On the automorphisms of the Drinfel'd double of a Borel Lie subalgebra

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Abstract

Let $\mathfrak{g}$ be a complex simple Lie algebra with Borel subalgebra $\mathfrak{b}$. Consider the semidirect product $I\mathfrak{b} = \mathfrak{b} \ltimes \mathfrak{b}^*$, where the dual $\mathfrak{b}^*$ of $\mathfrak{b}$, is equipped with the coadjoint action of $\mathfrak{b}$ and is considered as an abelian ideal of $I\mathfrak{b}$. We describe the automorphism group $\mathrm{Aut}(I\mathfrak{b})$ of the Lie algebra $I\mathfrak{b}$. In particular we prove that it contains the automorphism group of the extended Dynkin diagram of $\mathfrak{g}$. In type A_n, the dihedral subgroup was recently proved to be contained in $\mathrm{Aut}(I\mathfrak{b})$ by Dror Bar-Natan and Roland Van Der Veen in [Bv20] (where $I\mathfrak{b}$ is denoted by $I\mathfrak{u}_n$). Their construction is handmade and they ask for an explanation: this note fully answers the question.

1 Introduction

Given any complex Lie algebra $\mathfrak{a}$, one can form its "inhomogeneous version" $I\mathfrak{a} := \mathfrak{a} \ltimes \mathfrak{a}^*$. It is the semidirect product of $\mathfrak{a}$ with its dual $\mathfrak{a}^*$ where $\mathfrak{a}^*$ is considered as an abelian ideal and $\mathfrak{a}$ acts on $\mathfrak{a}^*$ via the coadjoint action.

As mentioned in [Bv20], for applications in knot theory and representation theory, the most important case is when $\mathfrak{a} = \mathfrak{b}$ is the Borel subalgebra of some simple Lie algebra $\mathfrak{g}$. It is precisely the situation studied here. In addition to [Bv20], several examples of these algebras appear with variations in the litterature. In [NW93], Nappi-Wittney use the case when $\mathfrak{g} = \mathfrak{sl}_2$ in conformal field theory. Several authors also consider $\mathfrak{b} \ltimes \mathfrak{n}^*$ where $\mathfrak{n}$ is the derived subalgebra of $\mathfrak{b}$. It is the quotient of $I\mathfrak{b}$ by its center. In [KZJ07], Knutson and Zinn-Justin meet this algebra for $\mathfrak{g} = \mathfrak{gl}_n$ in the associative setting, see below. In [Fei12, Fei11], Feigin uses $\mathfrak{b} \ltimes \mathfrak{n}^*$ in order to study degenerate flag varieties for $\mathfrak{g} = \mathfrak{sl}_n$. For a general semisimple Lie algebra $\mathfrak{g}$, in [PY12], Panyushev and Yakimova study the invariants of $\mathfrak{b} \ltimes \mathfrak{n}^*$ under the action of their adjoint group. Finally, in [PY13, Pho20], similar considerations are studied replacing $\mathfrak{b}$ by an arbitrary parabolic subalgebra of $\mathfrak{g}$.

The aim of this note is to give a new interpretation of $I\mathfrak{b}$ in the language of Kac-Moody algebras and to completely describe the automorphism group of $I\mathfrak{b}$.

Before describing this group, we introduce some notation. Let r denote the rank of $\mathfrak{g}$ and G the adjoint group with Lie algebra $\mathfrak{g}$. Let B be the Borel subgroup of G with $\mathfrak{b}$ as

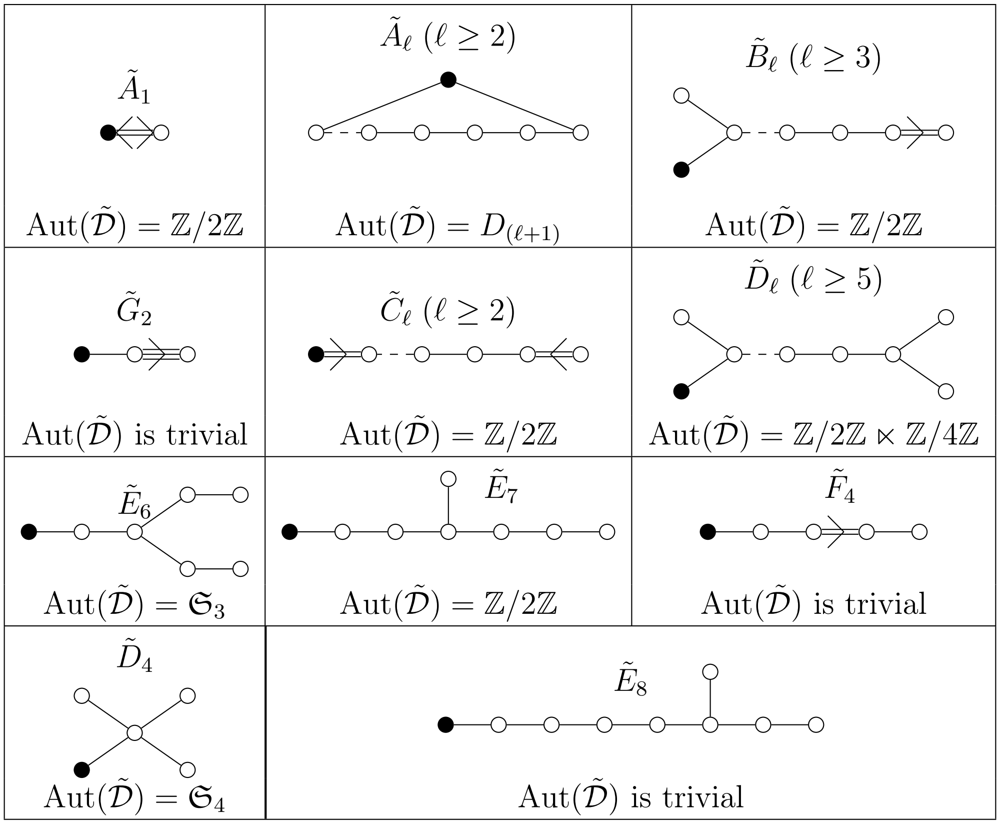

Figure 1: Extended Dynkin diagrams and their automorphisms

Lie algebra. Consider two abelian additive groups: the quotient $\mathfrak{g}/\mathfrak{b}$ and the space $\mathcal{M}_r(\mathbb{C})$ of square matrices.

An important ingredient is the extended Dynkin diagram of $\mathfrak{g}$. On Figure 1, these diagrams and their automorphisms are shortly recalled.

Theorem 1. *The neutral component* $\mathrm{Aut}(I\mathfrak{b})^\circ$ *of the automorphism group* $\mathrm{Aut}(I\mathfrak{b})$ *of the Lie algebra* $I\mathfrak{b}$ *decomposes as*

$$\mathbb{C}^* \ltimes \Big((B \ltimes \mathfrak{g}/\mathfrak{b}) \times \mathcal{M}_r(\mathbb{C}) \Big).$$

The group of components $\mathrm{Aut}(I\mathfrak{b})/\mathrm{Aut}(I\mathfrak{b})^\circ$ *is isomorphic to the automorphism group of the affine Dynkin diagram of* $\mathfrak{g}$ *and can be lift as a subgroup of* $\mathrm{Aut}(I\mathfrak{b})$*.*

The details of how these subgroups act on $I\mathfrak{b}$ are given in Section 3. Section 4 explain how the semidirect products are formed.

One of the amazing facts is that the extended Dynkin diagram of $\mathfrak{g}$ plays a crucial role in $\mathrm{Aut}(I\mathfrak{b})$. On one hand, we explain this by constructing the extended Cartan matrix of $\mathfrak{g}$ in terms of $I\mathfrak{b}$ in Section 3.1. On the other hand, this diagram is the Dynkin diagram of the untwisted affine Lie algebra constructed from the loop algebra of $\mathfrak{g}$. A second explanation is given by Theorem 2 that realizes $I\mathfrak{b}$ as a subquotient of the affine Lie algebra associated to $\mathfrak{g}$.

In [KZJ07], Knutson and Zinn-Justin defined a degeneration $\bullet$ of the standard associative product on $\mathcal{M}_n(\mathbb{C})$. Let $\mathfrak{b}$ denote the set of upper triangular matrices. Identifying the vector space $\mathcal{M}_n(\mathbb{C})$ with $\mathfrak{b} \times \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$ in a natural way one gets

$$(R, L) \bullet (V, M) = (RV, RM + LV),$$

for any $R, V \in \mathfrak{b}$ and $L, M \in \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$. The Lie algebra of the group $(\mathcal{M}_n(\mathbb{C}), \bullet)^\times$ of invertible elements of this algebra is $\mathfrak{b} \ltimes \mathcal{M}_n(\mathbb{C})/\mathfrak{b}$, where the product is defined similarly with that of $I\mathfrak{b}$. Note also that a cyclic automorphism (corresponding in our setting with the cyclic automorphism of the affine Dynkin diagram of type A_{n-1} and with the unexpected cyclic automorphism of [Bv20]) appears in [KZJ07]. Moreover [KZJ07, Proposition 2], which realizes $(\mathcal{M}_n(\mathbb{C}), \bullet)$ as a subquotient of $\mathcal{M}_n(\mathbb{C}[t])$, is similar with our Theorem 2.

Motivation and story of this work. In [Bv20], the authors constructed an "unexpected" cyclic automorphism of $I\mathfrak{b}$ when $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. The first version of this work was an explanation for this automorphism by using affine Lie algebras. Simultaneously with this first version, A. Knutson mentioned to Bar-Natan his earlier work [KZJ07] with Zinn-Justin.

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2 The Lie algebras $I\mathfrak{b}$, $\mathfrak{g}_+^\epsilon$ and $\mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$

2.1 Definitions of $I\mathfrak{b}$ and $\mathfrak{g}_+^\epsilon$

Let $\mathfrak{g}$ be a complex simple Lie algebra with Lie bracket denoted by $[\ ,\ ]$. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Let $\mathfrak{b}^-$ be the opposite Borel subalgebra of $\mathfrak{b}$ containing $\mathfrak{h}$. Set $\mathcal{V} = \mathfrak{b} \oplus \mathfrak{b}^-$ viewed as a vector space. In this section, we define the Lie bracket $[\ ,\ ]_\epsilon$ on $\mathcal{V}$ depending on the complex parameter ϵ, interpolating between $I\mathfrak{b}$ and the direct product $\mathfrak{g} \oplus \mathfrak{h}$.

Let $\mathfrak{n}$ and $\mathfrak{n}^-$ denote the derived subalgebras of $\mathfrak{b}$ and $\mathfrak{b}^-$ respectively. Fix $\epsilon \in \mathbb{C}$. Define the skew-symmetric bilinear bracket $[\ ,\ ]_\epsilon$ on $\mathcal{V}$ by

$$\begin{array}{lll} [x, x']_\epsilon = [x, x'] & \forall x, x' \in \mathfrak{b} & \\ [y, y']_\epsilon = \epsilon[y, y'] & \forall y, y' \in \mathfrak{b}^- & \\ [x, y]_\epsilon = (\epsilon X + \epsilon \frac{H}{2}, \frac{H}{2} + Y) & \forall x \in \mathfrak{b}\ y \in \mathfrak{b}^- & \text{where } [x, y] = X + H + Y \in \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^- \end{array}$$

Then $[\ ,\ ]_\epsilon$ satisfies Jacobi identity (see below for a proof). Endowed with this Lie bracket, $\mathcal{V}$ is denoted by $\mathfrak{g}_+^\epsilon$. Assume, for a momennt that ϵ in nonzero. The invertible linear map

$$\begin{array}{rccl} \varphi_\epsilon : & \mathfrak{b}\oplus\mathfrak{b}^- & \longrightarrow & \mathfrak{b}\oplus\mathfrak{b}^- \\ & (x,y) & \longmapsto & (x,\epsilon y) \qquad \text{for any } x\in\mathfrak{b},\ y\in\mathfrak{b}^- \end{array}$$

allows to interpret $\mathfrak{g}_+^\epsilon$ as an Inönü-Wigner contraction [IW53] of $\mathfrak{g}_+^1$. Indeed, for any nonzero ϵ, we have

$$[X,Y]_\epsilon = {\varphi_\epsilon}^{-1}([\varphi_\epsilon(X),\varphi_\epsilon(Y)]_1) \qquad \forall X,Y\in\mathcal{V}. \tag{1}$$

We now describe $\mathfrak{g}_+^1$. Using the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}\oplus\mathfrak{h}\oplus\mathfrak{n}^-, \tag{2}$$

one defines the injective linear map

$$\begin{array}{rccl} \iota_{\mathfrak{g}}^1 : & \mathfrak{g}=\mathfrak{n}\oplus\mathfrak{h}\oplus\mathfrak{n}^- & \longrightarrow & \mathfrak{g}_+^1 \\ & (\xi,\alpha,\zeta) & \longmapsto & (\xi+\frac{\alpha}{2},\frac{\alpha}{2}+\zeta) \end{array}$$

and checks that it is a Lie algebra homomorphism. Moreover, the image of

$$\begin{array}{rccl} \iota_{\mathfrak{h}}^1 : & \mathfrak{h} & \longrightarrow & \mathfrak{g}_+^1 \\ & \alpha & \longmapsto & (-\alpha,\alpha) \end{array}$$

is the center of $\mathfrak{g}_+^1$ and

$$\mathfrak{g}_+^1 = \iota_{\mathfrak{g}}^1(\mathfrak{g})\oplus\iota_{\mathfrak{h}}^1(\mathfrak{h}). \tag{3}$$

Observe that we never used the Jacobi identity for $[\ ,\ ]_1$ to prove the isomorphism (3). Hence we can deduce from it that $[\ ,\ ]_1$ satisfies the Jacobi identity. Then, the expression (1) with φ_ϵ of $[\ ,\ ]_\epsilon$ from $[\ ,\ ]_1$ implies that $[\ ,\ ]_\epsilon$ satisfies the Jacobi identity for any nonzero ϵ. Since this property is closed on the space of bilinear maps, it is satisfied by $[\ ,\ ]_0$ too.

Consider now $I\mathfrak{b}$ with its Lie bracket $[\ ,\ ]_{I\mathfrak{b}}$ defined by: $\mathfrak{b}^*$ is an abelian ideal on which $\mathfrak{b}$ acts by the coadjoint action. Denote by $\kappa : \mathfrak{g}\longrightarrow\mathfrak{g}^*$ the Killing form on $\mathfrak{g}$. Since the orthogonal of $\mathfrak{b}$ with respect to κ is $\mathfrak{n}$, $\mathfrak{b}^*$ identifies with $\mathfrak{g}/\mathfrak{n}$ as a $\mathfrak{b}$-module. Identify $\mathfrak{g}/\mathfrak{n}$ with $\mathfrak{b}^-$ in a canonical way (that is by $y\in\mathfrak{b}^-\longmapsto y+\mathfrak{n}$) and denote by $\pi : \mathfrak{g}\longrightarrow\mathfrak{b}^-$ the quotient map. Then $Ib=\mathfrak{b}\oplus\mathfrak{b}^*$ identifies with $\mathfrak{b}\oplus\mathfrak{b}^-=\mathcal{V}$. Let $[\ ,\ ]_I$ denote the Lie bracket transported to $\mathcal{V}$ from $[\ ,\ ]_{I\mathfrak{b}}$. Let $x,x'\in\mathfrak{b}$ and $y,y'\in\mathfrak{b}^-$ and decompose $[x,y']-[x',y]$ as $X+H+Y$ with respect to $\mathfrak{g}=\mathfrak{n}\oplus\mathfrak{h}\oplus\mathfrak{n}^-$. Then

$$[(x,y),(x',y')]_I = ([x,x'],H+Y). \tag{4}$$

We now describe $\mathfrak{g}_+^0$. The Lie bracket $[\ ,\ ]_0$ on $\mathcal{V}=\mathfrak{g}_+^0$ is given by

$$[(x,y),(x',y')]_0 = ([x,x'],\frac{H}{2}+Y). \tag{5}$$

Comparing (4) and (5), one gets that the following linear map η is a Lie algebra isomorphism:

$$\begin{array}{rccl} \eta : & \mathcal{V}=\mathfrak{b}\oplus(\mathfrak{h}\oplus\mathfrak{n}^-) & \longrightarrow & \mathfrak{b}\oplus\mathfrak{b}^*=Ib \\ & (x,h,y) & \longmapsto & (x,\kappa(2h+y,\Box)). \end{array}$$

2.2 The affine Kac-Moody Lie algebra

The affine Kac-Moody Lie algebra $\mathfrak{g}^{\mathrm{KM}}$ is constructed from the semisimple Lie algebra $\mathfrak{g}$. We refer to [Kum02, Chapters I and XIII] for the basic properties of $\mathfrak{g}^{\mathrm{KM}}$. Denote by $\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}})$ the one dimensional center of $\mathfrak{g}^{\mathrm{KM}}$. Consider the Borel subalgebra $\mathfrak{b}^{\mathrm{KM}}$ of $\mathfrak{g}^{\mathrm{KM}}$ and its derived subalgebra $\mathfrak{n}^{\mathrm{KM}}$. By killing the semi-direct product and the central extension from the construction of $\mathfrak{g}^{\mathrm{KM}}$, one gets

$$\begin{aligned} \tilde{\mathfrak{g}} &:= [\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}]/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) \\ &\cong \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}, \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathfrak{b}} &:= (\mathfrak{b}^{\mathrm{KM}} \cap [\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}])/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) \subset \tilde{\mathfrak{g}} \\ \tilde{\mathfrak{n}} &:= (\mathfrak{n}^{\mathrm{KM}} \cap [\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}])/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}]. \end{aligned}$$

Identify $\mathfrak{g}$ with the subspace $\mathbb{C} \otimes \mathfrak{g} \subset \tilde{\mathfrak{g}}$. Note that $\mathfrak{g}^{\mathrm{KM}}/\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) = \tilde{\mathfrak{g}} + \mathbb{C}d$ where d acts as the derivation with respect to t.

We consider the set of (positive) roots $\Phi^{(+)}$ (resp. $\tilde{\Phi}^{(+)}$) of $\mathfrak{g}$ (resp. $\mathfrak{g}^{\mathrm{KM}}$) and the set of simple roots Δ (resp. $\tilde{\Delta}$) with respect to $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ (resp. $\mathfrak{h} + \mathbb{C}d + \mathfrak{z}(\mathfrak{g}^{\mathrm{KM}}) \subset \mathfrak{b}^{\mathrm{KM}} \subset \mathfrak{g}^{\mathrm{KM}}$). We recall the following classical facts:

$$\mathfrak{n}^{KM} \cong \tilde{\mathfrak{n}} = \bigoplus_{\alpha \in \tilde{\Phi}^{+}} \tilde{\mathfrak{g}}_{\alpha}$$

where $\tilde{\mathfrak{g}}_{\alpha} \cong \mathfrak{g}^{KM}_{\alpha}$ is the root space associated to α. Moreover, $\tilde{\mathfrak{n}}$ is generated, as a Lie algebra by the subspaces $(\tilde{\mathfrak{g}}_{\alpha})_{\alpha \in \tilde{\Delta}}$. The identification of Δ with $\{\alpha \in \tilde{\Delta} \,|\, \alpha(d) = 0\}$ yields the above-described embedding $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. Denoting by δ the indivisible positive imaginary root in $\tilde{\Phi}$, we have

$$\tilde{\Phi} = \{n\delta + \alpha \,|\, \alpha \in \Delta \cup \{0\}, n \in \mathbb{Z}\} \setminus \{0\}$$

$$\tilde{\Delta} = \Delta \cup \{\alpha_0 + \delta\}$$

where $-\alpha_0$ is the highest root of Φ.

Finally, the extended Dynkin diagram can be reconstructed from the combinatorics of $\tilde{\Delta}$ in $\tilde{\Phi}$. Indeed, the nodes correspond to the elements of $\tilde{\Delta}$ and the non-diagonal entries $a_{\alpha,\beta}$ of the generalized Cartan matrix (encoding the arrows of the diagram) are $a_{\alpha,\beta} = -\max\{n \in \mathbb{N} | \beta + n\alpha \in \tilde{\Phi}\}$ by Serre relations.

We list in Figure 1 the extended Dynkin diagram $\tilde{\mathcal{D}}_{\mathfrak{g}}$ in each simple type. The black node corresponds to the simple root $\alpha_0 + \delta$. We also provide the automorphism group of $\tilde{\mathcal{D}}_{\mathfrak{g}}$. Note that by the definition of $\mathfrak{g}^{\mathrm{KM}}$ given in [Kum02, §1.1], any $\theta \in \mathrm{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ provides an automorphism $\theta^{KM} \in \mathrm{Aut}(\mathfrak{g}^{\mathrm{KM}})$ stabilizing both $\mathfrak{h}+\mathbb{C}d+\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}})$ and $\mathfrak{b}^{\mathrm{KM}}$ and permuting $\tilde{\Delta}$ [1] as θ does. Since $\mathfrak{z}(\mathfrak{g}^{\mathrm{KM}})$ and $[\mathfrak{g}^{\mathrm{KM}}, \mathfrak{g}^{\mathrm{KM}}]$ are characteristic in $\mathfrak{g}^{\mathrm{KM}}$, this yields an automorphism $\tilde{\theta} \in \mathrm{Aut}(\tilde{\mathfrak{g}})$.

[1] and even permuting the set of generators e_{α}, $\alpha \in \tilde{\Delta}$

2.3 Realisation of $\mathfrak{g}_+^\epsilon$

The Lie algebras $\tilde{\mathfrak{b}}$ and $\tilde{\mathfrak{n}}$ decompose as

$$\begin{aligned}\tilde{\mathfrak{b}} &= \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{n}^-,\\ \tilde{\mathfrak{n}} &= \mathbb{C}[t]\mathfrak{n} \oplus t\mathbb{C}[t]\mathfrak{b}^-.\end{aligned}$$

Moreover, $(t-\epsilon)\tilde{\mathfrak{n}}$ is an ideal of $\tilde{\mathfrak{b}}$, and $\tilde{\mathfrak{b}}/((t-\epsilon)\tilde{\mathfrak{n}})$ is a Lie algebra.

Theorem 2. *Let $\epsilon \in \mathbb{C}$. The Lie algebras $\mathfrak{g}_+^\epsilon$ and $\tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}}$ are isomorphic.*

Proof. From Section 2.1, we have $\mathfrak{g}_+^1 \overset{v.s.}{=} \mathfrak{b} \oplus \mathfrak{b}^-$. Elements of $\mathfrak{g}_+^1$ will be written as couples with respect to this decomposition.

Set $\widetilde{\mathfrak{g}_+^1} := \mathbb{C}[t^{\pm 1}] \otimes \mathfrak{g}_+^1$ and extend $\iota_{\mathfrak{g}}^1$ to an injective $\mathbb{C}[t^{\pm 1}]$-linear map $\tilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}_+^1}$. Consider the subspace $\mathfrak{w} := \mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$ that is a Lie subalgebra of $\widetilde{\mathfrak{g}_+^1}$. The Inönü-Wigner contraction on $\mathfrak{g}_+^1$ with respect to the decomposition $\mathfrak{b} \oplus \mathfrak{b}^-$ gives rise to $\mathfrak{g}_+^\epsilon$ ($\epsilon \in \mathbb{C}$). In particular, the linear map

$$\begin{array}{rcl}\mathfrak{g}_+^\epsilon & \longrightarrow & \mathfrak{w}/(t-\epsilon)\mathfrak{w} \\ (x,y) & \longmapsto & x + ty + (t-\epsilon)\mathfrak{w} \qquad \text{for any } x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-.\end{array} \tag{6}$$

is a Lie algebra isomorphism

Set $\mathfrak{b}_0^- := \iota_{\mathfrak{g}}^1(\mathfrak{b}^-) = \{(h,h) | h \in \mathfrak{h}\} \oplus \mathfrak{n}^-$. Observe that $t\mathfrak{b}_0^-$ is contained in $\mathfrak{w}$. Indeed, for any $h \in \mathfrak{h}$, the element $t(h,h) = t(h,0) + t(0,h)$ belongs to $\mathbb{C}[t]\mathfrak{b} \oplus t\mathbb{C}[t]\mathfrak{b}^-$. In particular, one gets a linear map induced by the inclusions of $\mathfrak{b}$ and $t\mathfrak{b}_0^-$ in $\mathfrak{w}$:

$$\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}.$$

One can easily check that it induces a linear isomorphism $\mathfrak{b} \oplus t\mathfrak{b}_0^- \longrightarrow \mathfrak{w}/(t-\epsilon)\mathfrak{w}$. Setting $\tilde{\mathfrak{b}}_{\mathfrak{w}} := \langle \mathfrak{b} \oplus t\mathfrak{b}_0^- \rangle_{Lie} \subset \mathfrak{w}$, we thus get a Lie algebra isomorphism.

$$\tilde{\mathfrak{b}}_{\mathfrak{w}}/((t-\epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}) \longrightarrow \mathfrak{w}/(t-\epsilon)\mathfrak{w}. \tag{7}$$

Since, $\mathfrak{b} = \{(h,0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\mathfrak{n})$ and $\langle \iota_{\mathfrak{g}}^1(\mathfrak{n}) \oplus \iota_{\mathfrak{g}}^1(t\mathfrak{b}^-) \rangle_{Lie} = \iota_{\mathfrak{g}}^1(\langle \mathfrak{n} \oplus t\mathfrak{b}^- \rangle_{Lie}) = \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$, we have

$$\tilde{\mathfrak{b}}_{\mathfrak{w}} = \{(h,0) | h \in \mathfrak{h}\} \oplus \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}}) \cong \iota_{\mathfrak{g}}^1(\tilde{\mathfrak{b}}) \cong \tilde{\mathfrak{b}}, \tag{8}$$

the middle Lie isomorphism being the identity on $\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ and sending $(h,0)$ to $\frac{1}{2}(h,h)$ for each $h \in \mathfrak{h}$. Moreover, $(t-\epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}} = (t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$. Indeed, $(t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ is contained in $(t-\epsilon)\mathfrak{w} \cap \tilde{\mathfrak{b}}_{\mathfrak{w}}$, and $\mathfrak{b} \oplus t\mathfrak{b}_0^-$ is complementary to $(t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}})$ in $\tilde{\mathfrak{b}}_{\mathfrak{w}}$.

We finally get the desired Lie isomorphism

$$\tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}} \overset{(8)}{\cong} \tilde{\mathfrak{b}}_{\mathfrak{w}}/(t-\epsilon)\iota_{\mathfrak{g}}^1(\tilde{\mathfrak{n}}) \overset{(7)}{\cong} \mathfrak{w}/(t-\epsilon)\mathfrak{w} \overset{(6)}{\cong} \mathfrak{g}_+^\epsilon$$

□

In addition, we can make explicit the isomorphism of Theorem 2:

$$\begin{array}{rcccl} \gamma_\epsilon : & \mathfrak{g}_+^\epsilon & \stackrel{\cong}{\longrightarrow} & \tilde{\mathfrak{b}}/(t-\epsilon)\tilde{\mathfrak{n}} & \\ & x & \longmapsto & x & \text{if } x \in \mathfrak{n} \\ & y & \longmapsto & ty & \text{if } y \in \mathfrak{n}^- \\ & (a,b) & \longmapsto & (a-\epsilon b)+2tb & \text{if } a,b \in \mathfrak{h} \end{array}$$

and its inverse map is induced by

$$\begin{array}{rcccl} \theta : & \tilde{\mathfrak{b}} & \longrightarrow & \mathcal{V} & \\ & Px & \longmapsto & P(\epsilon)x & \text{if } x \in \mathfrak{n} \\ & tRy & \longmapsto & R(\epsilon)y & \text{if } y \in \mathfrak{n}^- \\ & Qh & \longmapsto & (\frac{Q(\epsilon)+Q(0)}{2}h, \frac{Q(\epsilon)-Q(0)}{2\epsilon}h) & \text{if } h \in \mathfrak{h}\, (\epsilon \neq 0) \\ & & & (Q(0)h, \frac{1}{2}Q'(0)h) & \text{if } h \in \mathfrak{h}\, (\epsilon = 0) \end{array}$$

Note that, in order to prove Theorem 2, we could alternatively have checked directly that θ is a Lie algebra homomorphism from $\tilde{\mathfrak{b}}$ onto $\mathfrak{g}_+^\epsilon$ with Kernel $(t-\epsilon)\tilde{\mathfrak{n}}$.

3 Some subgroups of $\mathrm{Aut}(I\mathfrak{b})$

3.1 The roots of $I\mathfrak{b}$

From Sections 2.1 and 2.3, we can interpret the algebra $I\mathfrak{b}$ in the Kac-Moody world via the isomorphism

$$\begin{array}{rcl} I\mathfrak{b} & \longrightarrow & \tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}} \\ (x,y) & \longmapsto & x+ty \end{array} \qquad \left(\begin{array}{c} x \in \mathfrak{b}, \\ y \in \mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n} \stackrel{\kappa}{\cong} \mathfrak{b}^* \end{array} \right)$$

From now on, this identification will be made systematically. In particular, we write $I\mathfrak{b} = \mathfrak{b} \oplus t\mathfrak{b}^-$. We first describe some basic properties of $I\mathfrak{b}$ in this language.

Lemma 3. *1. The subalgebra $\mathfrak{c} := \mathfrak{h} \oplus t\mathfrak{h}$ is a Cartan subalgebra of $I\mathfrak{b}$. Namely, $\mathfrak{c}$ is abelian and equal to its normalizer.*

2. Under the action of $\mathfrak{c}$, $I\mathfrak{b}$ decomposes as

$$I\mathfrak{b} = \mathfrak{c} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^-} t\mathfrak{g}_\alpha.$$

For $\alpha \in \Phi^+$, $\mathfrak{c}$ acts on $\mathfrak{g}_\alpha$ with the weight $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$. It acts on $t\mathfrak{g}_\alpha$ with the weight $(\alpha, 0) \in \mathfrak{h}^* \times t\mathfrak{h}^*$, if $\alpha \in \Phi^-$. Here, we identified $\mathfrak{c}^*$ with $\mathfrak{h}^* \times t\mathfrak{h}^*$ in a natural way.

3. The set of ad*-nilpotent elements of $I\mathfrak{b}$ is $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}} = \mathfrak{n} \oplus t\mathfrak{b}^-$.*

4. The centre of $I\mathfrak{b}$ is $\mathfrak{z}(I\mathfrak{b}) = t\mathfrak{h}$.

5. *The derived subalgebra of* $I\mathfrak{b}$ *is* $[I\mathfrak{b}; I\mathfrak{b}] = \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$.

Proof. 1-2) The fact that $\mathfrak{c}$ is abelian is clear from the definition of $\tilde{\mathfrak{g}}$. The decomposition in eigenspaces also. The action of $t\mathfrak{h}$ is zero since it sends $\tilde{\mathfrak{n}}$ to $t\tilde{\mathfrak{n}}$ that vanishes itself in $I\mathfrak{b}$. The decomposition of $I\mathfrak{b}$ in weight spaces under the action of $\mathfrak{c}$ follows. Then this decomposition also implies that $N_{I\mathfrak{b}}(\mathfrak{c}) = \mathfrak{c}$.
3) The elements of $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ are clearly ad-nilpotent. From 2), an element with non-zero component in $\mathfrak{h}$ is not ad-nilpotent.
4) Since it acts as 0 on $\tilde{\mathfrak{n}}$ and on $\mathfrak{h}$, we have $t\mathfrak{h} \subset \mathfrak{z}(I\mathfrak{b})$. The decomposition in weight spaces implies the converse inclusion.
5) The inclusion $[I\mathfrak{b}, I\mathfrak{b}] \subset \tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is clear. On the other hand we deduce from the weight space decomposition that the subspaces $(\tilde{\mathfrak{g}}_\alpha)_{\alpha\in\tilde{\Delta}}$ belong to $[I\mathfrak{b}, I\mathfrak{b}]$. Since they generate $\tilde{\mathfrak{n}}$ in $\tilde{\mathfrak{g}}$, the result follows. □

From Lemma 3 (2), the set of nonzero weights $\Phi(I\mathfrak{b})$ of $\mathfrak{c}$ acting on $I\mathfrak{b}$ identifies with Φ. It is also useful to embed $\Phi(I\mathfrak{b})$ in $\tilde{\Phi}$ by

$$\begin{array}{rrcl} \varphi: & \Phi(I\mathfrak{b}) & \longrightarrow & \tilde{\Phi} \\ & \alpha \in \Phi^+ & \longmapsto & \alpha \\ & \alpha \in \Phi^- & \longmapsto & \delta + \alpha \end{array}$$

Indeed, the weight space $(I\mathfrak{b})_\alpha$ identifies with $\tilde{\mathfrak{g}}_{\varphi(\alpha)}$, for any $\alpha \in \Phi(I\mathfrak{b})$. In particular, for $\alpha, \beta \in \tilde{\Phi} \cup \{0\}$, we have $[I\mathfrak{b}_{\varphi^{-1}(\alpha)}, I\mathfrak{b}_{\varphi^{-1}(\beta)}] \subset I\mathfrak{b}_{\varphi^{-1}(\alpha+\beta)}$ with equality when $\alpha, \beta, \alpha+\beta \notin \{0, \delta\}$. Set also $\Delta(I\mathfrak{b}) = \varphi^{-1}(\tilde{\Delta}) = \Delta \cup \{-\alpha_0\}$.

Lemma 4. 1. *The derived subalgebra of* $I\mathfrak{b}^{(1)} := [I\mathfrak{b}, I\mathfrak{b}]$ *is*

$$I\mathfrak{b}^{(2)} = \bigoplus_{\alpha\in\Phi(I\mathfrak{b})\setminus\Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha$$

2. *Assume that* $\mathfrak{g}$ *is not* $\mathfrak{sl}_2$. *For* $\alpha, \beta \in \Delta(I\mathfrak{b})$ *(*$\alpha \neq \beta$*), the corresponding entry of the generalized Cartan Matrix of* $\mathfrak{g}^{\mathrm{KM}}$ *is given by*

$$a_{\alpha,\beta} = -\max\{n \in \mathbb{N} \mid \beta + n\alpha \in \Phi(I\mathfrak{b})\}.$$

Proof. 1) Recall that $\tilde{\mathfrak{n}}$ is generated as a Lie algebra by the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha\in\tilde{\Delta}}$. Thus, for weight reasons, the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha\in\tilde{\Phi}\setminus\tilde{\Delta}}$ are root spaces included in $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$. Since $\tilde{\Delta}$ is a linearly independant family, they are in fact the only root spaces not contained in $[\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}]$. Quotienting, this yields $\bigoplus_{\alpha\in\Phi(I\mathfrak{b})\setminus\Delta(I\mathfrak{b})} (I\mathfrak{b})_\alpha = I\mathfrak{b}^{(2)}$.

2)Recall that the statement is valid if we replace $\Phi(I\mathfrak{b})$ by $\tilde{\Phi}$, see Section 2.2. It is thus sufficient to show that

$$\beta + n\alpha \in \tilde{\Phi} \;\Rightarrow\; \beta + n\alpha \in \Phi(I\mathfrak{b}).$$

When $\alpha, \beta \in \Delta$, the statement is clear since $\Phi^+ \subset \Phi(I\mathfrak{b})$.
If $\beta = \delta + \alpha_0$, then $\beta + n\alpha \in \tilde{\Phi}$ means that $\alpha_0 + n\alpha \in \Phi$. Since $\alpha_0 + n\alpha$ has elements of $-\Delta$

in its support, it has to lie in Φ^-. Thus $\beta + n\alpha \in \Phi(I\mathfrak{b})$.
If $\alpha = \delta + \alpha_0$, then $\beta + n\alpha \in \tilde{\Phi}$ means that $\beta + n\alpha_0 \in \Phi$. For height reasons, we must have $n \in \{0, 1\}$. Then, $\beta + n\alpha \in \Phi(I\mathfrak{b})$. $\square$

Remark. One can observe that the first assertion of Lemma 4 is similar with

$$[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{\alpha \in \Phi^+ \setminus \Delta} \mathfrak{b}_\alpha.$$

3.2 The adjoint subgroup of $\mathrm{Aut}(I\mathfrak{b})$

Let G be the adjoint group with Lie algebra $\mathfrak{g}$. Let T and B be the subgroups of G with Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$. Consider now $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$ equipped with the addition as an abelian algebraic group. The adjoint action of B on $\mathfrak{g}$ stabilizes $\mathfrak{n}$ and induces a linear action on $\mathfrak{b}^- \cong \mathfrak{g}/\mathfrak{n}$ by group isomorphisms. We can perform the semidirect product:

$$IB := B \ltimes \mathfrak{b}^-.$$

By construction the Lie algebra of IB identifies with $I\mathfrak{b}$. The adjoint action of IB on $I\mathfrak{b}$ is given by

$$\begin{array}{ccl} IB \times I\mathfrak{b} & \longrightarrow & I\mathfrak{b} \\ ((b, f), x + ty) & \longmapsto & b \cdot x + t(b \cdot y + [f, x]) \quad \text{for } b \in B,\ f, x \in \mathfrak{b} \text{ and } y \in \mathfrak{b}^-, \end{array} \tag{9}$$

where $\cdot$ denotes the B action on $\mathfrak{b}$ and on $\mathfrak{b}^-$. It induces a morphism

$$\mathrm{Ad} : IB \longrightarrow \mathrm{Aut}(I\mathfrak{b})$$

with Kernel $Z(IB) \cong (1, \mathfrak{h})$. In particular, one gets:

Lemma 5. *The image* $\mathrm{Ad}(IB)$ *is isomorphic with* $B \ltimes \mathfrak{g}/\mathfrak{b}$.

Note also that $\mathrm{Ad}(IB) = H \ltimes (N \ltimes \mathfrak{g}/\mathfrak{b})$ where N and H are the connected subgroups of B with respective Lie algebras $\mathfrak{n}$ and $\mathfrak{h}$. Since $\mathfrak{n} + t\mathfrak{b}^-$ is the set of ad-nilpotent elements of $I\mathfrak{b}$, we get the following result from (9).

Lemma 6. 1. *The group of elementary automorphisms* $\mathrm{Aut}_e(I\mathfrak{b}) = \exp(\mathfrak{n} + t\mathfrak{b}^-)$ *coincides with* $N \ltimes \mathfrak{g}/\mathfrak{b}$.

2. $\mathrm{Ad}(IB) = \exp(I\mathfrak{b})$

3.3 An unipotent subgroup of $\mathrm{Aut}(I\mathfrak{b})$

Lemma 7. *The following map is an injective group homomorphism*

$$\begin{array}{ccc} \mathrm{Hom}(I\mathfrak{b}/[I\mathfrak{b},I\mathfrak{b}],\mathfrak{z}(I\mathfrak{b})) & \longrightarrow & \mathrm{Aut}(I\mathfrak{b}) \\ u & \longmapsto & \left(\begin{array}{cccc} \bar{u}: & I\mathfrak{b} & \longrightarrow & I\mathfrak{b} \\ & X & \longmapsto & X+u(X) \end{array} \right). \end{array}$$

We denote by $U \subset \mathrm{Aut}(I\mathfrak{b})$ the image of this map. From Lemma 3, we have $U \cong \mathcal{M}_r(\mathbb{C})$ where $r = \dim \mathfrak{h}$.

Proof. Let $X, Y \in I\mathfrak{b}$. On one hand, we have

$$[\bar{u}(X), \bar{u}(Y)] = [X+u(X), Y+u(Y)] = [X,Y],$$

since the image of u is contained in the center. On the other hand,

$$\bar{u}([X,Y]) = [X,Y],$$

since u vanishes on the derived subalgebra. It follows that $\bar{u}$ is a Lie algebra isomorphism.

Since $\mathfrak{z}$ is contained in $[I\mathfrak{b}, I\mathfrak{b}]$ the map of the lemma is a group homomorphism from $(\mathrm{Hom}(I\mathfrak{b}/[I\mathfrak{b},I\mathfrak{b}],\mathfrak{z}),+)$ to $(\mathrm{Aut}(I\mathfrak{b}),\circ)$. □

Note that the existence of this group of automorphisms is quite general. Indeed, the only useful property of $I\mathfrak{b}$ in this proof is $\mathfrak{z}(I\mathfrak{b}) \cap [I\mathfrak{b}, I\mathfrak{b}] = \{0\}$.

3.4 The loop subgroup

Lemma 8. *The following map is an injective group homomorphism*

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathrm{Aut}(I\mathfrak{b}) \\ \tau & \longmapsto & \left(\begin{array}{ccccl} \delta_\tau : & I\mathfrak{b} & \longrightarrow & I\mathfrak{b} & \\ & x & \longmapsto & x & \text{if } x \in \mathfrak{b} \\ & ty & \longmapsto & \tau ty & \text{if } y \in \mathfrak{b}^- \end{array} \right). \end{array}$$

We denote by $D \subset \mathrm{Aut}(I\mathfrak{b})$ the image of this map

Proof. It is a straightforward check on $\mathfrak{b} \ltimes t\mathfrak{b}^-$ that the δ_τ are automorphisms of $I\mathfrak{b}$. □

Remark. The map δ_τ corresponds with the variable changing $t \mapsto \tau t$ in the $\mathbb{C}[t]$-Lie algebra $\tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}}$. Moreover, the Lie algebra of D acts on $I\mathfrak{b}$ like $\mathbb{C}d$ where d is the derivation involved in the definition of $\mathfrak{g}^{KM}$.

3.5 Automorphisms stabilizing the Cartan subalgebra

For any $\alpha \in \Delta(I\mathfrak{b})$, fix a nonzero element X_α in the corresponding root space $(I\mathfrak{b})_\alpha$. Set

$$G := \left\{ \theta \in \mathrm{Aut}(I\mathfrak{b}) \,\middle|\, \begin{array}{l} \theta(\mathfrak{h}) \subset \mathfrak{h} \\ \theta(\{X_\alpha \,:\, \alpha \in \Delta(I\mathfrak{b})\}) = \{X_\alpha \,:\, \alpha \in \Delta(I\mathfrak{b})\} \end{array} \right\}.$$

Note that, since $\mathfrak{c}$ is the sum of $\mathfrak{h}$ with $\mathfrak{z}(I\mathfrak{b})$ and since the center is characteristic, the elements of G also stabilze $\mathfrak{c}$.

Proposition 9. *The group G is isomorphic to the automorphism group of the affine Dynkin diagram of $\mathfrak{g}$.*

Proof. The group G stabilizes $\mathfrak{c}$ and hence the set of weights of $\mathfrak{c}$ acting on $I\mathfrak{b}$. This yields an action of G on $\Phi(I\mathfrak{b})$.

Moreover, $I\mathfrak{b}^{(1)} = [I\mathfrak{b}, I\mathfrak{b}]$ and $I\mathfrak{b}^{(2)} = [I\mathfrak{b}^{(1)}, I\mathfrak{b}^{(1)}]$ are characteristic and stabilized by G. Now, Lemma 4 implies that G stabilizes $\Phi(I\mathfrak{b}) \setminus \Delta(I\mathfrak{b})$ and hence $\Delta(I\mathfrak{b})$. Moreover, by Lemma 4 (2), we have for $g \in G$ and $\alpha, \beta \in \Delta(I\mathfrak{b})$:

$$\begin{aligned} a_{\alpha,\beta} &= -\max\{n | (\mathrm{ad}\, X_\alpha)^n (X_\beta) \neq 0\} \\ &= -\max\{n | g((\mathrm{ad}\, X_\alpha)^n (X_\beta)) \neq 0\} \\ &= -\max\{n | (\mathrm{ad}\, X_{g(\alpha)})^n (X_{g(\beta)}) \neq 0\} = a_{g(\alpha),g(\beta)}. \end{aligned}$$

Hence g actually induces an automorphism of the extended Dynkin diagram[2] and we thus obtain a group homomorphism

$$\Theta \,:\, G \to \mathrm{Aut}(\tilde{\mathcal{D}}_\mathfrak{g}).$$

We claim that Θ is surjective. Indeed, fix a group automorphism θ of $\tilde{\mathcal{D}}_\mathfrak{g}$. As seen in Section 2.2, there exists $\tilde{\theta} \in \mathrm{Aut}(\tilde{\mathfrak{g}})$ which stabilizes both $\mathfrak{h}$ and $\tilde{\mathfrak{b}}$ and which permutes $\Delta(I\mathfrak{b})$ as θ does. Then it stabilizes $\tilde{\mathfrak{n}} = [\tilde{\mathfrak{b}}, \tilde{\mathfrak{b}}]$ and thus induces the desired element of $\mathrm{Aut}(\tilde{\mathfrak{b}}/t\tilde{\mathfrak{n}})$.

We now prove that Θ is injective. Let θ in its Kernel. By the definition of the group G, θ stabilizes $\mathfrak{h}$. Since the restrictions of the elements of $\Delta(I\mathfrak{b})$ span $\mathfrak{h}^*$, the restriction of θ to $\mathfrak{h}$ has to be the identity. In particular, θ acts trivially on $\Phi(I\mathfrak{b})$ and stabilizes each root space $(I\mathfrak{b})_\alpha$ for $\alpha \in \Phi(I\mathfrak{b})$. But θ stabilizes the set $\{X_\alpha \,:\, \alpha \in \Delta(I\mathfrak{b})\}$. Hence θ acts trivially on each $\tilde{\mathfrak{g}}_\alpha$ for $\alpha \in \Delta(I\mathfrak{b})$. Since $\tilde{\mathfrak{n}}$ is generated by the $(\tilde{\mathfrak{g}}_\alpha)_{\alpha \in \Delta(I\mathfrak{b})}$, the restriction of θ to $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is the identity map. Finally, θ is trivial and Θ is injective. □

Remark. [Bv20, Theorem 2] is the construction of an explicit order n automorphism of $\mathfrak{gl}^\epsilon_{n+}$. Theorem 2 and the above proof of the surjectivity of Θ also show the existence of such an automorphism for $\mathfrak{sl}^\epsilon_{n+}$ with nonzero ϵ. Hence we just got both an explanation and an extension (to any simple $\mathfrak{g}$) of the Bar-Natan-van der Veen's Theorem 2.

[2] If $\mathfrak{g}$ is $\mathfrak{sl}_2$, Lemma 4 (2) does not apply. However, any permutation of $\tilde{\Delta}$ is an automorphism of the extended Dynkin diagram in this case.

4 Description of $\mathrm{Aut}(I\mathfrak{b})$

In this section, we describe the structure of

$$\mathrm{Aut}(I\mathfrak{b}) = \{g \in \mathrm{GL}(I\mathfrak{b}) \,:\, \forall X, Y \in I\mathfrak{b} \qquad g([X,Y]) = [g(X), g(Y)]\}$$

in terms of the subgroups U, $\mathrm{Ad}(IB)$, D and G introduced in Section 3.

Observe that $\mathrm{Aut}(I\mathfrak{b})$ is a Zariski closed subgroup of the linear group $\mathrm{GL}(I\mathfrak{b})$.

Theorem 10. *We have the following decomposition*

$$\mathrm{Aut}(I\mathfrak{b}) = G \ltimes (D \ltimes (\mathrm{Ad}(IB) \times U)).$$

In particular, the neutral component is $\mathrm{Aut}(I\mathfrak{b})^\circ = D \ltimes (\mathrm{Ad}(IB) \times U)$ *and* $G \cong \mathrm{Aut}(\tilde{\mathcal{D}}_{\mathfrak{g}})$ *can be seen as the component group of* $\mathrm{Aut}(I\mathfrak{b})$*.*

The result is a consequence of the lemmas provided below. Indeed, by Lemma 12, the four subgroups generate $\mathrm{Aut}(I\mathfrak{b})$. By Lemma 11, the subgroup generated by U and $\mathrm{Ad}(IB)$ is a direct product $U \times \mathrm{Ad}(IB)$. Then the structure of $\mathrm{Aut}(I\mathfrak{b})$ follows from lemma 13.

Since D, $\mathrm{Ad}(IB)$ and U are connected and G is discrete, $\mathrm{Aut}(I\mathfrak{b}) = \bigsqcup_{g \in G} gD\mathrm{Ad}(IB)U$ is a finite disjoint union of irreducible subsets of the same dimension. They are thus the irreducible components of $\mathrm{Aut}(IB)$ and the remaining statements of Theorem 10 follow.

Lemma 11. *The subgroups* U *and* $\mathrm{Ad}(IB)$ *are normal in* $\mathrm{Aut}(I\mathfrak{b})$*. Moreover,* $U \cap \mathrm{Ad}(IB) = \{\mathrm{Id}\}$*.*

Proof. Recall that $\mathrm{Ad}(IB)$ is generated by the exponentials of $\mathrm{ad}(x)$ with $x \in I\mathfrak{b}$. Then for any $\theta \in \mathrm{Aut}(I\mathfrak{b})$,

$$\theta \mathrm{Ad}(IB) \theta^{-1} = \theta \exp(I\mathfrak{b}) \theta^{-1} = \exp(\theta(I\mathfrak{b})) = \exp(I\mathfrak{b}) = \mathrm{Ad}(IB).$$

Take now $u \in \mathrm{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$ so that $\bar{u} \in U$. Since $[I\mathfrak{b}, I\mathfrak{b}]$ (resp. $\mathfrak{z}(I\mathfrak{b})$) is a characteristic subspace of $I\mathfrak{b}$, it is stabilized by θ^{-1} (resp. θ). Hence $\theta u \theta^{-1}$ vanishes on $[I\mathfrak{b}, I\mathfrak{b}]$ and takes values in $\mathfrak{z}$. So $\theta \bar{u} \theta^{-1}$ is an element of U.

Let $(b, g) \in IB$ and $h \in \mathfrak{h}$. Then $\mathrm{Ad}(b,g)(h) = b \cdot h + t[g,h] \subset h + (\mathfrak{n} \oplus t\mathfrak{n}^-)$. As a consequence, whenever $\mathrm{Ad}(b,g)(\mathfrak{h}) \subset \mathfrak{h} + \mathfrak{z}$ we have $\mathrm{Ad}(b,g)(\mathfrak{h}) \subset \mathfrak{h}$. Hence $\mathrm{Ad}(IB) \cap U = \{\mathrm{Id}\}$. ☐

Lemma 12. *We have* $\mathrm{Aut}(I\mathfrak{b}) = GD\mathrm{Ad}(IB)U$*.*

Proof. Let $\theta \in \mathrm{Aut}(I\mathfrak{b})$. Since the two Cartan subalgebras $\mathfrak{c}$ and $\theta(\mathfrak{c})$ are Ad-conjugate (see [Bou75, §3, n° 3, th. 2]), there exists $\theta_1 \in \mathrm{Ad}(IB)\theta$ which stabilizes $\mathfrak{c}$.

Then $\theta_1(\mathfrak{h})$ is complementary to the center $t\mathfrak{h} = \theta_1(t\mathfrak{h})$ in $\mathfrak{c}$. Thus, there exists $\theta_2 \in U\theta_1$ such that θ_2 stabilizes $\mathfrak{h}$.

Since θ_2 stabilizes $\mathfrak{c}$, it acts on $\Phi(I\mathfrak{b})$. Arguing as in the proof of Proposition 9, we show that it stabilizes $\Delta(I\mathfrak{b})$ and that the induced permutation is actually an automorphism of the extended Dynkin diagram. Thus there exists $\theta_3 \in G\theta_2$ with the additional property that the induced permutation on $\Delta(I\mathfrak{b})$ and thus on $\Phi(I\mathfrak{b})$ are trivial. Then θ_3 acts on each $(I\mathfrak{b})_\alpha$ for $\alpha \in \Delta(I\mathfrak{b})$.

Since Δ is a basis of $\mathfrak{h}^*$, one can find $h \in H \subset B \subset IB$ such that $\mathrm{Ad}(h) \circ \theta_3$ acts trivially on each $(I\mathfrak{b})_\alpha$ for $\alpha \in \Delta$. Moreover, D acts trivially on these roots spaces and with weight 1 on $(I\mathfrak{b})_{\alpha_0}$. This yields $\theta_4 \in D\mathrm{Ad}(H)GU\mathrm{Ad}(IB)\theta$ which acts trivially on $\mathfrak{h}$ and on each $(I\mathfrak{b})_\alpha$, $\alpha \in \Delta(I\mathfrak{b})$.

Recall now that $\tilde{\mathfrak{n}}/t\tilde{\mathfrak{n}}$ is generated by the spaces $((I\mathfrak{b})_\alpha)_{\alpha\in\Delta(I\mathfrak{b})}$. Since θ_4 acts trivially on $\tilde{\mathfrak{n}}$ and on $\mathfrak{h}$, it has to be trivial. As a consequence, $\theta \in \mathrm{Ad}(IB)UG\mathrm{Ad}(H)D = GD\mathrm{Ad}(IB)U$, the last equality following from Lemma 11. □

Lemma 13. *The intersections $D \cap (\mathrm{Ad}(IB) \times U)$ and $G \cap (D \ltimes (\mathrm{Ad}(IB) \times U))$ are the trivial group $\{\mathrm{Id}\}$. Moreover, $(D \ltimes (\mathrm{Ad}(IB) \times U))$ is normal in $\mathrm{Aut}(I\mathfrak{b})$.*

Proof. Let $\tau \in \mathbb{C}^*$, $b \in B$, $f \in \mathfrak{g}/\mathfrak{n}$ and $u \in \mathrm{Hom}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}], \mathfrak{z}(I\mathfrak{b}))$ such that the elements associated $\delta_\tau \in D$, $(b, f) \in IB$ and $\tilde{u} \in U$ (see Section 3) satisfy $\delta_\tau = \mathrm{Ad}(b, f) \circ \tilde{u}$. For $b' \in \mathfrak{b}$, we have

$$b' = \delta_\tau(b') = (\mathrm{Ad}(b, f) \circ \bar{u})(b') = \mathrm{Ad}(b, f)(b' + u(b')) = b \cdot b' + (b \cdot u(b') + t([f, b'])).$$

In particular, $b \cdot b' = b'$ and, whenever $b' \in \mathfrak{n}$, $[f, b'] = 0$ in $\mathfrak{g}/\mathfrak{n}$. So $b \in B$ centralizes $\mathfrak{b}$ and $\mathrm{ad}_\mathfrak{g} f$ normalizes $\mathfrak{n}$. As a consequence, $b = 1_B$, f is 0 in $\mathfrak{g}/\mathfrak{b}$ and $u = 0$. Thus the only element of $D \cap (\mathrm{Ad}(IB) \times U)$ is the trivial one.

Since $[I\mathfrak{b}, I\mathfrak{b}]$ is characteristic in $I\mathfrak{b}$, we have a natural group morphism $p : \mathrm{Aut}(I\mathfrak{b}) \to \mathrm{Aut}(I\mathfrak{b}/[I\mathfrak{b}, I\mathfrak{b}])$. From the description of $[I\mathfrak{b}, I\mathfrak{b}]$ in Lemma 3, it is straightforward that D, $\mathrm{Ad}(IB)$ and U are included in $Ker(p)$ while $p_{|G}$ is injective. From Lemma 12, we then deduce that $D \ltimes (\mathrm{Ad}(IB) \times U) = Ker(p)$ and the desired properties follow. □

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