An example of a thick wall

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Among quotients associated to distinct G-linearized line bundles, those corresponding to chambers have a very good property: the fibers are orbits. Theorem 4.2.7 shows that between two relevant chambers the quotient is changed by a transformation similar to a Mori flip. Moreover, if G is a torus, then two quotients corresponding to chambers are linked by a finite sequence of such transformations. In this appendix, we show by an example that this can fail for arbitrary reductive group G. For this, we produce a linear action of G on a projective space, which admits a proper wall of codimension zero.

Let us fix some notation. We consider the connected reductive group $G = \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$. Let χ_0 be the character of G defined by $\chi_0(t,g) = t$. Then χ_0 generates the character group of G. If T_1 is the maximal torus of $\mathrm{SL}(2,\mathbb{C})$ consisting in diagonal matrices, then $T = \mathbb{C}^* \times T_1$ is a maximal torus of G. Its character group is freely generated by χ_0 comma before and χ_1 defined by the following formula:

$$\chi_1\left(t, \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\right) = u, \qquad t, u \in \mathbb{C}^*.$$

Let $W = \mathbb{C}^2$, $V = \mathbb{C}^8$. Let us choose an isomorphism $V \simeq \mathbb{C} \oplus \mathbb{C} \oplus W \oplus W \oplus W$. An element of V is thus represented by a 5-tuple $(x_-, x_0, v_-, v_0, v_+)$ where $x_-, x_0 \in \mathbb{C}$ and $v_-, v_0, v_+ \in W$. We define an action of G on V by the following formula:

$$(t,g) \star (x_-, x_0, v_-, v_0, v_+) = \left(t^{-2}x_-, x_0, t^{-8} g \cdot v_-, g \cdot v_0, t^2 g \cdot v_+\right) \qquad (1)$$

where $\cdot$ is the canonical action of $\mathrm{SL}(2,\mathbb{C})$ on W. From now on, we use the notation of Section 1.1.5.

We represent the set of weights of the action of T on V by Figure 1. The coordinates in the basis (χ_0, χ_1) of these weights are denoted by (a, b) in the figure. In addition, the convex hulls of some parts of $\mathrm{st}(V)$ are drawn with thick lines.

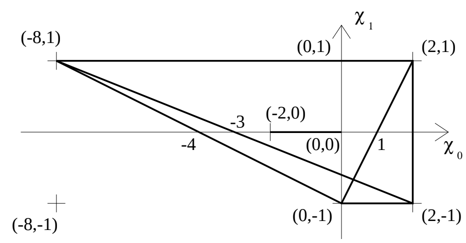

Figure 1: State of V.

Formula 1 defines an action of G on $X = \mathbb{P}(V)$ and a G-linearization on $\mathcal{O}_X(1)$ as well; we denote by $\mathcal{L}$ this G-linearized line bundle. According to Section 1.1.5, for $\mathcal{L}$, a point $x \in X$ is:

- semi-stable if and only if for all $g \in G$, the origin belongs to the set $\mathrm{Conv}(\mathrm{st}_V(g.x))$;
- stable if and only if for all $g \in G$, the origin belongs to the interior of $\mathrm{Conv}(\mathrm{st}_V(g.x))$;
- unstable if and only if there exists $g \in G$ such that the origin does not belong to $\mathrm{Conv}(\mathrm{st}_V(g.x))$.

Now we want to vary the ample G-linearized line bundle on X. We also denote by χ_0 the trivial line bundle over X where G acts on the fibers by χ_0. Since the group $\mathrm{NS}(X)$ is isomorphic to $\mathbb{Z}$, by [**KKV**] each G-linearized line bundle on X is isomorphic to $\mathcal{L}^{\otimes n} \otimes m\chi_0$ for some $(m, n) \in \mathbb{Z}^2$. It follows that the group $\mathrm{NS}^G(X)$ is isomorphic to $\mathbb{Z}^2$. From now on, we identify $\mathrm{NS}^G(X)$ with $\mathbb{Z}^2$, and so $\mathrm{NS}^G(X)_{\mathbb{R}}$ with $\mathbb{R}^2$. Note that the line bundle corresponding to $(m, n) \in \mathbb{Z}^2$ is ample if and only if n is positive. Since two ample G-linearized line bundles on the same half-line from the origin are GIT-equivalent, we can restrict our study to the points of $\mathrm{NS}^G(X)_{\mathbb{R}}$ of the form $(r, 1)$ with $r \in \mathbb{R}$. We call the set of these points *the horizontal line* and r the *abscissa* of the point $(r, 1)$. We use these conventions in Figure 2.

Let $r \in \mathbb{Q}$. There exists a power, say $\mathcal{L}^{\otimes n} \otimes m\chi_0$ (with $m = nr \in \mathbb{Z}$) of $\mathcal{L} \otimes r\chi_0$ which is the restriction (as a G-line bundle) of $\mathcal{O}(1)$ for an embedding of X into a G-module. The sets $\mathrm{st}(x)$ corresponding to this embedding are obtained from $\mathrm{st}_V(x)$ by applying a dilation of factor n followed by a translation of vector $(m, 0)$. So to study the stability for $\mathcal{L} \otimes r\chi_0$, we can move the origin along the horizontal line in Figure 1 by $-r$ and keep the weights of the action of V. Finally the stability for $\mathcal{L} \otimes r\chi_0$ of a point $x \in X$ depends on the relative position of the point $(-r, 0)$ and the convex hulls in $\mathcal{X}(T) \otimes \mathbb{R}$ of the sets $\mathrm{st}_V(g.x)$ with $g \in G$.

From now on, we denote by (e_1, e_2) the canonical basis of W. Let $x \in X$ and let $\tilde{x} = (x_-, x_0, v_-, v_0, v_+)$ be a representative of x in V. There exists $g \in \mathrm{SL}(2, \mathbb{C})$ such that $g.v_-$ is proportional to e_1. But now, if $r > 4$ the point $(-r, 0)$ does not belong to the convex hull of $\mathrm{st}(g.x)$ and x is not semi-stable for $\mathcal{L} \otimes r\chi_0$. So if $r > 4$, $X^{\mathrm{ss}}(\mathcal{L} \otimes r\chi_0)$ is empty. Analogously, we prove that if $r < -1$ then $\mathcal{L} \otimes r\chi_0$ is not effective.

Moreover, the "origins" of the form $(-r, 0)$ in Figure 1 which correspond to the intersection of the horizontal line and a wall belong to the boundary of a set $\mathrm{conv}(\mathrm{st}(x))$ for some $x \in X$. So the abscissa of the intersection of a wall and the horizontal line is $r = 4$, $r = 3$, $r = 2$, $r = 0$, $r = -1$ or the segment $0 \leq r \leq 2$.

Let $x = [0 : 0 : e_1 : e_2 : 0]$. There are seven distinct sets of the form $\mathrm{st}(g.x)$: two segments, four triangles and one rectangle. The point $(-4, 0)$ is either on the boundary or in the interior of these convex sets. So, $r = 4$ is the abscissa of the wall $H(x)$. In the same way, we show that $r = 3$ is the wall $H\left([0 : 0 : e_1 : 0 : e_2]\right)$ and $r = -1$ is the wall $H\left([0 : 0 : 0 : e_1 : e_2]\right)$.

Obviously, the walls $H\left([1 : 0 : 0 : 0 : 0]\right)$ and $H\left([0 : 1 : 0 : 0 : 0]\right)$ have $r = 2$ and $r = 0$ as their abscissa. Moreover, the intersection of the horizontal line and the wall $H\left([1 : 1 : 0 : 0 : 0]\right)$ is the interval $0 \leq r \leq 2$.

So we obtain six walls, three chambers and six cells in the G-ample cone (see Figure 2). The cone $\mathcal{C}^G(X)$ is partitioned into nine GIT-classes.

Theorem 4.2.7 compares quotients corresponding to two chambers C^+ and C^- relevant to a cell F. The starting point is that the set $X^{\mathrm{ss}}(F)$ contains both $X^{\mathrm{s}}(C^+)$ and $X^{\mathrm{s}}(C^-)$, and so defines two morphisms:

$$X^{\mathrm{ss}}(C^+)//G \overset{f_+}{\longrightarrow} X^{\mathrm{ss}}(F)//G \overset{f_-}{\longleftarrow} X^{\mathrm{ss}}(C^-)//G.$$

In the G-ample cone, the property $X^{\mathrm{ss}}(F) \supset X^{\mathrm{ss}}(C)$ means that F intersects the closure of C. Moreover, if we want to have $X^{\mathrm{s}}(F) = X^{\mathrm{s}}(C^+) \cap X^{\mathrm{s}}(C^-)$ it is natural to assume that C^+ and C^- are relevant to F. These explains why Theorem 4.2.7 concerns two relevant chambers to a face.

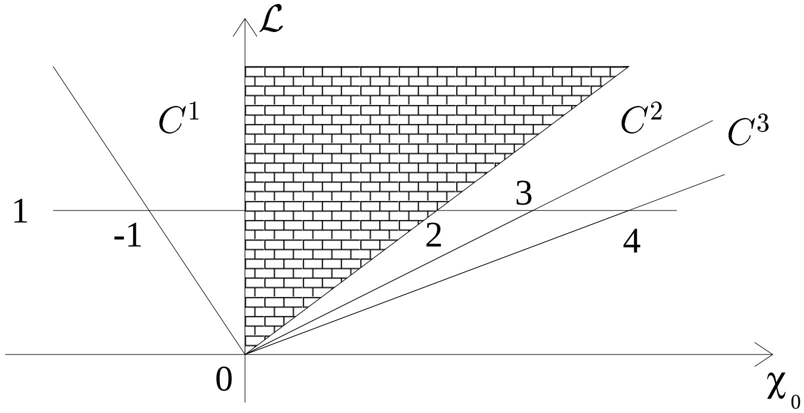

Figure 2: The G-ample cone.

On the other hand, if there is no codimension zero wall, then any two chambers can be joined by a chain of relevant chambers. So quotients corresponding to two arbitrary chambers are related by a sequence of birational transformations corresponding to relevant chambers.

Back to the example, if we want to relate the quotients associated to C^1 and C^2, we must look at the sequence of transformations:

$$\begin{array}{ccccccccc} X^{ss}(C^1)//G & & & & X^{ss}(\mathcal{L}\otimes\chi_0)//G & & & & X^{ss}(C^2)//G \\ & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & X^{ss}(\mathcal{L})//G & & & & X^{ss}(\mathcal{L}\otimes 2\chi_0)//G & & \end{array}$$

And so, we obtain $X^{\mathrm{ss}}(\mathcal{L}\otimes\chi_0)//G$ as a natural intervening quotient between $X^{\mathrm{ss}}(C^1)//G$ and $X^{\mathrm{ss}}(C^2)//G$.

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