Balanced configurations of $2n+1$ plane vectors

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Abstract. A configuration $\{v_1, v_2, \cdots, v_m\}$ (where m is a positive integer) of vectors of $\mathbb{R}^2$ is said to be uniform and balanced if for any index $i \in \{1, \cdots, m\}$ the set with multiplicities

$$\mathcal{D}_i = \{\det(v_i, v_j)\ :\ j \neq i\}$$

is symmetric around the origin and does not contain it. Solving a conjecture of E. Cattani and A. Dickenstein, we prove that the linear group $\mathrm{GL}_2(\mathbb{R})$ acts transitively on the set of uniform and balanced configurations of m vectors.

Keywords: Balanced configuration, polygon linearly regular, linear algebra

1. Introduction

A configuration $\{v_1, v_2, \cdots, v_m\}$ of vectors of $\mathbb{R}^2$ is said to be *balanced* if for any index $i \in \{1, \cdots, m\}$ the set with multiplicities

$$\{\det(v_i, v_j)\ :\ j \neq i\}$$

is symmetric around the origin. It is said to be *uniform* if every pair of vectors is linearly independent.

E. Cattani, A. Dickenstein and B. Sturmfels introduced the notion of balanced configuration in (CDS99; CD02) for its relationship with multivariable hypergeometric functions in the sense of Gel'fand, Kapranov and Zelevinsky (see (GKZ89; GKZ90)). These functions include, as particular example, the classical Gauss hypergeometric functions, as well as the multivariable generalizations of Appell, Horn, and Lauricella (see (EMOT81)).

Balanced planar configurations with at most six vectors have been classified in (CDS99). In a previous version of (CD02), E. Cattani and A. Dickenstein classified, with the help of a computer calculation, the balanced planar configurations of seven vectors. Moreover, they conjectured that any uniform balanced planar configuration is $\mathrm{GL}_2(\mathbb{R})$-equivalent to a regular $(2n+1)$-gon (where n is a positive integer). In this note, we prove this conjecture for all n. In (CD02), our method is adapted to the case $n = 3$.

In Section 2, we precisely state our result. The following section is the proof. To obtain a sketch of proof, one can read the begining of Paragraphs 3.1 to 3.6.

2. Statement of the result

We start with giving a precise definition of balanced configurations:

Definition. A planar configuration $\{v_1, \cdots, v_m\}$ is said to be *balanced* if for all $i = 1, \cdots, m$ and for all x in $\mathbb{R}$ the cardinality of the set $\{j \neq i \ : \ \det(v_i, v_j) = x\}$ equals those of the set $\{j \neq i \ : \ \det(v_i, v_j) = -x\}$.

Remark. Assume $\{v_1, \cdots, v_m\}$ is balanced and m is even. Then, the set $\{\det(v_1, v_j) \ : \ j = 2, \cdots, m\}$ is symmetric around 0 and its cardinality (with multiplicities) is odd; so it contains 0. Then, $\{v_1, \cdots, v_m\}$ is not uniform. In this note, we are only interested in uniform and balanced configurations. So, from now on, we assume that $m = 2n + 1$ for some positive integer n.

Let us identify $\mathbb{R}^2$ with the field $\mathbb{C}$ of complex numbers. To avoid any confusion with index-numbers, we denote by $\sqrt{-1}$ the complex number i. Denote by $\mathbb{U}_m$ the set of m^{th}-roots of 1. Set $\omega = e^{\frac{2\sqrt{-1}\pi}{m}}$. Then, $\mathbb{U}_m = \{w^k \ : \ k = 0, \cdots 2n\}$ is uniform and balanced. Indeed, for all integers k and a, we have

$$\det(\omega^k, \omega^{k+a}) = -\det(\omega^k, \omega^{k-a}). \tag{1}$$

One can notice that the group $\mathrm{GL}_2(\mathbb{R})$ acts naturally on the set of balanced (resp. uniform balanced) configurations of m vectors. Indeed, if $g \in \mathrm{GL}_2(\mathbb{R})$ then $\det(g.v_i, g.v_j) = \det(g)\det(v_i, v_j)$. The aim of this note is to prove the

THEOREM 1. *For any odd integer m, $\mathrm{GL}_2(\mathbb{R})$ acts transitively on the set of uniform balanced configurations of m vectors.*

In other words, modulo $\mathrm{GL}_2(\mathbb{R})$, $\mathbb{U}_m$ is the only uniform balanced configuration of m vectors.

3. The proof

3.1 — The set $\{0, \cdots, 2n\}$ is denoted by I. We denote by $\mathcal{P}_2(I)$ the set of pairs of elements of I. Let us fix a uniform planar configuration $\mathcal{C} = \{v_0, \cdots, v_{2n}\}$.

We now explain the first step of the proof in the case when for all distinct i, k and l in I $\det(v_i, v_k) \neq \det(v_i, v_l)$. Then, one easily check that $\mathcal{C}$ is balanced if and only if

$$\forall i,\, k \in I \quad \exists l \in I \quad \det(v_i, v_k + v_l) = 0. \tag{2}$$

The default of Condition (2) is that the quantifiers does not respect the symetry in k and l of the equation $\det(v_i, v_k + v_l) = 0$. Then following lemma show that Condition (2) is equivalent to the better following one:

$$\forall k,\, l \in I \quad \exists i \in I \quad \det(v_i, v_k + v_l) = 0.$$

LEMMA 1. *Let us recall that $\mathcal{C}$ is uniform. Then, $\mathcal{C}$ is balanced if and only if there exists a map $\phi : \mathcal{P}_2(I) \longrightarrow I$ such that*

1. $\forall \{k,l\} \in \mathcal{P}_2(I) \qquad \det(v_{\phi(\{k,l\})}, v_k + v_l) = 0$, *and*
2. $\forall \{k,l\}, \{k,l'\} \in \mathcal{P}_2(I) \qquad (\phi(\{k,l\}) = \phi(\{k,l'\}) \Rightarrow l = l')$.

Proof. If E is a finite set, we denote its cardinality by $|E|$. Assume that there exists ϕ as in the lemma.

If there exists a pair $\{i,l\}$ such that $\phi(\{i,l\}) = i$, then $\det(v_i, v_l) = -\det(v_i, v_i) = 0$; this contradicts the fact that $\mathcal{C}$ is uniform. Therefore, for all $i \in I$, $\phi^{-1}(\{i\})$ is formed by pairs of elements of $I - \{i\}$. Moreover, by Assertion 2 these pairs are pairwise disjoint. Therefore, $|\phi^{-1}(\{i\})| \leq n$. But, $|\mathcal{P}_2(I)| = mn$ and $|I| = m$. It follows that for all $i \in I$, $|\phi^{-1}(\{i\})| = n$. One easily deduces that $\phi^{-1}(\{i\})$ is a partition of $I - \{i\}$. It follows that $\mathcal{C}$ is balanced.

Conversely, let us assume that $\mathcal{C}$ is balanced. Then, for all $i \in I$, there exists a (a priori non unique) part $\mathcal{P}_2^i(I)$ of $\mathcal{P}_2(I)$ such that:

- $I - \{i\}$ is the disjoint union of the elements of $\mathcal{P}_2^i(I)$, and
- $\forall \{k,l\} \in \mathcal{P}_2^i(I) \qquad \det(v_i, v_k + v_l) = 0$.

Let $\{k,l\} \in \mathcal{P}_2(I)$. Since $\mathcal{C}$ is uniform, the set of vectors $v \in \mathbb{R}^2$ such that $\det(v, v_k + v_l) = 0$ is the line generated by $v_k + v_l$ and there exists at most one $i \in I$ such that $\det(v_i, v_k + v_l) = 0$. This means that for any $i \neq j$ the set $\mathcal{P}_2^i(I) \cap \mathcal{P}_2^j(I)$ is empty.

Moreover, $|\mathcal{P}_2^i(I)| = n$, for all $i \in I$. Therefore, the cardinality of $\bigcup_{i \in I} \mathcal{P}_2^i(I)$ equals nm, which is the cardinality of $\mathcal{P}_2(I)$. It follows that

$$\bigcup_{i \in I} \mathcal{P}_2^i(I) = \mathcal{P}_2(I).$$

Therefore, for all $\{k,l\} \in \mathcal{P}_2(I)$, there exists a unique $\phi(\{k,l\}) \in I$ such that $\{k,l\} \in \mathcal{P}_2^{\phi(\{k,l\})}(I)$. Then, ϕ satisfies Condition 1. Condition 2 follows from the fact that the elements of $\mathcal{P}_2^i$ are pairwise disjoint.

Notation. From now on, we identify I from $\mathbb{Z}/m\mathbb{Z}$, by $k \mapsto k + m\mathbb{Z}$. Since m is odd, 2 is invertible in the ring $\mathbb{Z}/m\mathbb{Z}$. For example, with this identification, we have $v_{\frac{1}{2}} = v_{n+1}$ and $v_{\frac{-1}{2}} = v_n$.

3.2 — We assume that $\mathcal{C}$ is uniform and balanced. The second step of the proof is to show that $\mathcal{C}$ has the same combinatorics as $\mathbb{U}_m$. Precisely, we show that by relabeling the vectors, we can assume that $\mathcal{C}$ satisfies the equations $\det(v_k, v_{k+a} + v_{k-a}) = 0$ similar to Equalities (1).

Definition. Each v_k has a unique polar form $v_k = \rho_k e^{\alpha_k \sqrt{-1}}$ with ρ_k in $]0; +\infty[$ and α_k in $[0; 2\pi[$. The set $\mathcal{C}$ is said to be *labeled by increasing arguments* if there exists $k \in I$ such that

$$\alpha_k < \alpha_{k+1} < \cdots < \alpha_{2n} < \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1}.$$

LEMMA 2. *If $\mathcal{C}$ is labeled by increasing arguments then the unique map ϕ satisfying Lemma 1 is $\phi(\{k,l\}) = \frac{k+l}{2}$ for all $\{k,l\} \in \mathcal{P}_2(I)$.*

Proof. By relabeling the vectors, it is sufficient to prove that $\phi(\{0,k\}) = \frac{k}{2}$, for all $k = -n, \cdots, -1, 1, \cdots, n$. By symmetry, we may assume that $k = 1, \cdots, n$. Let us fix such a k.

Notice that the set of $i \in I$ such that $\det(v_0, v_i)$ is positive (that is, such that $\alpha_i - \alpha_0 < \pi$) is of cardinality n. Therefore, since $\mathcal{C}$ is labelled by increasing arguments $\alpha_n - \alpha_0 < \pi < \alpha_{n+1} - \alpha_0$. In the same way, we have: $\alpha_{n+k} - \alpha_k < \pi < \alpha_{n+k+1} - \alpha_k$ (see Figure 2).

With our convention , we have $\alpha_{\frac{1}{2}} = \alpha_{n+1}$ and $\alpha_{\frac{2k-1}{2}} = \alpha_{n+k}$. Since $v_{\phi(\{0,k\})}$ belongs to $\mathbb{R}(v_0 + v_k)$, we have:

$$\phi(\{0,k\}) \in \{\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \cdots, \frac{2k-1}{2}\}. \tag{3}$$

This ends the proof when $k = 1$. We now proceed by induction on k. Let us assume that $\phi(\{i, i+j\}) = i + \frac{j}{2}$, for all $j = 1, \cdots, k-1$.

By Lemma 1, $\phi(\{0,k\}) \neq \phi(\{0,j\})$, for all $j = 1, \cdots, k-1$. Therefore,

$$\phi(\{0,k\}) \notin \{\frac{1}{2}, \frac{2}{2}, \cdots, \frac{k-1}{2}\}. \tag{4}$$

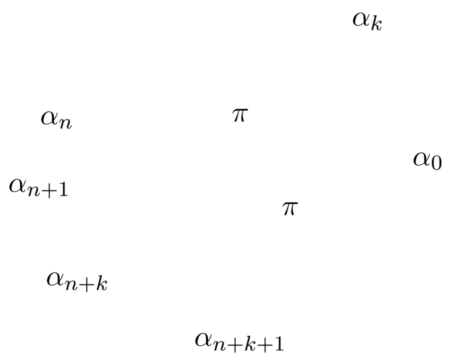

Figure 1. The positions of α_0, α_k, α_n, α_{n+1}, α_{n+k} and α_{n+k+1}

In the same way, $\phi(\{0, k\}) \neq \phi(\{k - j, k\})$, for all $j = 1, \cdots, k - 1$. Therefore,

$$\phi(\{0, k\}) \notin \{\frac{k+1}{2}, \frac{2}{2}, \cdots, \frac{2k-1}{2}\}. \tag{5}$$

The lemma follows from (3), (4) and (5).

We will say that $\mathcal{C}$ is *labeled in respect with Lemma 2* if the function $\phi : \mathcal{P}_2(I) \longrightarrow I = \mathbb{Z}/m\mathbb{Z}$, $\{k, l\} \longmapsto \frac{k+l}{2}$ satisfies Conditions 1 (and 2) of Lemma 1. Lemma 2 shows that if $\mathcal{C}$ is labeled by increasing arguments, then it is labeled in respect with Lemma 2.

3.3 — In this paragraph, we assume that $\mathcal{C}$ is labeled in respect with Lemma 2; and we give a method to construct $\mathcal{C}$ starting from v_n, v_0 and v_{-n}. Moreover, we have $\det(v_0, v_n + v_{-n}) = 0$. This leave us at most one degree of liberty to construct all uniform balanced configurations labeled in respect with Lemma 2 modulo the action of $\mathrm{GL}_2(\mathbb{R})$. Let us start with

LEMMA 3. *With above notation, for all $k \in I$ we have:*

$$\det(v_k, v_{k+1}) = \det(v_0, v_1) \quad \text{and} \quad \det(v_k, v_{k+\frac{1}{2}}) = \det(v_0, v_n).$$

Proof. For all $k \in I$, since $\phi(\{k, k+2\}) = k+1$, we have $\det(v_k, v_{k+1}) = \det(v_{k+1}, v_{k+2})$. The first assertion follows immediately. For all k, we also have $\det(v_k, v_{k+n}) = \det(v_{k+n}, v_{k+2n})$. Since n is prime with $m = 2n + 1$, this implies the second assertion.

Set $A_1 := \det(v_{\frac{-1}{2}}, v_{\frac{1}{2}})$ and $A_n := \det(v_0, v_{\frac{-1}{2}})$ (let us recall that $v_{\frac{-1}{2}} = v_n$).

LEMMA 4. *For all $k \in \mathbb{Z}/m\mathbb{Z}$, we have:*

$$v_{\frac{k}{2}} = -\frac{A_1}{A_n} v_{\frac{k-1}{2}} - v_{\frac{k-2}{2}}.$$

Proof. Since $\mathcal{C}$ is uniform there exist a and b in $\mathbb{R}$ such that $v_{\frac{k}{2}} = av_{\frac{k-1}{2}} + bv_{\frac{k-2}{2}}$. But, by Lemma 3, we have: $\det(v_{\frac{k-2}{2}}, v_{\frac{k-1}{2}}) = -A_n$, $\det(v_{\frac{k-2}{2}}, v_{\frac{k}{2}}) = A_1$ and $\det(v_{\frac{k}{2}}, v_{\frac{k-1}{2}}) = A_n$. One easily deduces that $a = -\frac{A_1}{A_n}$ and $b = -1$.

3.4 — Any uniform balanced configuration is $\mathrm{GL}_2(\mathbb{R})$-equivalent to one with $v_{\frac{-1}{2}} = (0,1)$ and $v_0 = (1,0)$. Then, since $\det(v_0, v_{\frac{-1}{2}} + v_{\frac{1}{2}}) = 0$, $v_{\frac{1}{2}} = (t, -1)$ for one t in $\mathbb{R}$. Moreover, by Lemma 4, we have:

$$v_{\frac{k}{2}} = tv_{\frac{k-1}{2}} - v_{\frac{k-2}{2}}.$$

Conversely, we define a sequence $(w_k(t))_{k \geq -1}$ of vector of $\mathbb{R}^2$ with a parameter $t \in \mathbb{R}$ as follows. We start with

$$w_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad w_1(t) = \begin{pmatrix} t \\ -1 \end{pmatrix},$$

and define $w_k(t)$ by induction:

$$\forall k \geq 2 \qquad\qquad w_k(t) = tw_{k-1}(t) - w_{k-2}(t). \tag{6}$$

Now, $\mathcal{C}$ is any uniform balanced configuration labeled in respect with Lemma 2. Formally, the above discussion gives

LEMMA 5. *There exits a unique $g_{\mathcal{C}} \in \mathrm{GL}_2(\mathbb{R})$ such that $g_{\mathcal{C}}.v_{\frac{k}{2}} = w_k$, for $k = -1, 0$ and a unique $t_{\mathcal{C}} \in \mathbb{R}$ such that $g_{\mathcal{C}}.v_{\frac{1}{2}} = w_1(t_{\mathcal{C}})$. Moreover, for all $k \geq -1$, we have:*

$$w_k(t_{\mathcal{C}}) = g_{\mathcal{C}}.v_{\frac{k}{2}}.$$

Proof. The existence and the uniqueness of $g_{\mathcal{C}}$ are obvious; those of $t_{\mathcal{C}}$ is a direct consequence of the equality $\det(v_0, v_{\frac{-1}{2}} + v_{\frac{1}{2}}) = 0$.

The configuration $\mathcal{C}' := \{g_{\mathcal{C}}.v_k\}_{k \in I}$ is uniform and balanced and labeled in respect with Lemma 2, since $\mathcal{C}$ be. Then, $\mathcal{C}'$ satisfies Lemma 4 with $A_1 = \det(g_{\mathcal{C}}.v_{\frac{-1}{2}}, g_{\mathcal{C}}.v_{\frac{1}{2}}) = -t$ and $A_n = \det(g_{\mathcal{C}}.v_0, g_{\mathcal{C}}.v_{\frac{-1}{2}}) = 1$. The lemma follows.

3.5 — To obtain all the $\mathrm{GL}_2(\mathbb{R})$-orbits of uniform balanced configurations, it remains to find the $t_{\mathcal{C}}$ which occur in Lemma 5. But, this lemma shows that $w_{2n+1}(t_{\mathcal{C}}) = w_0$. We will deduce that the parameter $t_{\mathcal{C}}$ must be choosen among at most n values:

LEMMA 6. *Denote by (x^*, y^*) the coordinate forms of $\mathbb{R}^2$.*

Then, for all $k \geq 0$, the function $x^*(w_k(t))$ (resp. $y^*(w_k(t))$) is a polynomial function of degree k (resp. $k-1$) and of same parity as k (resp. $k-1$). In particular, the equation $w_{2n+1}(t_{\mathcal{C}}) = w_0$ has at most n solutions.

Proof. One easily check the first assertion by induction on k. One can notice that $y^*(w_{2n+1}(0)) \neq 0$. Then, the equation $y^*(w_{2n+1}(t)) = 0$ has $2j$ solutions: $-t_j < \cdots < -t_1 < 0 < t_1 < \cdots < t_j$ with $j \leq n$. Since $x^*(w_{2n+1}(t))$ is an odd function, at most one element of a pair $\pm t_k$ is a solution of the equation $x^*(w_{2n+1}(t)) = 1$. This ends the proof of the lemma.

3.6 — To end the proof, we show that the equation $w_{2n+1}(t_{\mathcal{C}}) = w_0$ has indeed n solutions corresponding to various labeling of the regular d-gons where d runs over the divisors of $2n+1$.

LEMMA 7. *We have:*

$$\{t \in \mathbb{R} \ : \ w_{2n+1}(t) = w_0\} = \{\frac{1}{\sin(2k\pi/m)} \ : \ k = 1, \cdots n\}.$$

Proof. We are going to construct n balanced configurations labeled in respect with Lemma 2. Each one will give a solution of the equation $w_{2n+1}(t) = w_0$.

Consider $\mathbb{U}_m = \{\omega^k \ : \ k = 0, \cdots 2n\}$. Let us fix $k \in \{1, \cdots n\}$. Set $\zeta = \omega^k$ and denote by d the order of ζ in the group $\mathbb{U}_m$. Then, $\mathcal{C}_k := \{\zeta^l \ : \ l = 0, \cdots, d-1\}$ is a uniform balanced configuration. Moreover, $\mathcal{C}_k$ is labeled in respect with Lemma 2.

The integer d divides m and is odd. Let e be the positive integer such that $d = 2e+1$. Denote by g_k the unique element of $\mathrm{GL}_2(\mathbb{R})$ such that $g_k.\zeta^0 = w_0$ and $g_k.\zeta^e = w_{-1}$. Let t_k be the unique real number such that $g_k.\zeta^{-e} = w_1(t_k)$. One easily checks that $t_k = (\sin(2k\pi/m))^{-1}$.

But, by Lemma 5, we have for all $l \geq -1$, $w_l(t_k) = g_k.\zeta^{\frac{l}{2}}$. Since d divides $m = 2n+1$, we deduce that $w_{2n+1}(t_k) = w_0$.

3.7 —

Proof. [of Theorem 1] We may assume that $\mathcal{C} = \{v_0, \cdots, v_{2n}\}$ is labeled by increasing arguments. We define $g_{\mathcal{C}} \in \mathrm{GL}_2(\mathbb{R})$ and $t_{\mathcal{C}} \in \mathbb{R}$ as in Lemma 5. Then, by Lemma 7, there exists a unique $k_{\mathcal{C}} \in \{1, \cdots n\}$ such that $t_{\mathcal{C}} = (\sin(2k_{\mathcal{C}}\pi/m))^{-1}$. Set $\zeta = \omega^{k_{\mathcal{C}}}$. Let $g_{k_{\mathcal{C}}} \in \mathrm{GL}_2(\mathbb{R})$ defined as in the proof of Lemma 7. Then, by Lemma 5, for all $k = 0, \cdots, 2n$, we have $w_k(t_{\mathcal{C}}) = g_{\mathcal{C}}.v_{\frac{k}{2}} = g_{k_{\mathcal{C}}}\zeta^{\frac{k}{2}}$. Since $\mathcal{C}$ is uniform, it follows that the order of ζ in $\mathbb{U}_m$ is m. Theorem 1 is proved.

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