Distributions on homogeneous spaces and applications

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Abstract

Let G be a complex semisimple algebraic group. In 2006, Belkale-Kumar defined a new product $\odot_0$ on the cohomology group $\mathrm{H}^*(G/P, \mathbb{C})$ of any projective G-homogeneous space G/P. Their definition uses the notion of Levi-movability for triples of Schubert varieties in G/P.

In this article, we introduce a family of G-equivariant subbundles of the tangent bundle of G/P and the associated filtration of the De Rham complex of G/P viewed as a manifold. As a consequence one gets a filtration of the ring $\mathrm{H}^*(G/P, \mathbb{C})$ and prove that $\odot_0$ is the associated graded product. One of the aim of this more intrinsic construction of $\odot_0$ is that there is a natural notion of fundamental class $[Y]_{\odot_0} \in (\mathrm{H}^*(G/P, \mathbb{C}), \odot_0)$ for any irreducible subvariety Y of G/P.

Given two Schubert classes σ_u and σ_v in $\mathrm{H}^*(G/P, \mathbb{C})$, we define a subvariety Σ_u^v of G/P. This variety should play the role of the Richardson variety; more precisely, we conjecture that $[\Sigma_u^v]_{\odot_0} = \sigma_u \odot_0 \sigma_v$. We give some evidence for this conjecture, and prove special cases.

Finally, we use the subbundles of TG/P to give a geometric characterization of the G-homogeneous locus of any Schubert subvariety of G/P.

1 Introduction

Let G be a complex semisimple group and let P be a parabolic subgroup of G. In this paper, we are interested in the Belkale-Kumar product $\odot_0$ on the cohomology group of the flag variety G/P.

The Belkale-Kumar product. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P$. Let W and W_P denote respectively the Weyl groups of G and P. Let W^P be the set of minimal length representative in the cosets of W/W_P. For any $w \in W^P$, let X_w be the corresponding Schubert variety (that

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is, the closure of BwP/P) and let $[X_w] \in \mathrm{H}^*(G/P, \mathbb{C})$ be its cohomology class. The structure coefficients c_{uv}^w of the cup product are written as

$$[X_u].[X_v] = \sum_{w \in W^P} c_{uv}^w [X_w]. \tag{1}$$

Let L be the Levi subgroup of P containing T. This group acts on the tangent space $T_{P/P}G/P$ of G/P at the base point P/P. Moreover, this action is multiplicity free and we have a unique decomposition

$$T_{P/P}G/P = V_1 \oplus \cdots \oplus V_s, \tag{2}$$

as sum of irreducible L-modules. It turns out that, for any $w \in W^P$, the tangent space $T_w := T_{P/P}w^{-1}X_w$ of the variety $w^{-1}X_w$ at the smooth point P/P decomposes as

$$T_w = (V_1 \cap T_w) \oplus \cdots \oplus (V_s \cap T_w). \tag{3}$$

Set $T_w^i := T_w \cap V_i$. Since $[X_w]$ has degree $2(\dim(G/P) - \dim(T_w))$ in the graded algebra $\mathrm{H}^*(G/P)$, if $c_{uv}^w \neq 0$ then

$$\dim(T_u) + \dim(T_v) = \dim(G/P) + \dim(T_w), \tag{4}$$

or equivalently

$$\sum_{i=1}^{s} \Big(\dim(T_u^i) + \dim(T_v^i) \Big) = \sum_{i=1}^{s} \Big(\dim(V_i) + \dim(T_w^i) \Big). \tag{5}$$

The Belkale-Kumar product requires the equality (5) to hold term by term. More precisely, the structure constants $\tilde{c}_{uv}^w$ of the Belkale-Kumar product [BK06],

$$[X_u] \odot_0 [X_v] = \sum_{w \in W^P} \tilde{c}_{uv}^w [X_w] \tag{6}$$

can be defined as follows (see [RR11, Proposition 2.4]):

$$\tilde{c}_{uv}^w = \begin{cases} c_{uv}^w & \text{if } \forall 1 \leq i \leq s \quad \dim(T_u^i) + \dim(T_v^i) = \dim(V_i) + \dim(T_w^i), \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The product $\odot_0$ defined in such a way is associative and satisfies Poincaré duality. The Belkale-Kumar product was proved to be the more relevant product for describing the Littlewood-Richardson cone (see [BK06, Res10, Res11a]).

Motivations. If G/P is cominuscule then $T_{P/P}G/P$ is an irreducible L-module (that is, $s = 1$). In this case, the Belkale-Kumar product is simply the cup product. This paper is motivated by the guess that several known results for cominuscule G/P could be generalized to any G/P using the Belkale-Kumar product. In particular, it might be a first step toward a positive geometric uniform combinatorial rule for computing the coefficients $\tilde{c}_{uv}^w$. Indeed, we define a subvariety Σ_u^v which is encoded by combinatorial datum (precisely a subset of roots of G). We also define a Belkale-Kumar fundamental class $[\Sigma_u^v]_{\odot_0}$ and conjecture that $[\Sigma_u^v]_{\odot_0} = [X_u]\odot_0[X_v]$.

A geometric construction of the Belkale-Kumar ring. The first aim of this paper is to give a geometric construction of the Belkale-Kumar ring which does not deal with the Schubert basis. Consider the connected center Z of L and its character group $X(Z)$. The Azad-Barry-Seitz theorem (see [ABS90]) asserts that each V_i in the decomposition (2) is an isotipical component for the action of Z associated to some weight denoted by $\alpha_i \in X(Z)$. The group P acts on $T_{P/P}G/P$ but does not stabilize the decomposition (2). But, the group $X(Z)$ is endowed with a partial order $\succcurlyeq$ (see Section 3.1 for details), such that for any $\alpha \in X(Z)$ the sum

$$V^{\succcurlyeq\alpha} := \oplus_{\alpha_i\succcurlyeq\alpha} V_i \tag{8}$$

is P-stable. Since $V^{\succcurlyeq\alpha}$ is P-stable, it induces a G-homogeneous subbundle $T^{\succcurlyeq\alpha}G/P$ of the tangent bundle TG/P. We obtain a family of distributions indexed by $X(Z)$. This family forms a decreasing multi-filtration: if $\alpha\succcurlyeq\beta$ then $T^{\succcurlyeq\alpha}G/P$ is a subbundle of $T^{\succcurlyeq\beta}G/P$. Moreover, these distributions are globally integrable in the sense that

$$[T^{\succcurlyeq\alpha}G/P, T^{\succcurlyeq\beta}G/P] \subset T^{\succcurlyeq\alpha+\beta}G/P. \tag{9}$$

This allows us to define a filtration ("à la Hodge") of the De Rham complex and so of the algebra $\mathrm{H}^*(G/P,\mathbb{C})$ indexed by the group $X(Z)\times\mathbb{Z}$. We consider the associated graded algebra.

Theorem 1 *The $(X(Z)\times\mathbb{Z})$-graded algebra $\mathrm{Gr}\,\mathrm{H}^*(G/P,\mathbb{C})$ associated to the $(X(Z)\times\mathbb{Z})$-filtration is isomorphic to the Belkale-Kumar algebra $(\mathrm{H}^*(G/P,\mathbb{C}),\odot_0)$.*

The first step to get Theorem 1 is to give it a precise sense defining the orders on $X(Z)$ and $X(Z)\times\mathbb{Z}$ and the filtrations. The key point to prove the isomorphism is that the Schubert basis $([X_w])_{w\in W^P}$ of $\mathrm{H}^*(G/P,\mathbb{C})$ is adapted to the filtration. Indeed each subspace of the filtration is spanned by the Schubert classes it contains. To obtain this result, we use Kostant's harmonic forms [Kos61].

Theorem 1 is closed to [BK06, Theorem 43] obtained by Belkale-Kumar. In [BK06], the filtration on $\mathrm{H}^*(G/P,\mathbb{C})$ is defined using the Schubert basis. On the other hand, the filtration on $\mathrm{H}^*_{DR}(G/P,\mathbb{C})$ is defined using Kostant's K-invariant forms (where K is a compact form of G). Here, the filtration is defined independently of any basis or any choice of a compact form of G.

This "intrinsic" definition of the Belkale-Kumar also gives a pleasant interpretation of the functoriality result of [RR11, Theorem 1.1]. Indeed, let τ be a one-parameter subgroup of Z such that

$$\forall\alpha\in X(Z)\qquad \alpha\succcurlyeq 0 \;\Rightarrow\; \langle\tau,\alpha\rangle \geq 0,$$

and

$$\forall 1 \leq i \neq j \leq s \qquad \langle \tau, \alpha_i \rangle \neq \langle \tau, \alpha_j \rangle. \tag{10}$$

Setting for any $n \in \mathbb{Z}$

$$V^{\geq n} := \oplus_{\langle \tau, \alpha \rangle \geq n} V_i,$$

one gets a globally integrable family of distributions on G/P indexed by $\mathbb{Z}$. Then, one gets a $\mathbb{Z}$-filtration of the ring $\mathrm{H}^*(G/P, \mathbb{C})$. By (6) and (10), the associated $\mathbb{Z}$-graded ring is isomorphic to $\mathrm{Gr}\,\mathrm{H}^*(G/P, \mathbb{C})$. Then, [RR11, Theorem 1.1] is a direct consequence of the immediate lemma 12 below.

A conjecture. The main motivation to show Theorem 1 is to define the fundamental class for the Belkale-Kumar product of any irreducible subvariety Y of G/P. This class $[Y]_{\odot_0}$ which belongs to $\mathrm{Gr}\,\mathrm{H}^*(G/P, \mathbb{C})$ is defined in Section 4.4.

Let w_0 and w_0^P be the longest elements of W and W_P respectively. If $v \in W^P$ then $v^\vee := w_0 v w_0^P$ belongs to W^P and $[X_{v^\vee}]$ is the Poincaré dual class of $[X_v]$. Consider the weak Bruhat order $\lessdot$ on W^P. We are interested in the product $[X_u]\odot_0[X_v] \in \mathrm{H}^*(G/P, \mathbb{C})$, for given u and v in W^P. Lemma 24 below shows that if $[X_u]\odot_0[X_v] \neq 0$ then $v^\vee \lessdot u$. Assume that $v^\vee \lessdot u$ and consider the group

$$H_u^v := u^{-1} B u \cap w_0^P v^{-1} B v w_0^P. \tag{11}$$

It is a closed connected subgroup of G containing T; in particular, it can be encoded by its set Φ_u^v of roots. Let Σ_u^v denote the closure of the H_u^v-orbit of P/P:

$$\Sigma_u^v = \overline{H_u^v . P/P}. \tag{12}$$

Another characterization of this subvariety is given by the following statement.

Proposition 1 *The variety Σ_u^v is the unique irreducible component of the intersection $u^{-1}X_u \cap w_0^P v^{-1} X_v$ containing P/P. Moreover, this intersection is transverse along Σ_u^v.*

Our main conjecture can be stated as follow.

Conjecture 1 *If $v^\vee \lessdot u$ then*

$$[\Sigma_u^v]_{\odot_0} = [X_u]\odot_0[X_v] \in \mathrm{Gr}\,\mathrm{H}^*(G/P, \mathbb{C}).$$

Write

$$[\Sigma_u^v]_{\odot_0} = \sum_{w \in W^P} d_{uv}^w [X_w].$$

By Proposition 11 d_{uv}^w are integers. Moreover, Conjecture 1 is equivalent to $\tilde{c}_{uv}^w = d_{uv}^w$ for any $w \in W^P$. The first evidence is the following weaker result.

Proposition 2 *Then*

(i) $d_{uv}^w \neq 0 \iff \tilde{c}_{uv}^w \neq 0$;

(ii) $d_{uv}^w \leq \tilde{c}_{uv}^w$.

Known cases. Conjecture 1 generalizes another one for G/B. Indeed, if $G/P = G/B$ is a complete flag variety then Conjecture 1 is equivalent to the following one.

Conjecture 2 *For G/B and any u, v, and w in W, the structure constant $\tilde{c}_{uv}^w$ is equal to 1 if for any $1 \leq i \leq s$, $\dim(T_u^i) + \dim(T_v^i) = \dim(V_i) + \dim(T_w^i)$ and 0 otherwise.*

In particular, Conjecture 2 implies that we have a uniform combinatorial and geometric model for the Belkale-Kumar product. Conjecture 2 was explicitly stated in [DR09]. E. Richmond proved in [Ric09] and [Ric12] this conjecture for $G = \mathrm{SL}_n(\mathbb{C})$ or $G = \mathrm{Sp}_{2n}(\mathbb{C})$. In Section 7, we prove it for $G = \mathrm{SO}_{2n+1}(\mathbb{C})$ (this proof is certainly known from some specialists but I have shortly included it for convenience). Very rencently, Dimitrov-Roth got also a proof for classical groups and G2 [DR17]. Using [BK06, Corollary 44], we wrote a program [Res13] to check this conjecture: it is checked in type F_4 and E_6.

Conjecture 1 will be proved in type A in a forthcoming paper.

Combinatorial evidences. Consider the following degenerate version of Conjecture 1.

Conjecture 3 *The product $[X_u] \odot_0 [X_v]$ only depends on the set Φ_u^v.*

The expression of the Belkale-Kumar structure coefficients as products given in [Ric09] shows that Conjecture 3 holds in type A. Consider now the case $G = \mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ and P maximal. In this case, in [Res12], it is proved that the set of triples $(u, v, w) \in W^P$ such that $\tilde{c}_{uv}^w = 1$ only depends on Φ_u^v, according to Conjecture 3. If G/P is cominuscule $\tilde{c}_{uv}^w = c_{uv}^w$ for any $(u, v, w) \in W^P$. Then the Thomas-Young combinatorial rule [TY09] for c_{uv}^w implies that Conjecture 3 holds.

Distributions and Schubert varieties. In Section 3, we study the restriction of the distributions to the Schubert varieties X_u. More precisely, for any x in X_u and $\alpha \in X(Z)$ we are interested in $T_x X_u \cap T_x^{\succcurlyeq \alpha} G/P$. For α fixed, the dimension of $T_x X_u \cap T_x^{\succcurlyeq \alpha} G/P$ has a fixed value for $x \in X_u$ general and can jump for x in a strict subvariety of X_u. Consider the maximal open subset X_u^0 of X_u such that for any $\alpha \in X(Z)$ the dimension of $T_x X_u \cap T_x^{\succcurlyeq \alpha} G/P$ does not depend on $x \in X_u^0$. Consider the global stabilizer Q_u of X_u, that is, the set of $g \in G$ such that $g.X_u = X_u$. Since X_u is B-stable, Q_u is a standard parabolic subgroup of G.

Proposition 3 *With above notation, we have*

$$X_u^0 = Q_u.uP/P.$$

If G/P is cominuscule, the filtration is trivial and Proposition 3 asserts that Q_u acts transitively on the smooth locus of X_u. This was previously proved by Brion and Polo in [BP00]. Proposition 3 is in the philosophy to generalize known results from cominuscule homogeneous spaces to any homogeneous space G/P, using the Belkale-Kumar product.

Note that Proposition 3 is equivalent to [BKR12, Theorem 7.4]. Nevertheless, we think that the distributions give a pleasant interpretation of this result. In Section 3 we present a proof using the properties of the Peterson map.

Retruning to the setting of Conjecture 1, we assume moreover that the intersection $u^{-1}X_u \cap w_0^P v^{-1}X_v$ is proper. Then Conjecture 1 is implied by the fact that Σ_u^v is the only irreducible component of this intersection that has the same $X(Z)$-dimension (see Section 2.3). Proposition 3 is clearly related to this version of Conjecture 1.

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Contents

2 Infinitesimal filtrations

2.1 The case of a vector space

Ordered group. Let Γ be a finitely generated free abelian group whose the law is denoted by $+$. Consider the vector space $\Gamma\otimes_{\mathbb{Z}}\mathbb{Q}$. Assume that a closed strictly convex cone $\mathcal{C}$ in $\Gamma\otimes_{\mathbb{Z}}\mathbb{Q}$ of nonempty interior is given. Moreover, we assume that $\mathcal{C}$ is rational polyhedral, that is defined by finitely many linear rational inequalities, or equivalently generated, as a cone, by finitely many vectors in $\Gamma\otimes_{\mathbb{Z}}\mathbb{Q}$. We consider the partial order $\preccurlyeq$ on Γ defined by $\alpha\preccurlyeq\beta$ if and only if $\beta-\alpha$ belongs to $\mathcal{C}$. The group Γ endowed with the order $\preccurlyeq$ is an ordered group:

$$\forall\alpha,\,\beta,\,\gamma\in\Gamma\quad \alpha\preccurlyeq\beta\ \Rightarrow\ (\alpha+\gamma)\preccurlyeq(\beta+\gamma). \tag{13}$$

The order $\preccurlyeq$ satisfies the following version of the Ramsey theorem (see also Bolzano-Weirstrass' theorem).

Lemma 1 *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of pairwise distinct elements of Γ such that $\alpha_n\preccurlyeq 0$ for any n.*

Then there exists a subsequence $(\alpha_{\phi(n)})_{n\in\mathbb{N}}$ such that for any n

$$\alpha_{\phi(n+1)}\prec\alpha_{\phi(n)}.$$

Proof. Let $\varphi_1, \ldots, \varphi_s$ be elements of $\mathrm{Hom}(\Gamma, \mathbb{Z})$ such that $x \in \mathcal{C}$ if and only if $\varphi_i(x) \geq 0$ for any $i = 1, \ldots, s$.

Consider first the sequence $\varphi_1(\alpha_n)$ and set

$$I_1 = \{n \,|\, \forall m \geq n \;\; \varphi_1(\alpha_m) > \varphi_1(\alpha_n)\}.$$

Assume, for a contradiction that I_1 is infinite. Denoting by $\phi(k)$ the k^{th} element of I_1, we get an increasing subsequence of $(\varphi_1(\alpha_n))_{n\in\mathbb{N}}$. But $\varphi_1(\alpha_n) \in \mathbb{Z}$ and $\varphi_1(\alpha_n) \leq 0$: a contradiction. Hence I_1 is finite.

Up to taking a subsequence, we may assume that I_1 is empty; that is

$$\forall n \geq 0 \qquad \exists m > n \qquad \varphi_1(\alpha_m) \leq \varphi_1(\alpha_n).$$

This property allows to construct a nonincreasing subsequence of $\varphi_1(\alpha_n)$. Hence, by considering such a subsequence, we may assume that

$$\forall n \geq 0 \qquad \varphi_1(\alpha_{n+1}) \leq \varphi_1(\alpha_n).$$

By successively proceeding similarly, for $i = 2, \ldots, s$, one gets a subsequence $\alpha_{\psi(n)}$ such that

$$\forall i = 1, \ldots, n \qquad \forall n \qquad \varphi_i(\alpha_{\psi(n+1)}) \leq \varphi_1(\alpha_{\psi(n)}).$$

Since the α_n are pairwise distinct, we deduce that $\alpha_{\psi(n+1)} \preccurlyeq \alpha_{\psi(n)}$. □

Remark. Consider the cone $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \;:\; y \geq 0 \text{ and } \sqrt{2}x - y \geq 0\}$ and the group $\Gamma = \mathbb{Z}^2$. Lemma 1 does not hold for the induced order $\succcurlyeq$ showing the rationality assumption on $\mathcal{C}$ is necessary. Indeed, denote by $\pi \,:\, \mathbb{R}^2 \longrightarrow \mathbb{R}$ the linear projection on the line $y = 0$ with kernel the line $y = \sqrt{2}x$. Then $\pi(\mathbb{Z}^2)$ is dense as the group generated by 1 and $\frac{\sqrt{2}}{2}$. In particular, one can construct a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ such that $y_{n+1} < y_n < 0$ and $0 > \sqrt{2}x_{n+1} - y_{n+1} > \sqrt{2}x_n - y_n$. Then the elements of the sequence are pairwise incomparable for the partial order $\succcurlyeq$.

Γ-filtration. The group Γ is used here to index filtrations.

Definition. Let V be a finite dimensional real or complex vector space. A Γ-*filtration* of V is a collection $F^{\succcurlyeq\beta}V$ of linear subspaces of V indexed by $\beta \in \Gamma$ satisfying

(i) $\alpha \preccurlyeq \beta \;\Rightarrow\; F^{\succcurlyeq\beta}V \subset F^{\succcurlyeq\alpha}V$,

(ii) $\exists \beta_0 \in \Gamma$ s.t. $V = F^{\succcurlyeq\beta_0}V$,

(iii) if $F^{\succcurlyeq\alpha}V \neq \{0\}$ then $\alpha \preccurlyeq 0$.

Lemma 2 *Let $(F^{\succcurlyeq\beta}V)_{\beta\in\Gamma}$ be a Γ-filtration. Then the set $\{F^{\succcurlyeq\beta}V \,|\, \beta \in \Gamma\}$ of linear subspaces of V is finite.*

Proof. By contradiction, assume that there exists a sequence $F^{\succcurlyeq\alpha_n}V$ of pairwise distinct linear subspaces of V. By axiom (iii), $\alpha_n\preccurlyeq 0$ for any but eventually one n. Now, Lemma 1 implies that there exists a decreasing subsequence $\alpha_{\phi(k)}$. Since the linear subspaces $F^{\succcurlyeq\alpha_n}V$ are pairwise distinct, the subsequence $F^{\succcurlyeq\alpha_{\phi(k)}}V$ is increasing. This contradicts the assumption that V is finite dimensional. $\square$

Γ-filtrations coming from decompositions. For each $\beta\in\Gamma$, $\sum_{\alpha\succ\beta}F^{\succcurlyeq\alpha}V$ is a linear subspace of $F^{\succcurlyeq\beta}V$. Let us choose a supplementary subspace S^β:

$$F^{\succcurlyeq\beta}V = S^\beta \oplus \sum_{\alpha\succ\beta} F^{\succcurlyeq\alpha}V. \tag{14}$$

One of the motivation for axiom (iii) in the definition of Γ-filtration is the following lemma.

Lemma 3 *With above notation,*

$$F^{\succcurlyeq\beta}V = \sum_{\alpha\succcurlyeq\beta} S^\alpha. \tag{15}$$

Proof. It is clear that the sum is contained in $F^{\succcurlyeq\beta}V$. Conversely, since V is finite dimensional, we have

$$F^{\succcurlyeq\beta}V = S^\beta \oplus (F^{\succcurlyeq\alpha_1}V + \cdots + F^{\succcurlyeq\alpha_s}V),$$

for some $\alpha_i\in\Gamma$ such that $\alpha_i\succ\beta$. By axiom (iii), we may assume that for any $i=1,\ldots,s$ we have $\alpha_i\preccurlyeq 0$. If each $F^{\succcurlyeq\alpha_i}V$ satisfies the lemma, the lemma is proved for $F^{\succcurlyeq\beta}V$. Otherwise, we restart the proof with each α_i in place of β. Since Γ is discrete, the set of $\alpha\in\Gamma$ such that $0\succcurlyeq\alpha\succcurlyeq\beta$ is finite. In particular, the procedure ends by axiom (iii) of the definition of a Γ-filtration. $\square$

Conversely, assume that a linear subspace S^α of V is given for any $\alpha\in\Gamma$. If these linear subspaces satisfy ($S^\alpha\neq\{0\}\Rightarrow\alpha\preccurlyeq 0$), and there exist $\alpha_1,\ldots,\alpha_s$ such that $V=S^{\alpha_1}+\cdots+S^{\alpha_s}$ then the formula (15) defines a Γ-filtration of V. The Γ-filtration of V is said to *come from a decomposition* if there exists a decomposition

$$V=\bigoplus_{\alpha\in\Gamma} S^\alpha,\ \text{with } S^\alpha\neq\{0\}\ \Rightarrow\ \alpha\preccurlyeq 0, \tag{16}$$

such that (15) holds.

The *f-dimension vector* (f stand for filtered) of the Γ-filtration, is the vector $(fd^\beta(V))_{\beta\in\Gamma}$ of $\mathbb{N}^\Gamma$ defined by

$$\Gamma\longrightarrow\mathbb{N},\ \beta\longmapsto fd^\beta(V)=\dim(F^{\succcurlyeq\beta}V),$$

for any $\beta \in \Gamma$. Define the *grading associated to the* Γ*-filtration* by setting

$$\operatorname{Gr}^\beta V = \frac{F^{\succcurlyeq\beta}V}{\sum_{\alpha\succ\beta} F^{\succcurlyeq\alpha}V}, \text{ and } \operatorname{Gr} V = \bigoplus_{\beta\in\Gamma} \operatorname{Gr}^\beta V. \tag{17}$$

The *g-dimension vector* (g stands for graded) $(gd^\beta(V))_{\beta\in\Gamma}$ of the Γ-filtration is defined by

$$\Gamma \longrightarrow \mathbb{N},\ \beta \longmapsto gd^\beta(V) := \dim(\operatorname{Gr}^\beta V).$$

Lemma 4 *The* Γ*-filtration comes from a decomposition if and only if*

$$\dim(\operatorname{Gr} V) = \dim(V). \tag{18}$$

In this case, the g-dimension vector of V *only depends on the* f*-dimension vector of* V*.*

Proof. Assume first that the Γ-filtration comes from a decomposition. Fix linear subspaces S^α satisfying the conditions (16) and (15). For any $\beta \in \Gamma$, the identity (14) holds and $\dim(\operatorname{Gr}^\beta V) = \dim(S^\beta)$. Hence the lemma follows from the condition (16).

Conversely, assume that the condition (18) is fulfilled and choose linear subspaces S^β satisfying (14). Let $\beta_0 \in \Gamma$ such that $F^{\succcurlyeq\beta_0}V = V$. Lemma 3 implies that $V = \sum_{\beta\succcurlyeq\beta_0} S^\beta$. The condition (18) implies that the sum is direct. Moreover, it implies that $S^\gamma = \{0\}$ if $\gamma \not\succcurlyeq \beta_0$. Using Lemma 3 once again, we deduce that the filtration comes from the decomposition $V = \bigoplus_\beta S^\beta$.

If $fd^\beta = 0$ then $F^{\succcurlyeq\beta}V = \{0\}$, $\operatorname{Gr}^\beta = \{0\}$ and $gd^\beta = 0$. Let Γ_{max} be the set of maximal elements among the elements β in Γ satisfying $F^{\succcurlyeq\beta}V \neq \{0\}$. For $\beta \in \Gamma_{\max}$, we have $gd^\beta(V) = fd^\beta(V)$. Et caetera. □

Example. Consider the group $\mathbb{Z}^2$ endowed with the order $(a,b)\preccurlyeq(a',b')$ if and only if $a \le a'$ and $b \le b'$. Fix a two dimensional vector space V and three pairwise distinct lines l_1, l_2, and l_3 in V. Consider the following family $(S^\beta)_{\beta\in\mathbb{Z}^2}$ of linear subspaces of V: $S^{(-2,0)} = l_1$, $S^{(0,-2)} = l_2$, $S^{(-1,-1)} = l_3$, and $S^\beta = \{0\}$ if $\beta \notin \{(-2,0),(0,-2),(-1,-1)\}$. The filtration defined by the formula (15) does not come from a decomposition. More precisely, $\operatorname{Gr} V \simeq l_1 \oplus l_2 \oplus l_3$ has dimension three whereas V has dimension two.

Another useful notion is the *weight* $\rho(V)$ of the Γ-filtration of V defined by

$$\rho(V) = \sum_{\beta\in\Gamma} gd^\beta(V)\beta. \tag{19}$$

Filtrations induced on a linear subspace. Let W be a linear subspace of V. The Γ-filtration on V induces one on W by setting

$$\forall\beta \in \Gamma \quad F^{\succcurlyeq\beta}W := W \cap F^{\succcurlyeq\beta}V. \tag{20}$$

Lemma 5 *If the Γ-filtration on V comes from a decomposition then the induced Γ-filtration on W comes from a decomposition.*

Proof. Fix linear subspaces S_V^β and S_W^β of V such that

$$V = S_V^\beta \oplus F^{\succcurlyeq\beta}V, \quad W = S_W^\beta \oplus F^{\succcurlyeq\beta}W, \quad S_W^\beta \subset S_V^\beta.$$

Lemma 3 implies that

$$W = \sum_{\beta\in\Gamma} S_W^\beta. \tag{21}$$

Lemma 4 shows

$$V = \bigoplus_{\beta\in\Gamma} S_V^\beta.$$

Since $S_W^\beta \subset S_V^\beta$ it follows that the sum (21) is direct. $\square$

Filtrations induced on p-forms. Let p a nonnegative integer. A Γ-filtration of V induces a filtration on the space $\bigwedge^p V^*$ of skewsymmetric p-forms on V as follows.

Definition. Let $\beta \in \Gamma$. Denote by $F^{\preccurlyeq\beta}\bigwedge^p V^*$ the linear subspace of forms $\omega \in \bigwedge^p V^*$ such that for any any $\alpha_1, \ldots, \alpha_p \in \Gamma$, for any $v_i \in F^{\succcurlyeq\alpha_i}V$, we have

$$\alpha_1 + \cdots + \alpha_p \not\preccurlyeq \beta \Rightarrow \omega(v_1, \ldots, v_p) = 0. \tag{22}$$

The first properties of these linear subspaces are.

Proposition 4 *(i) If $\beta \preccurlyeq \gamma$ then $F^{\preccurlyeq\beta}\bigwedge^p V^* \subset F^{\preccurlyeq\gamma}\bigwedge^p V^*$.*

(ii) Let $\beta_0 \in \Gamma$ be such that $F^{\succcurlyeq\beta_0}V = V$. If $F^{\preccurlyeq\gamma}\bigwedge^p V^* \neq \{0\}$ then $\gamma \succcurlyeq p\beta_0$.

(iii) We have $F^{\preccurlyeq 0}\bigwedge^p V^* = \bigwedge^p V^*$.

(iv) For β and γ in Γ, we have $F^{\preccurlyeq\beta}\bigwedge^p V^* \wedge F^{\preccurlyeq\gamma}\bigwedge^q V^* \subset F^{\preccurlyeq\beta+\gamma}\bigwedge^{p+q} V^*$.

Proof. If $\beta \preccurlyeq \gamma$ then $\alpha_1 + \cdots + \alpha_p \not\preccurlyeq \gamma$ implies $\alpha_1 + \cdots + \alpha_p \not\preccurlyeq \beta$. Hence the conditions defining $F^{\preccurlyeq\gamma}\bigwedge^p V^*$ are conditions defining $F^{\preccurlyeq\beta}\bigwedge^p V^*$. The first assertion follows.

Let $\gamma \not\succcurlyeq p\beta_0$. The definition of $F^{\preccurlyeq\gamma}\bigwedge^p V^*$ with $\alpha_1 = \cdots = \alpha_p = \beta_0$ implies that $F^{\preccurlyeq\gamma}\bigwedge^p V^*$ is reduced to zero.

Let ω be any p-form. We want to prove that $\omega \in F^{\preccurlyeq 0}\bigwedge^p V^*$. Let $\alpha_1, \ldots, \alpha_p$ such that $\alpha_1 + \cdots + \alpha_p \not\preccurlyeq 0$. Then some i_0 satisfies $\alpha_{i_0} \not\preccurlyeq 0$. In particular, $F^{\succcurlyeq\alpha_{i_0}}V = \{0\}$. This implies that ω is zero on $F^{\succcurlyeq\alpha_1}V \times \cdots \times F^{\succcurlyeq\alpha_s}V$.

Let ω_1 and ω_2 belong to $F^{\preccurlyeq\beta}\bigwedge^p V^*$ and $F^{\preccurlyeq\alpha}\bigwedge^q V^*$ respectively. Let $\alpha_1,\dots,\alpha_{p+q}$ be such that $\alpha_1+\cdots+\alpha_{p+q}\not\preccurlyeq\beta+\gamma$. Let $v_i\in F^{\succcurlyeq\alpha_i}V$. Then

$$\begin{aligned}&(\omega_1\wedge\omega_2)(v_1,\dots,v_{p+q})=\\ &\qquad\tfrac{1}{(p+q)!}\textstyle\sum_{\sigma\in\mathcal{S}_{p+q}}\varepsilon(\sigma)\omega_1(v_{\sigma(1)},\dots,v_{\sigma(p)}).\omega_2(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)}).\end{aligned} \tag{23}$$

It is sufficient to prove that any term in the sum (23) is zero. Since $(\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(p)})+(\alpha_{\sigma(p+1)}+\cdots+\alpha_{\sigma(p+q)})\not\preccurlyeq\beta+\gamma$, either $(\alpha_{\sigma(1)}+\cdots+\alpha_{\sigma(p)})\not\preccurlyeq\beta$ or $(\alpha_{\sigma(p+1)}+\cdots+\alpha_{\sigma(p+q)})\not\preccurlyeq\gamma$. In the two cases, the product

$$\omega_1(v_{\sigma(1)},\dots,v_{\sigma(p)}).\omega_2(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})$$

is equal to zero. □

Remark. The three first assertions of Proposition 4 mean that $(F^{\preccurlyeq\beta}\bigwedge^p V^*)_{\beta\in\Gamma}$ is a Γ-filtration of $\bigwedge^p V^*$ up to the changing of index $\beta\mapsto p\beta_0-\beta$. Indeed, even for $p=1$, taking orthogonal reverses inclusions and exchanges $\{0\}$ with the whole space.

Filtrations coming from a decomposition.

Lemma 6 *Let p be a positive integer. If the Γ-filtration on V comes from a decomposition then the induced Γ-filtration $F^{\succcurlyeq p\beta_0-\beta}\bigwedge^p V^*$ on $\bigwedge^p V^*$ comes from a decomposition.*

Proof. Write

$$V=\bigoplus_{\alpha\in\Gamma}S^\alpha\quad\text{and}\quad F^{\succcurlyeq\beta}V=\bigoplus_{\alpha\succcurlyeq\beta}S^\alpha,$$

with $(S^\alpha V\neq\{0\}\Rightarrow\alpha\preccurlyeq 0)$. For any $\beta\in\Gamma$, denote by T^β the orthogonal of $\bigoplus_{\alpha\neq\beta}S^\alpha$ in V^*. It can be identified with the dual of S^β and

$$V^*=\bigoplus_{\beta\in\Gamma}T^\beta. \tag{24}$$

For any collection of subspaces $F_1,\dots,F_p$ of V^*, $\pi(F_1\otimes\cdots\otimes F_p)$ denotes the subspace of $\wedge^p V^*$ obtained by adding wedge products of elements of the subspaces F_i. For any $\theta\in\Gamma$, set

$$(\wedge^p V^*)^\theta:=\sum_{\beta_1+\cdots+\beta_p=\theta}\pi(T^{\beta_1}\otimes\cdots\otimes T^{\beta_p}).$$

It is clear that (24) implies that

$$\wedge^p V^*=\bigoplus_{\theta\in\Gamma}(\wedge^p V^*)^\theta.$$

Moreover, for any $\theta\in\Gamma$, $(\wedge^p V^*)^\theta$ is the set of p-forms ω such that for any $\alpha_i\in\Gamma$ and $v_i\in S^{\alpha_i}$ such that $\alpha_1+\cdots+\alpha_p\neq\theta$ we have $\omega(v_1,\dots,v_p)=0$.

We claim that

$$F^{\preccurlyeq\beta}\wedge^p V^* = \bigoplus_{\theta\preccurlyeq\beta}(\wedge^p V^*)^\theta. \tag{25}$$

Indeed $F^{\preccurlyeq\beta}\bigwedge^p V^*$ is the subspace of forms $\omega \in \bigwedge^p V^*$ such that for any $\alpha_1,\dots,\alpha_p\in\Gamma$, any $v_i\in S^{\alpha_i}$, we have

$$\alpha_1+\cdots+\alpha_p\not\preccurlyeq\beta \Rightarrow \omega(v_1,\dots,v_p)=0.$$

Let θ such that $(\wedge^p V^*)^\theta\neq\{0\}$. Then there exist $\beta_1,\dots,\beta_p$ in Γ such that $\beta_1+\cdots+\beta_p=\theta$ and $T^{\beta_i}\neq\{0\}$ for any i. Hence $S^{\beta_i}\neq\{0\}$ for any i and $\beta_i\preccurlyeq 0$. We deduce that $\theta\preccurlyeq 0$. □

2.2 The case of manifolds

Let M be a smooth connected manifold and let TM denote its tangent bundle. Here comes the central definition of this work.

Definition. An *infinitesimal* Γ*-filtration* of M is a collection $F^{\succcurlyeq\beta}TM$ of vector subbundles of TM indexed by $\beta\in\Gamma$ satisfying

(i) $\alpha\preccurlyeq\beta \;\Rightarrow\; F^{\succcurlyeq\beta}TM\subset F^{\succcurlyeq\alpha}TM$,

(ii) $\exists\beta_0\in\Gamma$ s.t. $TM=F^{\succcurlyeq\beta_0}TM$,

(iii) if $F^{\succcurlyeq\alpha}TM\neq\{\underline{0}\}$ then $\alpha\preccurlyeq 0$.

The *f-rank vector* of the infinitesimal filtration is the map

$$\beta\longmapsto \mathrm{rk}(F^{\succcurlyeq\beta}TM), \tag{26}$$

and belongs to $\mathbb{N}^\Gamma$.

Definition. An infinitesimal Γ-filtration is said to *come from a decomposition* if for any $x\in M$, the Γ-filtration of T_xM comes from a decomposition.

Remark. We do not require a Γ-decomposition of the tangent bundle TM but only for a punctual decomposition.

Lemma 7 *Consider an infinitesimal* Γ*-filtration on* M *coming from a decomposition. Then for any* β*, the sum* $\sum_{\alpha\succ\beta}F^{\succcurlyeq\alpha}TM$ *is a subbundle of* TM*.*

Proof. Fix x in M and a Γ-decomposition of $T_xM=\oplus_\alpha S^\alpha$ such that the identities (16) and (15) hold. Then $\sum_{\alpha\succ\beta}F^{\succcurlyeq\alpha}T_xM=\sum_{\alpha\succ\beta}S^\alpha$. In particular, its dimension only depends on the g-dimension vector of the filtration of T_yM. This g-dimension vector only depends on the f-dimension by Lemma 4. It follows that the dimension of $\sum_{\alpha \succ \beta} F^{\succcurlyeq \alpha} T_x M$ does not depend on x. Now, the lemma follows from classical properties of vector subdundles. $\square$

Define the *grading associated to the infinitesimal* Γ*-filtration coming from a decomposition* by setting

$$\mathrm{Gr}^\beta TM = \frac{F^{\succcurlyeq \beta} TM}{\sum_{\alpha \succ \beta} F^{\succcurlyeq \alpha} TM} \quad \text{and} \quad \mathrm{Gr}\, TM = \bigoplus_{\beta \in \Gamma} \mathrm{Gr}^\beta TM. \tag{27}$$

They are vector bundles on M. The *g-rank vector* $(gd^\beta(M))_{\beta \in \Gamma}$ of the Γ-filtration is defined by

$$\Gamma \longrightarrow \mathbb{N},\ \beta \longmapsto gd^\beta(M) := \mathrm{rk}(\mathrm{Gr}^\beta TM).$$

2.3 The case of varieties

Let X be a smooth complex irreducible variety. Consider the complex tangent bundle TX.

Definition. An infinitesimal Γ-filtration of X is said to be *algebraic* if each $F^{\succcurlyeq \beta} TX$ is a complex algebraic vector subbundle of TX.

Let Y be an irreducible subvariety of X. For $y \in Y$, the Zariski-tangent space $T_y Y$ of Y at the point y is a complex subspace of $T_y X$. Set

$$F^{\succcurlyeq \beta} T_y Y = F^{\succcurlyeq \beta} T_y X \cap T_y Y. \tag{28}$$

Even if Y is smooth, $F^{\succcurlyeq \beta} T_y Y$ does not define a subbundle of TY since its dimension depends on y.

Lemma 8 *For any $\beta \in \Gamma$ and $y \in Y$, there exists an open neighborhood U of y' in Y such that for any $y' \in U$ we have*

$$\dim(F^{\succcurlyeq \beta} T_y Y) \geq \dim(F^{\succcurlyeq \beta} T_{y'} Y). \tag{29}$$

Proof. Locally in $y \in Y$ the subspace $F^{\succcurlyeq \beta} T_y Y$ of $T_y X$ can be expressed as the kernel of a matrix whose coefficients depends algebraically on y. The lemma follows. $\square$

The point $y \in Y$ is said to be Γ*-regular* if

$$\forall \beta \in \Gamma \quad \dim(F^{\succcurlyeq \beta} T_y Y) = \min_{y' \in Y} \dim(F^{\succcurlyeq \beta} T_{y'} Y). \tag{30}$$

Since Γ is countable, Lemma 8 shows that a very general point in Y is Γ-regular. More precisely, Lemma 2 implies that the set of Γ-regular points in Y is open. The open set of Γ-regular points of Y is denoted by $Y^{\Gamma-\mathrm{reg}}$. If $x \in Y^{\Gamma-\mathrm{reg}}$, the g-dimesnion of $T_x Y$ is called the Γ-dimension of Y.

3 Infinitesimal filtration of G/P and Schubert varieties

3.1 Infinitesimal filtration of G/P

As in the introduction, G is a complex semisimple group, P is a parabolic subgroup of G, $T \subset B \subset P$ are a fixed maximal torus and a Borel subgroup. Moreover, L denotes the Levi subgroup of P containing T and Z denotes the neutral component of its center. The group of multiplicative characters of Z is denoted by $X(Z)$. Set $\Gamma = X(Z)$. Our main example is an infinitesimal $X(Z)$-filtration of G/P.

Let S be any torus. If V is any S-module then $\Phi(V, S)$ denotes the set of nonzero weights of S on V. For $\beta \in X(S)$, V_β denotes the eigenspace of weight β.

Denote by $\mathfrak{p}$ and $\mathfrak{g}$ the Lie algebras of P and G and consider the convex cone $\mathcal{C}$ generated by $\Phi(\mathfrak{p}, Z)$ in $X(Z) \otimes \mathbb{Q}$. It is a closed strictly convex polyhedral cone of nonempty interior in $X(Z)\otimes\mathbb{Q}$. The associated order on $X(Z)$ is denoted by $\succcurlyeq$. The decomposition of $\mathfrak{g}/\mathfrak{p}$ under the action of Z:

$$\mathfrak{g}/\mathfrak{p} = \bigoplus_{\alpha \in X(Z)} (\mathfrak{g}/\mathfrak{p})_\alpha \tag{31}$$

is supported on $-\mathcal{C} \cap X(Z)$. The group P acts on $\mathfrak{g}/\mathfrak{p}$ by the adjoint action but does not stabilize the decomposition (31). For any $\beta \in X(Z)$, the linear subspace

$$F^{\succcurlyeq\beta}\mathfrak{g}/\mathfrak{p} = \bigoplus_{\substack{\alpha \in X(Z) \\ \alpha \succcurlyeq \beta}} (\mathfrak{g}/\mathfrak{p})_\alpha \tag{32}$$

is P-stable. More precisely, the set of $F^{\succcurlyeq\beta}\mathfrak{g}/\mathfrak{p}$ forms a P-stable $X(Z)$-filtration of $\mathfrak{g}/\mathfrak{p}$ coming from the decomposition (31). The tangent bundle $T(G/P)$ identifies with the fiber bundle $G \times_P \mathfrak{g}/\mathfrak{p}$ over G/P. These remarks allow to define a G-equivariant infinitesimal $X(Z)$-filtration on G/P by setting for any $\beta \in X(Z)$

$$F^{\succcurlyeq\beta}T(G/P) = G \times_P F^{\succcurlyeq\beta}\mathfrak{g}/\mathfrak{p}. \tag{33}$$

Consider the set $\Phi(\mathfrak{g}/\mathfrak{p}, T)$ of weights of T acting on $\mathfrak{g}/\mathfrak{p}$. Then $\Phi(\mathfrak{g}/\mathfrak{p}, T)$ is a subset of Φ. Let w belong to W^P and consider the centered Schubert variety $w^{-1}X_w$. Then P/P belongs to the open $w^{-1}Bw$-orbit in $w^{-1}X_w$. In particular, it is $X(Z)$-regular. Denote by $\Phi(w)$ the set of weights of T acting on $T_{P/P}w^{-1}X_w$. Then $\Phi(w) = \Phi^- \cap w^{-1}\Phi^+$ is the inversion set of w. Moreover, $\Phi(w)$ is contained in $\Phi(\mathfrak{g}/\mathfrak{p}, T)$. Since P/P is $X(Z)$-regular in $w^{-1}X_w$, the g-dimension of X_w is equal to the g-dimension of $T_{P/P}w^{-1}X_w$. The following result follows directly:

Lemma 9 *The g-dimension of $gd(X_w)$ of X_w is equal to*

$$\begin{array}{ccc} X(Z) & \longrightarrow & \mathbb{Z}_{\geq 0} \\ \alpha & \longmapsto & \#\{\theta \in \Phi(w) \,:\, \theta_{|Z} = \alpha\}, \end{array}$$

where θ belongs to $X(T)$ and $\theta_{|Z}$ denotes its restriction to Z.

3.2 Peterson's application

Let V' be any T-module without multiplicity and let $\beta \in X(T)$. Under the action of $\operatorname{Ker}\beta \subset T$, V' decomposes

$$V' = \bigoplus_{\alpha \in X(T)/\mathbb{Z}\beta} \left(\oplus_{k\in\mathbb{Z}} V'_{\alpha+k\beta}\right). \tag{34}$$

A subset Λ of $\Phi(V', T)$ is said to be β*-convex* if

$$\alpha \in \Lambda,\ \alpha + \beta \in \Phi(V', T) \Rightarrow \alpha + \beta \in \Lambda. \tag{35}$$

For any submodule V of V', V^β denotes the unique sub-T-module of V' isomorphic to V as a $\operatorname{Ker}(\beta)$-module and such that $\Phi(V^\beta, T)$ is β-convex. In other words, on each line $\alpha + \mathbb{Z}\beta \cap \Phi(V', T)$, one pushes the elements of $\Phi(V, T)$ in the direction β to get $\Phi(V^\beta, T)$.

Let $w \in W$. The point wP/P is denoted by $\dot{w}$. Let V be a T-submodule of $T_{\dot{w}}G/P$. Let β be a root of (G, T). We are interested in the action of the unipotent one-parameter subgroup U_β associated to β on $\dot{w}$ and V. Consider the point $\dot{v} = \lim_{\tau\to\infty} U_\beta(\tau)\dot{w}$. For any $\tau \in \mathbb{C}$, $U_\beta(\tau)V$ is a linear subspace of $T_{U_\beta(\tau)\dot{w}}G/P$ of the same dimension as V. Hence it is a point of a bundle in Grassmannian over G/P. Consider the limit in this bundle

$$\tau(V, \beta) := \lim_{\tau\to\infty} U_\beta(\tau)V. \tag{36}$$

This limit $\tau(V, \beta)$ is a T-stable submodule of the T-module without multiplicity $T_{\dot{v}}G/P$.

We can now state a Peterson's result (see [CK03, Section 8]).

Lemma 10 *The T-submodule $s_\beta\tau(V, \beta)$ of $T_{\dot{w}}G/P$ is equal to $V^{-\beta}$.*

Proof. The set $\{U_\beta(\tau)\dot{w} \,:\, \tau \in \mathbb{C}\} \cup \dot{v}$ is a T-stable curve isomorphic to $\mathbb{P}^1$. The computation of $\tau(V, \beta)$ lies in a bundle in Grassmannians over this line. This computation can be made quite explicitly by trivializing this bundle on the two T-stable open affine subsets of $\mathbb{P}^1$. □

3.3 A lemma on T-varieties

The following result is used in this paper to characterize Schubert varieties in terms of their tangent spaces among the irreducible T-stable subvarieties of G/P.

Lemma 11 *Let V be a T-module. Let $\mathcal{C}$ be a strictly convex cone in $X(T)\otimes\mathbb{Q}$. Let Σ be a closed T-stable subvariety of V such that*

(i) Σ is smooth at 0;

(ii) $T_0\Sigma = \oplus_{\chi\in\mathcal{C}} V_\chi$.

Then $\Sigma = \oplus_{\chi\in\mathcal{C}} V_\chi$.

Proof. Since $\mathcal{C}$ is strictly convex and $\Phi(V,T)$ is finite, there exist finitely many one-parameter subgroups $\lambda_1,\dots,\lambda_k$ of T such that

$$\forall\chi\in X(T)\quad \chi\in\mathcal{C}\iff\forall i\ \ \langle\lambda_i,\chi\rangle>0.$$

For any i, there exists a T-stable neighborhood of 0 in Σ such that any point x in this neighborhood satisfies $\lim_{t\to 0}\lambda_i(t)x=0$. Consider the set W of $v\in V$ such that $\lim_{t\to 0}\lambda_i(t)v=0$, for any i. By the second condition, W is precisely $T_0\Sigma$. We just proved that $T_0\Sigma$ contains an open subset of Σ. But these two varieties are irreducible and of same dimension (since Σ is assumed to be smooth at 0). Hence $\Sigma=T_0\Sigma$. $\square$

3.4 Schubert varieties

Let Y be a subvariety of G/P. Let $G(X)$ denote the stabilizer of Y in G; it is the set of g in G such that $gY=Y$. If $G(Y)$ has an open orbit in Y then this orbit is called the *homogeneous locus* of Y; otherwise, the homogeneous locus of Y is defined to be empty. In other words, the homogeneous locus of Y is the biggest open subset of Y homogeneous under a subgroup of G; it is denoted by $Y^{\rm hom}$.

Recall that $X_w=\overline{BwP/P}$. If $Y=X_w$ (for some $w\in W^P$) then the group $G(X_w)$ contains B: it is a standard parabolic subgroup of G. In particular, it is characterized by a subset Δ_w of simple roots. Precisely we set

$$\Delta_w=\{\alpha\in\Delta\ :\ P_\alpha X_w=X_w\}.$$

Proposition 5 *We have*

$$X_w^{X(Z)-{\rm reg}}=X_w^{\rm hom}.$$

Proof. Since the infinitesimal filtration is G-invariant, it is clear that $X_w^{X(Z)-{\rm reg}}$ is $G(X_w)$-stable and contains $X_w^{\rm hom}$. Moreover Lemma 8 implies that $X_w^{X(Z)-{\rm reg}}$ is open in X_w.

Assume that $X_w^{X(Z)-{\rm reg}}-X_w^{\rm hom}$ is nonempty. Choose an open B-orbit in $X_w^{X(Z)-{\rm reg}}-X_w^{\rm hom}$ and a T-fixed point $\dot v$ on it.

Obviously v is smaller than w for the Bruhat order. Since the Bruhat order is generated by T-stable curves, there exists a positive root β such that $s_\beta v\in W^P$ and $v<s_\beta v<w$. Since $B.\dot v$ is dense in an irreducible component of $X_w^{X(Z)-{\rm reg}}-X_w^{\rm hom}$, $s_\beta\dot v$ belongs to $X_w^{\rm hom}$.

Since $s_\beta \dot v$ is a T-fixed point in $G(X_w).\dot w$, it is equal to $u\dot w$ for some $u \in W(G(X_w))$.

We claim that

$$s_\beta \in G(X_w)/T. \tag{37}$$

Let us first explain how the claim leads to a contradiction. Since u belongs to $G(X_w)/T$, the claim implies that $s_\beta u^{-1} X_w^{\rm hom} = X_w^{\rm hom}$. But $\dot v = s_\beta u^{-1}\dot w$ and $\dot w$ belongs to $X_w^{\rm hom}$. Hence $\dot v \in X_w^{\rm hom}$ which is a contradiction.

Consider $\gamma = \pm w^{-1}u\beta$ where the sign is chosen to make γ negative. Since $u \in G(X_w)/T$, Claim (37) is equivalent to $s_\beta u^{-1}X_w = u^{-1}X_w$ or to $s_{u\beta}X_w = X_w$ or to

$$s_\gamma.(w^{-1}X_w) = w^{-1}X_w. \tag{38}$$

Look these two varieties in a neighborhood of P/P. More precisely, consider the unique affine open T-stable neighborhood Ω of P/P in G/P. Then Ω is isomorphic as a T-variety to a T-module without multiplicity. Since the two varieties of (38) are irreducible, it is sufficient to prove that

$$\Omega \cap s_\gamma.(w^{-1}X_w) = \Omega \cap w^{-1}X_w. \tag{39}$$

Since $s_\gamma P/P \in w^{-1}X_w$, $\gamma \in \Phi(w)$. In particular, $w^{-1}X_w$ is U_γ-stable. But, $s_\gamma P/P$ and P/P are smooth points in $w^{-1}X_w$. Hence

$$\lim_{\tau\to\infty} U_\gamma(\tau) T_{P/P} w^{-1}X_w = T_{s_\gamma P/P} w^{-1}X_w.$$

Then Lemma 10 shows that

$$\begin{array}{ll}\Phi(T_{P/P}s_\gamma w^{-1}X_w, T) & = s_\gamma \Phi(T_{s_\gamma P/P}w^{-1}X_w, T)\\ & = s_\gamma \Phi\left(\lim_{\tau\to\infty} U_\gamma(\tau)T_{P/P}w^{-1}X_w, T\right)\\ & = \Phi\left((T_{P/P}w^{-1}X_w)^{-\gamma}, T\right).\end{array}$$

Since P/P is Γ-regular in $s_\gamma w^{-1}X_w$,

$$\forall \alpha \in X(Z) \quad \dim(F^{\succcurlyeq\alpha}(T_{P/P}w^{-1}X_w)^{-\gamma}) = \dim(F^{\succcurlyeq\alpha}(T_{P/P}w^{-1}X_w)). \tag{40}$$

But $\gamma \notin \Phi(P)$, hence $\gamma_{|Z}$ is non trivial. Then, equality (40) implies that $\Phi((T_{P/P}w^{-1}X_w)^{-\beta}, T) = \Phi((T_{P/P}w^{-1}X_w), T)$. Equality (39) follows and the theorem is proved. □

4 Infinitesimal filtration and cohomology

4.1 Filtration of differential forms on a manifold

In this subsection, M is a smooth connected manifold of dimension d endowed with an infinitesimal Γ-filtration. The notion that allows to control the differential relatively to the filtration is the following one.

Definition. An infinitesimal Γ-filtration of M is said to be *integrable* if for any α and β in Γ we have

$$[F^{\succcurlyeq\alpha}TM, F^{\succcurlyeq\beta}TM] \subset F^{\succcurlyeq\alpha+\beta}TM. \tag{41}$$

Example. Let L be an integrable distribution on M. We get an integrable infinitesimal $\mathbb{Z}$-filtration be setting

$$\begin{array}{ll} F^{\succcurlyeq a}TM = TM & \forall a \in \mathbb{Z}_{<0}, \\ F^{\succcurlyeq 0}TM = L, & \\ F^{\succcurlyeq a}TM = \underline{0} & \forall a \in \mathbb{Z}_{>0}. \end{array}$$

Example. Let L be any distribution on M. We get an integrable infinitesimal $\mathbb{Z}$-filtration be setting

$$\begin{array}{ll} F^{\succcurlyeq a}TM = TM & \forall a \leq -2, \\ F^{\succcurlyeq -1}TM = L, & \\ F^{\succcurlyeq a}TM = \underline{0} & \forall a \in \mathbb{Z}_{\geq 0}. \end{array}$$

Consider the sheaf Ω^p of differential p-forms on M and the De Rham differential $d_p : \Omega^p \longrightarrow \Omega^{p+1}$. The De Rham cohomology group is

$$\mathrm{H}^p_{DR}(M,\mathbb{R}) := \frac{\text{Ker } d_p(M)}{\text{Im } d_{p-1}(M)}.$$

The exterior product

$$\begin{array}{rccc} \wedge : & \Omega^p \times \Omega^{p'} & \longrightarrow & \Omega^{p+p'} \\ & (\omega, \omega') & \longmapsto & \omega \wedge \omega' \end{array}$$

induces a product $\wedge$ in cohomology since

$$d(\omega \wedge \omega') = (d\omega) \wedge \omega' + (-1)^p \omega \wedge d\omega'.$$

In particular, $\mathrm{H}^*_{DR}(M,\mathbb{R}) := \oplus_{k=0}^d \mathrm{H}^k_{DR}(M,\mathbb{R})$ is a graded algebra.

We now consider the Γ-filtration on the sheaf Ω^p induces by the infinitesimal Γ-filtration.

Definition. Let $\beta \in \Gamma$ and let U be an open subset of M. The subspace $F^{\preccurlyeq\beta}\Omega^p(U)$ of $\Omega^p(U)$ is defined to be the set of forms $\omega \in \Omega^p(U)$ such that for any $\alpha_1, \ldots, \alpha_p \in \Gamma$, for any $x \in U$ and for any $\xi_i \in F^{\succcurlyeq\alpha_i}T_xM$, we have

$$\alpha_1 + \cdots + \alpha_p \not\preccurlyeq \beta \Rightarrow \omega_x(\xi_1, \ldots, \xi_p) = 0. \tag{42}$$

A direct consequence of Proposition 4 is

Proposition 6 *(i) If $\beta \preccurlyeq \gamma$ then $F^{\preccurlyeq\beta}\Omega^p \subset F^{\preccurlyeq\gamma}\Omega^p$.*

(ii) Let $\beta_0 \in \Gamma$ be such that $F^{\succcurlyeq\beta_0}TM = TM$. If $F^{\preccurlyeq\gamma}\Omega^p \neq \{0\}$ then $\gamma \succcurlyeq p\beta_0$.

(iii) We have $F^{\preccurlyeq 0}\Omega^p = \Omega^p$.

(iv) For β and γ in Γ, we have $F^{\preccurlyeq\beta}\Omega^p \wedge F^{\preccurlyeq\gamma}\Omega^q \subset F^{\preccurlyeq\beta+\gamma}\Omega^{p+q}$.

The integrability is essential in the following result.

Proposition 7 *Assume that the infinitesimal filtration is Γ-integrable. Then for any $\beta \in \Gamma$*

$$d_p(F^{\preccurlyeq\beta}\Omega^p) \subset F^{\preccurlyeq\beta}\Omega^{p+1}.$$

Proof. Let U be an open subset of M and let $\omega \in F^{\preccurlyeq\beta}\Omega^p(U)$. Let $x \in U$ and let $\xi_i \in F^{\succcurlyeq\alpha_i}TM$ be defined in a neighborhood of x such that $\alpha_1 + \cdots + \alpha_{p+1} \not\preccurlyeq \beta$. It remains to prove that

$$d_p(\omega)_x(\xi_1, \ldots, \xi_{p+1}) = 0.$$

Cartan's formula implies

$$\begin{aligned} d_p(\omega)_x(\xi_1, \ldots, \xi_{p+1}) = \; & \textstyle\sum_i \pm \xi_i \cdot \omega(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}) \\ & + \textstyle\sum_{i<j} \pm \omega_x([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1}). \end{aligned}$$

Since $[\xi_i, \xi_j] \in F^{\succcurlyeq\alpha_i+\alpha_j}M$ and

$$(\alpha_i + \alpha_j) + \alpha_1 + \cdots + \hat{\alpha}_i + \cdots + \hat{\alpha}_j + \cdots + \alpha_{p+1} \not\preccurlyeq \beta,$$

the term $\omega_x([\xi_i, \xi_j], \xi_1, \ldots, \hat{\xi}_i, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1})$ is zero.

Consider now a term

$$\xi_i \cdot \omega(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}). \tag{43}$$

If $\alpha_i \not\preccurlyeq 0$ then $\xi_i = 0$ and the term (43) is zero. Assume now that $\alpha_i \preccurlyeq 0$. The weight of $\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}$ is $\theta := \sum_{j=1}^{p+1} \alpha_j - \alpha_i$. Since $\theta + \alpha_i \not\preccurlyeq \beta$ and $\alpha_i \preccurlyeq 0$, we have $\theta \not\preccurlyeq \beta$. Since ω belongs to $F^{\preccurlyeq\beta}\Omega^p(U)$, it follows that $\omega(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{p+1}) = 0$. □

4.2 Filtration of the cohomology

The Γ-filtration on M induces an increasing Γ-filtration on the cohomology. Indeed, Propositions 6 and 7 show that the De Rham complex is Γ-filtered. Namely, we set

$$F^{\preccurlyeq\beta}\,\mathrm{H}^p(M, \mathbb{R}) := \frac{\mathrm{Ker}(d_p) \cap F^{\preccurlyeq\beta}\Omega^p(M, \mathbb{R})}{d_{p-1}(\Omega^{p-1}(M, \mathbb{R})) \cap F^{\preccurlyeq\beta}\Omega^p(M, \mathbb{R})}. \tag{44}$$

Propositions 6 and 7 show the following one.

Proposition 8 *The sets $F^{\preccurlyeq \beta}\,\mathrm{H}^p(M,\mathbb{R})$ are canonically identified with subspaces of $\mathrm{H}^p(M,\mathbb{R})$.*

(i) If $F^{\succcurlyeq \beta_0}M = TM$ then $F^{\succcurlyeq p\beta_0-\beta}\,\mathrm{H}^p(M,\mathbb{R})$ is a Γ-filtration of $\mathrm{H}^p(M,\mathbb{R})$.

(ii) The filtration respects the structure of algebra. Namely, for β and γ in Γ, we have

$$F^{\preccurlyeq \beta}\,\mathrm{H}^p(M,\mathbb{R}) \wedge F^{\preccurlyeq \gamma}\,\mathrm{H}^q(M,\mathbb{R}) \subset F^{\preccurlyeq \beta+\gamma}\,\mathrm{H}^{p+q}(M,\mathbb{R}).$$

Remark. The Γ-filtration is defined at the level of the de Rham complex and not only at the level of the cohomology. In particular, it induces a spectral sequence which should be study to understand the relations between the ordinary and the Belkale-Kumar cohomologies. Here we only study the Belkale-Kumar product.

Consider now the $(\Gamma \times \mathbb{Z})$-graded algebra associated to this Γ-filtration of the $\mathbb{Z}$-graded (by degree) algebra $\mathrm{H}^*(M,\mathbb{R})$ by setting

$$Gr^{\beta}\,\mathrm{H}^p(M,\mathbb{R}) := \frac{F^{\preccurlyeq \beta}\,\mathrm{H}^p(M,\mathbb{R})}{\sum_{\gamma \prec \beta} F^{\preccurlyeq \gamma}\,\mathrm{H}^p(M,\mathbb{R})} \tag{45}$$

and

$$Gr^{\bullet}\,\mathrm{H}^*(M,\mathbb{R}) := \bigoplus_{\beta\in\Gamma,\, p\in\mathbb{N}} Gr^{\beta}\,\mathrm{H}^p(M,\mathbb{R}). \tag{46}$$

Then $Gr^{\bullet}\,\mathrm{H}^*(M,\mathbb{R})$ is a ring graded by $\Gamma \times \mathbb{Z}$.

Now, we observe the following easy functoriality result.

Lemma 12 *Let M and N be two smooth manifolds endowed with integrable infinitesimal Γ-filtrations. Let $\phi\, M \longrightarrow N$ a smooth map such that*

$$\forall \alpha \in \Gamma \qquad T\phi(F^{\geq \alpha}TM) \subset F^{\geq \alpha}TN.$$

Then the pullback $\phi^* : \mathrm{H}^*(N,\mathbb{R}) \longrightarrow \mathrm{H}^*(M,\mathbb{R})$ respects the Γ-filtration. In particular, it induces a Γ-graded morphism $Gr\phi^* : Gr\,\mathrm{H}^*(N,\mathbb{R}) \longrightarrow Gr\,\mathrm{H}^*(M,\mathbb{R})$.

4.3 Cohomology with complex coefficients

Recall that M is a connected manifold. Consider the cohomology group $\mathrm{H}^*(M,\mathbb{C})$ with complex coefficients and consider the following complex vector bundle on M

$$T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}.$$

A *complex infinitesimal* Γ*-filtration of* M is a family of complex subbundles

$$F^{\preccurlyeq \beta}T^{\mathbb{C}}M \subset T^{\mathbb{C}}M,$$

indexed by $\beta \in \Gamma$ satisfying the three assertions of Definition 2.2. A complex infinitesimal Γ-filtration is said to be Γ-integrable if for any β and γ in Γ, we have

$$[F^{\preccurlyeq\beta}T^{\mathbb{C}}M, F^{\preccurlyeq\gamma}T^{\mathbb{C}}M] \subset F^{\preccurlyeq\beta+\gamma}T^{\mathbb{C}}M. \tag{47}$$

A complex infinitesimal integrable Γ-filtration induces a filtration of the De Rham complex and of the groups $\mathrm{H}^p(M, \mathbb{C})$.

Example. Let M be an holomorphic manifold. Let J denote the complex structure on the tangent bundle TM. Since $J^2 = -\mathrm{Id}$, its eigenvalues acting on $TM \otimes \mathbb{C}$ are $\pm\sqrt{-1}$. Let $T^{1,0}M$ (resp. $T^{0,1}M$) denote the complex subbundle of $TM \otimes \mathbb{C}$ associated to the eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). There is a natural $\mathbb{C}$-linear isomorphism $\iota^{1,0} : TM \longrightarrow T^{1,0}M$. It is well known that $T^{1,0}M$ is an integrable distribution in $T^{\mathbb{C}}M$. Then we get a complex infinitesimal integrable $\mathbb{Z}$-filtration by setting

$$\begin{array}{ll} F^{\succcurlyeq a}T^{\mathbb{C}}M = T^{\mathbb{C}}M & \forall a \in \mathbb{Z}_{<0}, \\ F^{\succcurlyeq 0}T^{\mathbb{C}}M = T^{1,0}M, & \\ F^{\succcurlyeq a}T^{\mathbb{C}}M = \underline{0}, & \forall a \in \mathbb{Z}_{>0}. \end{array}$$

The $\mathbb{Z}$-filtration of $\mathrm{H}^p(M, \mathbb{C})$ is called the Hodge filtration of M (see for example [Voi07]).

4.4 The case of a smooth complex variety

Let M be a smooth complex irreducible variety endowed with an algebraic Γ-filtration. Assume that this filtration is integrable and comes from a decomposition (recall the definition from Section 2.2). Set $\tilde{\Gamma} := \Gamma \times \mathbb{Z}$ endowed with the order $(\beta, n) \succcurlyeq (\gamma, m)$ if and only if $\beta \succcurlyeq \gamma$ and $n \geq m$.

Define a complex $\tilde{\Gamma}$-filtration on $T^{\mathbb{C}}M$ by setting for any $\beta \in \Gamma$,

$$\begin{array}{ll} F^{\succcurlyeq(\beta,a)}T^{\mathbb{C}}M = T^{\mathbb{C}}M & \forall a \in \mathbb{Z}_{<0}, \\ F^{\succcurlyeq(\beta,0)}T^{\mathbb{C}}M = \iota^{1,0}(F^{\succcurlyeq\beta}TM), & \\ F^{\succcurlyeq(\beta,a)}T^{\mathbb{C}}M = \underline{0}, & \forall a \in \mathbb{Z}_{>0}. \end{array}$$

Integration along subvarieties. Let N be an irreducible subvariety of M. Denote by n the dimension of M and by d that of N. By Lemma 4, the dimension vector $(fd^{\beta}(T_xN))_{\beta\in\Gamma}$ does not depend on $x \in N$ general. This general value of the dimension vector is by definition the *f-dimension vector of* N and is denoted by $fd^{\beta}(N)$. For any x in N, the Γ-filtration of T_xN comes from a decomposition by Lemma 5. In particular, Lemma 4 shows that the g-dimensional vector of T_xN does not depend on x in N general. This remark allows to define the g-dimension vector of N. Then the weight $\rho(N) \in \Gamma$ of N is defined by the formula

$$\rho(N) = \sum_{\beta \in \Gamma} gd^{\beta}(N)\beta. \tag{48}$$

Consider the extended notions to $\tilde{\Gamma}$: $\widetilde{gd}^{(\beta,0)}(N) = gd^{(\beta,0)}(T_xN \otimes \mathbb{C}) = gd^{\beta}(N)$, $\widetilde{gd}^{(0,-1)}(N) = d$ and $\widetilde{gd}^{(\beta,a)}(N) = 0$ otherwise. Note that $\tilde{\rho}(N) = (\rho(N), -d)$.

Consider now the linear map

$$\begin{array}{ccc} \Omega^{2d}(M,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ \omega & \longmapsto & \int_N \omega_{|N}. \end{array}$$

The following lemma relies the filtration and the integration.

Lemma 13 *Let $\beta \in \Gamma$ and let $e \in \mathbb{Z}$ such that $(\beta, e) \not\succcurlyeq \tilde{\rho}(N)$. If $\omega \in F^{\preccurlyeq(\beta,e)}\Omega^{2d}(M,\mathbb{C})$ then*

$$\int_N \omega_{|N} = 0.$$

Proof. Let $x \in N$ be a general point. By Lemma 5, the Γ-filtration on T_xN comes from a decomposition. Then there exists a basis $(\xi_1, \dots, \xi_d)$ of T_xN such that for any $\beta \in \Gamma$, the set of ξ_i which belong to $F^{\succcurlyeq\beta}T_xN$ spans $F^{\succcurlyeq\beta}T_xN$. Such a basis exists since by Lemma 5, the Γ-filtration on T_xN comes from a decomposition. Let α_i be the maximal element of Γ such that ξ_i belongs to $F^{\succcurlyeq\alpha_i}T_xN$.

Consider the basis $(\iota^{(1,0)}(\xi_1), \dots, \iota^{(1,0)}(\xi_d), \iota^{(0,1)}(\xi_1), \dots, \iota^{(0,1)}(\xi_d))$ of $T_xN \otimes \mathbb{C}$. Since x is any general point on N, it is sufficient to prove that

$$\omega(\iota^{(1,0)}(\xi_1), \dots, \iota^{(1,0)}(\xi_d), \iota^{(0,1)}(\xi_1), \dots, \iota^{(0,1)}(\xi_d)) = 0.$$

But $\iota^{(1,0)}(\xi_i) \in F^{\succcurlyeq(\alpha_i,0)}T^{\mathbb{C}}N$ and $\iota^{(0,1)}(\xi_i) \in F^{\succcurlyeq(0,-1)}T^{\mathbb{C}}N$. Hence the weight of $(\iota^{(1,0)}(\xi_1), \dots, \iota^{(1,0)}(\xi_d), \iota^{(0,1)}(\xi_1), \dots, \iota^{(0,1)}(\xi_d))$ is $\sum_{i=1}^{d}(\alpha_i, 0) + d(0,-1) = \tilde{\rho}(N)$. The lemma follows. $\square$

The restriction of the map $\omega \longmapsto \int_N \omega_{|N}$ to the closed $2d$-forms is zero on the exact forms and induces a linear map

$$\int_N : \mathrm{H}^{2d}(M,\mathbb{C}) \longrightarrow \mathbb{C}.$$

Consider now the restriction of this map to $F^{\preccurlyeq\tilde{\rho}(N)}\,\mathrm{H}^{2d}(M,\mathbb{C})$. By Lemma 13, this restriction induces a linear map

$$\int_N : \mathrm{Gr}^{\tilde{\rho}(N)}\,\mathrm{H}^{2d}(M,\mathbb{C}) \longrightarrow \mathbb{C}.$$

Poincaré pairing. Assume that M is compact and recall that it is orientable since it is holomorphic. Let p be an integer such that $0 \leq p \leq 2d$. The integration allows to define a paring

$$\begin{array}{ccc}\mathrm{H}^p(M,\mathbb{C})\times\mathrm{H}^{2d-p}(M,\mathbb{C}) & \longrightarrow & \mathbb{C}\\ ([\omega_1],[\omega_2]) & \longmapsto & \int_M \omega_1\wedge\omega_2.\end{array}\tag{49}$$

By Poincaré duality, this bilinear form is non degenerated. In particular, $\mathrm{H}^p(M,\mathbb{C})$ and $\mathrm{H}^{2d-p}(M,\mathbb{C})$ have the same dimension.

Let $\tilde\alpha\in\tilde\Gamma$. Consider the following restriction of the bilinear form (49):

$$\begin{array}{ccc}F^{\preccurlyeq\tilde\alpha}\,\mathrm{H}^p(M,\mathbb{C})\times F^{\preccurlyeq\tilde\rho(M)-\tilde\alpha}\,\mathrm{H}^{2d-p}(M,\mathbb{C}) & \longrightarrow & \mathbb{C}\\ ([\omega_1],[\omega_2]) & \longmapsto & \int_M \omega_1\wedge\omega_2.\end{array}\tag{50}$$

Since

$$\begin{array}{l}\tilde\alpha\succcurlyeq\tilde\rho(M)\Rightarrow F^{\preccurlyeq\tilde\alpha}\Omega^{2d}(M)=\Omega^{2d}(M),\ \text{and}\\ \tilde\alpha\not\succcurlyeq\tilde\rho(M)\Rightarrow F^{\preccurlyeq\tilde\alpha}\Omega^{2d}(M)=0,\end{array}$$

Lemma 13 shows that

$$\tilde\alpha+\tilde\beta\not\succcurlyeq\tilde\rho(M)\Rightarrow F^{\preccurlyeq\tilde\alpha}\,\mathrm{H}^p(M,\mathbb{C})\wedge F^{\preccurlyeq\tilde\beta}\,\mathrm{H}^{2d-p}(M,\mathbb{C})=\{0\}.\tag{51}$$

In particular, the pairing (50) passes to the quotient and induces a pairing

$$\begin{array}{ccc}\mathrm{Gr}^{\tilde\alpha}\,\mathrm{H}^p(M,\mathbb{C})\times\mathrm{Gr}^{\tilde\rho(M)-\tilde\alpha}\,\mathrm{H}^{2d-p}(M,\mathbb{C}) & \longrightarrow & \mathbb{C}\\ ([\omega_1],[\omega_2]) & \longmapsto & \int_M \omega_1\wedge\omega_2.\end{array}\tag{52}$$

Definition. The $\tilde\Gamma$-filtration of $\mathrm{H}^*(M,\mathbb{C})$ is said to be *compatible with Poincaré duality* if for any integer $0\le p\le 2d$ and for any $\tilde\alpha\in\tilde\Gamma$, the pairing (52) is non degenerate.

Lemma 14 *The $\tilde\Gamma$-filtration of $\mathrm{H}^*(M,\mathbb{C})$ is compatible with Poincaré duality if and only if for any nonnegative integer p and any $\tilde\alpha\in\tilde\Gamma$, we have*

$$\dim(\mathrm{Gr}^{\tilde\alpha}\,\mathrm{H}^p(M,\mathbb{C}))=\dim(\mathrm{Gr}^{\tilde\rho(M)-\tilde\alpha}\,\mathrm{H}^{2d-p}(M,\mathbb{C}))\tag{53}$$

Proof. If the $\tilde\Gamma$-filtration of $\mathrm{H}^*(M,\mathbb{C})$ is compatible with Poincaré duality we obviously have the equalities of dimensions.

Assume now that (53) hold. In a basis adapted to the filtration, implication (51) implies that the matrix A of the pairing (49) is upper triangular. Moreover, the matrices (in the induced basis) of the pairings (52) are the diagonal blocs of A. But equalities (53) imply that these blocs are square. Since A is invertible, it follows that any bloc is invertible. $\square$

Definition. Let N be an irreducible subvariety of a compact smooth irreducible complex variety M endowed with an integrable infinitesimal Γ-filtration coming from a decomposition. Assume that the $\tilde\Gamma$-filtration is compatible with Poincaré duality. Define $[N]_{\odot_0}\in\mathrm{Gr}^{\tilde\rho(M)-\tilde\rho(N)}\,\mathrm{H}^{2(n-d)}(M,\mathbb{C}))$ to satisfy the following formula

$$\int_N[\omega]=\int_M[N]_{\odot_0}\wedge[\omega],\tag{54}$$

for any $[\omega]\in\mathrm{Gr}^{\tilde\rho(N)}\,\mathrm{H}^{2d}(M,\mathbb{C})$.

On can refer to Proposition 11 for a more concerte characterization of $[N]_{\odot_0}$ and in particular its relation with $[N]$, in the case when $M=G/P$.

5 Isomorphism with the Belkale-Kumar product

5.1 The Belkale-Kumar product

In this section, we recall the Belkale-Kumar notion of Levi-movability (see [BK06]).

The cycle class of the Schubert variety X_w in $\mathrm{H}^*(G/P,\mathbb{C})$ is denoted by σ_w and it is called a Schubert class. The degree of σ_w is $2(\dim G/P - l(w)$, where $l(w) = \sharp\Phi(w)$ is the length of w. The Schubert classes form a basis of the cohomology of G/P:

$$\mathrm{H}^*(G/P,\mathbb{C}) = \bigoplus_{w\in W^P} \mathbb{C}\sigma_w. \tag{55}$$

The Poincaré dual of σ_w is denoted by $\sigma_w^\vee$. Note that σ_e is the class of the point. Let σ_u, σ_v, σ_w be three Schubert classes (with $u, v, w \in W^P$). If there exists an integer d such that $\sigma_u.\sigma_v.\sigma_w = d\sigma_e$ then we set $c_{uvw} = d$; we set $c_{uvw} = 0$ otherwise. These coefficients are the (symmetrized) structure coefficients of the cup product on $\mathrm{H}^*(G/P,\mathbb{C})$ in the Schubert basis in the following sense:

$$\sigma_u.\sigma_v = \sum_{w\in W^P} c_{uvw}\sigma_w^\vee$$

and $c_{uvw} = c_{vuw} = c_{uwv}$.

Consider the tangent space T_u of the orbit $u^{-1}BuP/P$ at the point P/P; and, similarly consider T_v and T_w. Using the transversality theorem of Kleiman, Belkale and Kumar showed in [BK06, Proposition 2] the following important lemma.

Lemma 15 *The coefficient c_{uvw} is nonzero if and only if there exist $p_u, p_v, p_w \in P$ such that the natural map*

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{p_uT_u} \oplus \frac{T_P(G/P)}{p_vT_v} \oplus \frac{T_P(G/P)}{p_wT_w}$$

is an isomorphism.

Then Belkale-Kumar defined Levi-movability.

Definition. The triple $(\sigma_u, \sigma_v, \sigma_w)$ is said to be *Levi-movable* if there exist $l_u, l_v, l_w \in L$ such that the natural map

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{l_uT_u} \oplus \frac{T_P(G/P)}{l_vT_v} \oplus \frac{T_P(G/P)}{l_wT_w}$$

is an isomorphism.

Belkale-Kumar set

$$c^{\odot_0}_{uvw} = \begin{cases} c_{uvw} & \text{if } (\sigma_u,\, \sigma_v,\, \sigma_w) \text{ is Levi} - \text{movable;} \\ 0 & \text{otherwise.} \end{cases}$$

They defined on the group $\mathrm{H}^*(G/P, \mathbb{C})$ a bilinear product $\odot_0$ by the formula

$$\sigma_u \odot_0 \sigma_v = \sum_{w \in W^P} c^{\odot_0}_{uvw} \sigma^\vee_w.$$

Theorem 2 (Belkale-Kumar 2006) *The product $\odot_0$ is commutative, associative and satisfies Poincaré duality.*

[RR11, Proposition 2.4] gives an equivalent characterization of Levi-movability. It can be formulated as follows.

Proposition 9 *Let $u,\, v,\, w \in W^P$ such that $c_{uvw} \neq 0$. Then $(\sigma_u,\, \sigma_v,\, \sigma_w)$ is Levi-movable if and only if*

$$2gd(G/P) = gd(X_u) + gd(X_v) + gd(X_w).$$

5.2 The statements

The first aim of this section is to prove (see Section 5.4.10) the following result of compatibility between the basis of Schubert classes and the $\tilde{\Gamma}$-filtration on $\mathrm{H}^*(G/P, \mathbb{C})$.

Proposition 10 *For any $\tilde{\beta} \in \tilde{\Gamma}$ and for any integer p, the linear subspace $F^{\preccurlyeq \tilde{\beta}}\, \mathrm{H}^p(G/P, \mathbb{C})$ is spanned by the Schubert classes it contains.*

More precisely, $F^{\preccurlyeq \tilde{\beta}}\, \mathrm{H}^p(G/P, \mathbb{C})$ is spanned by the Schubert classes $\sigma_{w^\vee}$ where $w \in W^P$ satisfies $(\rho(X_w), -l(w)) \preccurlyeq \tilde{\beta}$.

For any $w \in W^P$, denote by $\overline{\sigma_{w^\vee}}$ the class of $\sigma_{w^\vee} \in F^{\preccurlyeq (\rho(X_w), -l(w))}\, \mathrm{H}^{l(w)}(G/P, \mathbb{C})$ in $\mathrm{Gr}^{(\rho(X_w), -l(w))}\, \mathrm{H}^{l(w)}(G/P, \mathbb{C})$. Proposition 10 implies that $(\overline{\sigma_{w^\vee}})_{w \in W^P}$ is a basis of $\mathrm{Gr}\, \mathrm{H}^*(G/P, \mathbb{C})$. Consider now the obvious linear isomorphism

$$\begin{array}{cccll} \Psi : & \mathrm{H}^*(G/P, \mathbb{C}) & \longrightarrow & \mathrm{Gr}\, \mathrm{H}^*(G/P, \mathbb{C}) & \\ & \sigma_{w^\vee} & \longmapsto & \overline{\sigma_{w^\vee}} & \text{for any } w \in W^P. \end{array}$$

Theorem 3 *The linear isomorphism Ψ from the algebra $(\mathrm{H}^*(G/P, \mathbb{C}), \odot_0)$ onto the algebra $\mathrm{Gr}\, \mathrm{H}^*(G/P, \mathbb{C})$ is an isomorphism of algebras.*

The theorem is proved in Section 5.5 after some preparation. The first consequence concerns Poincaré duality (see Section 5.4.10).

Corollary 1 *The $(X(Z) \times \mathbb{Z})$-filtration of $\mathrm{H}^*(G/P, \mathbb{C})$ is compatible with Poincaré duality.*

This corollary allows to define the graded Schubert classes by setting, for any $w \in W^P$,

$$\sigma_w^{\odot_0} := [X_w]_{\odot_0}. \tag{56}$$

Finally, we get, by applying Proposition 11 to $Y = X_w$, the following result of compatibility.

Lemma 16 *For any $w \in W^P$, we have*

$$\Psi(\sigma_w) = \sigma_w^{\odot_0}.$$

5.3 An upper bound for $\dim(F^{\preccurlyeq\tilde{\alpha}}\,\mathrm{H}^p(G/P,\mathbb{C}))$

For any $w \in W$, as a consequence of the relation $\Phi^- = (\Phi^- \cap w^{-1}\Phi^+) \cup (\Phi^- \cap w^{-1}\Phi^-)$, we have (see [Kum02, 1.3.22.3])

$$\sum_{\alpha \in \Phi^- \cap w^{-1}\Phi^+} \alpha = w^{-1}\rho - \rho. \tag{57}$$

Assume that $w \in W^P$. Since P/P is $X(T)$-regular and T acts on $T_{P/P}w^{-1}X_w$ without multiplicities and with weights $\Phi^- \cap w^{-1}\Phi^+$, we have

$$\rho(X_w) = \rho(w^{-1}X_w) = \left(\sum_{\alpha \in \Phi^- \cap w^{-1}\Phi^+} \alpha \right)_{|Z} = \left(w^{-1}\rho - \rho\right)_{|Z}. \tag{58}$$

In particular

$$\rho(G/P) = 2\left(\rho_L - \rho\right)_{|Z} = -2\rho_{|Z}, \tag{59}$$

since ρ_L is trivial on Z. Hence

$$\rho(G/P) - \rho(X_w) = \left(-\rho - w^{-1}\rho\right)_{|Z}. \tag{60}$$

Lemma 17 *For any $w \in W^P$, we have*

$$\rho(G/P) - \rho(X_w) = \rho(X_{w^\vee}).$$

Proof. Remark that

$$((w^\vee)^{-1}\rho)_{|Z} = ((w_0^P w^{-1} w_0 \rho)_{|Z} = -w_0^P(w^{-1}\rho)_{|Z} = -(w^{-1}\rho)_{|Z},$$

since w_0^P belongs to L and acts trivially on Z. The lemma follows. $\square$

Lemma 18 *Let n denote the dimension of G/P. The dimension of $F^{\preccurlyeq\beta}\,\mathrm{H}^{2(n-d)}(G/P,\mathbb{C})$ is less or equal to the number of $w \in W^P$ such that $\rho(G/P) - \rho(X_w) \preccurlyeq \beta$ and $l(w) = d$.*

Proof. For each $w \in W^P$ such that $\rho(G/P) - \rho(X_w) \not\preccurlyeq \beta$ and $l(w) = d$, consider the linear form

$$\int_{X_{w^\vee}} : \mathrm{H}^{2(n-d)}(G/P, \mathbb{C}) \longrightarrow \mathbb{C}.$$

By Lemmas 17 and 13, this linear form is zero on $F^{\preccurlyeq\beta}\,\mathrm{H}^{2(n-d)}(G/P, \mathbb{C})$. But by Poincaré duality these linear forms are linearly independent. This implies that the codimension of $F^{\preccurlyeq\beta}\,\mathrm{H}^{2(n-d)}(G/P, \mathbb{C})$ in $\mathrm{H}^{2(n-d)}(G/P, \mathbb{C})$ is at least the number of $w \in W^P$ such that $\rho(G/P) - \rho(X_w) \not\preccurlyeq \beta$ and $l(w) = d$. The lemma follows. $\square$

5.4 Kostant's harmonic forms

5.4.1 The role of Kostant's harmonic forms in this paper

Let w in W^P. In 1963, B. Kostant constructed an explicit $\mathbb{C}$-valued closed differential form ω_w on G/P such that the associated cohomology class $[\omega_w]$ is equal to σ_w up to a scalar multiplication. Kostant's form ω_w is used here to localize the Schubert class relatively to the filtration.

Lemma 19 *The Schubert class $\sigma_{w^\vee}$ belongs to $F^{\preccurlyeq(\rho(X_w), -l(w))}\,\mathrm{H}^{l(w)}(G/P, \mathbb{C})$.*

Before proving Lemma 19 in Section 5.4.10, we recall Kostant's construction.

5.4.2 Restriction to K-invariant forms

Let K be a maximal compact subgroup of G such that $T \cap K$ is a maximal torus of K. Then K is a connected compact Lie group.

Consider the subcomplex of K-invariant forms:

$$d_p : \Omega^p(G/P, \mathbb{C})^K \longrightarrow \Omega^{p+1}(G/P, \mathbb{C})^K,$$

and its cohomology $\mathrm{H}^*_{DR}(G/P, \mathbb{C})^K$. The identity $d_{p-1}(\Omega^{p-1}(G/P, \mathbb{C})^K) = d_{p-1}(\Omega^{p-1}(G/P, \mathbb{C})) \cap \Omega^p(G/P, \mathbb{C})^K$ allows to define a morphism

$$\mathrm{H}^*_{DR}(G/P, \mathbb{C})^K \longrightarrow \mathrm{H}^*_{DR}(G/P, \mathbb{C}),$$

which is an isomorphism.

Since K acts transitively on G/P, the restriction map to the tangent space at P/P provides a linear isomorphism

$$\Omega^p(G/P, \mathbb{C})^K \longrightarrow \left(\bigwedge^p \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{p}, \mathbb{C})\right)^{K\cap L}. \tag{61}$$

Let $\mathfrak{k}$ denote the Lie algebra of K. This compact form $\mathfrak{k}$ determines a real structure $\square^*$ on $\mathfrak{g}$. More precisely, $\square^*$ is a $\mathbb{C}$-antilinear endomorphism of $\mathfrak{g}$ such that $\mathfrak{k}$ is the set of $\xi \in \mathfrak{g}$ such that $\xi^* = -\xi$.

Consider now the complex dual $(\mathfrak{g}/\mathfrak{l})^*$ of the complex vector space $\mathfrak{g}/\mathfrak{l}$. Since $\mathfrak{l}$ is stable by $\square^*$, $\mathfrak{g}/\mathfrak{l}$ is endowed with a real structure still denoted by $\square^*$. Then

$(\mathfrak{g}/\mathfrak{l})^*$ is also endowed with a real structure by setting $\varphi^* = \overline{\varphi(\square^*)}$, for any $\varphi \in (\mathfrak{g}/\mathfrak{l})^*$. Define a morphism

$$\begin{array}{rccc} \theta \ : & \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{p}, \mathbb{C}) & \longrightarrow & (\mathfrak{g}/\mathfrak{l})^* \\ & \varphi + \psi & \longmapsto & \varphi + \psi(\square^*), \end{array}$$

where φ is $\mathbb{C}$-linear and ψ is $\mathbb{C}$-antilinear. One checks that θ is a $\mathbb{C}$-linear isomorphism and that it commutes with the real structure and the actions of $K \cap L$. Note that L also acts on $(\mathfrak{g}/\mathfrak{l})$. Since $K \cap L$ is Zariski dense in L, we have

$$\left(\bigwedge^p (\mathfrak{g}/\mathfrak{l})^*\right)^{K\cap L} = \left(\bigwedge^p (\mathfrak{g}/\mathfrak{l})^*\right)^L. \tag{62}$$

Finally we get an isomorphism

$$\Omega^p(G/P, \mathbb{C})^K \longrightarrow \left(\bigwedge^p (\mathfrak{g}/\mathfrak{l})^*\right)^L. \tag{63}$$

5.4.3 The Lie algebra $\mathfrak{r}$

Let $\mathfrak{u}$ and $\mathfrak{u}^-$ be the algebras of the unipotent radicals of P and its opposite parabolic subgroup P^-. Consider the sum

$$\mathfrak{r} = \mathfrak{u}^- \oplus \mathfrak{u} \tag{64}$$

endowed with a Lie algebra structure $[\cdot,\cdot]_{\mathfrak{r}}$ defined by keeping the brackets on $\mathfrak{u}^-$ and $\mathfrak{u}$ unchanged and by setting $[\mathfrak{u}^-, \mathfrak{u}]_{\mathfrak{r}} = 0$. The L-equivariant linear isomorphism $\mathfrak{r} \simeq \mathfrak{g}/\mathfrak{l}$ and its transpose $(\mathfrak{g}/\mathfrak{l})^* \simeq \mathfrak{r}^*$ induce isomorphisms

$$\Omega^\bullet(G/P, \mathbb{C})^K \simeq \left(\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}/\mathfrak{p}, \mathbb{C})\right)^L \simeq \left(\bigwedge^\bullet \mathfrak{r}^*\right)^L \simeq \left(\bigwedge^\bullet (\mathfrak{u}^-)^* \otimes \bigwedge^\bullet \mathfrak{u}^*\right)^L. \tag{65}$$

The term $\bigwedge^\bullet (\mathfrak{u}^-)^*$ corresponds to holomorphic forms on G/P and the term $\bigwedge^\bullet \mathfrak{u}^*$ corresponds to antiholomorphic forms.

Combining $\square^*$ and the Killing form $(\cdot,\cdot)$ one obtains an Hermitian form $\{\cdot,\cdot\}$ on $\mathfrak{g}$. Explicitly,

$$\{\xi, \eta\} = -(\xi, \eta^*),$$

for any $\xi, \eta \in \mathfrak{g}$. Denote by $\{\cdot,\cdot\}_{\mathfrak{r}}$ its restriction to $\mathfrak{r}$. The decomposition $\mathfrak{u}^- \oplus \mathfrak{u} = \mathfrak{r}$ is orthogonal for $\{\cdot,\cdot\}_{\mathfrak{r}}$. Consider now the graded exterior algebra $\wedge^\bullet \mathfrak{r}^* = \oplus_p \wedge^p \mathfrak{r}^*$ and extend the bilinear form $\{\cdot,\cdot\}_{\mathfrak{r}}$ on $\wedge^\bullet \mathfrak{r}^*$. The decomposition $\mathfrak{r} = \mathfrak{u}^- \oplus \mathfrak{u}$ induces a $\mathbb{N}^2$-grading $\wedge^\bullet \mathfrak{r}^* = \oplus_{(p,q)\in\mathbb{N}^2} \wedge^{p,q} \mathfrak{r}^*$ by setting

$$\wedge^{p,q}\mathfrak{r}^* = \wedge^p(\mathfrak{u}^-)^* \otimes \wedge^q(\mathfrak{u})^*.$$

Moreover, the sum $\oplus_{(p,q)\in\mathbb{N}^2} \wedge^{p,q} \mathfrak{r}^*$ is orthogonal for $\{\cdot,\cdot\}_{\mathfrak{r}}$.

Let $b \in \mathrm{End}(\wedge^\bullet \mathfrak{r}^*)$ be the Chevalley-Eilenberg coboundary operator of the Lie algebra $\mathfrak{r}$. It has degree $+1$, more precisely

$$b(\wedge^{p,q}\mathfrak{r}^*) \subset \wedge^{p+1,q}\mathfrak{r}^* \oplus \wedge^{p,q+1}\mathfrak{r}^*.$$

Set $b = b^{1,0} + b^{0,1}$ according to this decomposition. Let $\partial \in \mathrm{End}(\wedge^\bullet \mathfrak{r})$ denote the Chevalley-Eilenberg boundary operator. Using the Killing form, we identify $\mathfrak{r}$ and $\mathfrak{r}^*$ and transport ∂ to an operation $\partial^* \in \mathrm{End}(\wedge^\bullet \mathfrak{r}^*)$ of degree -1. Decompose $\partial^* = \partial^{-1,0} + \partial^{0,-1}$ according to the decomposition $\wedge^{p-1,q}\mathfrak{r}^* \oplus \wedge^{p,q-1}\mathfrak{r}^*$. Set

$$\mathcal{L} = \partial^* \circ b + b \circ \partial^*. \tag{66}$$

[Kos63, Proposition 4.2] gives an alternative expression of $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}(\partial^{0,-1} \circ b^{0,1} + b^{0,1} \circ \partial^{0,-1}). \tag{67}$$

5.4.4 The $(X(Z) \times \mathbb{Z})$-filtration of $\wedge^\bullet \mathfrak{r}^*$

Consider the action of $Z \times \mathbb{C}^*$ on $\mathfrak{r}$ given by

$$(z, \tau).(\xi^- + \xi) = (\tau z \xi^-, \xi), \qquad \forall z \in Z,\, \tau \in \mathbb{C}^*,\, \xi^- \in \mathfrak{u}^-,\, \xi \in \mathfrak{u}. \tag{68}$$

Then the group $Z \times \mathbb{C}^*$ acts on $\wedge^\bullet \mathfrak{r}^*$ and induces a $\tilde{\Gamma}$-decomposition

$$\wedge^\bullet \mathfrak{r}^* = \bigoplus_{\tilde{\beta} \in X(Z) \times \mathbb{Z}} (\wedge^\bullet \mathfrak{r}^*)_{\tilde{\beta}}. \tag{69}$$

Note that the weights of Z acting on $(\mathfrak{u}^-)^*$ are the weights of Z acting on $\mathfrak{u}$; in particular, they are positive for the order $\succcurlyeq$. As a consequence, we have

$$(\wedge^\bullet \mathfrak{r}^*)_{\tilde{\beta}} \neq \{0\} \ \Rightarrow\ \tilde{\beta} \succcurlyeq 0. \tag{70}$$

Set

$$F^{\preccurlyeq \tilde{\beta}}(\wedge^\bullet \mathfrak{r}^*) = \oplus_{\tilde{\alpha} \preccurlyeq \tilde{\beta}} (\wedge^\bullet \mathfrak{r}^*)_{\tilde{\alpha}}. \tag{71}$$

Consider now, like in the formula (65), the diagonal action of L on $\mathfrak{r}$:

$$l.(\xi^- + \xi) = l\xi^- + l\xi, \qquad \forall l \in L,\, \xi^- \in \mathfrak{u}^-,\, \xi \in \mathfrak{u}.$$

Since Z is contained in the center of L; the action (68) of $Z \times \mathbb{C}^*$ and the above action of L commute. In particular the decomposition (69) is L-stable. Set $C = (\wedge^\bullet \mathfrak{r}^*)^L$ and $C_{\tilde{\beta}} = C \cap (\wedge^\bullet \mathfrak{r}^*)_{\tilde{\beta}}$. The $(Z \times \mathbb{C}^*)$-module C decomposes as follows

$$C := \bigoplus_{\tilde{\beta} \in \tilde{\Gamma}} C_{\tilde{\beta}}. \tag{72}$$

The associated filtration of C is:

$$F^{\preccurlyeq \tilde{\beta}} C = F^{\preccurlyeq \tilde{\beta}}(\wedge^\bullet \mathfrak{r}^*) \cap C.$$

5.4.5 Action of L on $\wedge^\bullet(\mathfrak{u}^-)^*$

We now recall results of Kostant in [Kos61] on the action of T on $\wedge^\bullet(\mathfrak{u}^-)^*$.

Theorem 4 *(i) The set of vertices of the convex hull of the weights of T acting on $\bigwedge^\bullet(\mathfrak{u}^-)^*$ is the set of $\rho - w^{-1}\rho$ where $w \in W^P$.*

These weights are multiplicity free and the eigenline corresponding to $\rho - w^{-1}\rho$ is generated by

$$\phi_w := \phi_{\alpha_1} \wedge \cdots \wedge \phi_{\alpha_p},$$

where $\{\alpha_1, \dots, \alpha_p\} = \Phi^+ \cap w^{-1}\Phi^-$; and $\phi_{\alpha_i} \in (\mathfrak{u}^-)^*$ is a vector of weight α_i.

(ii) For any $w \in W^P$, the vector ϕ_w is an highest weight vector for L. Denote by M_w the simple L-module generated by ϕ_w.

5.4.6 A first differential form

We are now ready to define a first K-invariant differential form on G/P. Set

$$h_w = \mathrm{Id}_w \in M_w \otimes M_w^* \subset \left(\wedge^p(\mathfrak{u}^-)^* \otimes \wedge^p \mathfrak{u}^*\right)^L, \tag{73}$$

where p is the length of w, that is s the codimension of $X_{w^\vee}$. Since Z is central in L, Z acts with weight $(\rho - w^{-1}\rho)_{|Z}$ on M_w. In particular,

$$h_w \in C_{((\rho - w^{-1}\rho)_{|Z}, -p)}. \tag{74}$$

If G/P is cominuscule then h_w corresponds by the isomorphism (65) to the wanted closed differential form representing σ_w. In general, more work is useful.

5.4.7 An Hermitian product on $\mathfrak{r}$

Recall that the Hermitian product $\{\cdot,\cdot\}_\mathfrak{r}$ on $\mathfrak{r}$ induces Hermitian inner products on $\wedge^\bullet\mathfrak{r}$ and $\wedge^\bullet\mathfrak{r}^*$ still denoted by $\{\cdot,\cdot\}_\mathfrak{r}$.

Lemma 20 *For any nonnegative integer p, the $(X(Z) \times \mathbb{Z})$-decomposition (72) is $\{\cdot,\cdot\}_\mathfrak{r}$-orthogonal.*

Proof. It is sufficient to prove that the decomposition

$$\mathfrak{r} = \mathfrak{u} \oplus \bigoplus_{\alpha \in X(Z)} \mathfrak{u}^-_\alpha \tag{75}$$

is $\{\cdot,\cdot\}_\mathfrak{r}$-orthogonal. Since $\mathfrak{u}^* = \mathfrak{u}^-$ and the Killing form vanishes on $\mathfrak{u}^-$, $\mathfrak{u}$ and $\mathfrak{u}^-$ are $\{\cdot,\cdot\}_\mathfrak{r}$-orthogonal. Let now fix $\xi \in \mathfrak{u}^-_\beta$ and $\eta \in \mathfrak{u}^-_{\beta'}$ with $\beta \neq \beta' \in X(Z)$. Consider the adjoint action of Z on $\mathfrak{g}$, the induced one on $\mathrm{End}(\mathfrak{g})$ and the corresponding decomposition

$$\mathrm{End}(\mathfrak{g}) = \oplus_{\alpha \in X(Z)} \mathrm{End}(\mathfrak{g})_\alpha.$$

Note that for any $A \in \mathrm{End}(\mathfrak{g})_\alpha$ with $\alpha \neq 0$, we have $\mathrm{tr}(A) = 0$. The endomorphism $\mathrm{Ad}(\eta^*)$ belongs to $\mathrm{End}(\mathfrak{g})_{-\beta'}$. It follows that $\mathrm{Ad}(\eta^*) \circ \mathrm{Ad}(\xi)$ belongs to $\mathrm{End}(\mathfrak{g})_{\beta-\beta'}$ and that $\{\xi, \eta\} = -(\xi, \eta^*) = -\mathrm{tr}(\mathrm{Ad}(\eta^*) \circ \mathrm{Ad}(\xi)) = 0$. □

5.4.8 Operators on $\wedge^\bullet(\mathfrak{r}^*)$

Recall, from the formula (66), the definition of the operator $\mathcal{L} \in \mathrm{End}(\wedge^\bullet\mathfrak{r}^*)$.

Lemma 21 *The operator $\mathcal{L}$ stabilizes $C_{(\alpha,p)}$ for any integer p and any $\alpha \in X(Z)$.*

Proof. By [Kos63, Proposition 3.4], $b^{0,1}(C_{(\alpha,p)})$ is contained in $C_{(\alpha,p+1)}$. By [Kos63, formula 3.5.3], $\partial^{0,-1}(C_{(\alpha,p+1)})$ is contained in $C_{(\alpha,p)}$. We deduce that $(\partial^{0,-1} \circ b^{0,1})(C_{(\alpha,p)})$ is contained in $C_{(\alpha,p)}$. Similarly, $(b^{0,1} \circ \partial^{0,-1})(C_{(\alpha,p)})$ is contained in $C_{(\alpha,p)}$. We conclude using the formula (67). □

Note that $\mathcal{L}$ is an Hermitian operator. In particular, we have a $\{\cdot,\cdot\}_{\mathfrak{r}}$-orthogonal decomposition $\mathrm{Ker}\,\mathcal{L} \oplus \mathrm{Im}\,\mathcal{L} = \wedge^\bullet\mathfrak{r}^*$. Consider the quasiinverse $\mathcal{L}_0$ of $\mathcal{L}$: $\mathcal{L}_0$ is the Hermitian endomorphism of $\wedge^\bullet\mathfrak{r}^*$ such that $\mathrm{Ker}\,\mathcal{L}_0 = \mathrm{Ker}\,\mathcal{L}$ and $\mathcal{L}_{0|\,\mathrm{Im}\,\mathcal{L}} = (\mathcal{L}_{|\,\mathrm{Im}\,\mathcal{L}})^{-1}$.

Let $\pi : \mathfrak{r} \longrightarrow \mathrm{End}(\wedge^\bullet\mathfrak{r}^*)$ be induced by the coadjoint action. Let f_i be eigenvectors in $\mathfrak{u}^-$ for the action of Z that form a basis of $\mathfrak{u}^-$. Let g_j be the basis of $\mathfrak{u}$ defined by the conditions $(f_i, g_j) = \delta_i^j$ (the Kronecker symbol). Set

$$\mathcal{E} := 2\sum_i \pi(g_i) \circ \pi(f_i) \in \mathrm{End}(\wedge^\bullet\mathfrak{r}^*). \tag{76}$$

Kostant defined a third operator

$$\mathcal{R} := -\mathcal{L}_0 \circ \mathcal{E} \in \mathrm{End}(\wedge^\bullet\mathfrak{r}^*), \tag{77}$$

he proved that $\mathcal{R}$ is nilpotent and he defined

$$s_w = (\mathrm{Id} - \mathcal{R})^{-1}(h_w) = h_w + \mathcal{R}(h_w) + \mathcal{R}^2(h_w) + \cdots . \tag{78}$$

Here, we need the following improvement of [Kos63, Lemma 4.6] that proves the nilpotency of $\mathcal{R}$.

Lemma 22 *For any integer p and $\alpha \in X(Z)$, we have*

$$\mathcal{R}(C_{(\alpha,p)}) \subset \bigoplus_{\beta \prec \alpha} C_{(\beta,p)}.$$

Proof. Lemma 21 asserts that $\mathcal{L}$ stabilizes the $(X(Z) \times \mathbb{Z})$-decomposition of C. Since this decomposition is $\{\cdot,\cdot\}_{\mathfrak{r}}$-orthogonal by Lemma 20, this implies that $\mathcal{L}_0$ also stabilizes the $\tilde{\Gamma}$-decomposition of C. By the formula (77), it remains to prove that $\mathcal{E}(C_{\alpha,p)}) \subset \bigoplus_{\beta\prec\alpha} C_{(\beta,p)}$.

But each $\pi(f_i)$ vanishes on $\wedge^\bullet\mathfrak{u}^*$ and each $\pi(g_i)$ respects the degree. It follows that $\mathcal{E}(C_{(\alpha,p)}) \subset \bigoplus_{\beta\in X(Z)} C_{(\beta,p)}$. But $\pi(g_i)$ vanishes on $\wedge^\bullet(\mathfrak{u}^-)^*$. Moreover, f_i belongs to $\mathfrak{u}^-$ and has a weight $\gamma \preccurlyeq 0$. It follows that $\pi(f_i)(\wedge^\bullet(\mathfrak{u}^-)^*_\beta) \subset \wedge^\bullet(\mathfrak{u}^-)^*_{\beta-\gamma}$. The claim follows. □

5.4.9 Kostant's theorem

Theorem 5 *([Kos63]) Let $w \in W^P$. The element $s_w \in \bigwedge^\bullet \mathfrak{r}^*$ defined by* (78) *is L-invariant. In particular, s_w corresponds by the isomorphism* (65) *to a K-invariant form ω_w on G/P.*

Then the form ω_w is closed and its class $[\omega_w]$ in $\mathrm{H}^*_{DR}(G/P, \mathbb{C})$ is equal to the Schubert class $\sigma_w^\vee$, up to a positive real scalar.

5.4.10 Application

We can now prove Lemma 19.

Proof.[of Lemma 19] By Theorem 5, it is sufficient to prove that ω_w belongs to $F^{\preccurlyeq \tilde\rho(w)}\Omega^{l(w)}(G/P, \mathbb{C})$. But ω_w and the filtration are K-invariant on the K-homogeneous space G/P. Hence it is sufficient to prove that s_w belongs to $F^{\preccurlyeq \tilde\rho(w)}C$. This is a consequence of the property (74) and Lemma 22. □

Proof.[of Proposition 10] Let $\tilde\beta \in \tilde\Gamma$ and let p be an integer such that $0 \le p \le \dim(G/P)$. Consider $F^{\preccurlyeq\tilde\beta}\,\mathrm{H}^{2p}(G/P, \mathbb{C})$. On one hand, Lemma 18 shows that the dimension of $F^{\preccurlyeq\tilde\beta}\,\mathrm{H}^{p}(G/P, \mathbb{C})$ is not more than the cardinality of the set

$$W(\tilde\beta, p) = \{w \in W^P \;:\; \tilde\rho(G/P) - \tilde\rho(X_w) \preccurlyeq \tilde\beta \text{ and } l(w) = n - p\}.$$

On the other hand, Lemma 19 shows that $F^{\preccurlyeq\tilde\beta}\,\mathrm{H}^{p}(G/P, \mathbb{C})$ contains the classes $\sigma_{w^\vee}$ for w in the set

$$W'(\tilde\beta, p) = \{w \in W^P \;:\; \tilde\rho(X_w) \preccurlyeq \tilde\beta \text{ and } l(w) = p\}.$$

But Lemma 17 implies that the Poincaré duality $w \mapsto w^\vee$ induces a bijection between $W(\tilde\beta, p)$ and $W'(\tilde\beta, p)$. Since the family $(\sigma_{w^\vee})_{w \in W'(\tilde\beta, p)}$ is linearly independant the proposition follows. □

Proof.[of Corollary 1] The corollary is a direct consequence of Lemma 14 and the above proof of Proposition 10. □

5.5 Proof of Theorem 3

Let u and v be elements of W^P. Consider the following product in the ordinary cohomology ring $\mathrm{H}^*(G/P, \mathbb{C})$

$$\sigma_u.\sigma_v = \sum_{w \in W^P} c_{u\,v}^{w} \sigma_w.$$

By Lemma 19 and Lemma 17, σ_u belongs to $F^{\preccurlyeq \tilde\rho(G/P) - \tilde\rho(X_u)}\,\mathrm{H}^{l(w_0 w_o^P) - l(u)}(G/P, \mathbb{C})$. Similarly, σ_v belongs to $F^{\preccurlyeq \tilde\rho(G/P) - \tilde\rho(X_v)}\,\mathrm{H}^{l(w_0 w_o^P) - l(v)}(G/P, \mathbb{C})$. Now Proposition 8 shows that

$$\sigma_u.\sigma_v \in F^{\preccurlyeq 2\tilde\rho(G/P) - \tilde\rho(X_u) - \tilde\rho(X_v)}\,\mathrm{H}^{2l(w_0 w_o^P) - l(u) - l(v)}(G/P, \mathbb{C}).$$

By Proposition 10, this means that

$$c_{u\,v}^w \neq 0 \quad \Rightarrow \quad \tilde{\rho}(G/P) - \tilde{\rho}(X_w) \preccurlyeq 2\tilde{\rho}(G/P) - \tilde{\rho}(X_u) - \tilde{\rho}(X_v), \tag{79}$$
$$\Rightarrow \quad \tilde{\rho}(X_u) + \tilde{\rho}(X_v) \preccurlyeq \tilde{\rho}(X_w) + \tilde{\rho}(G/P). \tag{80}$$

Proposition 10 implies also that

$$\overline{\sigma_u} \,.\, \overline{\sigma_v} = \sum_{\substack{w\in W^P \\ \tilde{\rho}(X_u)+\tilde{\rho}(X_v)=\tilde{\rho}(X_w)+\tilde{\rho}(G/P)}} c_{u\,v}^w \overline{\sigma_w}. \tag{81}$$

On the other hand, Proposition 9 shows that

$$\sigma_u \odot_0 \sigma_v = \sum_{\substack{w\in W^P \\ gd(X_u)+gd(X_v)=gd(X_w)+gd(G/P)}} c_{u\,v}^w \sigma_w. \tag{82}$$

Comparing the identities (81) and (82), it remains to prove, under the assumption $c_{u\,v}^w \neq 0$, that the equivalence

$$\tilde{\rho}(X_u)+\tilde{\rho}(X_v) = \tilde{\rho}(X_w)+\tilde{\rho}(G/P) \iff gd(X_u)+gd(X_v) = gd(X_w)+gd(G/P)$$

holds.

The implication "$\Leftarrow$" is an immediate consequence of the definition (19) of $\rho(\cdot)$. Conversely, assume that $\tilde{\rho}(X_u) + \tilde{\rho}(X_v) = \tilde{\rho}(X_w) + \tilde{\rho}(G/P)$. Since $c_{u\,v}^w \neq 0$, the Belkale-Kumar numerical criterion of Levi-movability (see [BK06, Theorem 15]) implies that $\sigma_u \odot_0 \sigma_v \odot_0 \sigma_{w^\vee} = c_{u\,v}^w[pt]$. In particular, Proposition 9 implies that $gd(X_u) + gd(X_v) = gd(X_w) + gd(G/P)$. The theorem is proved.

5.6 The Belkale-Kumar fundamental class

Recall from Section 4.4 the definition of the Belkale-Kumar fundamental class of any subvariety of G/P. We can now give a simple characterization of this class using the notion of $X(Z)$-dimension.

Proposition 11 *Let Y be an irreducible subvariety of G/P of dimension d. Consider the expansion of its fundamental class in the Schubert basis*

$$[Y] = \sum_{w\in W^P} d_w \sigma_w.$$

Then the expansion of its $\odot_0$-fundamental class in the Schubert basis is

$$[Y]_{\odot_0} = \sum_{\substack{w\in W^P \\ \rho(X_w)=\rho(Y)}} d_w \sigma_w^{\odot_0}.$$

Proof. It remains to prove that for any $[\omega] \in \mathrm{Gr}^{\tilde{\rho}(Y)}\, \mathrm{H}^{2d}(G/P, \mathbb{C})$,

$$\int_Y \omega = [\omega]_{\odot_0} \odot_0 \Big(\sum_{\substack{w\in W^P \\ \rho(X_w)=\rho(Y)}} d_w \sigma_w \Big).$$

Since the two members of the equality depend linearly on $[\omega]$, it is sufficient to prove it for $[\omega] = \sigma_{u^\vee}$, for any $u \in W^P$ such that $\rho(X_u) = \rho(Y)$ and $l(u) = d$. By ordinary Poincaré duality, this case is equivalent to the following equality

$$\sigma_{u^\vee}.(\sum_{\substack{w \in W^P \\ l(w)=n-d}} d_w \sigma_w) = \sigma_{u^\vee} \odot_0 (\sum_{\substack{w \in W^P \\ \rho(X_w)=\rho(Y)}} d_w \sigma_w).$$

Since the only product $\sigma_{u^\vee}.\sigma_w$ that is nonzero in the above formula is $\sigma_{u^\vee}.\sigma_u$, the proposition follows. □

6 Intersecting Schubert varieties

Given $u, v \in W^P$ such that $v^\vee < u$, we construct in this section a familly of varieties containing both the Richardson variety $X_u \cap w_0 X_v$ (up to translation) and the variety Σ_u^v. We prove (see Proposition 13) that Conjecture 4 holds for Σ_u^v if and only if it holds for all these varieties. To end this section, we show that Conjecture 4 is equivalent to a formula using the Kostant harmonic forms that looks like a Fubini formula.

6.1 Products on $\mathrm{H}^*(G/P, \mathbb{C})$ and Bruhat orders

The Bruhat order on W^P is defined by

$$u < v \iff X_u \subset X_v.$$

This order is generated by $u < v$ if $l(v) = l(u)+1$ and $v = s_\alpha u$ for some positive root α. The weak Bruhat order on W^P is generated by the relation $u \lessdot v$ if $l(v) = l(u)+1$ and $v = s_\alpha u$ for some simple root α. The relation between these two orders is

$$u \lessdot v \Rightarrow u < v. \tag{83}$$

A useful characterization of the weak Bruhat order is given by the following result (see [Bou68]).

Lemma 23 *Let u and v in W^P. Then $u \lessdot v$ if and only if $\Phi(u)$ is contained in $\Phi(v)$.*

The following relation between the cup product and the Bruhat order is well known

$$\sigma_u.\sigma_v \neq 0 \iff v^\vee < u.$$

We have the following relation between the Belkale-Kumar product and the weak Bruhat order.

Lemma 24 *Let u and v in W^P. If $\sigma_u \odot_0 \sigma_v \neq 0$ then $v^\vee \lessdot u$.*

Proof. By assumption, there exists $w \in W^P$ such that (u, v, w) is Levi-movable and $l(u) + l(v) + l(w) = l(w_0 w_0^P)$. Hence, for (l_1, l_2, l_3) in a nonempty open subset of L^3:

$$l_1 T_u \cap l_2 T_v \cap l_3 T_w = \{0\}.$$

In particular, $l_1 T_u + l_2 T_v = T_{P/P} G/P$. Since $\Delta L.(B, w_0^P B_L)$ is open in L^2, there exist $l \in L$, $b_1, b_2 \in B_L$ such that $l b_1 T_u + l w_0^P b_2 T_v = T_{P/P} G/P$. But T_u and T_v are B_L-stable and $T_{P/P} G/P$ is L-stable, hence

$$T_u + w_0^P T_v = T_{P/P} G/P.$$

It follows that $\Phi(u) \cup w_0^P \Phi(v) = \Phi(G/P)$. But $\Phi(v^\vee) = \Phi(G/P) - w_0^P \Phi(v)$. Hence $\Phi(v^\vee) \subset \Phi(u)$ and $v^\vee \lessdot u$. $\square$

Remark. The converse of the implication of Lemma 24 does not hold. Indeed consider $\mathrm{SL}_3(\mathbb{C})$ with its usual maximal torus and Borel subgroup B. Denote the two simple reflections of W by s_1 and s_2. Then $\sigma_{s_1 s_2} \odot_0 \sigma_{s_2 s_1} = 0$ while $(s_2 s_1)^\vee = s_2 \lessdot s_1 s_2$.

6.2 Like Richardson's varieties

Let $u, v \in W^P$. The Richardson variety X_u^v is defined by

$$X_u^v = X_u \cap w_0 X_v.$$

It is well known that X_u^v is irreducible, normal and satisfies $[X_u^v] = \sigma_u.\sigma_v$. In particular, X_u^v is empty if and only if $v^\vee < u$.

Assume now that $v^\vee \lessdot u$. Fix $y \in W^P$ such that $v^\vee \lessdot y \lessdot u$. Consider the intersection

$$I_u^v(y) := y^{-1} X_u \cap w_0^P v^{-1}.X_v. \tag{84}$$

The first example $I_u^v(v^\vee) = (v^\vee)^{-1} X_u^v$ is just a translated Richardson variety.

By the relation (83), the point yP/P belongs to X_u. It follows that P/P belongs to $y^{-1} X_u$. Since vP/P belongs to X_v, P/P belongs to $w_0^P v^{-1}.X_v$. It follows that

$$P/P \in I_u^v(y). \tag{85}$$

The following lemma shows that the variety $I_u^v(y)$ contains a translated Richardson variety.

Lemma 25 *Let u, v, and y in W^P such that $v^\vee \lessdot y \lessdot u$. Then $I_u^{y^\vee}(y)$ is contained in $I_u^v(y)$.*

Proof. It remains to prove that $y^{-1} X_u \cap w_0^P (y^\vee)^{-1}.X_{y^\vee}$ is contained in $y^{-1} X_u \cap w_0^P v^{-1}.X_v$. It is sufficient to prove that $(y^\vee)^{-1}.X_{y^\vee}$ is contained in $v^{-1}.X_v$. But $(y^\vee)^{-1}.X_{y^\vee} = \overline{((y^\vee)^{-1} B y^\vee).P/P}$ and $v^{-1}.X_v = \overline{(v^{-1} B v).P/P}$. Hence it is sufficient to prove that $\Phi(\mathfrak{g}/\mathfrak{p},T)\cap(y^\vee)^{-1}\Phi^+$ is contained in $\Phi(\mathfrak{g}/\mathfrak{p},T)\cap v^{-1}\Phi^+$. But $v^\vee \lessdot y$ and hence $y^\vee \lessdot v$. Lemma 23 allows to conclude. $\square$

The fact that X_u and X_v are B-stable implies that the group $H_u^v(y) := y^{-1}By \cap w_0^P v^{-1}Bvw_0^P$ acts on $I_u^v(y)$. Set $y' = y(v^\vee)^{-1}$ in such a way that $y = y'v^\vee$. Note that $yw_0^P v^{-1} = y'w_0$ and that

$$H_u^v(y) = (v^\vee)^{-1}(y'^{-1}By' \cap B^-)v^\vee. \tag{86}$$

The group $H_u^v(y)$ is a connected subgroup of G, containing T and acting on $I_u^v(y)$. Consider now the group $U(y') = y'^{-1}Uy' \cap U^-$.

Let $\overset{\circ}{G/P} = B^-P/P$ denote the open T-stable affine cell containing P/P. Set $\overset{\circ}{I}{}_u^v(y) = \overset{\circ}{G/P} \cap I_u^v(y)$; it is an open T-stable affine neighborhood of P/P in $I_u^v(y)$. The following statement describes the geometry of this neighborhood.

Theorem 6 *Let u, v, and y in W^P such that $v^\vee \lessdot y \lessdot u$. Then the following morphism*

$$\begin{array}{cccc} \Psi \ : & U(y') \times \overset{\circ}{I}{}_u^{y^\vee}(y) & \longrightarrow & \overset{\circ}{I}{}_u^v(y) \\ & (u,x) & \longmapsto & (v^\vee)^{-1}uv^\vee.x \end{array}$$

is an isomorphism.

Proof. The weights of T acting on the Lie algebra of the group $U(y) = U^- \cap y^{-1}Uy$ are $\Phi(y) = \Phi^- \cap y^{-1}\Phi^+$. The weights of T acting on the tangent space at the point P/P of the variety $w_0^P(y^\vee)^{-1}X_{y^\vee}$ are $\Phi(\mathfrak{g}/\mathfrak{p},T)\cap y^{-1}\Phi^-$. But $\overset{\circ}{G/P}$ is isomorphic as a T-variety to the affine space $\mathfrak{g}/\mathfrak{p}$. It follows that the map

$$\begin{array}{ccc} U(y) \times [w_0^P(y^\vee)^{-1}X_{y^\vee} \cap \overset{\circ}{G/P}] & \longrightarrow & \overset{\circ}{G/P} \\ (u,x) & \longmapsto & ux \end{array} \tag{87}$$

is an isomorphism. The variety $y^{-1}X_u$ is stable by $y^{-1}By$ and so by $U(y)$. It follows that the map

$$\begin{array}{ccc} U(y) \times [w_0^P(y^\vee)^{-1}X_{y^\vee} \cap \overset{\circ}{G/P} \cap y^{-1}X_u] & \longrightarrow & \overset{\circ}{G/P} \cap y^{-1}X_u \\ (u,x) & \longmapsto & ux \end{array}$$

is an isomorphism.

Since $v^\vee \lessdot y$ and $y = y'v^\vee$, the set $\Phi(y)$ is the disjoint union of $\Phi(v^\vee)$ and $(v^\vee).\Phi(y')$ (see for example [Bou02]). Then the map

$$\begin{array}{ccc} U(y') \times U(v^\vee) & \longrightarrow & U(y) \\ (u',u) & \longmapsto & (v^\vee)^{-1}u'v^\vee u \end{array}$$

is an isomorphism. Note that in the above expression we have fixed representative (still denoted by $v^\vee$) of $v^\vee$ in the normalizer of the torus T. Composing these isomorphisms gives the following one:

$$\begin{array}{ccc} U(y') \times U(v^\vee) \times \overset{\circ}{I}{}_u^{y^\vee}(y) & \longrightarrow & \overset{\circ}{G/P} \cap y^{-1}X_u \\ (u',u,x) & \longmapsto & (v^\vee)^{-1}u'v^\vee ux. \end{array}$$

Since $\Phi(y')$ is contained in $(v^\vee)^{-1}\Phi^-$, and $w_0^P v^{-1} X_v = \overline{((v^\vee)^{-1}B^- v^\vee).P/P}$, the variety $w_0^P v^{-1} X_v$ is stable under the action of $U(y')$. Hence

$$\begin{array}{ccc} U(y') \times \left[(U(v^\vee) \cdot \mathring{I}_u^{y^\vee}(y)) \cap w_0^P v^{-1} X_v \right] & \longrightarrow & \mathring{I}_u^v(y) \\ (u', x) & \longmapsto & (v^\vee)^{-1} u' v^\vee x \end{array}$$

is an isomorphism. It remains to prove that

$$(U(v^\vee) \cdot \mathring{I}_u^{y^\vee}(y)) \cap w_0^P v^{-1} X_v = \mathring{I}_u^{y^\vee}(y).$$

Let $u \in U(v^\vee)$ and $x \in \mathring{I}_u^{y^\vee}(y)$ such that ux belongs to $w_0^P v^{-1} X_v$. It is sufficient to prove that $u = e$. Replacing $y^\vee$ by v in the morphism (87), we get an isomorphism

$$\begin{array}{cccc} \Theta \ : & U(v^\vee) \times [w_0^P v^{-1} X_v \cap \mathring{G/P}] & \longrightarrow & \mathring{G/P} \\ & (u', x') & \longmapsto & u'x'. \end{array}$$

One can easily check that x belongs to $w_0^P v^{-1} X_v \cap \mathring{G/P}$ and that $\Theta(u, x) = \Theta(e, ux)$. Now, the injectivity of Θ implies that $u = e$. $\square$

An important consequence of Theorem 6 for our purpose is the following statement.

Corollary 2 *The variety $I_u^v(y)$ is normal at the point P/P. In particular, there exists an unique irreducible component $\Sigma_u^v(y)$ of $I_u^v(y)$ which contains P/P.*

Proof. The corollary follows from the theorem and the fact that the Richardson varieties are irreducible and normal (see [KWY13] for a short proof). $\square$

If $y = v^\vee$ then Theorem 6 is trivial. In the opposite situation when $y = u$ it implies the following result.

Corollary 3 *Let u and v in W^P such that $v^\vee \lessdot u$. The orbit $H_u^v(u).P/P$ is open in $I_u^v(u)$. In other words, $\Sigma_u^v(u)$ is the closure of $H_u^v(u).P/P$.*

Proof. If $y = u$ then the translated Richardson variety $I_u^{y^\vee}(y) = I_u^{u^\vee}(u)$ is reduced to the point P/P. The corollary follows immediately. $\square$

6.3 A conjecture

Here comes our main conjecture.

Conjecture 4 *Let $u, v \in W^P$ such that $v^\vee \lessdot u$. Then*

$$[\Sigma_u^v(u)]_{\odot_0} = \sigma_u^{\odot_0} \odot_0 \sigma_v^{\odot_0}.$$

Some observations on this conjecture are collected in the following propositions.

Proposition 12 *Expand $[\Sigma_u^v(u)]_{\odot_0}$ and $\sigma_u \odot_0 \sigma_v$ in the Schubert basis:*

$$[\Sigma_u^v(u)]_{\odot_0} = \sum_{w\in W^P} d_{uv}^w \sigma_w^{\odot_0}, \text{ and}$$

$$\sigma_u^{\odot_0} \odot_0 \sigma_v^{\odot_0} = \sum_{w\in W^P} \tilde{c}_{uv}^w \sigma_w^{\odot_0}.$$

Then, for any $w \in W^P$,

(i) $\tilde{c}_{uv}^w \geq d_{uv}^w$*;*

(ii) $\tilde{c}_{uv}^w \neq 0 \iff d_{uv}^w \neq 0$.

Proof. Write $[\Sigma_u^v(u)] = \sum_{w\in W^P} e_{uv}^w \sigma_w$ and $\sigma_u.\sigma_v = \sum_{w\in W^P} c_{uv}^w \sigma_w$ in ordinary cohomology. Since $\Sigma_u^v(u)$ is an irreducible component of the intersection $I_u^v(u)$ and that this intersection is proper along this component, the inequality

$$c_{uv}^w \geq e_{uv}^w \tag{88}$$

holds for any $w \in W^P$. Consider now a coefficient d_{uv}^w for some fixed $w \in W^P$. If $d_{uv}^w = 0$ then the first assertion of the proposition is obvious. Assume $d_{uv}^w \neq 0$. By Proposition 11, $d_{uv}^w = e_{uv}^w$. Comparing the inequality (88) and the first assertion, one observes that it is sufficient to prove that $\tilde{c}_{uv}^w = c_{uv}^w$; that is that $\tilde{c}_{uv}^w \neq 0$.

Since $d_{uv}^w \neq 0$, Proposition 11 implies that $\rho(X_w) = \rho(\Sigma_u^v(u))$. Since P/P belongs to the open $H_u^v(u)$-orbit in $\Sigma_u^v(u)$ and is $X(Z)$-regular. In particular

$$\rho(\Sigma_u^v(u)) = \sum_{\gamma\in X(Z)} \dim[(T_{P/P}\Sigma_u^v(u))_\gamma]\gamma,$$

where $(T_{P/P}\Sigma_u^v(u))_\gamma$ is the weight space of weight γ of the Z-module $T_{P/P}\Sigma_u^v(u)$. But $T_{P/P}\Sigma_u^v(u)$ is the transverse intersection of $T_{P/P}u^{-1}X_u$ and $T_{P/P}w_0^P v^{-1}X_v$. It follows that $\rho(\Sigma_u^v(u)) = \rho(u^{-1}X_u) + \rho(w_0^P v^{-1}X_v) = \rho(X_u) + \rho(X_v)$. Finally $\rho(X_w) = \rho(X_u) + \rho(X_v)$ and Proposition 11 shows that $\tilde{c}_{uv}^w = c_{uv}^w$.

Assuming that $d_{uv}^w \neq 0$, the first assertion implies that $\tilde{c}_{uv}^w \neq 0$. Assume conversely that $\tilde{c}_{uv}^w \neq 0$ in other words that $(u, v, w^\vee)$ is Levi-movable. Arguing like in the proof of Lemma 24, one can check that there exists $l \in L$ such that $u^{-1}X_u$, $w_0^P v^{-1}X_v$ and $l(w^\vee)^{-1}X_{w^\vee}$ intersect transversally at P/P. It follows immediately that $\Sigma_u^v(u)$ and $l(w^\vee)^{-1}X_{w^\vee}$ intersect transversally at P/P. Hence $e_{uv}^w \neq 0$.

It remains to prove that $e_{uv}^w = d_{uv}^w$. The condition $\tilde{c}_{uv}^w \neq 0$ in the $X(Z)$-graded algebra $\operatorname{Gr}\mathrm{H}^*(G/P, \mathbb{C})$ implies that $\rho(X_w) = \rho(X_u) + \rho(X_v)$. But $\rho(X_u) + \rho(X_v) = \rho(\Sigma_u^v(u))$. Proposition 11 shows that $e_{uv}^w = d_{uv}^w$. $\square$

Proposition 13 *Let $u, v \in W^P$ such that $v^\vee \lessdot u$.*

(i) Conjecture 4 holds if $\Sigma_u^v(u)$ has dimension 0, 1 *or* 2.

(ii) Conjecture 4 holds if and only if for any $y \in W^P$ such that $v^\vee \lessdot y \lessdot u$ we have $[\Sigma_u^v(y)]_{\odot_0} = \sigma_u \odot_0 \sigma_v$.

Proof. If $\Sigma_u^v(u)$ has dimension 0 then $u = v^\vee$. In particular $[\Sigma_u^v(y)]_{\odot_0} = [pt] = \sigma_u \odot_0 \sigma_v$.

If $\Sigma_u^v(u)$ has dimension 1 then $u = s_\alpha v^\vee$, for some simple root α. Moreover, $l(u) = l(v^\vee)+1$. This implies that X_u is stable by the action of P_α (the minimal parabolic subgroup associated to α). In particular $s_\alpha X_u = X_u$. It follows that $u^{-1}X_u = u^{-1}s_\alpha X_u = (v^\vee)^{-1}X_u$. In particular $I_u^v(u) = I_u^v(v^\vee)$ is a translated Richardson variety and is irreducible. Moreover, $\sigma_u.\sigma_v = [I_u^v(u)] = [\Sigma_u^v(u)]$. Proposition 11 implies that $\sigma_u \odot_0 \sigma_v = [\Sigma_u^v(u)]_{\odot_0}$.

Assume now that $u = s_\alpha s_\beta v^\vee$, for some simple roots α and β such that $l(u) = l(v^\vee) + 2$. Then (note that $s_\alpha X_u = X_u$)

$$\begin{array}{rcl} I_u^v(u) & = & u^{-1}X_u \cap w_0^P v^{-1} X_v \\ & = & (v^\vee)^{-1}(s_\beta s_\alpha X_u \cap w_0 X_v) \\ & = & (v^\vee)^{-1} s_\beta (s_\alpha X_u \cap w_0 s_{\beta^*} X_v), \end{array}$$

where $\beta^* = -w_0\beta$. But the condition $v^\vee \lessdot s_\beta v^\vee$ implies that $s_{\beta^*} v \lessdot v$ (see for example Lemma 23). Then $s_{\beta^*}X_v = X_v$ and $I_u^v(u)$ is obtained by translation from the Richardson variety $s_\alpha X_u \cap w_0 s_{\beta^*} X_v$. The first assertion of the proposition follows.

Let α be a simple root such that $y \lessdot s_\alpha y \lessdot u$. Set $\beta = -y^{-1}\alpha$ and set $U_\beta : \mathbb{C} \longrightarrow G$, the associated additive one-parameter subgroup. Consider the flat limit $\lim_{t\to\infty} U_\beta(t)\Sigma_u^v(y)$. Since $U_\beta(t)y^{-1}B/B$ tends to $y^{-1}s_\alpha B/B$ when t goes to infinity, $\lim_{t\to\infty} U_\beta(t)y^{-1}X_u = y^{-1}s_\alpha X_u$. Since $v^\vee \lessdot y \lessdot s_\alpha y$, $\beta \in \Phi(s_\alpha y) - \Phi(v^\vee)$ and $w_0^P\beta \in \Phi(v)$. In particular, $w_0^P v^{-1} X_v$ is U_β-stable. But $\Sigma_u^v(s_\alpha y)$ is an irreducible component of the intersection $y^{-1}s_\alpha X_u \cap w_0^P v^{-1} X_v$; and, this intersection is transverse along this component. It follows that $\Sigma_u^v(s_\alpha y)$ is an irreducible component of $\lim_{t\to\infty} U_\beta(t)\Sigma_u^v(y)$. Writing

$$[\Sigma_u^v(y)] = \sum_{w\in W^P} d'_w \sigma_w \quad \text{and} \quad [\Sigma_u^v(s_\alpha y)] = \sum_{w\in W^P} d''_w \sigma_w,$$

we deduce that

$$d''_w \leq d'_w \qquad \forall w \in W^P. \tag{89}$$

Write now

$$[\Sigma_u^v(v^\vee)] = \sum_{w\in W^P} d_w \sigma_w \quad \text{and} \quad [\Sigma_u^v(u)] = \sum_{w\in W^P} e_w \sigma_w.$$

Since $\Sigma_u^v(v^\vee)$ is a translated of the Richardson variety $X_u \cap w_0 X_v$,

$$\sigma_u.\sigma_v = \sum_{w\in W^P} d_w \sigma_w.$$

By an immediate induction, we deduce from (89) that

$$e_w \leq d'_w \leq d_w \qquad \forall w \in W^P.$$

Conjecture (4) holds for $y = u$ if and only if for any $w \in W^P$ such that (u, v, w) is Levi-movable $e_w = d_w$. Then, $d'_w = d_w$ for any such $w \in W^P$ and $[\Sigma_u^v(y)]_{\odot_0} = \sigma_u \odot_0 \sigma_v$. $\square$

6.4 Interpretation in terms of harmonic forms

Kostant's harmonic forms allow to formulate Conjecture 4 as an identity of integrals.

Proposition 14 *Let u and v in W^P such that $v^\vee \lessdot u$. Then $\sigma_u \odot_0 \sigma_v = [\Sigma_u^v(u)]_{\odot_0}$ if and only if for any w in W^P such that (u, v, w) is Levi-movable, we have*

$$\int_{(u^\vee)^{-1} X_{u^\vee}} \omega_{u^\vee} . \int_{(v^\vee)^{-1} X_{v^\vee}} \omega_{v^\vee} . \int_{\Sigma_u^v(u)} \omega_{w^\vee} = \int_{(P^u)^-} \omega_{u^\vee} \wedge \omega_{v^\vee} \wedge \omega_{w^\vee}.$$

Proof. For any $w \in W^P$, consider the Kostant's harmonic form ω_w and the nonzero complex number λ_w (see Theorem 5) such that

$$[\omega_w] = \lambda_w \sigma_w^\vee. \tag{90}$$

Then

$$\lambda_w = \int_{w^{-1} X_w} \omega_w. \tag{91}$$

By Propositions 11 and 12, Conjecture 4 is equivalent to the fact that for any $w \in W^P$ such that (u, v, w) is Levi-movable, we have

$$\sigma_u . \sigma_v . \sigma_w = [\Sigma_u^v(u)] . \sigma_w. \tag{92}$$

But on one hand

$$\sigma_u . \sigma_v . \sigma_w = \frac{1}{\lambda_{u^\vee} \lambda_{v^\vee}} [\omega_{u^\vee} \wedge \omega_{v^\vee}] . \sigma_w = \frac{\int_{w^{-1} X(w)} \omega_{u^\vee} \wedge \omega_{v^\vee}}{\lambda_{u^\vee} \lambda_{v^\vee}}. \tag{93}$$

And on the other hand

$$[\Sigma_u^v(u)] . \sigma_w = \frac{\int_{\Sigma_u^v(u)} \omega_{w^\vee}}{\lambda_{w^\vee}}. \tag{94}$$

In particular the equality (92) is equivalent to

$$\lambda_{w^\vee} . \int_{w^{-1} X_w} \omega_{u^\vee} \wedge \omega_{v^\vee} = \lambda_{u^\vee} . \lambda_{v^\vee} . \int_{\Sigma_u^v(u)} \omega_{w^\vee}; \tag{95}$$

which is, by (91), equivalent to

$$\lambda_{w^\vee}.\int_{w^{-1}X_w}\omega_{u^\vee}\wedge\omega_{v^\vee}=\int_{(u^\vee)^{-1}X_{u^\vee}}\omega_{u^\vee}.\int_{(v^\vee)^{-1}X_{v^\vee}}\omega_{v^\vee}.\int_{\Sigma_u^v(u)}\omega_{w^\vee}. \tag{96}$$

We claim that

$$\lambda_{w^\vee}.\int_{w^{-1}X_w}\omega_{u^\vee}\wedge\omega_{v^\vee}=\int_{(P^u)^-}\omega_{u^\vee}\wedge\omega_{v^\vee}\wedge\omega_{w^\vee}. \tag{97}$$

Let d be the positive integer such that $\sigma_u.\sigma_v.\sigma_w=d[pt]$. We have

$$d=\int_{G/P}\frac{\omega_{u^\vee}\wedge\omega_{v^\vee}\wedge\omega_{w^\vee}}{\lambda_{u^\vee}\lambda_{v^\vee}\lambda_{w^\vee}}.$$

Since $\sigma_u.\sigma_v=d\sigma_{w^\vee}$, we also have

$$d=\int_{w^{-1}X(w)}\frac{\omega_{u^\vee}\wedge\omega_{v^\vee}}{\lambda_{u^\vee}\lambda_{v^\vee}}.$$

Claim (97) is obtained by identifying these two expressions of d.

The proposition follows now from the equations (97) and (96). $\square$

Remark. Observe that $(P^u)^-$ is isomorphic to the product of the three T-stable affine neighborhoods of P/P in $(u^\vee)^{-1}X_{u^\vee}$, $(v^\vee)^{-1}X_{v^\vee}$ and $\Sigma_u^v(u)$. With this observation the equality of Proposition 14 looks like a Fubini formula.

7 The case of the complete flag varieties

Given u in W, set $\Phi(u)^c:=\Phi^- - \Phi(u)$. Let u,v, and w in W. For the complete flag variety G/B the Levi-movability is easy to understand. Indeed T_u, T_v, and T_w are $L=T$-stable. In particular, $(\sigma_u,\sigma_v,\sigma_w)$ is Levi-movable if and only if the natural map $T_{B/B}(G/B)\longrightarrow\frac{T_{B/B}(G/B)}{T_u}\oplus\frac{T_{B/B}(G/B)}{T_v}\oplus\frac{T_{B/B}(G/B)}{T_w}$ is an isomorphism. This is equivalent to the fact that Φ^- is the disjoint union of $\Phi(u)^c$, $\Phi(v)^c$, and $\Phi(w)^c$. Since $\Phi(w)^c=\Phi(w^\vee)$, one gets the following equivalence

$$\tilde c_{uv}^w\neq 0\iff \Phi(w)^c=\Phi(u)^c\sqcup\Phi(v)^c.$$

Conjecture 4 generalizes a classical one on G/B.

Proposition 15 *Let G be a semisimple group and consider the Belkale-Kumar cohomology of G/B. Let u and v belong to W. Then $\sigma_u\odot_0\sigma_v=[\Sigma_u^v(u)]_{\odot_0}$ if and only if $\sigma_u\odot_0\sigma_v$ is either equal to zero or to σ_w for some $w\in W$.*

Proof. Assume that $\sigma_u\odot_0\sigma_v=[\Sigma_u^v(u)]_{\odot_0}$.
Case 1. Suppose there exists $w\in W$ such that $\Phi(w)=\Phi(H_u^v(u))$.
Then (see for example Lemma 11) $\Sigma_u^v(u)=w^{-1}X_w$; hence $[\Sigma_u^v(u)]_{\odot_0}=\sigma_w$. In particular $\sigma_u\odot_0\sigma_v=\sigma_w$.

Case 2. Suppose there exists no $w \in W$ such that $\Phi(w) = \Phi(H_u^v(u))$.
Since $\Phi(H_u^v(u)) = \Phi(u) \cap \Phi(v)$, there is no $w \in W$ such that $\Phi(w)^c = \Phi(u)^c \sqcup \Phi(v)^c$. Hence there is no $w \in W$ such that $(\sigma_u, \sigma_v, \sigma_{w^\vee})$ is Levi-movable. Then $\sigma_u \odot_0 \sigma_v = 0$. Moreover Proposition 10 implies that $\mathrm{Gr}^{\tilde\rho(G/P) - \tilde\rho(\Sigma_u^v(u))}\, \mathrm{H}^*(G/P, \mathbb{C}) = \{0\}$. In particular, $[\Sigma_u^v(u)]_{\odot_0} = 0$.

Assume now that $\sigma_u \odot_0 \sigma_v = \sigma_w$ for some $w \in W$.
Since $\Phi(w)^c = \Phi(u)^c \sqcup \Phi(v)^c$, Lemma 11 shows that $w^{-1} X_w = \Sigma_u^v(u)$. Hence $\sigma_u \odot_0 \sigma_v = [\Sigma_u^v(u)]_{\odot_0}$.

Assume finally that $\sigma_u \odot_0 \sigma_v = 0$.
It remains to prove that $[\Sigma_u^v(u)]_{\odot_0} = 0$. Since $\Phi(H_u^v(u)) = \Phi(u) \cap \Phi(v)$, $[\Sigma_u^v(u)]_{\odot_0}$ belongs to $\mathrm{Gr}^{\rho(X_u)+\rho(X_v)}\, \mathrm{H}^*(G/B, \mathbb{C})$. If there is no w in W such that $\rho(X_w) = \rho(X_u) + \rho(X_v)$ then Proposition 10 shows that $\mathrm{Gr}^{\rho(X_u)+\rho(X_v)}\, \mathrm{H}^*(G/B, \mathbb{C}) = \{0\}$. In particular $[\Sigma_u^v(u)]_{\odot_0} = 0$. Assume now that there exists w in W such that $\rho(X_w) = \rho(X_u) + \rho(X_v)$. Then $[\Sigma_u^v] = d\sigma_w + \cdots$ for some integer d. If $d = 0$ there is nothing to prove. If $d \neq 0$ then $\sigma_u.\sigma_v = e\sigma_w + \cdots$ for some integer $e \geq d$. The numerical criterium [BK06, Theorem 15] shows that $\sigma_u \odot_0 \sigma_v = e\sigma_w$. This contradicts the assumption $\sigma_u \odot_0 \sigma_v = 0$. $\square$

Proposition 15 shows that, for G/B, Conjecture 4 is equivalent to the following one.

Conjecture 5 *Let u, v, and w in W such that $\Phi(w)^c = \Phi(u)^c \sqcup \Phi(v)^c$. Then $\sigma_u \odot_0 \sigma_v = \sigma_w$ in $\mathrm{H}^*(G/B, \mathbb{C})$.*

Conjecture 5 was stated by Dimitrov and Roth in [DR09]. If $G = \mathrm{SL}_n(\mathbb{C})$ then Conjecture 5 was proved by Richmond in [Ric09]. If $G = \mathrm{Sp}_{2n}(\mathbb{C})$ then Conjecture 5 was proved independently in [Ric12] and [Res11b]. Dimitrov and Roth have a proof for each simple classical G, but it is not published. Here we include a proof for the group $\mathrm{SO}_{2n+1}(\mathbb{C})$.

Proposition 16 *Conjecture 5 holds for the group* $\mathrm{SO}_{2n+1}(\mathbb{C})$.

Proof. Let V be a $(2n+1)$-dimensional complex vector space and let $\mathcal{B} = (x_1, \dots, x_{2n+1})$ be a basis of V^*. Let G be the special orthogonal group associated to the quadratic form $Q = x_{n+1}^2 + \sum_{i=1}^n x_i x_{2n+2-i}$. Consider the maximal torus $T = \{\mathrm{diag}(t_1, \dots, t_n, 1, t_n^{-1}, \dots, t_1^{-1}) \,:\, t_i \in \mathbb{C}^*\}$ of G. Let B be the Borel subgroup of G consisting of upper triangular matrices in the dual base of $\mathcal{B}$. Consider W, Φ, Φ^+ associated to $T \subset B \subset G$.

Let u, v, and w in W such that $\sigma_u \odot_0 \sigma_v \odot_0 \sigma_w = d[pt]$ for some positive integer d. It remains to prove that $d = 1$. The Levi-movability implies that $\Phi^- = \Phi(u)^c \sqcup \Phi(v)^c \sqcup \Phi(w)^c$.

Consider the linear group $\hat G = \mathrm{GL}(V)$. Let $\hat T$ denote the subgroup of $\hat G$ consisting of diagonal matrices and let $\hat B$ denote the subgroup of $\hat G$ consisting of upper triangular matrices in $\hat G$. Consider $\hat W$, $\hat\Phi$, $\hat\Phi^+$ associated to $\hat T \subset \hat B \subset \hat G$.

Since T is a regular torus in $\hat{G}$, the group W identifies with a subgroup of $\hat{W}$. In particular, u, v, and w belong to $\hat{W}$. One can easily check that the similar property of Φ^- implies that $\hat{\Phi}^- = \hat{\Phi}(u)^c \sqcup \hat{\Phi}(v)^c \sqcup \hat{\Phi}(w)^c$. Consider now the three Schubert varieties $\hat{X}_u$, $\hat{X}_v$, and $\hat{X}_w$ in $\hat{G}/\hat{B}$. The fact that Conjecture 5 holds for $\hat{G}$ implies that

$$u^{-1}\hat{X}_u \cap v^{-1}\hat{X}_v \cap w^{-1}\hat{X}_w = \{\hat{B}/\hat{B}\}. \tag{98}$$

Consider now the inclusion $G/B \subset \hat{G}/\hat{B}$. Then X_u is contained in $\hat{X}_u$ (and similar inclusions hold for v and w). In particular, the condition (98) implies that

$$u^{-1}X_u \cap v^{-1}X_v \cap w^{-1}X_w = \{B/B\}. \tag{99}$$

Moreover, the condition on Φ^- implies that the intersection in (99) is transverse. It follows that $d = 1$. □

Proposition 17 *Conjecture 5 holds for the groups of type F_4 and E_6.*

Proof. For $w \in W$ set

$$p(w) = \prod_{\alpha \in \Phi^+ \cap w\Phi^+} (\rho, \alpha),$$

where $(\cdot, \cdot)$ is a W-invariant scalar product and ρ is the half sum of the positive roots. Let u, v, and w in W such that $\Phi(w)^c = \Phi(u)^c \sqcup \Phi(v)^c$. By [BK06, Corollary 44],

$$\sigma_u \odot_0 \sigma_v = \frac{p(u).p(v)}{p(w)} \sigma_w$$

in $\mathrm{H}^*(G/B, \mathbb{C})$. To prove the proposition, it is sufficient to check that $p(w) = p(u).p(v)$. This is checked by a Sage program (see [Res13]). For example, in type F_4, if

$$u^\vee = s_3s_2s_3s_2, \quad v^\vee = s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4 \quad \text{and} \\ w^\vee = s_1s_2s_3s_4s_2s_3s_1s_2s_3s_4s_3s_2s_3s_2$$

then

$$p(u) = \frac{3}{2} \quad p(v) = 113400 \quad p(w) = 170100.$$

And, in type E_6, if

$$u = s_6s_5s_4s_3s_2s_4s_5s_6s_5s_3 \quad v = s_4s_3s_2s_4s_5s_4s_3s_2s_4s_2 \\ w = s_6s_5s_4s_3s_2s_4s_5s_6s_5s_4s_3s_2s_4s_5s_4s_3s_2s_4s_3s_2$$

then

$$p(u) = 20160 \quad p(v) = 4320 \quad p(w) = 87091200.$$

□

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