Reading
"The space of Rational maps on $\mathbb{P}^1$"
by J.H. Silverman
An Introduction to Geometric Invariant Theory

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Chapter 1

Reductive groups and Affine quotients

1.1 Affine algebraic variety

1.1.1 — Let us fix an algebraically closed field $\mathbf{k}$. We endow $\mathbf{k}^n$ with the Zariski topology. If U is open in $\mathbf{k}^n$, we set:

$$\mathcal{O}_{\mathbf{k}^n}(U) := \{\frac{f}{g} \;:\; U \longrightarrow \mathbf{k} \mid f, g \in \mathbf{k}[x_1, \cdots, x_n] \text{ with } g(x) \neq 0 \;\; \forall x \in U\}.$$

Then $\mathcal{O}_{\mathbf{k}^n}$ is a sheaf on $\mathbf{k}^n$. The ringed space $(\mathbf{k}^n, \mathcal{O}_{\mathbf{k}^n})$ is denoted $\mathbb{A}^n$ and called the *affine space*.

1.1.2 — Let I be an ideal of $\mathbf{k}[x_1, \cdots, x_n]$ and $X = V(I)$ be the associated closed subset of $\mathbb{A}^n$ endowed with the induced topology. Consider on X the following sheaf:

$$\mathcal{O}_X(U) := \{\frac{f}{g} \;:\; U \longrightarrow \mathbf{k} \mid f, g \in \mathbf{k}[x_1, \cdots, x_n] \text{ with } g(x) \neq 0 \;\; \forall x \in U\}.$$

Definition. An *affine variety* is a ringed space isomorphic to some $(X, \mathcal{O}_X)$ as above.

1.1.3 — The first fundamental result of algebraic geometry is

Theorem 1 (Hilbert's Nullstellensatz) *If* $\sqrt{I} = \{f \in \mathbf{k}[x_1, \cdots, x_n] \;:\; \exists n > 0 \;\; f^n \in I\}$*, we have*

$$\mathcal{O}_X(X) \simeq \mathbf{k}[x_1, \cdots, x_n]/\sqrt{I}.$$

As a consequence, $\mathcal{O}_X(X)$ is a finitely generated $\mathbf{k}$-algebra without non zero nilpotent element. Conversely, we have:

Proposition 1 *Let A be a finitely generated $\mathbf{k}$-algebra without non zero nilpotent element. Then, there exists a unique affine algebraic variety X such that $\mathcal{O}_X(X) \simeq A$.*

Proof. Since A is finitely generated there exist a surjective morphism ϕ : $\mathbf{k}[x_1, \cdots, x_n] \longrightarrow A$. Since A has no non zero nilpotent element, the kernel I of ϕ satisfies $I = \sqrt{I}$. Then, $X = V(I)$ works. □

1.1.4 — By restriction, any open subset U in an affine variety, is endowed with a sheaf. We have:

Proposition 2 *Let X be an affine variety and $f \in \mathcal{O}_X(X)$ be a regular function. Consider $U := \{x \in X \mid f(x) \neq 0\}$.*

Then, U is an affine algebraic variety with $\mathcal{O}_U(U) = \mathcal{O}_X(X)[1/f]$.

Proof. Let Γ be the subset of $X \times \mathbf{k}$ defined by $xf(x) = 1$. It is easy to see that Γ is an affine variety isomorphic to U as a ringed space. □

1.2 Affine algebraic groups

1.2.1 — Consider $\mathrm{GL}_n(\mathbf{k}) \subset \mathrm{M}_n(\mathbf{k})$ as an affine variety. Notice that the product $\mathrm{GL}_n(\mathbf{k}) \times \mathrm{GL}_n(\mathbf{k}) \longrightarrow \mathrm{GL}_n(\mathbf{k})$ is a morphism. Moreover, by Kramer's formula, the inverse map $\mathrm{GL}_n(\mathbf{k}) \longrightarrow \mathrm{GL}_n(\mathbf{k})$ is also a morphism. So, $\mathrm{GL}_n(\mathbf{k})$ is the first example of affine algebraic group.

Definition. An affine algebraic group is a closed (in Zariski topology) subgroup of $\mathrm{GL}_n(\mathbf{k})$ endowed with its structure of affine variety and group.

Remark. Actually, one can prove that any affine variety H with a law of group which is given by morphisms is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbf{k})$.

The first examples are $\mathrm{GL}_n(\mathbf{k})$, $U_n(\mathbf{k})$, $B_n(\mathbf{k})$, $T_n(\mathbf{k})$, $\mathrm{SL}_n(\mathbf{k})$. In particular, the additive and multiplicative groups $\mathbb{G}_a$ and $\mathbb{G}_m$ are affine algebraic groups. The finite groups are also affine algebraic groups.

1.2.2 — Let G be an affine algebraic group and X be an (affine) algebraic variety. An action $\theta : G \times X \longrightarrow X$ is said to be *algebraic* if it θ is a morphism.

If $x \in X$, we denote by $G.x$ and G_x the orbit and the stabilizer of x. Let X^G denote the set of fixed points of G in X.

The first result about algebraic actions is

Proposition 3 *The G-orbits in X are open in their closure.*

Proof. Let $x \in X$. Consider the morphism $\theta : G \longrightarrow X$, $g \longmapsto g.x$. It is a general fact about algebraic morphisms that its image contains an open subset Ω of its closure. Then, the image of θ is the union of the $g.\Omega$ and hence is open in its closure. □

1.2.3 — A representation of an algebraic group G in a finite dimensional vector space V is a morphism $\rho : G \longrightarrow \mathrm{GL}(V)$. A rational representation of G is a vector space W (possibly of infinite dimension) endowed with a linear action of W and covered by representations (of finite dimension) of G. A fundamental rational representation come from actions on affine varieties:

Proposition 4 *The algebra $\mathcal{O}_X(X)$ is a rational representation of G.*

Proof. Consider the action $\sigma : G \times X \longrightarrow X$ and the induced morphism $\sigma^* : \mathcal{O}_X(X) \longrightarrow \mathcal{O}_G(G) \otimes \mathcal{O}_X(X)$. Let $f \in \mathcal{O}_X(X)$. We have to prove that the orbit of G is contained in a finite dimensional subspace of $\mathcal{O}_X(X)$. Set $\sigma^*(f) = \sum a_i \otimes f_i$. One easily checks that $G.f$ is contained in the subspace spanned by the f_i's. □

1.3 Reductive groups

1.3.1 Linearly reductive groups

1.3.1.1 — The first definition comes from representation theory.
Definition. An algebraic group is said to be *linearly reductive* if for any representation V and any non zero invariant vector $v \in V$ there exists an invariant linear form $\phi \in V^*$ such that $\phi(v) = 1$.

1.3.1.2 — In this paragraph, we assume that $\mathbf{k} = \mathbb{C}$ is the field of complex numbers. Let K be a compact Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. Consider the Zariski closure $K_\mathbb{C}$ of K in $\mathrm{GL}_n(\mathbb{C})$. Indeed, one can prove that $K_\mathbb{C}$ only depends on K and not on its embedding in $\mathrm{GL}_n(\mathbb{R})$.

The group $K_\mathbb{C}$ is linearly reductive.

Such examples are $\mathbb{C}^*$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $SO(n, \mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$. Actually, any complex reductive group equals $K_\mathbb{C}$ for some compact Lie group K.

1.3.1.3 — Consider an algebraically closed field $\mathbf{k}$ of positive characteristic p. The group $G = \mathbb{Z}/p\mathbb{Z}$ acts on $V = \mathbf{k}^p$ by $\tau.(x_1, \cdots, x_p) = (x_p, x_1, \cdots, x_{p-1})$, where τ the class of 1 in G. The vector $v = (1, \cdots, 1)$ is fixed by G. The set of G-invariant vectors of V^* is the line generated by $\varphi = \sum x_i$. But, $\phi(v) = 0$. So, G is not linearly reductive.

1.3.1.4 — Representations of linearly reductive groups have very useful properties:

Proposition 5 *Let G be a linearly reductive group. Then,*

1. *Any (rational) representation of G is completely reducible.*
2. *Let V be a (rational) representation of G. Then, V^G has a unique stable supplementary.*

3. *Let $\phi : V \longrightarrow W$ be a surjective G-equivariant linear map between two (rational) representations. Then, $\phi(V^G) = W^G$.*

Proof. Let S be G-stable subspace of a representation V of G. Consider $W := V \otimes S^* \simeq \mathrm{Hom}(S, V)$. Let $\theta \in W$ corresponding to the inclusion of S in V. Since G is linearly reductive, there exists $\phi \in V^* \otimes S$ such that $\phi(\theta) = 1$. The kernel of ϕ considered as a linear map from V to S is G-stable and supplementary to S.

By induction, on the dimension one easily deduce that any representation is completely reducible.

With notation of the second assertion, we fix a supplementary S of V^G. We claim that S is the sum of non trivial irreducible submodules of V. The claim trivially implies the unicity of S. Consider the G-equivariant projection $p : V \longrightarrow V^G$ with kernel S. Let W be a non trivial irreducible submodule of V. Since W is irreducible, $W \cap S$ is $\{0\}$ or W. If it were $\{0\}$, the restriction of p to W is injective and W is trivial: contradiction. So $W \subset S$.

The last point is an easy consequence of the fact the above proof. $\square$

1.3.1.5 — The following representation $t.(x, y) = (x, y + tx)$ of $\mathbb{G}_a$ is not completely reducible. So, $\mathbb{G}_a$ is not linearly reductive.

Definition. A group G is said to be *reductive* if it does not contain $\mathbb{G}_a$ as a normal subgroup.

Theorem 2 (Weyl) *Assume the characteristic of* $\mathbf{k}$ *is zero. Then, G is reductive if and only if G is linearly reductive.*

1.3.2 Geometrically reductive groups

In positive characteristic, the good notion is:

Definition. A group G is said to be *geometrically reductive* if for any representation V of G and any non zero fix point $v \in V$ there exists a G-invariant homogeneous polynomial f on V of positive degree such that $f(v) = 1$.

The following representation $t.(x, y) = (x, y + tx)$ of $\mathbb{G}_a$ shows that $\mathbb{G}_a$ is not geometrically reductive.

Theorem 3 (Haboush, [Hab75]) *An algebraic group G is reductive if and only if it is geometrically reductive.*

1.4 Hilbert-Nagata's Theorem

1.4.1 — Let us consider an action of an affine algebraic group G over an affine algebraic variety X. One goal of Geometric Invariant Theory is to construct a quotient Z of X under the action of G. If we ask Z for being an affine algebraic variety, it is equivalent by Proposition 1 to construct $\mathcal{O}_Z(Z)$. A natural candidate to be $\mathcal{O}_Z(Z)$ is $\mathcal{O}_X(X)^G$. So, the first question is: Is $\mathcal{O}_X(X)^G$ finitely generated ?

1.4.2 — Now, one can state the:

Theorem 4 (Hilbert-Nagata-Haboush, [Nag65]) *Let A be a finitely generated $\mathbf{k}$-algebra endowed with a rational action of a reductive group G.*

Then, A^G is finitely generated.

Proof.[in characteristic zero]
A reduction. We firstly prove that it is sufficient to prove the theorem for $A = \mathbf{k}[V]$ for a finite dimensional G-representation V.

Let W be a finite dimensional G-submodule of A which generates A as a $\mathbf{k}$-algebra. It exists since A is finitely generated and rational. The inclusion $W \subset A$ extend to a surjective equivariant morphism of algebras:

$$\varphi \ : \ S(W) = \mathbf{k}[W^*] \longrightarrow A.$$

Since G is linearly reductive, φ induces a surjective map $\mathbf{k}[W^*]^G \longrightarrow A^G$. So, it is equivalent to prove the theorem for $A = \mathbf{k}[W^*]$.

From now on, $A = \mathbf{k}[V]$ for a finite dimensional G-representation V. Consider the unique G-equivariant projection:

$$\rho \ : \ A \longrightarrow A^G.$$

Firstly A^G is noetherian since A is noetherian and for any ideal $I \subset A^G$, we have $(I.A)^G = I$.

Consider $A^{G,+}$ the ideal of A^G of elements of positive valuation. Let $a_1, \cdots, a_s$ be generators of the ideal $A^{G,+}$. We claim that $a_1, \cdots, a_s$ generate A^G as a $\mathbf{k}$-algebra. Let B denote the $\mathbf{k}$-algebra generated by $a_1, \cdots, a_s$. Let $f \in A^G$ homogeneous of degree d. We will prove that $f \in B$ by induction on the degree d of f. If $d = 0$, it is obvious. Else, $f \in A^{G,+}$ and $f = \sum a_i f_i$ with $a_i \in A^G$ of strictly less degrees. We can conclude by induction. $\square$

Actually, we just proved the theorem for linearly reductive groups: this proof is essentially due to Hilbert. Nagata proved that this proof can be modified to work for geometrically reductive groups: Nagata's proof is much more difficult.

1.5 The categorical quotient

1.6 Definition

Let G be a geometrically reductive group acting on an affine variety X. Consider $Y := \mathrm{Spec}(\mathcal{O}_X(X)^G)$ and the morphism $\pi \ : \ X \longrightarrow Y$ induced by inclusion.

Theorem 5 *With above notation, we have:*

1. *(Y, π) is a categorical quotient.*

2. *The morphism π is surjective.*

3. *Let F_1 and F_2 be two closed and G-stable subsets of X. If F_1 and F_2 are disjoint then $\pi(F_1)$ and $\pi(F_2)$ are disjoint.*

4. *If F is closed and G-stable in X, then $\pi(F)$ is closed in Y.*

5. *Each fiber of π contains exactly one closed orbit.*

Proof. The first point is obvious with affine varieties. The general case is obtained by gluing.

Let us prove that π is surjective. It is sufficient to prove that for any ideal I in $\mathcal{O}_X(X)^G$ such that $I.\mathcal{O}_X(X) = \mathcal{O}_X(X)$, we have $I = \mathcal{O}_X(X)^G$. In characteristic zero this follows from the decomposition as $\mathcal{O}_X(X)^G$-module : $\mathcal{O}_X(X) = \mathcal{O}_X(X)^G \oplus S$. In general, this is more difficult (see [FSR05, Lemma 11.4.2]).

Let F_1 and F_2 be as in the statement.

First proof in characteristic zero. Let I_1 and I_2 be the ideals of F_1 and F_2. Since F_1 and F_2 are disjoint, $I_1 + I_2 = \mathbf{k}[X]$. By Proposition 5, $I_1^G + I_2^G = \mathbf{k}[X]^G$. The statement follows.

Proof in arbitrary characteristic. One can find $\phi \in \mathcal{O}_X(X)$ such that $\phi_{|Z_1} = 0$ and $\phi_{|Z_1} = 0$.

Consider the subspace W spanned by the G-orbit of ϕ. Let $\phi_1, \cdots, \phi_s \in G.\phi$ be a base of W. These regular functions define a regular G-equivariant morphism:

$$\begin{array}{cccc} \theta : & X & \longrightarrow & W^* \\ & x & \longmapsto & (\phi_1(x), \cdots, \phi_s(x)). \end{array}$$

Moreover, $\theta(Z_1) = (0, \cdots, 0)$ and $\theta(Z_2) = (1, \cdots, 1)$. Since G is geometrically reductive there exists $F \in (S^q W)^G$ (with $q > 0$) such that $F(1, \cdots, 1) = 1$. Let f be the element of $\mathcal{O}_X(X)$ corresponding to F. Then, $f \in \mathcal{O}_Y(Y)$ and $f(\pi(Z_1)) = 0$ and $f(\pi(Z_2)) = 1$.

Let F be a closed and G-stable in X. Assume that $\pi(F)$ is not closed: there exists $y \in \overline{\pi(Z)} - \pi(Z)$. Since π is surjective, $Z_2 = \pi^{-1}(y)$ is a non empty closed G-stable subset of X disjoint from Z_1. Now, the proof of the last point implies a contradiction. □

1.7 Examples

1.7.1 — Consider the action of the symmetric group S_n over $\mathbb{Z}[x_1, \cdots, x_n]$.

Theorem 6 *We have: $\mathbb{Z}[x_1, \cdots, x_n]^{S_n} = \mathbb{Z}[\sigma_1, \cdots, \sigma_n]$. Moreover, the σ_i's are algebraically independent.*

Proof. By induction on n. □

This implies that $\mathbb{A}^n/S_n \simeq \mathbb{A}^n$.

1.7.2 — Consider the action of the group with two elements on $\mathbf{k}^2$ by multiplication by -1. The invariants are generated by x^2, y^2 and xy. In particular, the quotient $\mathbf{k}^2/\!/ \pm 1 \simeq \mathrm{Spec}(\mathbf{k}[X,Y,Z]/(Z^2 - XY)$ which is not smooth and even non factorial.

1.7.3 — Let us consider the action of $\mathrm{SL}_n(\mathbf{k})$ by conjugacy on $\mathrm{M}_n(\mathbf{k})$. One can prove that the quotient is the map $\pi : \mathrm{M}_n(\mathbf{k}) \longrightarrow \mathbf{k}^n$ which maps a matrix on the coefficients of its characteristic polynomial.

Each fiber contains a unique diagonalisable orbit. By Dunford decomposition, this is the closed orbit of the fiber.

1.8 Local properties of the quotient

Proposition 6 *Let G be reductive group acting on an affine variety X.*

1. *If X is irreducible $X/\!/G$ is.*
2. *If X is normal, $X/\!/G$ is.*
3. *Assume G is semi-simple (reductive connected with finite center). If X is factorial, $X/\!/G$ is.*

Proof. The two first points are obvious. A key point for the last assertion is the fact that a semisimple group has no non trivial character. Let $f \in \mathbf{k}[X]^G$. Let $f = p_1.\cdots.p_k$ be a decomposition as a product of irreducible elements of $\mathbf{k}[X]$. We claim that each p_k is G-invariant. By unicity of the decomposition and connectedness of G, for any $g \in G$ and $g.p_i \in \mathbf{k}[X]^*.p_i$. Let us fix $x \in X$. If $p_i(x) = 0$, then $g.p_i(x) = 0$. Else, for any $g \in G$ there exists a unique $\lambda_g \in \mathbf{k}^*$ such that $g.p_i(x) = \lambda_g p_i(x)$. One easily checks that $g \longmapsto \lambda_g$ is a character of G; so, is trivial. Finally, p_i is G-invariant. □

1.8.1 — Let p, q and n be positive integers. Consider the following action of GL_n on $\mathrm{M}_{pn} \times \mathrm{M}_{nq}$:

$$g.(A,B) = (gA, Bg^{-1}).$$

We have an obvious invariant morphism $\pi : \mathrm{M}_{pn} \times \mathrm{M}_{nq} \longrightarrow \mathrm{M}^n$, $(A,B) \longmapsto BA$, where M^n is the subvariety of M_{pq} of matrices of rank less or equal n.

It is a classical result of GIT that this morphism is the GIT-quotient (see [KP00]). In particular, this implies that M^n is normal.

1.9 Stable points

Proposition 7 *Let G be reductive group acting on an affine variety X. We assume that there exists points in X with finite isotropy. Consider $\pi : X \longrightarrow X/\!/G$. Let $x \in X$. The following are equivalent:*

1. $\pi^{-1}(\pi(x)) = G.x$*;*

2. $G.x$ *is closed in* X *and* G_x *is finite.*

Proof. Assume that $G.x$ is closed in X and G_x is finite. By Theorem 5, $G.x$ is the only closed orbit in $\pi^{-1}(\pi(x))$: for any $y \in \pi^{-1}(\pi(x))$ $\overline{G.y} \supset G.x$. Since the dimension of $G.x$ is maximal, Proposition 3 implies that $G.y = G.x$.

Conversely, assume that $\pi^{-1}(\pi(x)) = G.x$. The set of points in X with finite isotropy is open. In particular, generically the fibers of π have dimension greater than dim G. By the semicontinuity Theorem for the fibers, the dimension of $\pi^{-1}(\pi(x))$ is greater than dim G. Since this fiber is $G.x$, the second assertion follows. $\square$

A point satisfying the second assertion of the proposition is said to be *stable*. We denote by X^{s} the set of stable points.

Proposition 8 *The subset* X^{s} *of* X *is open.*

Proof. Let F be the set of $x \in X$ such that G_x is NOT finite. Applying the semicontinuity Theorem to $G \times X \longrightarrow X \times X$, we see that F is closed and obviously G-stable in X. Since $X - X^{\mathrm{s}} = \pi^{-1}(\pi(F))$, Theorem 5 shows that X^{s} is open. $\square$

Chapter 2

Projective Quotients

2.1 The case $\mathbb{P}(V)$

2.1.1 The construction

Let V be a representation of a reductive group G. We consider the action of G on $\mathbb{P}(V)$.

Consider the graduation by degree: $\mathbf{k}[V] = \bigoplus_d \mathbf{k}[V]_d$, and $\mathbf{k}[V]^G = \bigoplus_d \mathbf{k}[V]^G_d$.

Lemma 1 *There exists a positive d_0 and homogeneous G-invariant polynomials $f_0, \cdots, f_s$ such that*

$$\mathbf{k}[f_0, \cdots, f_s] = \bigoplus_{r \geq 0} \mathbf{k}[V]^G_{rd_0}.$$

Proof. It is a very classical result. See [Gro61, Lemma 2.1.6] or [Bou61]. $\square$

Consider $\pi_{d_0} : \mathbb{P}(V) - -- > \mathbb{P}^s,\ x \longmapsto [f_0 : \cdots : f_s]$. Set $\mathbb{P}(V)^{\rm ss}_{d_0} := \{[v] \in \mathbb{P}(V) \,|\, \exists i \ \ f_i(v) \neq 0\}$ and Y_{d_0} the closure of $\pi_{d_0}(\mathbb{P}(V)^{\rm ss}_{d_0})$. We also denote by π_{d_0} the restriction:

$$\pi_{d_0} : \mathbb{P}(V)^{\rm ss}_{d_0} \longrightarrow Y_{d_0}.$$

The properties of this construction are

Theorem 7 *1. $\mathbb{P}(V)^{\rm ss}_{d_0}$ is the set of $[v] \in \mathbb{P}(V)$ such that there exists a G-invariant homogeneous polynomial f of positive degree such that $f(v) \neq 0$.*

2. The morphism π_{d_0} is an affine morphism and for all open affine subset $U \subset Y$ we have $\mathbf{k}[U] = \mathbf{k}[\pi_{d_0}^{-1}(U)]^G$.

Before giving the proof we explain a consequence:

Corollary 1 *1. Actually, $\mathbb{P}(V)^{\rm ss}_{d_0}$, π_{d_0} and Y_{d_0} does not depend on d_0. So, we will forget d_0.*

2. *The morphism π is a categorical quotient of $\mathbb{P}(V)^{\mathrm{ss}}$. It is surjective.*

We will denote Y by $\mathbb{P}(V)^{\mathrm{ss}}/\!/G$.

Proof.[of the theorem]
The first assertion is easy. It is sufficient to prove the second one for an open and affine covering of Y. Set $U_i := (x_i \neq 0) \cap Y$ (with obvious notation). One can check that $\pi_{d_0}^{-1}(U_i) = \mathbb{P}(V)_{f_i}$ is affine and $\mathbf{k}[\mathbb{P}(V)_{f_i}]^G = \mathbf{k}[U_i]$. □

2.1.2 First Examples

2.1.2.1 — Consider the following action of $\mathbf{k}^*$ on $\mathbf{k}^{n+2} = V$:

$$t.(x_0, \cdots, x_{n+1}) = (t^2.x_0, \cdots, t^2.x_n, x_{n+1}).$$

We have $\mathbf{k}[V]^{\mathbf{k}^*} = \mathbf{k}[x_{n+1}]$. So, Y is a point, $\mathbb{P}(V)^{\mathrm{ss}}$ is the affine open subset defined by $x_{n+1} \neq 0$.

Consider the following action of $\mathbf{k}^*$ on $\mathbf{k}^{n+2} = V$:

$$t.(x_0, \cdots, x_{n+1}) = (x_0, \cdots, x_n, t^{-2}x_{n+1}).$$

Note that this action on V induces the same action of $\mathbf{k}^*$ on $\mathbb{P}(V)$ as above. We have $\mathbf{k}[V]^{\mathbf{k}^*} = \mathbf{k}[x_0, \cdots, x_n]$. So, $Y = \mathbb{P}^n$, $\mathbb{P}(V)^{\mathrm{ss}} = \mathbb{P}(V) - \{[0 : \cdots : 0 : 1]\}$ and π is the projection from $[0 : \cdots : 0 : 1]$.

Consider the following action of $\mathbf{k}^*$ on $\mathbf{k}^{n+2} = V$:

$$t.(x_0, \cdots, x_{n+1}) = (t.x_0, \cdots, t.x_n, t^{-1}x_{n+1}).$$

Note that this action on V induces again the same action of $\mathbf{k}^*$ on $\mathbb{P}(V)$ as above. The ring $\mathbf{k}[V]^{\mathbf{k}^*}$ is generated by the $x_i.x_{n+1}$'s for $i = 0, \cdots, n$. So, $Y = \mathbb{P}^n$, $\mathbb{P}(V)^{\mathrm{ss}} = \mathbb{P}(V) - \mathbb{P}(V)^{\mathbf{k}^*}$ identifies with $\mathbf{k}^{n+1} - \{0\}$. Then, $\pi : \mathbf{k}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$ is the usual quotient map.

These examples shows that whereas the construction does not depends on d_0 it depends really on the linear action of G on V and not only on the induced action on $\mathbb{P}(V)$.

2.1.2.2 — For an action of a finite group, one have $X = X^{\mathrm{ss}} = X^{\mathrm{s}}$. So, GIT constructs orbit spaces. A very simple example is:

$$(\mathbb{P}^1)^n / S_n \simeq \mathbb{P}^n.$$

2.2 The general case

2.2.1 — If X is a closed G-stable subvariety of $\mathbb{P}(V)$, we set $X^{\mathrm{ss}} = \mathbb{P}(V)^{\mathrm{ss}} \cap X$ and restrict π to obtain

$$\pi : X^{\mathrm{ss}} \longrightarrow X^{\mathrm{ss}}/\!/G.$$

2.2.2 — We denote by X^{us} the complementary of X^{ss} in X. Let $\tilde{X}$ ($\tilde{X}^{\mathrm{us}}$) be the affine cone over X (resp. $\tilde{X}^{\mathrm{us}}$) in V. Actually, $X^{\mathrm{ss}}/\!/G$ is the quotient by $\mathbf{k}^*$ of the quotient of the affine cone $\tilde{X}$ by G. More precisely, we have the commutative diagram

$$\begin{tikzcd}
\tilde{X} - \tilde{X}^{\mathrm{us}} \arrow[r, "\pi"] \arrow[d] & \tilde{X}/\!/G - \{\pi(0)\} \arrow[d, "/\!/\mathbf{k}^*"] \\
X^{\mathrm{ss}} \arrow[r] & X^{\mathrm{ss}}/\!/G.
\end{tikzcd}$$

2.2.3 — If $\mathcal{L}$ is the restriction of $\mathcal{O}(1)$ to X, we have: $\Gamma(X, \mathcal{L}^{\otimes d}) = \mathbf{k}[\tilde{X}]_d$. More generally, if $\mathcal{L}$ is an **ample** G-linearized line bundle on X, we set

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X \,:\, \exists d > 0 \text{ and } \sigma \in \Gamma(X, \mathcal{L}^{\otimes d})^G \quad \sigma(x) \neq 0\}.$$

We also consider the G-invariant morphism

$$\pi \,:\, X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow \mathrm{Proj}(\oplus_d \Gamma(X, \mathcal{L}^{\otimes d})^G)$$

associated to the inclusion of $\oplus_d \Gamma(X, \mathcal{L}^{\otimes d})^G$ in $\oplus_d \Gamma(X, \mathcal{L}^{\otimes d})$.

Theorem 8 *With above notation, we have:*

1. *π is a categorical quotient. We denote* $\mathrm{Proj}(\oplus_d \Gamma(X, \mathcal{L}^{\otimes d})^G)$ *by $X^{\mathrm{ss}}(\mathcal{L})/\!/G$.*
2. *The morphism π is affine and surjective.*
3. *Let F_1 and F_2 be two closed and G-stable subsets of $X^{\mathrm{ss}}(\mathcal{L})$. If F_1 and F_2 are disjoint then $\pi(F_1)$ and $\pi(F_2)$ are disjoint.*
4. *If F is closed and G-stable in X, then $\pi(F)$ is closed in Y.*
5. *Each fiber of π contains exactly one closed orbit in $X^{\mathrm{ss}}(\mathcal{L})$.*
6. *The sheaf $\mathcal{O}_{X^{\mathrm{ss}}(\mathcal{L})/\!/G}$ is $\pi_*(\mathcal{O}^G_{X^{\mathrm{ss}}(\mathcal{L})})$.*

2.2.4 — Points of $X^{\mathrm{ss}}(\mathcal{L})$ are said to be *semistable*. A point which is not semistable is said to be *unstable*. A point $x \in X^{\mathrm{ss}}(\mathcal{L})$ is said to be *stable* if $G.x$ is closed in $X^{\mathrm{ss}}(\mathcal{L})$ and G_x is finite. We have the same properties for the set $X^{\mathrm{s}}(\mathcal{L})$ of stable points as in the affine case:

Proposition 9 *Let G be reductive group acting on an affine variety X. Let $\mathcal{L}$ be an ample G-linearized line bundle. We assume that there exists points in X with finite isotropy. Consider $\pi \,:\, X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G$. Let $x \in X^{\mathrm{ss}}(\mathcal{L})$.*

1. *$\pi^{-1}(\pi(x)) = G.x$ if and only if x is stable.*
2. *$X^{\mathrm{s}}(\mathcal{L})$ is open in X.*

2.3 Hilbert-Mumford's Theorem

2.3.1 The statement

2.3.1.1 — Let V be a representation of a reductive group G. Let X be a closed G-stable subvariety of $\mathbb{P}(V)$. We denote by $\tilde{X}$ the affine cone over X in V.

Lemma 2 *Let $v \in V$ be a non zero vector.*

1. $[v] \in X^{\mathrm{ss}} \iff 0 \notin \overline{G.v}$.

2. $[v] \in X^{\mathrm{s}} \iff G.v$ *is closed and* G_v *is finite.*

Proof. Consider the quotient $\tilde{\pi} : V \longrightarrow V//G$. The point $[v]$ is unstable iff $\tilde{\pi}(v) = \tilde{\pi}(0)$; that is, by Theorem 5 $0 \in \overline{G.v}$.

If G_v is finite and $[v]$ is semistable, $G_{[v]}$ is finite. Else, $G_{[v]}$ acts non trivially on the line $[v]$ in V and 0 belongs to the closure of $G.v$. The second assertion follows easily. □

2.3.1.2 — A one parameter subgroup of G is a morphism of groups $\lambda : \mathbb{G}_m \longrightarrow G$. The set of one parameter subgroups of G is denoted by $Y(G)$. Let Y be a G-variety (actually, V, $\mathbb{P}(V)$, $\tilde{X}$ or X) and y be a point in Y. Consider $\sigma : \mathbf{k}^* \longrightarrow Y$, $t \longmapsto \lambda(t).y$. Since Y is separated, if σ can be extended to a morphism $\tilde{\sigma} : \mathbb{A}^1 \longrightarrow Y$ the value $\tilde{\sigma}(0)$ does not depends on the extension. This value is denoted by $\lim_{t\to 0} \lambda(t).y$. If Y is complete the limit always exists.

2.3.1.3 — We can now state Hilbert-Mumford's Theorem:

Theorem 9 (Hilbert-Mumford verion 1) *With above notation,we have:*

1. $x = [v] \in X - X^{\mathrm{ss}} \iff \exists \lambda \in Y(G) \quad \lim_{t\to 0} \lambda(t).v = 0$.

2. $x = [v] \in X - X^{\mathrm{s}} \iff \exists \lambda \in Y(G) \quad \lim_{t\to 0} \lambda(t).v$ *exists in* V.

2.3.2 Maximal tori

An algebraic *torus* is a group isomorphic to a product of copies of $\mathbb{G}_m$. Over $\mathbb{C}$ they are the complexifications of the tori $(S^1)^s$. A fundamental theorem about the structure of algebraic groups is (see [Hum75])

In an algebraic group all the maximal tori are conjugated.

Let T be a maximal torus in a group G. A consequence of this statement is that any one parameter subgroup of G is conjugated to a one parameter subgroup of T.

For $G = \mathrm{GL}_n(\mathbf{k})$, the set of diagonal matrices is a maximal torus.

2.3.3 Iwahori Decomposition

2.3.3.1 — Let $\mathbf{k}[[t]]$ denote the ring of formal series. Note that an element $f = \sum_{i\geq 0} a_i t^i$ is invertible in $\mathbf{k}[[t]]$ if and only if a_0 is non zero. The fraction field of $\mathbf{k}[[t]]$ is $\mathbf{k}((t)) = \{t^{-k}f \ : \ \text{for } k \in \mathbb{N} \text{ and } f \in \mathbf{k}[[t]]\}$.

For any affine algebraic group G we set:

$$G(\mathbf{k}[[t]]) = \mathrm{Hom}(\mathrm{Spec}(\mathbf{k}[[t]]), G) = \mathrm{Hom}(\mathbf{k}[G], \mathbf{k}[[t]]),$$

and

$$G(\mathbf{k}((t))\,) = \mathrm{Hom}(\mathrm{Spec}(\mathbf{k}((t))\,), G) = \mathrm{Hom}(\mathbf{k}[G], \mathbf{k}((t))\,).$$

For $G = \mathrm{GL}_n(\mathbf{k})$, $G(\mathbf{k}[[t]])$ identifies with the set of matrices with coefficient in $\mathbf{k}[[t]]$ and invertible determinant, and $G(\mathbf{k}((t))\,) = \mathrm{GL}_n(\mathbf{k}((t))\,)$.

2.3.3.2 — Note that $\mathbf{k}[\mathbf{k}^*] = \mathbf{k}[t, t^{-1}]$ is contained in $\mathbf{k}((t))$ as a sub-$\mathbf{k}$-algebra. In particular, $G(\mathbf{k}[t, t^{-1}]) = \mathrm{Hom}(\mathrm{Spec}(\mathbf{k}[t, t^{-1}]), G) = \mathrm{Hom}(\mathbf{k}[G], \mathbf{k}[t, t^{-1}])$ is contained in $G(\mathbf{k}((t))\,)$. In particular, $Y(G)$ is contained in $G(\mathbf{k}((t))\,)$.

Theorem 10 (Iwahori Decomposition) *Let G be a reductive group and T be a maximal torus of G. Then,*

$$G(\mathbf{k}((t))\,) = G(\mathbf{k}[[t]]).Y(T).G(\mathbf{k}[[t]]).$$

Proof.[for $G = \mathrm{GL}_n$] Let $A \in \mathrm{GL}_n(\mathbf{k}((t))\,)$. We write $A = t^{-r}.A'$, where A' is a matrix with its coefficients in $\mathbf{k}[[t]]$. Note that A' does not always belong to $\mathrm{GL}_n(\mathbf{k}[[t]])$. Since $\mathbf{k}[[t]]$ is euclidean, there exists P, $Q \in \mathrm{GL}_n(\mathbf{k}[[t]])$ and a diagonal matrix D (with entries in $\mathbf{k}[[t]]$) such that $A' = P.D.Q$. (Moreover, one may assume that each diagonal coefficient divides the following one). Since the determinant of D is not zero (in $\mathbf{k}[[t]]$), one can write $D = D'.\Delta$, where $D' \in Y(T)$ and $\Delta \in \mathrm{GL}_n(\mathbf{k}[[t]])$. Then, $A = P.(D'.t^{-r}).(\Delta Q)$ and the theorem follows. $\square$

2.3.3.3 — We can now prove the theorem.

Proof.[of Theorem 9.] Let us assume that $0 \in \overline{G.v}$. Then by the valuative criterion of properness, there exists $\phi \in G(\mathbf{k}((t))\,)$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Spec}(\mathbf{k}((t))\,) \arrow[r, "\phi"] \arrow[d] & G \arrow[r, "g \mapsto g.v"] & V \\
\mathrm{Spec}(\mathbf{k}[[t]]) \arrow[urr, "\bar{\phi}"'] & &
\end{tikzcd}$$

and $\bar{\phi}((t)) = 0$. This means that $\lim_{t\to 0} \phi(t).v = 0$!

We now apply Iwahori's Theorem to ϕ: $\phi = \psi_1.\lambda.\psi_2$. Since ψ_1 has a limit in G when $t \to 0$, $\lim_{t\to 0} \lambda(t).\psi_2(t).v = 0$. Set $g_2 = \lim_{t\to 0} \psi_2(t) \in G$ and

$\lambda' = g_2^{-1}.\lambda.g_2$. We have: $\lim_{t\to 0}\lambda'(t).(g_2^{-1}\psi_2(t)).v = 0$; that one may assume that $g_2 = e$. In this case, $\psi_2(t).v = v + tw(t)$, with $w \in V(\mathbf{k}[[t]])$. So, we have; $\lambda(t)\psi_2(t)v = \lambda(t)v + t\lambda(t).w(t)$. Using a base where λ acts diagonally one easily deduces that $\lim_{t\to 0}\lambda(t)v = 0$.

Let us prove the second point. Let $x = [v]$ be a non stable point. The morphism $\sigma : G \longrightarrow V$, $g \mapsto g.v$ is not proper. So, by the valuative criterion of properness, there exists $\phi \in G(\mathbf{k}((t)))$ such that $\sigma\circ\phi$ has a limit at 0 in V. We can conclude by the same argument as above. □

2.3.4 Action of one parameter subgroups

A one parameter subgroup acts on V diagonally: there exists a decomposition $V = \oplus_i V_i$ of V and pairwise distinct integers r_i such that $t.v_i = t^{r_i}v_i$ for all $v_i \in V_i$. Let $v = \sum_i v_i$ be a non zero vector. Let μ be the opposite of the minimum of the r_i's such that $v_i \neq 0$ and i_0 its corresponding index. Then, we have:

$$\begin{array}{ll} \lim_{t\to 0}\lambda(t)[v] = [v_{i_0}] & \\ \lim_{t\to 0}\lambda(t)v = 0 & \iff \mu < 0 \\ \lim_{t\to 0}\lambda(t)v \text{ exists} & \iff \mu \leq 0 \end{array}$$

2.3.5 The numerical criterion

2.3.5.1 — Definition. Let $\mathcal{L}$ be a G-linearized line bundle over a projective G-variety X. Let $x \in X$. Set $z = \lim_{t\to 0}\lambda(t).x$. The group $\mathbb{G}_m$ fixes z and so acts on the line $\mathcal{L}_z$ linearly: this gives an integer denoted by $\mu^{\mathcal{L}}(x,\lambda)$.

In the case when $\mathcal{L} = \mathcal{O}(1)$, we have: $\mu^{\mathcal{L}}([v],\lambda) = \mu$ (with the notation of the preceding paragraph). One can now state a second version of Hilbert-Mumford Theorem:

Theorem 11 (Hilbert-Mumford version 2) *Let $\mathcal{L}$ be an ample G-linearized line bundle over a projective G-variety X. Then, we have:*

1. $x = [v] \in X^{\rm ss}(\mathcal{L}) \iff \forall\lambda \in Y(G) \quad \mu^{\mathcal{L}}(x,\lambda) \leq 0.$
2. $x = [v] \in X^{\rm s}(\mathcal{L}) \iff \forall\lambda \in Y(G)$ *non trivial* $\quad \mu^{\mathcal{L}}(x,\lambda) < 0.$

2.3.5.2 — Here we give a simple geometric interpretation of the sign of $\mu^{\mathcal{L}}(x,\lambda)$.

Lemma 3 *Let λ be a one parameter subgroup of G. Let $x \in X$. Set $z = \lim_{t\to 0}\lambda(t).x$. Let $\tilde{x}$ be a non zero point in $\mathcal{L}_x$.*

Then, we have,

1. $\mu^{\mathcal{L}}(x,\lambda) > 0 \iff \lim_{t\to 0}\lambda(t)\tilde{x} = z.$
2. $\mu^{\mathcal{L}}(x,\lambda) = 0 \iff \lim_{t\to 0}\lambda(t)\tilde{x}$ *exists and is a non zero element of* $\mathcal{L}_z$.

3. $\mu^{\mathcal{L}}(x,\lambda) < 0 \iff \lim_{t\to 0}\lambda(t)\tilde{x}$ *does not exists.*

Proof. Consider $Y := \{\lambda(t).x \,|\, t \in \mathbf{k}^*\} \cup \{z\}$. Consider $\theta : \mathrm{H}^0(Y, \mathcal{L}_{|Y}) \longrightarrow \mathbf{k}$, $\sigma \longmapsto \sigma(z)$. Since Y is affine, a general Serre's Theorem shows that θ is surjective. Moreover, θ is $\mathbb{G}_m$-equivariant. Since $\mathbb{G}_m$ is reductive, there exists $\sigma \in \mathrm{H}^0(Y, \mathcal{L}_{|Y})$ such that $\mathbf{k}.\sigma$ is $\mathbb{G}_m$-stable and $\sigma(z) = 1$. Then, the set of $y \in Y$ such that $\sigma(y) = 0$ is closed, $\mathbb{G}_m$-stable and does not contain y: it is empty. So, $\mathcal{L}_{|Y}$ is trivial. The lemma follows easily. □

2.4 Example: Actions of a torus

Consider a torus $T = \mathbb{G}_m^r$ action linearly on a finite dimensional vector space V. This action is diagonalisable, and the diagonal entries are elements of $\mathrm{Hom}(\mathbb{G}_m^r, \mathbb{G})$. This group is called the group of characters of T and denoted by $X(T)$. It is isomorphic to $\mathbb{Z}^r$.

For $\chi \in X(T)$, we set $V_\chi := \{v \in V \,:\, t.v = \chi(t)v\}$. Then, $V = \oplus_{\chi \in X(T)} V_\chi$. If $[v] \in \mathbb{P}(V)$, we write $v = \sum v_\chi$ and denote by $\mathrm{St}(v)$ the set of χ's such that $v_\chi \neq 0$.

Note also that $Y(T)$ is also isomorphic to $\mathbb{Z}^r$ and that the composition induces a perfect paring $X(T) \times Y(T) \longrightarrow \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$, denoted by $\langle \chi, \lambda \rangle$.

We have:

$$\mu([v], \lambda) = \min_{\chi \in \mathrm{St}(v)} \langle \chi, \lambda \rangle.$$

Consider $\mathcal{P}(v)$ the convex hull of $\mathrm{St}(v)$ in $X(T) \otimes \mathbb{R}$. By Hahn-Banach's Theorem, we have:

$[v]$ is semistable (resp. stable) if and only if 0 belong to $\mathcal{P}(v)$ (resp. the interior of $\mathcal{P}(v)$.

Chapter 3

The space of rational maps on $\mathbb{P}^1$ over a field

3.1 Introduction

Let us fix a field $\mathbf{k}$ and an integer $d \geq 2$. Let P and Q be two polynomial of degree less than d. In the usual coordinate on $\mathbb{P}^1$ we consider: $\phi(z) = P(z)/Q(z)$. If P and Q are coprime, ϕ is a morphism from $\mathbb{P}^1$ to $\mathbb{P}^1$. If moreover at least one is of degree d, ϕ is said to be a *rational morphism of* $\mathbb{P}^1$ *of degree* d. Let Rat_d denote the set of rational morphisms of degree d.

The group $\mathrm{SL}_2(\mathbf{k})$ acts on $\mathbb{P}^1$. So, it acts on Rat_d by:

$$g.\phi = g \circ \phi \circ g^{-1}.$$

We are interested to Rat_d modulo $\mathrm{SL}_2(\mathbf{k})$.

Let us rewrite in a more intrinsic way. From now on, we assume that $\mathbf{k}$ is algebraically closed. Let V be a fixed $\mathbf{k}$-vector space of dimension two. We denote by $\mathbb{P}^1$, the projective space $\mathbb{P}(V)$. Consider $\mathbb{P}(\mathbf{k}[V]_d \otimes V)$ endowed with the natural action $\mathrm{SL}(V)$. The resultant *Res* of (P, Q) can be thought as an homogeneous polynomial function on $\mathbf{k}[V]_d \otimes V$ which is $\mathrm{SL}(V)$-invariant. Let Rat_d denote the open subset of $\mathbb{P}(\mathbf{k}[V]_d \otimes V)$ defined by $Res \neq 0$. Then, Rat_d is a smooth affine variety. We are interested in

$$M_d := \mathrm{Rat}_d /\!/ \mathrm{SL}_2(\mathbf{k}) \quad \text{and} \quad M_d^{\mathrm{ss}} := \mathbb{P}(\mathbf{k}[V]_d \otimes V)^{\mathrm{ss}} /\!/ \mathrm{SL}_2(\mathbf{k}).$$

First, by Theorem 7 M_d^{ss} is a projective compactification of M_d.

3.2 The semistable points

A point $(P, Q) \in \mathbf{k}[V]_d \otimes V$ induces a linear map $\theta_{(P,Q)} : V^* \longrightarrow \mathbf{k}[V]_d$; its image is the subspace spanned by P and Q. Let $[P : Q] \in \mathbb{P}(\mathbf{k}[V]_d \otimes V)$. Let D

denote the gcd of (P, Q). Then $[P : Q]$ induces a rational map $\phi' : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree $d - \deg(D)$.

Proposition 10 *Let $(P, Q) \in \mathbb{P}(\mathbf{k}[V]_d \otimes V)$.*

1. *Assume $d = 2r$ even. A point $[P : Q] \in \mathbb{P}(\mathbf{k}[V]_d \otimes V)$ is unstable if and only if it is not stable if and only if either*
   (a) *there exists a $\zeta \in \mathbb{P}^1$ which is a root of P and Q of order $r + 1$; or*
   (b) *there exists a fix point $\zeta \in \mathbb{P}^1$ of ϕ' which is a root of P and Q of order r.*
2. *Assume $d = 2r + 1$ is odd.*
   A point $[P : Q]$ is unstable if and only if either
   (a) *there exists a $\zeta \in \mathbb{P}^1$ which is a root of P and Q of order $r + 2$; or*
   (b) *there exists a fix point $\zeta \in \mathbb{P}^1$ of ϕ' which is a root of P and Q of order $r + 1$.*
3. *$d = 2r + 1$. A point $[P : Q]$ is not stable if and only if*
   (a) *there exists a $\zeta \in \mathbb{P}^1$ which is a root of P and Q of order $r + 1$; or*
   (b) *there exists a fix point $\zeta \in \mathbb{P}^1$ of ϕ' which is a root of P and Q of order r.*

Proof. Consider $\lambda(t) = \mathrm{diag}(t, t^{-1}) \in Y(\mathrm{SL}_2(\mathbf{k}))$. Since any one parameter subgroup of $\mathrm{SL}_2(\mathbf{k})$ is conjugated to a positive multiple of λ it is sufficient to understand $\lim_{t \to 0} \lambda(t).(P, Q)$. Details are left to the reader. □

Note that the proposition implies that $\mathrm{Rat}_d \subset \mathbb{P}(\mathbf{k}[V]_d \otimes V)^{\mathrm{s}}$. In particular, points in M_d correspond to SL_2-orbits in Rat_d.

This proposition also implies that for d even, $\mathbb{P}(\mathbf{k}[V]_d \otimes V)^{\mathrm{ss}} = \mathbb{P}(\mathbf{k}[V]_d \otimes V)^{\mathrm{s}}$. In this case, M_d^{ss} is the space of the orbits of $\mathrm{SL}_2(\mathbf{k})$ in $\mathbb{P}(\mathbf{k}[V]_d \otimes V)^{\mathrm{ss}}$.

3.2.1 First Invariants

3.2.1.1 — The aim of this section is to construct elements in $\mathbf{k}[\mathrm{Rat}_d]^{\mathrm{SL}_2(\mathbf{k})}$. We firstly construct these functions a applications $\mathrm{Rat}_d \longrightarrow \mathbf{k}$. We will prove after that these functions are regular.

Let us fix $\phi = P/Q \in \mathrm{Rat}_d$. Let us start with a fix point ζ_i of ϕ. Consider the tangent maps of ϕ at ζ_i: this is an endomorphism of $T_{\zeta_i}\mathbb{P}^1$; so, its determinant is a well defined element μ_i of $\mathbf{k}$.

Consider the set of fixed points of ϕ, that is the roots of $YP - XQ$ with multiplicities. This gives a well defined point $(\zeta_1, \cdots, \zeta_{d+1})$ in $(\mathbb{P}^1)^{d+1}/S_{d+1}$. To each ζ_i is associated a μ_i; so, we have a well defined point in $(\mu_1, \cdots, \mu_{d+1}) \in \mathbf{k}^{d+1}/S_{d+1}$. The elementary functions of the μ_i are well defined functions

$$\sigma_i : \mathrm{Rat}_d \longrightarrow \mathbf{k}.$$

By construction, it is clear that σ_i is $\mathrm{SL}_2(\mathbf{k})$-invariant.

3.2.1.2 — We have now to prove:

Proposition 11 *The functions σ_i are regular and $\mathrm{SL}_2(\mathbf{k})$-invariant.*

Proof. We fix a base of V and so coordinated (X, Y). Consider the closed subvariety $F \subset \mathbb{P}^1 \times \mathrm{Rat}_d$ defined by $Fix := YP - XQ$. Notice that Fix is an homogeneous element of degree $d+1$ in $\mathrm{Rat}_d[X, Y]$. Consider the projection $p : F \longrightarrow \mathrm{Rat}_d$.

The variety Rat_d is covered by open subsets of the form $U_\zeta = \{\phi \mid \phi(\zeta) \neq \zeta\}$. So, it is sufficient to prove that σ_i is regular on the U_ζ; finally on U_∞.

Let $P = a_0X^d + a_1X^{d-1}Y + \cdots + a_dY^d$ and $Q = b_0X^d + b_1X^{d-1}Y + \cdots + b_dY^d$. The point $[P : Q] \in \mathrm{Rat}_d$ belongs to U_∞ iff $b_0 \neq 0$. So, we may assume that $b_0 = 1$. Moreover, $p^{-1}(U_\infty) \subset \mathbf{k} \times \mathrm{Rat}_d$ defined by : $X^{d+1} + (b_1 - a_0)X^d + \cdots = 0$. One easily deduces that $\mathbf{k}[p^{-1}(U_\infty)]$ is a free $\mathbf{k}[U_\infty]$-module of rank $d+1$.

Consider now the following morphism:

$$\Theta : \mathbb{P}^1 \times \mathrm{Rat}_d \longrightarrow \mathbb{P}^1 \times \mathrm{Rat}_d,\ (\zeta, \phi) \longmapsto (\phi(\zeta), \phi).$$

Since $\mathbb{P}^1 \times \mathrm{Rat}_d$ is smooth, one can consider the tangent bundle $T(\mathbb{P}^1 \times \mathrm{Rat}_d)$ and the tangent map $T\Theta$. Since Θ restricts to F as the identity, $T\Theta$ induces a endomorphism of the vector bundle $T(\mathbb{P}^1 \times \mathrm{Rat}_d)_{|F}$. Consider $p_1 : F \longrightarrow \mathbb{P}^1$. By restriction and projection, $T\Theta$ induces an endomorphism θ of $p_1^*(\mathbb{P}^1) = \mathcal{L}$. In other words θ is a section of $\mathcal{L}^* \otimes \mathcal{L}$; that is, a regular function on F.

The function $\theta'_{p^{-1}(U_\infty)}$ is an element of $\mathbf{k}[p^{-1}(U_\infty)]$. The functions $\sigma_{i|U_\infty}$ are just the coefficients of the characteristic polynomial of the multiplication by $\theta'_{p^{-1}(U_\infty)}$ in $\mathbf{k}[p^{-1}(U_\infty)]$ viewed as a free $\mathbf{k}[U_\infty]$-module of rank $d+1$. In particular, it is an element of $\mathbf{k}[U_\infty]$. □

3.2.2 More invariants

The idea to produce new invariants is to apply the preceding proposition to $\phi^{\circ n}$. This obviously produces invariant σ_j^n. But, the set of fixed points of $\phi^{\circ n}$ contains the set of fixed points of $\phi^{\circ m}$ for any $m|n$. It would be better to consider only the points of order exactly n.

Set $F_n \subset \mathbb{P}^1 \times \mathrm{Rat}_d$ be the set of fixed points of $\phi^{\circ n}$. It is an hypersurface of $\mathbb{P}^1 \times \mathrm{Rat}_d$. Obviously, $F_m \subset F_n$ for any $m|n$. Consider the closure F_n^* of $F_n - \cup_{m|n} F_m$: it is an union of irreducible components of F_m.

Using F_m^* in place of F in Proposition 11, one obtain new invariants $\sigma_i^{(n)}$ for $i = 1, \cdots, \deg(F_m^*)$.

3.3 The case $d = 2$

3.3.1 The affine case

3.3.1.1 — We now consider the case $d = 2$. Consider:

$$\varphi : M_2 \longrightarrow \mathbf{k}^2, \, (\sigma_1, \sigma_2).$$

One of the main Silverman's results is that φ is actually an isomorphism. He also proved that φ can be extended to an isomorphism from M_d^{ss} onto $\mathbb{P}^2$.

3.3.1.2 — One step of Silverman's proof is to show that φ is bijective, that is, $\mathrm{Rat}_2 \longrightarrow \mathbf{k}^2$, (σ_1, σ_2) separates the orbits and is surjective. To do this, one has to understand the orbits in Rat_d:

Lemma 4 *Let $\phi \in \mathrm{Rat}_d$.*

1. *Assume that ϕ has at least two fixed points. Then, there exists a_1, $b_1 \in \mathbf{k}$ such that ϕ is conjugated to*

   $$\phi_0 := \frac{z^2 + a_1 z}{b_1 z + 1}.$$

   Moreover, the multiplier of ϕ_0 at 0 (resp. at ∞) equals a_1 (resp. b_1). The third fixed point is $\frac{a_1-1}{b_1-1}$ and its multiplier is $\frac{a_1+b_1-2}{a_1b_1-1}$. In particular, $Res(\phi_0) = a_1b_1 - 1$ (with an abuse of notation), $\sigma_1(\phi_0) = (a_1^2b_1 + a_1b_1^2 - 2)/Res$ and $\sigma_2(\phi_0) = (a_1^2b_1^2 + a_1b_1 - 2b_1 + b_1^2 + a_1^2 - 2a_1)/Res$.

2. *If ϕ a a unique fixed point; it is equivalent to $z + \frac{1}{z}$. Moreover, $\sigma_1 = \sigma_2 = 3$ en ϕ; and, the multiplier of ϕ at infinity is 1.*

Moreover, $\sigma_3 = \sigma_1 - 2$.

Proof. If ϕ has at least two fixed points, an element of its orbits fixes 0 and ∞. So, we may assume that $\phi = (az^2 + a_1z)/(b_1z + b_2)$. Since the numerator and denominator of ϕ are coprime, $a \neq 0$ and $b_2 \neq 0$; so, one may assume that $a = 1$. The action of the diagonal elements in $\mathrm{SL}_2(\mathbf{k})$ (that is the stabilizer of 0 and ∞) allows to assume that $b_2 = 1$. The first assertion follows after some easy computation.

If $\phi = P/Q$ has a unique fixed point, we may assume that it is ∞. Then, $zP - Q$ has to be a constant polynomial in z; so, $\phi = z + 1/z$. □

Lemma 5 *The map $\varphi : \mathrm{M}_2 \longrightarrow \mathbf{k}^2$ is bijective.*

Proof. First, notice that the knowledge of (σ_1, σ_2) is equivalent to the knowledge of the three multipliers.

First, we will prove that φ is injective. In the first case of Lemma 4, if the two multipliers a_1 and b_1 equal 1 then $\phi_0 = (z^2 + z)/(z + 1) \notin \mathrm{Rat}_2$. This implies that the image by φ of the orbits of the first case of Lemma 4 does not contain the image of $z + 1/z$. Now the injectivity of ϕ is obvious.

Let us prove the surjectivity. Let us fix σ_1 and σ_2 which determines the μ_i's (up to order). If $\mu_1 = \mu_2 = \mu_3 = 1$ then $(\sigma_1, \sigma_2) = \varphi(z + 1/z)$. Assume now that $\mu_1 \neq 1$ and set $\phi = (z^2 + \mu_1 z)/(\mu_2 + 1)$. Notice that, ϕ belongs to Rat_d excepted if $\mu_1\mu_2 = 1$. But, since $\mu_1 + \mu_2 + \mu_3 = \mu_1\mu_2\mu_3 + 2$, if $\mu_1\mu_2 = 1$ then $\mu_1 = \mu_2 = \mu_3 = 1$; which is a contradiction. So, ϕ belongs to Rat_2. Moreover, its image by φ is necessary (σ_1, σ_2). □

Now, we are ready to prove one of the Silverman's result if the characteristic of $\mathbf{k}$ is zero.

Theorem 12 *The morphism φ is an isomorphism.*

Proof.[in characteristic zero] Since the characteristic is zero and φ is bijective, it is birational. Now, φ is a birational bijective morphism from M_d onto $\mathbb{A}^2$ (which is normal). Zariski's Main Theorem (see [GD66, §8.12]) proves that φ is an isomorphism. □

3.3.2 Working over $\mathbb{Z}$

3.3.2.1 — The above proof of Theorem 12 does not work over a field $\mathbf{k}$ of positive characteristic. Indeed, the map $\mathbf{k} \longrightarrow \mathbf{k}, x \longmapsto x^p$ is bijective but not birational.

Silverman's idea to avoid this problem is to work over $\mathbb{Z}$. Actually, an isomorphism over $\mathbb{Z}$ implies an isomorphism over any field. This idea implies several changes.

3.3.2.2 — An affine group scheme over $\mathbb{Z}$ is a affine scheme $G = \mathrm{Spec}(A)$ endowed with two morphisms: $G \times G \longrightarrow G$ and $G \longrightarrow G$ and a $\mathbb{Z}$-point $\mathrm{Spec}\mathbb{Z} \longrightarrow G$ which satisfy properties of a product, inverse and neutral element of a group. An action of G over an affine scheme $X = \mathrm{Spec}(B)$ is a morphism $\sigma : G \times X \longrightarrow X$ satisfying usual properties of an action. Consider the corresponding morphism $\sigma^* : B \longrightarrow A \otimes B$. An element $f \in B$ is said to be invariant if $\sigma^*(f) = 1 \otimes f$.

3.3.2.3 — Since the resultant is defined over $\mathbb{Z}$, Rat_d can be thought as a scheme $\mathbb{R}\mathrm{at}_d$ over $\mathbb{Z}$. Moreover, $\mathrm{SL}_2 = \mathrm{Spec}\mathbb{Z}[a, b, c, d]/(ad - bc = 1)$ is a group scheme. Actually, Seshadri proved in [Ses77] that GIT works in this context (see also [MFK94, Appendix Chapter 1]); in particular, the set of invariants $\mathcal{O}_{\mathbb{R}\mathrm{at}_d}(\mathbb{R}\mathrm{at}_d)^G$ is finitely generated ring. Let $\mathbb{M}_d$ denote the associate affine scheme.

We have to prove that the σ_i's are defined over $\mathbb{Z}$: in the proof of Proposition 11, one has just to replace tangent bundles by the sheaf of relative differential forms (which has a better behavior in the schematic context).

The advantage of working over $\mathbb{Z}$ is in the proof of Theorem 12. Since the fraction field of $\mathbb{Z}[x, y]$ is of characteristic zero, to prove that φ is birational it is sufficient to prove that φ is bijective on the geometric points; that is over any field ! Lemma 5 precisely proves this. After, Zariski Main's Theorem can be applied directly (see [GD66, §8.12]).

The method used here is slightly different from the original one by Silverman. I hope this proof is simpler. It is actually a direct adaptation of Proposition 0.2 of [MFK94]. Two properties of the candidate quotient $\mathbb{A}^2$ are particularly important in this proof: it is normal and the characteristic of the residual field of its generic point is zero.

We obtain the following statement:

Theorem 13 *Let $\mathbb{R}\mathrm{at}_d$ denote the affine scheme over $\mathbb{Z}$ of rational maps of degree d over $\mathbb{P}^1$; that is, $\mathbb{R}\mathrm{at}_d = \mathrm{Spec}((\mathbb{Z}[a_i, b_i][\frac{1}{\mathrm{Res}}])_0)$. Let G be the group scheme $\mathrm{Spec}(\mathbb{Z}[a,b,c,d]/(ad - bc = 1))$. The action $G \times \mathbb{R}\mathrm{at}_d \longrightarrow \mathbb{R}\mathrm{at}_d$ is a morphism of schemes over $\mathbb{Z}$. Set $\mathbb{M}_d = \mathbb{R}\mathrm{at}_d//G = \mathrm{Spec}((\mathbb{Z}[a_i, b_i][\frac{1}{\mathrm{Res}}])_0^G)$. By Seshadri's Theorem, is an affine scheme over $\mathbb{Z}$ of finite type. Consider the morphism $\varphi : \mathbb{M}_d \longrightarrow \mathbb{A}^2_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x, y])$ corresponding to the morphism $\mathbb{Z}[x, y] \longrightarrow (\mathbb{Z}[a_i, b_i][\frac{1}{\mathrm{Res}}])_0^G$, $x \mapsto \sigma_1$, $y \mapsto \sigma_2$.*

The morphism φ is an isomorphism of schemes over $\mathbb{Z}$.

3.3.3 The projective case

3.3.3.1 — Let us recall that $\mathbf{k}$ is any algebraically closed field. By Lemma 4, for $i = 1$ or 2, $\sigma_i = \frac{\tilde{\sigma}_i}{Res}$ for well defined homogeneous polynomial $\tilde{\sigma}_i$ in the coefficients of P and Q. So, we can consider the following rational map:

$$\theta : \mathbb{P}(\mathbf{k}[V]_2 \otimes V) - -- > \mathbb{P}^2, [Res : \tilde{\sigma}_1 : \tilde{\sigma}_2].$$

Consider Ω the open subset of $\mathbb{P}(\mathbf{k}[V]_2 \otimes V)$ defined by $(Res, \tilde{\sigma}_1, \tilde{\sigma}_2) \neq (0, 0, 0)$.

Theorem 14 *In fact, Ω equals $\mathbb{P}(\mathbf{k}[V]_2 \otimes V)^{\mathrm{s}}$ and θ is the projective GIT-quotient.*

We will prove Theorem 14 using the same method than in the affine case.

3.3.3.2 — The first step is analogous to Lemma 4.

Lemma 6 *Let $\phi \in \mathbb{P}(\mathbf{k}[V]_2 \otimes V)^{\mathrm{s}} - \mathrm{Rat}_2$. We have:*

1. *Let $(a, b) \in \mathbf{k}^2$. The point $[az : z + b] \in \mathbb{P}(\mathbf{k}[V]_2 \otimes V) - \mathrm{Rat}_2$ is stable if and only if $(a, b) \neq (0, 0)$.*
2. *There exists $(a, b) \in \mathbf{k}^2 - \{(0, 0)\}$ such that $[az : z + b]$ belongs to the orbit of ϕ.*
3. *For $(a, b) \in \mathbf{k}^2 - \{(0, 0)\}$, we denote by $[a : b]$ the corresponding point in $\mathbb{P}^1$. Two points $[az : z+b]$ and $[a'z : z+b']$ belong to the same $\mathrm{SL}_2(\mathbf{k})$-orbit if and only if $[a : b] = [a' : b']$ or $[a : b] = [a' : b']$.*

Proof. The first assertion is easy using Proposition 10. Set $\phi = [P : Q]$. Since $\phi \notin \mathrm{Rat}_2$, P and Q have a common root ζ. Since $\phi = [P : Q]$ is stable, ϕ' (with notation of Proposition 10) have a fix point $\xi \neq \zeta$. Let $g \in \mathrm{SL}_2(\mathbf{k})$ be such that $g\zeta = \infty$ and $g\xi = 0$. Then, $g.\phi = [az : cz + b]$ for a, b and c in $\mathbf{k}$. Moreover, ∞ cannot be fixed by $\frac{az}{cz+b}$, so $c \neq 0$. The second assertion follows.

The last assertion need more computation. The formula

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.[az : z + b] = [u^{-2}z : z + u^{-2}z] \tag{3.1}$$

implies that the orbit of $[az : z + b]$ only depends on the point $[a : b] \in \mathbb{P}^1$. This orbit will be denoted by $\mathcal{O}_{[a:b]}$. It remains to prove that $\mathcal{O}_{[a:b]} = \mathcal{O}_{[a':b']}$ if and only if $[a' : b']$ equals $[a : b]$ or $[b : a]$. The formula

$$\begin{pmatrix} 1 & b-a \\ 0 & 1 \end{pmatrix}.[az : z + b] = [bz : z + az] \tag{3.2}$$

implies the "if part". Let us assume that $\mathcal{O}_{[a:b]} = \mathcal{O}_{[a':b']}$. Notice that a or b is zero if and only if the degree of the gcd of the numerators and denominators (view as elements of $\mathbf{k}[V]_2$) equals two. In this case, a' or b' equals zero and $[a' : b']$ equals $[a : b]$ or $[b : a]$. Now we may assume that a and b are non zero. Then a' and b' are non zero. Let $g \in \mathrm{SL}_2(\mathbf{k})$ such that $g.[az : z+b] = [a'z : z+b']$. In this case, ∞ is the only common root to az and $z + b$ (viewed as elements of $\mathbf{k}[V]_2$): so, $g.\infty = \infty$. The rational map $[az : z+b]$ has exactly two fixed points 0 and $a - b$. So, $g^{-1}.0$ equals 0 or $a - b$. It $g.0 = 0$, g is diagonal and the above Formula 3.1 shows that $[a' : b'] = [a : b]$. If $g^{-1}.0 = a - b$, g is the product of a diagonal elements $\mathrm{SL}_2(\mathbf{k})$ and the matrix of Formula 3.2. So, the two formulas imply that $[a' : b'] = [a : b]$. □

3.3.3.3 — Proof.[of Theorem 14] It is clear that Ω is contained in the locus of stable points. Moreover, direct computations show that

$$\tilde{\sigma}_1([az : z + b]) = -ab \qquad \text{and} \qquad \tilde{\sigma}_2([az : z + b]) = -a^2 - b^2.$$

With Lemma 6, this implies easily that any stable point in $\mathbb{P}(\mathbf{k}[V]_2 \otimes V)$ belongs to Ω. These formulas also proves that θ induces a bijection from $(\mathbb{P}(\mathbf{k}[V]_2 \otimes V)^{\mathrm{s}} - \mathrm{Rat}_2)/\mathrm{SL}_2(\mathbf{k})$ onto $\mathbb{P}^1$. With Theorem 12, θ is bijective.

Since $\mathbb{P}^2$ is normal (since smooth), if the characteristic of $\mathbf{k}$ is zero, one can conclude exactly as in the affine case. If the characteristic is not zero, one can easily prove that θ is actually an isomorphism over $\mathbb{Z}$. □

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