# Proof of Theorem 3 for $F_4$

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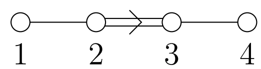
**Abstract**

This is a companion document of the paper untitled "Intersection multiplicity one for the Belkale-Kumar product in $G/B$". We present a proof of Theorem 3 for the rrot system $F_4$ assuming it known for root systems of rank 3.

$F_4$ 


There are 24 positive roots:

| | | | |
|---|---|---|---|
| $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ |
| $\alpha_3+\alpha_4$ | $\alpha_1+\alpha_2$ | $\alpha_2+2\alpha_3$ | $\alpha_2+\alpha_3$ |
| $\alpha_2+\alpha_3+\alpha_4$ | $\alpha_1+\alpha_2+\alpha_3$ | $\alpha_1+\alpha_2+2\alpha_3$ | $\alpha_2+2\alpha_3+2\alpha_4$ |
| $\alpha_1+\alpha_2+\alpha_3+\alpha_4$ | $\alpha_2+2\alpha_3+\alpha_4$ | $\alpha_1+2\alpha_2+2\alpha_3$ | $\alpha_1+\alpha_2+2\alpha_3+2\alpha_4$ |
| $\alpha_1+\alpha_2+2\alpha_3+\alpha_4$ | $\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4$ | $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4$ | $\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4$ |
| $\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$ | $\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$ | $\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$ | $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$ |

A root $a\alpha_1+b\alpha_2+c\alpha_3+d\alpha_4$ is written as $abcd$. Hence the positive roots are

| | | | |
|---|---|---|---|
| 1000 | 0100 | 0010 | 0001 |
| 0011 | 1100 | 0120 | 0110 |
| 0111 | 1110 | 1120 | 0122 |
| 1111 | 0121 | 1220 | 1122 |
| 1121 | 1222 | 1221 | 1242 |
| 1342 | 1231 | 1232 | 2342 |

Consider 3 convex anf coconvex subsets $\Phi_1, \Phi_2$ and $\Phi_3$ in $\Phi^+$ such that

$$\Phi_3 = \Phi_1 \sqcup \Phi_2 \tag{1}$$

eq:1

Let $\beta \leq \varphi \leq \gamma$ in $\Phi^+$ such that

$$\beta \in \Phi_1, \quad \gamma \notin \Phi_3, \quad \gamma + \beta \in \Phi_1. \tag{2}$$

eq:2

We assume that $\varphi \in \Phi_2$ and look for a contradiction. Observe that by Lemma 29 in the paper the last condition is equivalent to $\gamma + \beta \in \Phi_3$.

There are 17 pairs $(\beta \leq \gamma) \in (\Phi^+)^2$ such that $\gamma + \beta \in \Phi^+$:

| $\beta$ | $\gamma$ |
|---|---|
| $\alpha_4$ | $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ |
| $\alpha_3$ | $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ |
| $\alpha_2$ | $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$ |
| $\alpha_4$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ |
| $\alpha_2$ | $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ |
| $\alpha_3$ | $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ |
| $\alpha_3 + \alpha_4$ | $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$ |
| $\alpha_2$ | $\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ |
| $\alpha_2 + 2\alpha_3$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$ |
| $\alpha_2 + \alpha_3$ | $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ |
| $\alpha_2 + \alpha_3 + \alpha_4$ | $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4$ |
| $\alpha_3$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ |
| $\alpha_2 + \alpha_3$ | $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$ |
| $\alpha_3$ | $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4$ |
| $\alpha_2 + \alpha_3$ | $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4$ |
| $\alpha_1 + \alpha_2$ | $\alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ |
| $\alpha_1 + \alpha_2 + \alpha_3$ | $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ |

We consider successively these 17 cases.

Case A: $\beta = \alpha_4,\ \gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = 1121,\ \gamma + \beta = 1122.$

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0011, 1111, 0121$.

One element $\varphi = 0111$ gives an irreducible triple.

$$\begin{array}{llll}
\varphi = \beta + 0110 & \longmapsto & 0110 \in \Phi_2 & \\
1232 = \gamma + \varphi & \longmapsto & 1232 \notin \Phi_1 & \\
1232 = 1111 + 0121 & \longmapsto & 1232 \notin \Phi_2 & \text{by rank 3} \\
1232 \notin \Phi_3 & & & \\
1232 = (\gamma + \beta) + 0110 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case B: $\beta = \alpha_3,\ \gamma = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1111,\ \gamma + \beta = 1121.$

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0011, 1110$.

Two elements $\varphi = 0110, 0111$ give irreducible triples.

Case B1: $\varphi = 0110$.

$$\begin{array}{lll}
\varphi = \beta + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 \\
1110 = \varphi + \alpha_1 & \longmapsto & \alpha_1 \notin \Phi_2 \\
1221 = (\gamma + \beta) + \alpha_2 & \longmapsto & 1221 \in \Phi_3 \\
1221 = \gamma + \varphi & \longmapsto & 1221 \notin \Phi_1 \\
1221 \in \Phi_2 & & \\
1221 = 1110 + 0111 & \longmapsto & 0111 \in \Phi_2 \\
\gamma = 0111 + \alpha_1 & \longmapsto & \alpha_1 \notin \Phi_3 \\
1110 = \alpha_1 + \varphi & \longmapsto & 1110 \notin \Phi_1 \\
1110 \notin \Phi_3 & & \\
\gamma + \beta = \alpha_1 + 0121 & \longmapsto & 0121 \in \Phi_1 \\
1221 = 1100 + 0121 & \longmapsto & 1100 \in \Phi_2 \\
1110 = 1100 + \beta & \longmapsto & 1110 \in \Phi_3.
\end{array}$$

Contradiction.

Case B 2: $\varphi = 0111$.

$$\begin{array}{llll}
\varphi = 0011 + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by rank 3} \\
1221 = (\gamma + \beta) + \alpha_2 & \longmapsto & 1221 \in \Phi_3 & \\
1221 = \gamma + 0110 & \longmapsto & 1221 \notin \Phi_2 & \text{by Case B1} \\
1221 \in \Phi_1 & & & \\
\varphi = \alpha_4 + 0110 & \longmapsto & \alpha_4 \in \Phi_2 & \\
\gamma = 1110 + \alpha_4 & \longmapsto & 1110 \notin \Phi_3 & \\
1221 = \varphi + 1110 & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case C: $\beta = \alpha_2,\ \gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = 1122,\ \gamma + \beta = 1222$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0011, 0111, 0122, 1111$.

Six elements $\varphi = 0110, 1110, 0121, 1121, 0120, 1120]$ give irreducible triples.

Observe that

$$\begin{array}{lll}
1232 = (\gamma + \beta) + \alpha_3 & \longmapsto & 1232 \in \Phi_3, \\
1232 = \gamma + \varphi & \longmapsto & 1232 \notin \Phi_1;
\end{array}$$

hence $1232 \in \Phi_2$.

Case C1: $\varphi = 0110$.

$$\varphi = \beta + \alpha_3 \quad \longmapsto \quad \alpha_3 \in \Phi_2$$

We now distinguish the two cases when 0111 belongs to $\Phi_1$ or not.

Case C1a: $0111 \in \Phi_1$.

$$\begin{array}{llll}
0111 = \varphi = \beta + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_1 & \\
1232 = 1121 + 0111 & \longmapsto & 1121 \in \Phi_2 & \\
\gamma = \alpha_4 + 1121 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case C1b: $0111 \notin \Phi_1$.

By rank 3, $0111 \notin \Phi_2$; hence $0111 \notin \Phi_3$.

$$\begin{array}{lll} 0111=\varphi+\alpha_4 & \longmapsto & \alpha_4 \notin \Phi_3 \\ \gamma+\beta=0111+1111 & \longmapsto & 1111 \in \Phi_1 \\ 1111=\alpha_4+1110 & \longmapsto & 1110 \in \Phi_1 \\ 1232=0122+1110 & & \qquad\qquad \text{contradicts } \Phi_2 \text{ coconvex.} \end{array}$$

Case C2: $\varphi = 1110$.

$$\begin{array}{llll} \varphi=1100+\alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{rk3} \\ \varphi=0110+\alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\ \gamma=\alpha_1+0122 & \longmapsto & 0122 \notin \Phi_3 & \\ 1232=\varphi+0122 & & & \text{contradicts } \Phi_1 \text{ coconvex.} \end{array}$$

Case C3: $\varphi = 0121$.

$$\begin{array}{llll} \varphi=0111+\alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{rk3} \\ \varphi=0011+0110 & \longmapsto & 0011 \in \Phi_2 & \\ \gamma=0011+1111 & \longmapsto & 1111 \notin \Phi_3 & \\ 1232=\varphi+1111 & & & \text{contradicts } \Phi_3 \text{ coconvex.} \end{array}$$

Case C4: $\varphi = 1121$.

$$\begin{array}{llll} \varphi=0121+\alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case C3} \\ \varphi=1110+0011 & \longmapsto & 0011 \in \Phi_2 & \\ \gamma=0011+1111 & \longmapsto & 1111 \notin \Phi_3 & \\ 1111=0011+\beta+\alpha_1 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case C5: $\varphi = 0120$.

$$\begin{array}{llll} \varphi=1110+\alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case C2} \\ 1242=\gamma+\varphi & \longmapsto & 1242 \notin \Phi_1 & \\ 1242=0121+1121 & \longmapsto & 1242 \notin \Phi_2 & \\ 1242=(\gamma+\beta)+2\alpha_3 & \longmapsto & 1242 \in \Phi_3. & \end{array}$$

Contradiction.

Case C6: $\varphi = 1120$.

$$\begin{array}{llll} \varphi=1110+\alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case C2} \\ 1242=(\gamma+\beta)+2\alpha_3 & \longmapsto & 1242 \in \Phi_3 & \\ 1242=0121+1121 & \longmapsto & 1242 \notin \Phi_2 & \\ \gamma=\beta+0122 & \longmapsto & 0122 \notin \Phi_3 & \\ 1242=\varphi+0122+1121 & \longmapsto & 1242 \notin \Phi_1 & \end{array}$$

Contradiction.

Case D: $\beta = \alpha_4,\ \gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = 1221,\ \gamma + \beta = 1222.$

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0111, 1111, 0121, 1121$.

One element $\varphi = 0011$ gives an irreducible triple.

$$\begin{array}{llll}
1232 = 0111 + 1121 & \longmapsto & 1232 \notin \Phi_2 & \text{by rank 3} \\
1232 = \gamma + \varphi & \longmapsto & 1232 \notin \Phi_1 & \\
1232 \notin \Phi_3 & & & \\
\varphi = \alpha_3 + \beta & \longmapsto & \alpha_3 \in \Phi_2 & \\
1232 = \alpha_3 + (\gamma + \beta) & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

<u>Case E:</u> $\beta = \alpha_2,\ \gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = 1121,\ \gamma + \beta = 1221$.

Here, each element $\varphi$ of $]\beta;\gamma[= \{0011, 0110, 0111, 1110, 1111, 0121, 1120, 0120\}$ gives an irreducible triple.

<u>Case E1:</u> $\varphi = 1100$.

$$\begin{array}{llll}
\varphi = \beta + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\
\gamma + \beta = \varphi + 0121 & \longmapsto & 0121 \in \Phi_1 & \\
\gamma = \alpha_1 + 0121 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

<u>Case E2:</u> $\varphi = 0110$.

$$\begin{array}{llll}
\varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \\
\gamma + \beta = \varphi + 1111 & \longmapsto & 1111 \in \Phi_1 & \\
\gamma = \alpha_3 + 1111 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

<u>Case E3:</u> $\varphi = 0111$.

$$\begin{array}{llll}
\varphi = \beta + 0011 & \longmapsto & 0011 \in \Phi_2 & \\
\gamma + \beta = \varphi + 1110 & \longmapsto & 1110 \in \Phi_1 & \\
\gamma = 0011 + 1110 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

<u>Case E4:</u> $\varphi = 1110$.

$$\begin{array}{llll}
\varphi = 0110 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case E2} \\
\varphi = 1100 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case E1} \\
\gamma + \beta = \varphi + 0111 & \longmapsto & 0111 \in \Phi_1 & \\
\gamma = 0111 + \alpha_1 + \alpha_3 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

<u>Case E5:</u> $\varphi = 1111$.

$$\begin{array}{llll}
\varphi = 1100 + 0011 & \longmapsto & 0011 \in \Phi_2 & \text{by Case E1} \\
\varphi = 0111 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case E3} \\
\gamma + \beta = \varphi + 0110 & \longmapsto & 0110 \in \Phi_1 & \\
\gamma = 0011 + 1100 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

<u>Case E6:</u> $\varphi = 0121$.

$$\begin{array}{llll}
\varphi = 0111 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case E3} \\
\varphi = 0110 + 0011 & \longmapsto & 0011 \in \Phi_2 & \text{by Case E2} \\
\gamma + \beta = \varphi + 1100 & \longmapsto & 1100 \in \Phi_1 & \\
\gamma = 0011 + 1100 + \alpha_3 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case E7: $\varphi = 1120$.

$$\begin{array}{llll}
\varphi = 1110 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case E4} \\
1231 = 1110 + 0121 & \longmapsto & 1231 \notin \Phi_2 & \\
1231 = (\gamma + \beta) + \alpha_3 & \longmapsto & 1231 \in \Phi_3 & \\
1231 \in \Phi_1 & & & \\
\gamma = \alpha_3 + 1111 & \longmapsto & 1111 \notin \Phi_3 & \\
1231 = 0120 + 1111 & \longmapsto & 0120 \in \Phi_1 & \\
1231 = \varphi + 0111 & \longmapsto & 0111 \in \Phi_1 & \\
1111 = 0111 + \alpha_1 & \longmapsto & \alpha_2 \notin \Phi_3 & \\
\varphi = 0120 + \alpha_1 + \alpha_3 & & & \text{contradicts } \Phi_2 \text{ coconvex.}
\end{array}$$

Case E8: $\varphi = 0120$.

$$\begin{array}{llll}
\varphi = \beta + 2\alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \\
1231 = 0111 + 1120 & \longmapsto & 1231 \notin \Phi_2 & \text{previous cases} \\
1231 = (\gamma + \beta) + \alpha_3 & \longmapsto & 1231 \in \Phi_3 & \\
1231 \in \Phi_1 & & & \\
1231 = \varphi + 1111 & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case F: $\beta = \alpha_3,\ \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4.\ \gamma + \beta = 1242$.

The elements of $]\beta; \gamma[$ reducible to rank at most 3 are: $0011, 0110, 1110, 0121, 1121, 1231$.

Nine elements $\varphi = 0120, 0111, 1120, 0122, 1111, 1220, 1122, 1222, 1221$ give irreducible triples.

Case F1: $\varphi = 1221$.

$$\begin{array}{llll}
\varphi = 1111 + 0110 & \longmapsto & 1111 \in \Phi_2 & \text{rk3} \\
\varphi = 0111 + 1110 & \longmapsto & 0111 \in \Phi_2 & \text{rk3} \\
\gamma - \beta = 1111 + 0111 & \longmapsto & \gamma - \beta \in \Phi_2 & \\
\gamma = (\gamma - \beta) + \beta & \longmapsto & \gamma - \beta \notin \Phi_3. &
\end{array}$$

Contradiction.

Case F2: $\varphi = 0120$.

$\varphi = \beta + 0110 \notin \Phi_2$, by rank 3.

Case F3: $\varphi = 1220$.

$\varphi = 0110 + 1110 \notin \Phi_2$, by rank 3.

Case F4: $\varphi = 1120$.

$\varphi = \beta + 1110 \notin \Phi_2$, by rank 3.

Case F5: $\varphi = 1111$.

$$\begin{array}{llll}
1121 = \varphi + \beta & \longmapsto & 1121 \in \Phi 3 & \\
1121 \notin \Phi_2 & & & \text{by rank 3} \\
1121 \in \Phi_1 & & & \\
1121 = \alpha_1 + 0121 & \longmapsto & \alpha_1 \in \Phi_1 & \text{by rank 3} \\
\gamma - \beta = 0111 + \varphi & \longmapsto & 0111 \notin \Phi_3 & \\
\varphi = 0111 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_1. &
\end{array}$$

Contradiction.

Case F6: $\varphi = 1122$.

$\varphi = 1111 + 0011 \notin \Phi_2$ by rank 3 and Case F4.

Case F7: $\varphi = 0111$.

$$\begin{array}{llll} \varphi = 0110 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by rank 3} \\ \gamma = \alpha_4 + 1231 & \longmapsto & 1231 \notin \Phi_3 & \\ 1342 = \gamma + 0110 & \longmapsto & 1342 \notin \Phi_2 & \text{by rank 3} \\ 1342 = (\gamma + \beta) + \alpha_2 & \longmapsto & 1342 \in \Phi_3 & \\ 1342 \in \Phi_1 & & & \\ 1342 = \varphi + 1231 & & & \text{contradicts } \Phi_1 \text{ coconvex.} \end{array}$$

Case F8: $\varphi = 0111$.

$$\begin{array}{llll} 1342 = (\gamma - \beta) + 0120 & \longmapsto & 1342 \notin \Phi_2 & \text{previous cases} \\ \varphi = \alpha_2 + 0011 & \longmapsto & \alpha_2 \in \Phi_2 & \\ mbox by rank 3 & & & \\ 1342 = (\gamma + \beta) + \alpha_2 & \longmapsto & 1342 \in \Phi_3 & \\ 1342 \in \Phi_2 & & & \\ \varphi = \alpha_4 + 0110 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by rank 3} \\ \gamma = 1231 + \alpha_4 & \longmapsto & 1231 \notin \Phi_3 & \\ 1342 = 1231 + \varphi & & & \text{contradicts } \Phi_1 \text{ coconvex.} \end{array}$$

Case F9: $\varphi = 0122$.

$\varphi = 0111 + 0011 \notin \Phi_2$ by rank 3 and Case F8.

Case G: $\beta = \alpha_3 + \alpha_4,\ \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4.\ \gamma + \beta = 1242$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0111, 1111, 0121, 1121$.

One element $\varphi = 1221$ gives an irreducible triple.

$$\begin{array}{llll} \varphi = 1111 + 0110 & \longmapsto & 0110 \in \Phi_2 & \text{previous cases} \\ \varphi = 0111 + 1110 & \longmapsto & 1110 \in \Phi_2 & \text{previous cases} \\ \gamma = \beta + 1220 & \longmapsto & 1220 \notin \Phi_3 & \\ 1220 = 0110 + 1110 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case H: $\beta = \alpha_2,\ \gamma = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4.\ \gamma + \beta = 1342$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0120, 0110, 0111, 0122, 1220, 1222, 1231, 1232$.

Eight elements $\varphi = 0121, 1221, 0011, 1110, 1111, 1121, 1120, 1122$ give irreducible triples.

Case H1: $\varphi = 0121$.

$$\begin{array}{llll} \varphi = \alpha_4 + 0120 & \longmapsto & \alpha_4 \in \Phi_2 & \\ 0122 = \varphi + \alpha_4 & & & \text{contradicts } \Phi_2 \text{ convex.} \end{array}$$

Case H2: $\varphi = 1221$.

$$\begin{array}{llll} \varphi = \alpha_4 + 1220 & \longmapsto & \alpha_4 \in \Phi_2 & \\ 1222 = \varphi + \alpha_4 & & & \text{contradicts } \Phi_2 \text{ convex.} \end{array}$$

Case H3: $\varphi = 0011$.

$$\begin{array}{lcll}
\varphi = \beta + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\
2342 = (\gamma + \beta) + \alpha_1 & \longmapsto & 2342 \in \Phi_3 & \\
2342 = \gamma + \varphi & \longmapsto & 2342 \notin \Phi_1 & \\
2342 \in \Phi_2 & & & \\
2342 = 1222 + 1120 & \longmapsto & 1120 \in \Phi_3 & \\
1220 = \beta + 1120 & \longmapsto & 1220 \in \Phi_3 & \\
1220 \in \Phi_1 & & & \text{by rank 3} \\
2342 = 1220 + 1122 & \longmapsto & 1122 \in \Phi_2 & \\
\gamma = 1122 + 0120 & \longmapsto & 0120 \notin \Phi_3 & \\
1220 = \varphi + 0120 & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case H4: $\varphi = 1110$.

$$\begin{array}{lcll}
\varphi = 0110 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\
\alpha_0 = (\gamma + \beta) + \alpha_1 & \longmapsto & \alpha_0 \in \Phi_3 & \\
\alpha_0 = \gamma + 1100 & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by Case H3} \\
\alpha_0 \in \Phi_1 & & & \\
\varphi = 1100 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \\
\gamma = \alpha_3 + 1232 & \longmapsto & 1232 \notin \Phi_3 & \\
\alpha_0 = 1232 + \varphi & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case H5: $\varphi = 1111$.

$$\begin{array}{lcll}
\varphi = 1100 + 0011 & \longmapsto & 0011 \in \Phi_2 & \\
\gamma = 0011 + 1231 & \longmapsto & 1231 \notin \Phi_3 & \\
\alpha_0 = \gamma + 1100 & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by Case H3} \\
\varphi = 0111 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\
\alpha_0 = (\gamma + \beta) + \alpha_1 & \longmapsto & \alpha_0 \in \Phi_3 & \\
\alpha_0 \in \Phi_1 & & & \\
\alpha_0 = 1231 + \varphi & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case H6: $\varphi = 1121$.

$$\begin{array}{lcll}
\varphi = 1111 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case H5} \\
0110 = \beta + \alpha_3 & \longmapsto & 0110 \in \Phi_3 & \\
\varphi = 1110 + 0011 & \longmapsto & 0011 \in \Phi_2 & \text{by Case H4} \\
\gamma = \varphi + 0121 & \longmapsto & 0121 \notin \Phi_3 & \\
0121 = 0011 + 0110 & \longmapsto & 0121 \in \Phi_3. &
\end{array}$$

Contradiction.

Case H7: $\varphi = 1120$.

$$
\begin{array}{llll}
\varphi = 1110 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case H4} \\
\gamma = \alpha_3 + 1232 & \longmapsto & 1232 \notin \Phi_3 & \\
\varphi = 0120 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\
\alpha_0 = (\gamma + \beta) + \alpha_1 & \longmapsto & \alpha_0 \in \Phi_3 & \\
\alpha_0 = \gamma + 1100 & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by Case H3} \\
\alpha_0 \in \Phi_1 & & & \\
\alpha_0 = \varphi + 1222 & \longmapsto & 1222 \in \Phi_1 & \\
1232 = \alpha_3 + 1222 & \longmapsto & 1222 \notin \Phi_3. &
\end{array}
$$

Contradiction.

<u>Case H8:</u> $\varphi = 1122$.

$$
\begin{array}{llll}
\varphi = 1121 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by Case H6} \\
\gamma = \varphi + 0120 & \longmapsto & 0120 \notin \Phi_3 & \\
0121 = \alpha_4 + 0120 & \longmapsto & 0121 \notin \Phi_1 & \\
0121 \notin \Phi_2 & & & \text{by Case H1} \\
0121 \notin \Phi_3 & & & \\
0122 = \alpha_4 + 0121 & \longmapsto & 0122 \notin \Phi_1 & \\
\varphi = 1111 + 0011 & \longmapsto & 0011 \in \Phi_2 & \\
0122 = 2 \times 0011 + \beta & \longmapsto & 0122 \in \Phi_3 & \\
0122 \notin \Phi_2 & & & \text{by rank 3.}
\end{array}
$$

Contradiction.

<u>Case I:</u> $\beta = \alpha_2 + 2\alpha_3,\ \gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = 1222,\ \gamma + \beta = 1342.$

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0122, 0121, 1220, 1221$.

Three elements $\varphi = 1120, 1122, 1121$ give irreducible triples.

<u>Case I1:</u> $\varphi = 1120$.

$$
\begin{array}{llll}
\varphi = \beta + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \\
2342 = (\gamma + \beta) + \alpha_1 & \longmapsto & 2342 \in \Phi_3 & \\
2342 = \gamma + \varphi & \longmapsto & 2342 \notin \Phi_1 & \\
2342 \in \Phi_2 & & & \\
2342 = 1220 + 1122 & \longmapsto & 1122 \in \Phi_2 & \text{by rank 3} \\
\gamma = 1122 + \alpha_2 & \longmapsto & \alpha_2 \notin \Phi_3 & \\
\gamma + \beta = 1220 + 0122 & \longmapsto & 0122 \in \Phi_1 & \\
\gamma + \beta = \alpha_2 + 1242 & \longmapsto & 1242 \in \Phi_1 & \\
2342 = 1242 + 1100 & \longmapsto & 1100 \in \Phi_2 & \\
\gamma = 0122 + 1100 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}
$$

<u>Case I2:</u> $\varphi = 1121$.

$$\begin{array}{llll}
\varphi = 1120 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by Case I1} \\
\varphi = 0121 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by rank 3} \\
\gamma = 1221 + \alpha_4 & \longmapsto & 1221 \notin \Phi_3 & \\
2342 = (\gamma + \beta) + \alpha_1 & \longmapsto & 2342 \in \Phi_3 & \\
2342 = \gamma + 1120 & \longmapsto & 2342 \notin \Phi_2 & \\
2342 \in \Phi_2 & & & \\
2342 = \varphi + 1221 & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case I3: $\varphi = 1122$.

$$\begin{array}{llll}
\varphi = 1121 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by Case I2} \\
\gamma = 1221 + \alpha_4 & \longmapsto & 1221 \notin \Phi_3 & \\
1221 = \alpha_4 + 1220 & \longmapsto & 1220 \notin \Phi_3 & \\
\varphi = 0122 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by rank 3} \\
2342 = (\gamma + \beta) + \alpha_1 & \longmapsto & 2342 \in \Phi_3 & \\
2342 = 1121 + 1221 & \longmapsto & 2342 \notin \Phi_2 & \\
2342 \in \Phi_1 & & & \\
2342 = \varphi + 1220 & & & \text{contradicts } \Phi_1 \text{ coconvex.}
\end{array}$$

Case J: $\beta = \alpha_2 + \alpha_3,\ \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. Here $\gamma + \beta = 1342$ and $\gamma - \beta = 1122$.

The elements of $]\beta; \gamma[$ reducible to rank at most 3 are: $0111, 1110, 0121, 1221, 1231$.

Eight elements $\varphi = 1122, 1220, 0120, 1222, 1121, 1111, 0122, 1120$ give irreducible triples.

Case J1: $\varphi = 1122$.

Here $1122 = \gamma - \beta$. But $\gamma = (\gamma - \beta) + \beta$ implies $\gamma - \beta \notin \Phi_3$.

Case J2: $\varphi = 1220$.

$\varphi = \beta + 1110 \notin \Phi_2$ by rank 3 and coconvexity of $\Phi_2$.

Case J3: $\varphi = 0120$.

$$\begin{array}{llll}
\varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \\
\gamma + \beta = \varphi + 1222 & \longmapsto & 1222 \in \Phi_1 & \\
\gamma = 1222 + \alpha_3 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case J4: $\varphi = 1222$.

$$\begin{array}{lll}
\varphi = (\gamma - \beta) + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 \\
\gamma = \varphi + \alpha_3 & \longmapsto & \alpha_3 \notin \Phi_3 \\
\beta = \alpha_2 + \alpha_3 & \text{contradicts } \Phi_1 \text{ coconvex.} &
\end{array}$$

Case J5: $\varphi = 1121$.

$$\begin{array}{llll}
1231 = \beta + \varphi & \longmapsto & 1231 \in \Phi_3 & \\
1231 \in \Phi_1 & & & \text{by rank 3} \\
\gamma = \varphi + 0111 & \longmapsto & 0111 \notin \Phi_3 & \\
1231 = 0111 + 1120 & \longmapsto & 1120 \in \Phi_1 & \\
\gamma - \beta = \varphi + \alpha_4 & \longmapsto & \alpha_4 \notin \Phi_3 & \\
\varphi = 1120 + \alpha_4 & \longmapsto & 1120 \in \Phi_2. &
\end{array}$$

Contradiction.

Case J6: $\varphi = 1111$.

$$\begin{array}{llll} 1221 = \beta + \varphi & \longmapsto & 1221 \in \Phi_3 & \\ 1221 \in \Phi_1 & & & \text{by rank 3} \\ \gamma = \varphi + 0121 & \longmapsto & 0121 \notin \Phi_3 & \\ \gamma - \beta = \varphi + 0011 & \longmapsto & 0011 \notin \Phi_3 & \\ \varphi = 1100 + 0011 & \longmapsto & 1100 \in \Phi_2 & \\ 1221 = 1100 + 0121 & \longmapsto & 1100 \in \Phi_1. & \end{array}$$

Contradiction.

Case J7: $\varphi = 0122$.

$$\begin{array}{lll} \varphi = 0111 + 0011 & \longmapsto & 0011 \in \Phi_2 \\ \varphi = 0121 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 \\ \gamma + \beta = \varphi + 1220 & \longmapsto & 1220 \in \Phi_1 \\ 1231 = 0011 + 1220 & \longmapsto & 1231 \in \Phi_3 \\ \gamma = 1231 + \alpha_4 & \text{contradicts } \Phi_3 \text{ convex.} & \end{array}$$

Case J8: $\varphi = 1120$.

$$\begin{array}{llll} \varphi = 1110 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by rank 3} \\ \varphi = 0120 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{previous cases} \\ \gamma = 1222 + \alpha_3 & \longmapsto & 1222 \notin \Phi_3 & \\ 2342 = (\gamma + \beta) + \alpha_1 & \longmapsto & 2342 \in \Phi_3 & \\ 2342 = \gamma + 1110 & \longmapsto & 2342 \notin \Phi_2 & \\ 2342 \in \Phi_1 & & & \\ 2342 = \varphi + 1222 & \text{contradicts } \Phi_1 \text{ coconvex.} & & \end{array}$$

Case K: $\beta = \alpha_2 + \alpha_3 + \alpha_4,\ \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4 1231,\ \gamma + \beta = 1342$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $1111, 0121, 1221$.

One element $\varphi = 1121$ gives an irreducible triple.

Here $\gamma - \beta = 1120 \in \Phi$.

$$\begin{array}{lll} \varphi = (\gamma - \beta) + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 \\ \gamma = 0110 + \varphi & \longmapsto & 0110 \notin \Phi_3 \\ \beta = 0110 + \alpha_4 & & \text{contradicts } \Phi_1 \text{ convex.} \end{array}$$

Case L: $\beta = \alpha_3,\ \gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 = 1221,\ \gamma + \beta = 1231$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $0110, 1110, 0121, 1121$.

Six elements $\varphi = 0120, 1120, 1220, 0011, 0111, 1111$ give irreducible triples.

Case L1-3: $\varphi = 0120, 1120, 1220$.

$$\begin{array}{l} 0120 = 0110 + \beta \notin \Phi_2, \\ 1120 = 1110 + \beta \notin \Phi_2, \\ 1220 = 1110 + 0110 \notin \Phi_2, \end{array}$$

by rank 3 and coconvexity of $\Phi_2$.

Case L4: $\varphi = 0011$.

$$\begin{array}{lcll} \varphi = \beta + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \\ \gamma + \beta = \varphi + 1220 & \longmapsto & 1220 \in \Phi_1 & \\ \gamma = 1220 + \alpha_4 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case L5: $\varphi = 0111$.

$$\begin{array}{lcll} \varphi = 0011 + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by Case L4} \\ \varphi = 0110 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by rank 3} \\ \gamma + \beta = \varphi + 1120 & \longmapsto & 1120 \in \Phi_1 & \\ \gamma = 1120 + \alpha_3 + \alpha_4 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case L6: $\varphi = 1111$.

$$\begin{array}{lcll} \varphi = 1110 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by rank 3} \\ \varphi = 0011 + 1100 & \longmapsto & 1100 \in \Phi_2 & \text{by Case L4} \\ \varphi = 0111 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case L5} \\ 1110 = \beta + 1100 & \longmapsto & 1110 \in \Phi_3 & \\ 1110 \in \Phi_1 & & & \\ \gamma + \beta = \varphi + 1120 & \longmapsto & 1120 \in \Phi_1 & \\ \gamma = \varphi + 0110 & \longmapsto & 0110 \notin \Phi_3 & \\ 1110 = \alpha_1 + 0110 & & & \text{contradicts } \Phi_1 \text{ coconvex.} \end{array}$$

Case M: $\beta = \alpha_2 + \alpha_3,\ \gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = 1121,\ \gamma + \beta = 1231$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $1110, 0121$.

Four elements $\varphi = 0111, 0120, 1111, 1120$ give irreducible triples.

Case M1: $\varphi = 0111$.

$$\begin{array}{lcll} \gamma + \beta = \varphi + 1120 & \longmapsto & 1120 \in \Phi_1 & \\ \varphi = \beta + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \\ \gamma = 1120 + \alpha_4 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case M2: $\varphi = 0120$.

$$\begin{array}{lcll} \gamma + \beta = \varphi + 1111 & \longmapsto & 1111 \in \Phi_1 & \\ \varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \\ \gamma = 1111 + \alpha_3 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case M3: $\varphi = 1111$.

$$\begin{array}{lcll} \varphi = 0111 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case M1} \\ \varphi = 1110 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by rank 3} \\ \gamma + \beta = \varphi + 0120 & \longmapsto & 0120 \in \Phi_1 & \\ \gamma = 0120 + \alpha_1 + \alpha_4 & & & \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case M4: $\varphi = 1120$.

$$\begin{array}{lcll}
\varphi = 0120 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case M2} \\
\varphi = 1110 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by rank 3} \\
\gamma + \beta = \varphi + 0111 & \longmapsto & 0111 \in \Phi_1 & \\
\gamma = 0111 + \alpha_1 + \alpha_3 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N: $\beta = \alpha_3,\ \gamma = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = 1222,\ \gamma + \beta = 1232$.

Here, each element $\varphi$ in $]\beta;\gamma[ = \{0011, 0110, 0121, 0120, 1110, 1120, 1220, 1121, 1221, 0111, 0122, 1122, 111$ gives an irreducible triple.

Case N1: $\varphi = 0011$.

$$\begin{array}{lcll}
\gamma + \beta = \varphi + 1221 & \longmapsto & 1221 \in \Phi_1 & \\
\varphi = \beta + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \\
\gamma = 1221 + \alpha_4 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N2: $\varphi = 0110$.

$$\begin{array}{lcll}
\gamma + \beta = \varphi + 1122 & \longmapsto & 1122 \in \Phi_1 & \\
\varphi = \beta + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 & \\
\gamma = 1122 + \alpha_2 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N3-4: $\varphi = 0121$ and $0120$.

$$\begin{array}{l}
0121 = 0011 + 0110 \notin \Phi_2 \\
0120 = \beta + 0110 \notin \Phi_2
\end{array}$$

by previous cases and rank at most 3.

Case N5: $\varphi = 1110$.

$$\begin{array}{lcll}
\gamma + \beta = \varphi + 0122 & \longmapsto & 0122 \in \Phi_1 & \\
\varphi = \beta + 1100 & \longmapsto & 1100 \in \Phi_2 & \\
\gamma = 0122 + 1100 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N6-8: $\varphi = 1120, 1220$ and $1121$.

$$\begin{array}{l}
1220 = 1110 + 0110 \notin \Phi_2 \\
1120 = \beta + 1110 \notin \Phi_2 \\
1121 = 1110 + 0011 \notin \Phi_2
\end{array}$$

by previous cases and rank at most 3.

Case N9: $\varphi = 1221$.

$$\begin{array}{lcll}
\varphi = 0110 + 1111 & \longmapsto & 1111 \in \Phi_2 & \\
\varphi = 1110 + 0111 & \longmapsto & 0111 \in \Phi_2 & \\
\gamma = 1111 + 0111 & & & \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N10: $\varphi = 0111$.

$$\begin{array}{lll}
\varphi = 0110 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 \\
\gamma + \beta = \varphi + 1121 & \longmapsto & 1121 \in \Phi_1 \\
1122 = 1121 + \alpha_4 & \longmapsto & 1122 \in \Phi_3 \\
\varphi = 0011 + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 \\
\gamma = \alpha_2 + 1122 & & \qquad \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N11: $\varphi = 0122$.
$\varphi = 0111 + 0011 \notin \Phi_2$ by Cases N1 and N10.
Case N12: $\varphi = 1122$.

$$\begin{array}{lll}
\varphi = 0011 + 1111 & \longmapsto & 1111 \in \Phi_2 \\
\varphi = 1120 + 2\alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 \\
\gamma + \beta = \varphi + 0110 & \longmapsto & 0110 \in \Phi_1 \\
\gamma = 0111 + 1111 & \longmapsto & 0111 \notin \Phi_3 \\
0111 = \alpha_4 + 0110 & & \qquad \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case N13: $\varphi = 1111$.

$$\begin{array}{lll}
\varphi = 0011 + 1100 & \longmapsto & 1100 \in \Phi_2 \\
\gamma + \beta = \varphi + 0121 & \longmapsto & 0121 \in \Phi_1 \\
1221 = 1100 + 0121 & \longmapsto & 1221 \in \Phi_3 \\
\varphi = 1110 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 \\
\gamma = 1221 + \alpha_4 & & \qquad \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case O: $\beta = \alpha_2 + \alpha_3,\ \gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 = 1122,\ \gamma + \beta = 1232$.

Here, each element $\varphi$ in $]\beta;\gamma[ = \{0111, 1110, 0121, 1121, 1111, 0120, 0122, 1120\}$ gives an irreducible triple.

Case O1: $\varphi = 0111$.

$$\begin{array}{lll}
\varphi = \beta + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 \\
\gamma + \beta = \varphi + 1121 & \longmapsto & 1121 \in \Phi_1 \\
\gamma = \alpha_4 + 1121 & & \qquad \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case O2: $\varphi = 1110$.

$$\begin{array}{lll}
\varphi = \beta + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 \\
\gamma + \beta = \varphi + 0122 & \longmapsto & 0122 \in \Phi_1 \\
\gamma = \alpha_1 + 0122 & & \qquad \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case O3: $\varphi = 0121$.

$$\begin{array}{lll}
\varphi = \beta + 0011 & \longmapsto & 0011 \in \Phi_2 \\
\gamma + \beta = \varphi + 1111 & \longmapsto & 1111 \in \Phi_1 \\
\gamma = 0011 + 1111 & & \qquad \text{contradicts } \Phi_3 \text{ convex.}
\end{array}$$

Case O4: $\varphi = 1121$.

| | | | |
|---|---|---|---|
| $\varphi = 0121 + \alpha_1$ | $\longmapsto$ | $\alpha_1 \in \Phi_2$ | by Case O3 |
| $\varphi = 1110 + 0011$ | $\longmapsto$ | $011 \in \Phi_2$ | by Case O2 |
| $\gamma + \beta = \varphi + 0111$ | $\longmapsto$ | $0111 \in \Phi_1$ | |
| $0122 = 0111 + 0011$ | $\longmapsto$ | $0122 \in \Phi_3$ | |
| $\gamma = \alpha_1 + 0122$ | contradicts $\Phi_3$ convex. | | |

Case O5: $\varphi = 1111$.

| | | | |
|---|---|---|---|
| $\varphi = 0111 + \alpha_1$ | $\longmapsto$ | $\alpha_1 \in \Phi_2$ | previous cases |
| $\varphi = 1110 + \alpha_4$ | $\longmapsto$ | $\alpha_4 \in \Phi_2$ | previous cases |
| $\gamma + \beta = \varphi + 0121$ | $\longmapsto$ | $0121 \in \Phi_1$ | |
| $\gamma = 0121 + \alpha_1 + \alpha_4$ | | | contradicts $\Phi_3$ convex. |

Case O6: $\varphi = 0120$.

| | | | |
|---|---|---|---|
| $\varphi = \beta + \alpha_3$ | $\longmapsto$ | $\alpha_3 \in \Phi_2$ | |
| $1242 = 0121 + 1121$ | $\longmapsto$ | $1242 \notin \Phi_2$ | previous cases |
| $1242 = \gamma + \varphi$ | $\longmapsto$ | $1242 \notin \Phi_1$ | |
| $1242 = (\gamma + \beta) + \alpha_3$ | | | contradicts $\Phi_3$ convex. |

Case O7: $\varphi = 0122$.

| | | | |
|---|---|---|---|
| $\varphi = 0121 + \alpha_4$ | $\longmapsto$ | $\alpha_4 \in \Phi_2$ | previous cases |
| $\varphi = 0111 + 0011$ | $\longmapsto$ | $0011 \in \Phi_2$ | previous cases |
| $\gamma + \beta = \varphi + 1110$ | $\longmapsto$ | $1110 \in \Phi_2$ | |
| $1121 = 0011 + 1110$ | $\longmapsto$ | $1121 \in \Phi_3$ | |
| $\gamma = 1121 + \alpha_4$ | | | contradicts $\Phi_3$ convex. |

Case O8: $\varphi = 1120$.

| | | | |
|---|---|---|---|
| $\varphi = 1110 + \alpha_3$ | $\longmapsto$ | $\alpha_3 \in \Phi_2$ | previous cases |
| $\varphi = 0120 + \alpha_1$ | $\longmapsto$ | $\alpha_1 \in \Phi_2$ | previous cases |
| $1242 = (\gamma + \beta) + \alpha_3$ | $\longmapsto$ | $1242 \in \Phi_3$ | |
| $1242 = 0121 + 1121$ | $\longmapsto$ | $1242 \notin \Phi_2$ | previous cases |
| Hence | | $1242 \in \Phi_1$ | |
| $\gamma = \alpha_1 + 0122$ | $\longmapsto$ | $0122 \notin \Phi_3$ | |
| $1242 = \varphi + 0122$ | | | contradicts $\Phi_1$ coconvex. |

Case P: $\beta = \alpha_1 + \alpha_2,\ \gamma = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 = 1242,\ \gamma + \beta = \alpha_0 = 2342$.

The elements of $]\beta; \gamma[$ reducible to rank at most 3 are: $0111, 1110, 0122, 0121, 1220, 1121, 1342, 1231, 1120$

Two elements $\varphi = 1121, 1221$ give irreducible triples.

Case P1: $\varphi = 1121$.

$\varphi = 1120 + \alpha_4 \longmapsto \alpha_4 \in \Phi_2$.

$1222 = \varphi + \alpha_4 \in \Phi_2$ contradicts the rank at most 3.

Case P2: $\varphi = 1221$.

$1122 = \varphi + \alpha_4 \in \Phi_2$ contradicts the rank at most 3.

Case Q: $\beta = \alpha_1 + \alpha_2 + \alpha_3,\ \gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = 1232$. Here $\gamma + \beta = 2342 = \alpha_0$ and $\gamma - \beta = 0122$.

The elements of $]\beta;\gamma[$ reducible to rank at most 3 are: $1111, 1121, 1221, 1231$.

Four elements $\varphi = 1222, 1122, 1120, 1220$ give irreducible triples.

Case Q1: $\varphi = 1222$.

$$\begin{array}{lll} \varphi = (\gamma - \beta) + 1100 & \longmapsto & 1100 \in \Phi_2 \\ \gamma = \varphi + \alpha_3 & \longmapsto & \alpha_3 \notin \Phi_3 \\ \beta = 1100 + \alpha_3 & & \qquad\qquad \text{contradicts } \Phi_1 \text{ coconvex.} \end{array}$$

Case Q2: $\varphi = 1122$.

$$\begin{array}{lll} \varphi = (\gamma - \beta) + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 \\ \gamma = \varphi + 0110 & \longmapsto & 0110 \notin \Phi_3 \\ \beta = 0110 + \alpha_1 & & \qquad\qquad \text{contradicts } \Phi_1 \text{ coconvex.} \end{array}$$

Case Q3: $\varphi = 1120$.

$$\begin{array}{lll} \varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 \\ \gamma + \beta = \varphi + 1222 & \longmapsto & 1222 \in \Phi_1 \\ \gamma = 1222 + \alpha_3 & & \qquad\qquad \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

Case Q4: $\varphi = 1220$.

$$\begin{array}{lll} \varphi = \beta + 0110 & \longmapsto & 0110 \in \Phi_2 \\ \gamma + \beta = \varphi + 1122 & \longmapsto & 1122 \in \Phi_1 \\ \gamma = 1122 + 0110 & & \qquad\qquad \text{contradicts } \Phi_3 \text{ convex.} \end{array}$$

The 36 matrices $M$ with assumptions H1 and H2 for D4 : $[2,1][2]21$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$[4,2,3,1,2,4,2,3][4,2,3,1,2,4,2]21$ :

$$\left(\begin{array}{ccccccc}
0 & 0 & 0 & 0 & -\frac{1}{2}x_3x_{11}-x_9 & \frac{1}{3}x_0x_3x_{11}+\frac{1}{2}x_0x_9-\frac{1}{2}x_1x_{11}-x_6 & -\frac{1}{12}x_0x_3^2x_{11}+x_3x_6-  \\
0 & 1 & -x_3 & \frac{1}{2}x_0x_3-x_1 & \frac{1}{2}x_3x_5+x_4 & -\frac{1}{3}x_0x_3x_5-\frac{1}{2}x_0x_4+\frac{1}{2}x_1x_5+x_2 & \frac{1}{12}x_0x_3^2x_5-x_2x_3+ \\
0 & 0 & 0 & 0 & x_{11} & -x_0x_{11} & \frac{1}{3}x_0x_3x_{11}-\frac{1}{2}x_0x_9+\frac{1}{2}x_1x_{11}- \\
0 & 0 & 1 & -x_0 & -x_5 & x_0x_5 & -\frac{1}{3}x_0x_3x_5+\frac{1}{2}x_0x_4-\frac{1}{2}x_1x_5- \\
0 & 0 & 0 & 0 & 1 & -x_0 & \frac{1}{2}x_0x_3- \\
0 & 0 & 0 & 1 & 0 & -x_5 & \frac{1}{2}x_3x_5- \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

$[1,2,4,1,2,3,2,1][1,2,4,3,1,2,1]21$ :

$$\left(\begin{array}{ccccccc}
1 & 0 & x_3 & \frac{1}{2}x_3x_5+x_4 & -\frac{1}{2}x_3x_{11}-x_9 & \frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_{11}-x_{10} & -\frac{1}{12}x_3^2x_5x_{11}+x_4x_9+ \\
0 & 0 & 0 & \frac{1}{2}x_0x_3-x_1 & 0 & \frac{1}{3}x_0x_3x_{11}+\frac{1}{2}x_0x_9-\frac{1}{2}x_1x_{11}-x_6 & -\frac{1}{12}x_0x_3^2x_{11}+x_3x_6- \\
0 & 0 & 1 & x_5 & -x_{11} & x_5x_{11} & -\frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_{11} \\
0 & 0 & 0 & -x_0 & 0 & -x_0x_{11} & \frac{1}{3}x_0x_3x_{11}-\frac{1}{2}x_0x_9+\frac{1}{2}x_1x_{11} \\
0 & 0 & 0 & 1 & 0 & x_{11} & -\frac{1}{2}x_3x_{11} \\
0 & 0 & 0 & 0 & 1 & -x_5 & \frac{1}{2}x_3x_5 \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

and $[2,4][2]21$ :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$[2,3][2]21$ :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$[4,2,3,1,2,4,2,1][4,2,3,1,2,4,2]21$ :

$$\left(\begin{array}{ccccccc}
0 & 0 & 0 & -\frac{1}{2}x_3x_{11}-x_9 & 0 & \frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_{11}-x_{10} & -\frac{1}{12}x_3^2x_5x_{11}+x_4x_9- \\
0 & 1 & -x_3 & \frac{1}{2}x_0x_3-x_1 & \frac{1}{2}x_3x_5+x_4 & -\frac{1}{3}x_0x_3x_5-\frac{1}{2}x_0x_4+\frac{1}{2}x_1x_5+x_2 & \frac{1}{12}x_0x_3^2x_5-x_2x_3 \\
0 & 0 & 0 & -x_{11} & 0 & x_5x_{11} & -\frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_1 \\
0 & 0 & 1 & -x_0 & -x_5 & x_0x_5 & -\frac{1}{3}x_0x_3x_5+\frac{1}{2}x_0x_4-\frac{1}{2}x_1 \\
0 & 0 & 0 & 1 & 0 & -x_5 & \frac{1}{2}x_3 \\
0 & 0 & 0 & 0 & 1 & -x_0 & \frac{1}{2}x_0 \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

$[1,2][1]21$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$[4,2,3,1,2,4,3,2][4,2,3,1,2,4,3]21$

$$\left(\begin{array}{cccccccc} 0 & 0 & -x_{11} & x_5x_{11} & x_0x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3x_5x_{11} - \frac{1}{3}x_0x_5x_9 - \frac{1}{3}x_0x_4 \\ 0 & 1 & -x_3 & \frac{1}{2}x_3x_5 + x_4 & \frac{1}{2}x_0x_3 - x_1 & -\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2 \\ 0 & 0 & 0 & x_{11} & 0 & -x_0x_{11} \\ 0 & 0 & 1 & -x_5 & -x_0 & x_0x_5 \\ 0 & 0 & 0 & 1 & 0 & -x_0 \\ 0 & 0 & 0 & 0 & 1 & -x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

and
$[4,2][4]21$ :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$[1,2,4,2,3,1,2,1][1,2,4,2,3,2,1]21$ :

$$\left(\begin{array}{cccccccc} 1 & 0 & x_3 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} - x_{10} \\ 0 & 0 & -x_0 & -x_0x_5 & x_0x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3x_5x_{11} - \frac{1}{3}x_0x_5x_9 - \frac{1}{3}x_0 \\ 0 & 0 & 1 & x_5 & -x_{11} & x_5x_{11} \\ 0 & 0 & 0 & -x_0 & 0 & -x_0x_{11} \\ 0 & 0 & 0 & 1 & 0 & x_{11} \\ 0 & 0 & 0 & 0 & 1 & -x_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$[3,1,2,4,1,2,3,2][3,1,2,4,1,2,3]21$ :

$$\left(\begin{array}{cccccccc} 0 & 0 & x_5 & -x_0x_5 & x_5x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3x_5x_{11} - \frac{1}{3}x_0x_5x_9 - \frac{1}{3}x_0 \\ 1 & 0 & -x_3 & \frac{1}{2}x_0x_3 - x_1 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_0x_3x_{11} + \frac{1}{2}x_0x_9 - \frac{1}{2}x_1x_{11} - x_6 \\ 0 & 0 & 1 & -x_0 & x_{11} & -x_0x_{11} \\ 0 & 0 & 0 & 0 & -x_5 & x_0x_5 \\ 0 & 0 & 0 & 1 & 0 & x_{11} \\ 0 & 0 & 0 & 0 & 1 & -x_0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right.$$

$[3,2,4,1,2,3,1,2][3,2,4,1,2,3,1]21:$

$$\left(\begin{array}{ccccccc}
0 & 0 & x_5 & -x_0x_5 & x_5x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3x_5x_{11} - \frac{1}{3}x_0x_5x_9 - \frac{1}{3}x_0 \cdots \\
0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_0x_3x_{11} + \frac{1}{2}x_0x_9 - \frac{1}{2}x_1x_{11} - x_6 & \\
0 & 0 & 0 & x_5 & 0 & x_5x_{11} & \\
0 & 0 & 1 & -x_0 & x_{11} & -x_0x_{11} & \\
0 & 0 & 0 & 1 & 0 & x_{11} & \\
0 & 0 & 0 & 0 & 1 & -x_0 & \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

and $[2,3,1,2,4,1,2,3][3,2,4,1,2,3,2]21:$

$$\left(\begin{array}{ccccccc}
0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_0x_3x_{11} + \frac{1}{2}x_0x_9 - \frac{1}{2}x_1x_{11} - x_6 & -\frac{1}{12}x_0x_3^2x_{11} + x_3x_6 - x \cdots \\
0 & 0 & 0 & 0 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2 & \frac{1}{12}x_0x_3^2x_5 - x_2x_3 + x \cdots \\
0 & 0 & 1 & -x_0 & x_{11} & -x_0x_{11} & \frac{1}{3}x_0x_3x_{11} - \frac{1}{2}x_0x_9 + \frac{1}{2}x_1x_{11} - \cdots \\
0 & 0 & 0 & 0 & -x_5 & x_0x_5 & -\frac{1}{3}x_0x_3x_5 + \frac{1}{2}x_0x_4 - \frac{1}{2}x_1x_5 - \cdots \\
0 & 0 & 0 & 1 & 0 & x_{11} & -\frac{1}{2}x_3x_{11} - \cdots \\
0 & 0 & 0 & 0 & 1 & -x_0 & \frac{1}{2}x_0x_3 - \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

$[3,2,4,1,2,3,2,1][3,2,4,1,2,3,2]21:$

$$\left(\begin{array}{ccccccc}
0 & 0 & 0 & \frac{1}{2}x_3x_5 + x_4 & 0 & \frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} - x_{10} & -\frac{1}{12}x_3^2x_5x_{11} + x_4x_9 + \cdots \\
0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_0x_3x_{11} + \frac{1}{2}x_0x_9 - \frac{1}{2}x_1x_{11} - x_6 & -\frac{1}{12}x_0x_3^2x_{11} + x_3x_6 \cdots \\
0 & 0 & 0 & x_5 & 0 & x_5x_{11} & -\frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_1 \cdots \\
0 & 0 & 1 & -x_0 & x_{11} & -x_0x_{11} & \frac{1}{3}x_0x_3x_{11} - \frac{1}{2}x_0x_9 + \frac{1}{2}x_1x \cdots \\
0 & 0 & 0 & 1 & 0 & x_{11} & -\frac{1}{2}x_3x \cdots \\
0 & 0 & 0 & 0 & 1 & -x_0 & \frac{1}{2}x_0 \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

$[2,3,1,2,4,3,2,1][1,2,4,3,1,2,1]21:$

$$\left(\begin{array}{ccccccc}
1 & 0 & x_3 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} - x_{10} & -\frac{1}{12}x_3^2x_5x_{11} + x_4x_9 + x \cdots \\
0 & 0 & 0 & 0 & \frac{1}{2}x_0x_3 - x_1 & -\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2 & \frac{1}{12}x_0x_3^2x_5 - x_2x_3 + \cdots \\
0 & 0 & 1 & x_5 & -x_{11} & x_5x_{11} & -\frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} + \cdots \\
0 & 0 & 0 & 0 & -x_0 & x_0x_5 & -\frac{1}{3}x_0x_3x_5 + \frac{1}{2}x_0x_4 - \frac{1}{2}x_1x_5 \cdots \\
0 & 0 & 0 & 1 & 0 & x_{11} & -\frac{1}{2}x_3x_{11} \cdots \\
0 & 0 & 0 & 0 & 1 & -x_5 & \frac{1}{2}x_3x_5 \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.$$

and

[3, 2][3]21 :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[4, 2, 3, 1, 2, 4, 1, 2][4, 2, 3, 1, 2, 4, 1]21 :

$$\begin{pmatrix} 0 & 0 & -x_{11} & x_0x_{11} & x_5x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3x_5x_{11} - \frac{1}{3}x_0x_5x_9 - \frac{1}{3}x_0x_4 \\ 0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2 & \\ 0 & 0 & 0 & -x_{11} & 0 & x_5x_{11} & \\ 0 & 0 & 1 & -x_0 & -x_5 & x_0x_5 & \\ 0 & 0 & 0 & 1 & 0 & -x_5 & \\ 0 & 0 & 0 & 0 & 1 & -x_0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

[3, 1, 2, 4, 3, 1, 2, 1][3, 1, 2, 4, 3, 2, 1]21 :

$$\begin{pmatrix} 1 & 0 & x_3 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} - x_{10} & \\ 0 & 0 & -x_0 & -x_0x_5 & x_0x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3x_5x_{11} - \frac{1}{3}x_0x_5x_9 - \frac{1}{3}x_0 \\ 0 & 0 & 1 & x_5 & -x_{11} & x_5x_{11} & \\ 0 & 0 & 0 & 0 & -x_0 & x_0x_5 & \\ 0 & 0 & 0 & 1 & 0 & x_{11} & \\ 0 & 0 & 0 & 0 & 1 & -x_5 & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

[2, 4, 1, 2, 1][2, 4, 2, 1]21 :

$$\begin{pmatrix} 1 & 0 & x_3 & -\frac{1}{2}x_3x_{11} - x_9 \\ 0 & 0 & -x_0 & x_0x_{11} \\ 0 & 0 & 1 & -x_{11} \\ 0 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[1, 2, 4, 1, 2][1, 2, 4, 1]21 :

$$\begin{pmatrix} 0 & 0 & -x_{11} & x_0x_{11} \\ 0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 \\ 0 & 0 & 0 & -x_{11} \\ 0 & 0 & 1 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[1, 2, 3, 1, 2][1, 2, 3, 1]21 :

$$\begin{pmatrix} 0 & 0 & x_5 & -x_0x_5 \\ 0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 \\ 0 & 0 & 0 & x_5 \\ 0 & 0 & 1 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[4, 2, 3, 1, 2, 4, 2, 3, 1, 2, 1][4, 2, 3, 1, 2, 4, 2, 3, 2, 1]21$ :

$$\begin{pmatrix} 1 & 0 & 0 & x_3 & 0 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{2}x_3x_{11} - x_9 & 0 & \frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} - x_{10} & \\ 0 & 0 & 0 & -x_0 & -x_{11} & -x_0x_5 & x_0x_{11} & x_5x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x \\ 0 & 0 & 1 & 0 & -x_3 & 0 & \frac{1}{2}x_0x_3 - x_1 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2 & \\ 0 & 0 & 0 & 1 & 0 & x_5 & -x_{11} & 0 & x_5x_{11} & \\ 0 & 0 & 0 & 0 & 0 & -x_0 & 0 & x_{11} & -x_0x_{11} & \\ 0 & 0 & 0 & 0 & 1 & 0 & -x_0 & -x_5 & x_0x_5 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_{11} & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -x_5 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

$[1, 2, 4, 2, 1][4, 1, 2, 1]21$ :

$$\begin{pmatrix} 1 & 0 & x_3 & -\frac{1}{2}x_3x_{11} - x_9 \\ 0 & 0 & 0 & \frac{1}{2}x_0x_3 - x_1 \\ 0 & 0 & 1 & -x_{11} \\ 0 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[1, 2, 4, 2, 1][1, 2, 4, 2]21$ :

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}x_3x_{11} - x_9 \\ 0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 \\ 0 & 0 & 0 & -x_{11} \\ 0 & 0 & 1 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[3, 2, 4, 2, 3][4, 2, 3, 2]21$ :

$$\begin{pmatrix} 0 & 1 & -x_3 & -\frac{1}{2}x_3x_{11} - x_9 \\ 0 & 0 & 0 & \frac{1}{2}x_3x_5 + x_4 \\ 0 & 0 & 1 & x_{11} \\ 0 & 0 & 0 & -x_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[3, 2, 4, 2, 3][3, 2, 4, 2]21$ :

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}x_3x_{11} - x_9 \\ 0 & 1 & -x_3 & \frac{1}{2}x_3x_5 + x_4 \\ 0 & 0 & 0 & x_{11} \\ 0 & 0 & 1 & -x_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[1, 2, 3, 2, 1][3, 1, 2, 1]21 :$

$$\begin{pmatrix} 1 & 0 & x_3 & \frac{1}{2}x_3x_5 + x_4 \\ 0 & 0 & 0 & \frac{1}{2}x_0x_3 - x_1 \\ 0 & 0 & 1 & x_5 \\ 0 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[1, 2, 3, 2, 1][1, 2, 3, 2]21 :$

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}x_3x_5 + x_4 \\ 0 & 1 & -x_3 & \frac{1}{2}x_0x_3 - x_1 \\ 0 & 0 & 0 & x_5 \\ 0 & 0 & 1 & -x_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$[4, 2, 3, 1, 2, 4, 1, 2, 3, 2, 1][3, 1, 2, 4, 1, 2, 3, 1, 2, 1]21 :$

$$\begin{pmatrix} 1 & 0 & 0 & x_3 & 0 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{2}x_3x_{11} - x_9 & 0 & \frac{1}{3}x_3x_5x_{11} + \frac{1}{2}x_5x_9 + \frac{1}{2}x_4x_{11} - x_{10} \\ 0 & 0 & 1 & 0 & -x_3 & \frac{1}{2}x_0x_3 - x_1 & 0 & -\frac{1}{2}x_3x_{11} - x_9 & \frac{1}{3}x_0x_3x_{11} + \frac{1}{2}x_0x_9 - \frac{1}{2}x_1x_{11} - x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}x_0x_3 - x_1 & \frac{1}{2}x_3x_5 + x_4 & -\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2 \\ 0 & 0 & 0 & 1 & 0 & x_5 & -x_{11} & 0 & x_5x_{11} & -\frac{1}{3}x_3 \\ 0 & 0 & 0 & 0 & 1 & -x_0 & 0 & x_{11} & -x_0x_{11} & \frac{1}{3}x \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_0 & -x_5 & x_0x_5 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -x_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$[4, 2, 3, 1, 2, 4, 1, 2, 3, 2, 1][4, 2, 3, 1, 2, 4, 3, 1, 2, 1]21 :$

$$\left(\begin{array}{ccccccccc}
1 & 0 & 0 & x_3 & 0 & \frac{1}{2}x_3x_5+x_4 & -\frac{1}{2}x_3x_{11}-x_9 & 0 & \frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_{11}-x_{10} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}x_0x_3-x_1 & 0 & -\frac{1}{2}x_3x_{11}-x_9 & \frac{1}{3}x_0x_3x_{11}+\frac{1}{2}x_0x_9-\frac{1}{2}x_1x_{11}-x_6 \\
0 & 0 & 1 & 0 & -x_3 & 0 & \frac{1}{2}x_0x_3-x_1 & \frac{1}{2}x_3x_5+x_4 & -\frac{1}{3}x_0x_3x_5-\frac{1}{2}x_0x_4+\frac{1}{2}x_1x_5+x_2 \\
0 & 0 & 0 & 1 & 0 & x_5 & -x_{11} & 0 & x_5x_{11} \\
0 & 0 & 0 & 0 & 0 & -x_0 & 0 & x_{11} & -x_0x_{11} \\
0 & 0 & 0 & 0 & 1 & 0 & -x_0 & -x_5 & x_0x_5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -x_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

$[4,2,3,1,2,4,1,2,3,2,1][4,2,3,1,2,4,1,2,3,2]21:$

$$\left(\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & \frac{1}{2}x_3x_5+x_4 & 0 & -\frac{1}{2}x_3x_{11}-x_9 & \frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_{11}-x_{10} \\
0 & 1 & 0 & -x_3 & 0 & \frac{1}{2}x_0x_3-x_1 & -\frac{1}{2}x_3x_{11}-x_9 & 0 & \frac{1}{3}x_0x_3x_{11}+\frac{1}{2}x_0x_9-\frac{1}{2}x_1x_{11}-x_6 \\
0 & 0 & 1 & 0 & -x_3 & 0 & \frac{1}{2}x_3x_5+x_4 & \frac{1}{2}x_0x_3-x_1 & -\frac{1}{3}x_0x_3x_5-\frac{1}{2}x_0x_4+\frac{1}{2}x_1x_5+x_2 \\
0 & 0 & 0 & 0 & 0 & x_5 & 0 & -x_{11} & x_5x_{11} \\
0 & 0 & 0 & 1 & 0 & -x_0 & x_{11} & 0 & -x_0x_{11} \\
0 & 0 & 0 & 0 & 1 & 0 & -x_5 & -x_0 & x_0x_5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -x_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right.$$

$[2,4,2,3,2][2,4,2,3]21$

$$\begin{pmatrix}
0 & 0 & x_5 & x_5x_{11} \\
1 & 0 & -x_3 & -\frac{1}{2}x_3x_{11}-x_9 \\
0 & 0 & 1 & x_{11} \\
0 & 0 & 0 & -x_5 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$[2,3,1,2,1][2,3,2,1]21$

$$\begin{pmatrix}
1 & 0 & x_3 & \frac{1}{2}x_3x_5+x_4 \\
0 & 0 & -x_0 & -x_0x_5 \\
0 & 0 & 1 & x_5 \\
0 & 0 & 0 & -x_0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

$[4,2,3,1,2,4,1,2,3,1,2][4,2,3,1,2,4,1,2,3,1]21$

$$
\left(\begin{array}{cccccccccc}
0 & 0 & 0 & x_5 & -x_{11} & -x_0x_5 & x_5x_{11} & x_0x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0x_3 \\
0 & 1 & 0 & -x_3 & 0 & \frac{1}{2}x_0x_3-x_1 & -\frac{1}{2}x_3x_{11}-x_9 & 0 & \frac{1}{3}x_0x_3x_{11}+\frac{1}{2}x_0x_9-\frac{1}{2}x_1x_{11}-x_6 & \\
0 & 0 & 1 & 0 & -x_3 & 0 & \frac{1}{2}x_3x_5+x_4 & \frac{1}{2}x_0x_3-x_1 & -\frac{1}{3}x_0x_3x_5-\frac{1}{2}x_0x_4+\frac{1}{2}x_1x_5+x_2 & \\
0 & 0 & 0 & 0 & 0 & x_5 & 0 & -x_{11} & x_5x_{11} & \\
0 & 0 & 0 & 1 & 0 & -x_0 & x_{11} & 0 & -x_0x_{11} & \\
0 & 0 & 0 & 0 & 1 & 0 & -x_5 & -x_0 & x_0x_5 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_{11} & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -x_0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.
$$

$[3, 2, 4, 3, 2][3, 2, 4, 3]21$

$$
\begin{pmatrix}
0 & 0 & -x_{11} & x_5x_{11} \\
0 & 1 & -x_3 & \frac{1}{2}x_3x_5+x_4 \\
0 & 0 & 0 & x_{11} \\
0 & 0 & 1 & -x_5 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

$[2, 3, 1, 2, 4, 1, 2, 3, 1, 2, 1][2, 3, 1, 2, 4, 1, 2, 3, 2, 1]21$ :

$$
\left(\begin{array}{cccccccccc}
1 & 0 & 0 & x_3 & 0 & \frac{1}{2}x_3x_5+x_4 & -\frac{1}{2}x_3x_{11}-x_9 & 0 & \frac{1}{3}x_3x_5x_{11}+\frac{1}{2}x_5x_9+\frac{1}{2}x_4x_{11}-x_{10} & \\
0 & 0 & 0 & -x_0 & x_5 & -x_0x_5 & x_0x_{11} & x_5x_{11} & -x_0x_5x_{11} & \frac{1}{4}x_0 \\
0 & 1 & 0 & 0 & -x_3 & \frac{1}{2}x_0x_3-x_1 & 0 & -\frac{1}{2}x_3x_{11}-x_9 & \frac{1}{3}x_0x_3x_{11}+\frac{1}{2}x_0x_9-\frac{1}{2}x_1x_{11}-x_6 & \\
0 & 0 & 0 & 1 & 0 & x_5 & -x_{11} & 0 & x_5x_{11} & \\
0 & 0 & 0 & 0 & 1 & -x_0 & 0 & x_{11} & -x_0x_{11} & \\
0 & 0 & 0 & 0 & 0 & 0 & -x_0 & -x_5 & x_0x_5 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & x_{11} & \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -x_5 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 &
\end{array}\right.
$$