_r \cap U_{2n}(\mathbb{C})).A$.

Fix A, B and C in $Lie(L_r \cap U_{2n})$ such that

$$A + B + C = 0,$$

and $(\xi(A), \xi(B), \xi(C))$ is the point of $\mathcal{LR}(L_r)$ corresponding to $\rho_{IJK}(\lambda, \mu, \nu)$. The matrices A, B and C are as in formula (20) for some ; $A_1, B_1, C_1 \in u_r(\mathbb{C})$ and $A_2, B_2, C_2 \in u_{2(n-r)}(\mathbb{C}) \cap \mathrm{Lie}(\mathrm{Sp}_{2(n-r)})$.

Consider now the three following matrices of $\mathrm{Sp}_{2r} \cap U_{2r}$:

$$\bar{A} = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & -(J_r\, {}^t A_1\, J_r) \end{array} \right), \ \bar{B} = \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & -(J_r\, {}^t B_1\, J_r) \end{array} \right),$$

and,

$$\bar{C} = \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & -(J_r\, {}^t C_1\, J_r) \end{array} \right).$$

Obviously, $\bar{A} + \bar{B} + \bar{C} = 0$ and the spectrum of these matrices yield a point of $\Gamma(\mathrm{Sp}_{2r} \cap U_{2r})$. We claim that the corresponding point (by Theorem 18) is regular and belongs to $\mathcal{F}_2$.

Let $\alpha = (\alpha_1, \cdots, \alpha_n, -\alpha_n, \cdots, -\alpha_1)$ be the spectrum of $\sqrt{-1}A$; it satisfies $\alpha_1 > \cdots > \alpha_r$ and $\alpha_{r+1} > \cdots > \alpha_n$. Recall that $w_I \in S_{2n}$. Moreover, $w_I\alpha$ corresponds to λ and so is strictly dominant. Consider now, $w_{I_2} \in S_{2r}$. Since λ is dominant, so is its restriction $\bar{\lambda}$ to $T \cap Sp(2r)$. So, the coordinates of $w_{I_2}(\alpha_1, \cdots, \alpha_r, -\alpha_r, \cdots, \alpha_1)$ form a decreasing sequence. This implies that

$$\sum_{i \in I_2} (\bar{\lambda})_i = \sum_{i=1}^{r} \alpha_i = tr(A_1).$$

Similarly, we have

$$\sum_{i \in J_2} (\bar{\mu})_i = tr(B_1) \text{ and } \sum_{i \in K_2} (\bar{\nu})_i = tr(C_1),$$

with obvious notation. Now, the relation $\bar{A} + \bar{B} + \bar{C} = 0$ implies that

$$\sum_{i \in I_2} (\bar{\lambda})_i + \sum_{i \in J_2} (\bar{\mu})_i + \sum_{i \in K_2} (\bar{\nu})_i = 0.$$

So, $(\bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a regular point in $\mathcal{F}_2$ and Corollary 1 implies that $d_{I_2 J_2 K_2} = 1$.

In a similar way,

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) + \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right) + \left(\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_2 \end{array}\right) = 0,$$

provides a regular point in $\mathcal{F}_0$. So, Corollary 1 implies that $c_{I_0 J_0 K_0} = 1$.

We now prove that assertion (ii) implies assertion (iii). This implication is only concerned about $\mathbb{G}(r, 2r)$ and $\mathbb{G}_\omega(r, 2r)$: we may assume that $r = n$. Let us assume that $d_{IJK} = d_{IJK}^{\odot_0} = 1$. By [BK10, Corollary 11], the following product in $\mathrm{H}^*(\mathbb{G}(n, 2n))$ is nonzero:

$$\sigma_I(\mathbb{G}(n, 2n)).\sigma_J(\mathbb{G}(n, 2n)).\sigma_K(\mathbb{G}(n, 2n)) \neq 0.$$

Now, by Corollary 1 it is sufficient to prove that the face $\mathcal{F}^A$ of $\mathcal{L}R(\mathrm{SL}_{2n})$ corresponding to (I, J, K) contains regular points. Let $\mathcal{F}^C$ be the face of $\mathcal{L}R(\mathrm{Sp}_{2n})$ corresponding to $d_{IJK} = 1$. By Theorems 16 and 18, there exist $A, B, C \in u_n(\mathbb{C})$ such that

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & -J_n{}^t A J_n \end{array}\right) + \left(\begin{array}{c|c} B & 0 \\ \hline 0 & -J_n{}^t B J_n \end{array}\right) + \left(\begin{array}{c|c} C & 0 \\ \hline 0 & -J_n{}^t C J_n \end{array}\right) = 0,$$

and the spectrum (α, β, γ) of these three matrices give a regular point in $\mathcal{F}^C$. Since

$$\mathrm{tr}(A) + \mathrm{tr}(B) + \mathrm{tr}(C) = \sum_I \alpha_i + \sum_J \beta_i + \sum_K \gamma_i = 0,$$

we just obtained a regular point in $\mathcal{F}^A$.

Still assuming that $r=n$, we now want to prove that assertion (iii) implies assertion (ii). Consider the inclusion of $\mathbb{G}_\omega(n,2n)$ in $\mathbb{G}(n,2n)$. Let $\Omega_I(\mathbb{G}(n,2n))$, $\Omega_J(\mathbb{G}(n,2n))$ and $\Omega_K(\mathbb{G}(n,2n))$ be the three Schubert varieties of $\mathbb{G}(n,2n)$ corresponding to I, J and K and the standard flag in the basis of Paragraph 5.3.1. Since $c_{IJK}=1$, [Sot10, Theorem 2] implies that for general g, g' and g'' in Sp_{2n} the intersection $g\Omega_I(\mathbb{G}(n,2n))\cap g'\Omega_J(\mathbb{G}(n,2n))\cap g''\Omega_K(\mathbb{G}(n,2n))$ is transverse and reduced to one point F. Let us consider the orthogonal $F^{\perp_\omega}$ of F for ω. Since $g\in\mathrm{Sp}_{2n}$, $F^{\perp_\omega}$ belongs to $g\Omega_I(\mathbb{G}(n,2n))$; and finally to the intersection. We deduce that $F=F^{\perp_\omega}$ belongs to $\mathbb{G}_\omega(n,2n)$. So, the intersection $g\Omega_I(\mathbb{G}_\omega(n,2n))\cap g'\Omega_J(\mathbb{G}_\omega(n,2n))\cap g''\Omega_K(\mathbb{G}_\omega(n,2n))$ is reduced to one point F for general g, g' and g'' in Sp_{2n}. We deduce that $d_{IJK}=1$.

It remains to prove that assertion (iii) implies assertion (i). By the preceding argue, assertion (ii) holds. Since $\mathbb{G}_\omega(r,2r)$ is cominuscule, we may assume that $r<n$. Now, Proposition 7 implies that $d_{IJK}\neq 0$. It remains to prove that the corresponding face $\mathcal{F}(r,I,J,K)$ of $\mathcal{LR}(\mathrm{Sp}_{2n})$ contains regular points. Let us consider the three Schubert classes $\sigma_{(I,\bar{I}^c)}(\mathcal{F}l_{2n}(r,2n-r))$, $\sigma_{(J,\bar{J}^c)}(\mathcal{F}l_{2n}(r,2n-r))$ and $\sigma_{(K,\bar{K}^c)}(\mathcal{F}l_{2n}(r,2n-r))$ of $H^*(\mathcal{F}l_{2n}(r,2n-r))$. Since $c_{I_2J_2K_2}\neq 0$ and $c_{I_0\,J_0\,K_0}\neq 0$, the triple $((I,\bar{I}^c),(J,\bar{J}^c),(K,\bar{K}^c))\in\mathcal{S}(\mathcal{F}l_{2n}(r,2n-r))$ is Levi-movable. Let d be the positive integer such that

$$\sigma_{(I,\bar{I}^c)}(\mathcal{F}l_{2n}(r,2n-r))\odot_0\sigma_{(J,\bar{J}^c)}(\mathcal{F}l_{2n}(r,2n-r))\odot_0\sigma_{(K,\bar{K}^c)}(\mathcal{F}l_{2n}(r,2n-r))=d[\mathrm{pt}].$$

By [Ric09] (see also [Ric08] or [Res08a]), d is the product of $c_{I_0\,J_0\,K_0}$ and another Littlewood-Richardson coefficient c. The fact that $c_{I_0\,J_0\,K_0}=1$ allows to apply Theorem 12 to c: $c=c_{I_2J_2K_2}.c_{I_0\,J_0\,K_0}$. We deduce that $d=1$.

By [Res10b], by saturating the two inequalities φ_{IJK} and $\varphi_{\overline{I}^c\,\overline{J}^c\,\overline{K}^c}$, one obtains a face $\mathcal{F}$ of $\mathcal{LR}(\mathrm{SL}_{2n})$ intersecting the strictly dominant chamber and of codimension two.

Let T^A be the diagonal maximal torus of SL_{2n}. Let θ be the $\mathbb{Z}$-linear involution of $X(T^A)$ mapping ε_i on $-\varepsilon_{2n+1-i}$, with notation of Paragraph 5.2.1. Since θ corresponds to duality for representations, $\mathcal{LR}(\mathrm{SL}_{2n})$ is stable by the automorphism (θ,θ,θ) of $X(T^A)^3\otimes\mathbb{Q}$. Note that the character group of the maximal torus of Sp_{2n} defined in Paragraph 5.3.1 identifies by restriction with the set of θ-fixed points in $X(T^A)$. Moreover, by [BK10, Theorem 1], $\mathcal{LR}(\mathrm{Sp}_{2n})$ is precisely the set of points in $\mathcal{LR}(\mathrm{Sl}_{2n})$ fixed by (θ,θ,θ).

Since $\varphi_{IJK} \circ (\theta, \theta, \theta) = \varphi_{\overline{I}^c \overline{J}^c \overline{K}^c}$, $\mathcal{F}$ is stable by (θ, θ, θ). By convexity $\mathcal{F}$ contains regular θ-fixed points. We deduce using [BK10, Theorem 1], that $\mathcal{F}(r, I, J, K)$ contains regular points. □

8.1 Examples

We now give some examples performed with the Anders Buch's quantum calculator [Buc].

8.1.1 — Several multiplicative formulas for structure constants of $\odot_0$ are known (see [Ric08, Ric09, Res08a, KP10]). The formula $d^{\odot_0}_{IJK} = d_{I_2 J_2 K_2}.c_{I_0\, J_0\, K_0}$ could explain Theorem 19. Unfortunately, this last formula is not satisfied:

if $r = 3$, $n = 5$ and $I = J = K = \{3,\ 7,\ 10\}$ then $d^{\odot_0}_{IJK} = 2$, $d_{I_2 J_2 K_2} = 2$ and $c_{I_0\, J_0\, K_0} = 2$.

8.1.2 — We now consider $\mathbb{G}_\omega(n, 2n)$ and observe relations between d_{IJK} and c_{IJK} for $I, J, K \in \mathcal{S}(\mathbb{G}_\omega(n, 2n)) \subset \mathcal{P}(n, 2n) = \mathcal{S}(\mathbb{G}(n, 2n))$. Since $\mathbb{G}_\omega(n, 2n)$ and $\mathbb{G}(n, 2n)$ are cominuscule, the Belkale-Kumar product and the ordinary one coincide here. Let δ_I denote the number of diagonal elements in $\Lambda_I(\mathbb{G}_\omega(n, 2n))$. Theorem 19 shows that

$$d_{IJK} = 1 \iff c_{IJK} = 1.$$

Assume that $d_{IJK} = 1$. The fact that c_{IJK} is nonzero implies that the sum of the codimensions of the three corresponding Schubert varieties of $\mathbb{G}(n, 2n)$ equals the dimension of $\mathbb{G}(n, 2n)$. One can easily check that this means that $\delta_I + \delta_J + \delta_K = n$. The following example shows that this is not true if d_{IJK} is only assumed to be nonzero:

Set $n = 4$, $I = \{1,\ 2,\ 4,\ 6\}$ and $J = K = \{4,\ 6,\ 7,\ 8\}$. Then $d_{IJK} = 2$ and $\delta_I + \delta_J + \delta_K = 3 + 1 + 1 = 5$. In particular, $c_{IJK} = 0$.

8.1.3 — For I, J, K in $\mathcal{S}(\mathbb{G}_\omega(n, 2n))$ such that $c_{IJK} = 1$, we obviously have $\delta_I + \delta_J + \delta_K = n$. The following example shows that this is not true if c_{IJK} is only assumed to be nonzero.

Set $n = 4$, $I = J = \{2,\ 4,\ 6,\ 8\}$ and $K = \{3,\ 4,\ 7,\ 8\}$. Then $c_{IJK} = 2$ and $\delta_I + \delta_J + \delta_K = 6$. In particular, $d_{IJK} = 0$.

8.1.4 — We now assume that $\delta_I + \delta_J + \delta_K = n$ and $|\Lambda_I(\mathbb{G}_\omega(n, 2n))| + |\Lambda_J(\mathbb{G}_\omega(n, 2n))| + |\Lambda_K(\mathbb{G}_\omega(n, 2n))| = \frac{n(n+1)}{2}$. The Belkale-Kumar-Sottile theorem (see [Sot10, Theorem 2]) implies that

$$c_{IJK} \geq d_{IJK} \text{ and}$$
$$c_{IJK} - d_{IJK} \text{ is even.}$$

We already noticed that c_{IJK} and d_{IJK} can be different for dimension reasons. The following example shows that they can be different for other reasons.

Set $n=5$, $I=J=\{2,\ 4,\ 6,\ 8,\ 10\}$ and $K=\{3,\ 6,\ 7,\ 9,\ 10\}$. Then

$$d_{IJK}=4 \text{ and } c_{IJK}=6.$$

9 About the cohomology of $\mathbb{G}_Q(r,2n+1)$

This section is concerned by coefficient structures of the cohomology of ordinary and orthogonal Grassmanians. To avoid any confusion, those concerning ordinary and isotropic Grassmanians will be denoted with c and e respectively.

With notation of Section 5.4, consider the injective maps

$$\begin{array}{ccc} \mathcal{S}(\mathbb{G}_Q(r,2n+1)) & \hookrightarrow & \mathcal{S}(\mathcal{F}l_{2n+1}(r,2n+1-r)). \\ I & \longmapsto & (I,\bar{I}^c), \end{array} \tag{21}$$

and

$$\begin{array}{ccc} \mathcal{S}(\mathcal{F}l_{2n+1}(r,2n+1-r)) & \hookrightarrow & \mathcal{S}(\mathbb{G}(r,2n+1-r))\times\mathcal{S}(\mathbb{G}(2(n-r)+1,2n+1-r))\times\mathcal{S}(\mathbb{G}(r,2r)). \\ (I^1,I^2) & \longmapsto & (I_0,I_1,I_2) \end{array} \tag{22}$$

By composing these two injective maps and then forgetting I_1, we obtain an injective map

$$\begin{array}{ccc} \mathcal{S}(\mathbb{G}_Q(r,2n+1)) & \hookrightarrow & \mathcal{S}(\mathbb{G}(r,2n+1-r))\times\mathcal{S}(\mathbb{G}(r,2r)). \\ I & \longmapsto & (I_0,I_2) \end{array} \tag{23}$$

The aim of this section is to prove that this immersion is relevant relatively to the Belkale-Kumar product. We will also use the following particular case of the construction in Paragraph 5.3.4:

$$\begin{array}{ccc} \mathcal{S}(\mathbb{G}_\omega(r,2r)) & \hookrightarrow & \mathcal{S}(\mathbb{G}(r,2r)). \\ I & \longmapsto & I=I_2 \end{array} \tag{24}$$

9.1 — The following is [BK10, Theorem 41] :

Proposition 8 *Let I, J, $K\in\mathcal{S}(\mathbb{G}_Q(r,2n+1))$ such that $|\Lambda_I|+|\Lambda_J|+|\Lambda_K|=\dim\mathbb{G}_Q(r,2n+1)$. The following are equivalent:*

(i) $e^{\odot_0}_{IJK} \neq 0$;

(ii) $|\Lambda_{I_0}| + |\Lambda_{J_0}| + |\Lambda_{K_0}| = r(2n+1-2r)$ *and* $e_{IJK} \neq 0$;

(iii) $e_{I_2 J_2 K_2} \neq 0$ *and* $c_{I_0\,J_0\,K_0} \neq 0$.

9.2 — Here comes our main result about cohomology of $\mathbb{G}_Q(r, 2n+1)$; it allows to characterize the condition $e^{\odot_0}_{IJK} = 1$ in terms of the Littlewood-Richardson coefficients.

Theorem 20 *Let* I, J, $K \in \mathcal{S}(\mathbb{G}_Q(r, 2n+1))$ *such that* $|\Lambda_I| + |\Lambda_J| + |\Lambda_K| = \dim \mathbb{G}_Q(r, 2n+1)$. *The following are equivalent:*

(i) $e^{\odot_0}_{IJK} = 1$;

(ii) $e_{I_2 J_2 K_2} = 1$ *and* $c_{I_0\,J_0\,K_0} = 1$;

(iii) $c_{I_2 p'_2 K_2} = 1$ *and* $c_{I_0\,J_0\,K_0} = 1$.

Proof. The proof which is similar to those of Theorem 19 is left to the reader. □

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