A short geometric proof of a conjecture of Fulton

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Abstract

We give a new geometric proof of a Fulton conjecture about the Littlewood-Richardson coefficients. This conjecture was firstly proved by Knutson, Tao and Woodward using the Honeycomb theory (see [KTW04]). A geometric proof was given by Belkale in [Bel07b]. Our proof is based on the geometry of the Horn cones.

1 Introduction

1.1 The Horn conjecture

We start by a question first considered by H. Weyl ([Wey12]) in 1912:

What can be said about the eigenvalues of a sum of two Hermitian matrices, in terms of the eigenvalues of the summands?

Let $H(n)$ denote the set of n by n Hermitian matrices. For $A \in H(n)$, we denote its spectrum by $\alpha(A) = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$ repeated according to multiplicity and ordered such that $\alpha_1 \geq \cdots \geq \alpha_n$. We set

$$\Delta(n) := \{(\alpha(A), \alpha(B), \alpha(C)) \in \mathbb{R}^{3n} \;:\; A,\, B,\, C \in H(n) \text{ s.t. } A+B+C=0\}.$$

In 1962, Horn conjectured in [Hor62] an inductive description of $\Delta(n)$. We now introduce notation to state the Horn conjecture. Set $E(n) = \mathbb{R}^{3n}$, let $E(n)^+$ denote the set of $(\alpha_i, \beta_i, \gamma_i) \in E(n)$ such that $\alpha_i \geq \alpha_{i+1}$, $\beta_i \geq \beta_{i+1}$ and $\gamma_i \geq \gamma_{i+1}$ for all $i = 1, \cdots, n-1$. Because of trace, the points $(\alpha,\, \beta,\, \gamma)$ in $\Delta(n)$ satisfy $\sum_{i=1}^n (\alpha_i + \beta_i + \gamma_i) = 0$. Let $E_0(n)$ denote the hyperplane of $E(n)$ defined by this condition.

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Let $\mathcal{P}(r,n)$ denote the set of subsets of $\{1,\cdots,n\}$ with r elements. To any $I=\{i_1<\cdots<i_r\}\in\mathcal{P}(r,n)$ we usually associate (see Section 4 for details) a partition $\lambda_I=(i_r-r\geq\cdots\geq i_1-1)$. Note that λ_I has (at most) r parts and its first one is less or equal than $n-r$.

The Horn conjecture.([Hor62]) *The point $(\alpha_i,\beta_i,\gamma_i)\in E_0(n)\cap E(n)^+$ belongs to $\Delta(n)$ if and only if for every $r=1,\cdots,n-1$, for every $(I,J,K)\in\mathcal{P}(r,n)^3$ such that*

$$(\lambda_I,\lambda_J,\lambda_K-(n-r)1^r)\in\Delta(r), \tag{1}$$

the following inequality holds:

$$\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j+\sum_{k\in K}\gamma_k\leq 0. \tag{2}$$

Note that this conjecture implies that $\Delta(n)$ is a closed convex polyhedral cone. This fact is a consequence of convexity results in Hamiltonian geometry (see [Kir84]). The combination of a Klyachko theorem ([Kly98]) and of a Knutson-Tao theorem ([KT99]) implies this conjecture (see Section 2 for details).

1.2 Littlewood-Richardson coefficients

Recall that the irreducible representations of $\mathrm{Gl}_r(\mathbb{C})$ (or $U_r(\mathbb{C})$ if one wants to work with a compact Lie group) are indexed by sequences $\lambda=(\lambda_1\geq\cdots\geq\lambda_r)\in\mathbb{Z}^r$ (see for example [FH91, Lecture 6]). Denote the representation corresponding to λ by V_λ. As any representation of $\mathrm{GL}_r(\mathbb{C})$, the tensor product $V_\lambda\otimes V_\mu$ of two given irreducible representations V_λ and V_μ is a sum of irreducible representations. We define the Littlewood-Richardson coefficients $c^\nu_{\lambda\,\mu}\in\mathbb{N}$ as the corresponding multiplicities:

$$V_\lambda\otimes V_\mu=\sum_\nu c^\nu_{\lambda\,\mu}V_\nu. \tag{3}$$

The Knutson-Tao theorem [KT99] was previously known as the

Saturation conjecture. *If, for some $n>0$, $c^{n\nu}_{n\lambda\,n\mu}\neq 0$ then $c^\nu_{\lambda\mu}\neq 0$.*

This note is about another relation between the Horn conjecture and the stretchered Littlewood-Richardson coefficients; namely the following

Fulton conjecture. For any $n > 0$, $c_{\lambda\,\mu}^{\nu} = 1 \;\Rightarrow\; c_{n\lambda\,n\mu}^{n\nu} = 1$.

This conjecture was firstly proved by Knutson, Tao and Woodward [KTW04] using the Honeycomb theory. A geometric proof was given by Belkale in [Bel07b]. The aim of this note is to give a short proof of this conjecture based on the geometry of Horn cones.

The proof of the Horn conjecture is much more involved than its statement. In Section 2, we give an idea of the history of this proof and the subjects interplaying with it. Section 3 is concerned by the codimension one faces of the Horn cones. Sections 2 and 3 are mainly expository; we give proofs only when elementary linear algebra allows it. The last section is our proof of Fulton's conjecture.

2 Schubert calculus and the Horn conjecture

2.1 Schubert calculus

Let $\mathrm{Gr}(a, b)$ be the Grassmann variety of a-dimensional subspaces L of $\mathbb{C}^{a+b}$. Let $F_\bullet$: $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{a+b} = V$ be a complete flag of $\mathbb{C}^{a+b}$ (i.e. F_i is a i-dimensional subspace of V). The relative position of $L \in \mathrm{Gr}(a, b)$ and $F_\bullet$ defines a partition of $\mathrm{Gr}(a, b)$ which is a cellular decomposition and allows to describe the topology of $\mathrm{Gr}(a, b)$. More precisely, for any subset $I = \{i_1 < \cdots < i_a\}$ of cardinal a in $\{1, \cdots, a + b\}$, we define the Schubert variety $\Omega_I(F_\bullet)$ in $\mathrm{Gr}(a, b)$ by

$$\Omega_I(F_\bullet) = \{L \in \mathrm{Gr}(a, b) \;:\; \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The open subset of $\Omega_I(F_\bullet)$ defined by $\dim(L \cap F_{i_j}) = j$ for any j is denoted by $\Omega_I^\circ(F_\bullet)$; it is isomorphic to some affine space. The Poincaré dual of the homology class of $\Omega_I(F_\bullet)$ does not depend on $F_\bullet$; it is denoted by σ_I. The σ_I's form a $\mathbb{Z}$-basis for the cohomology group:

$$H^*(\mathrm{Gr}(a, b), \mathbb{Z}) = \bigoplus_{I \in \mathcal{P}(a, a+b)} \sigma_I.$$

Now, let I, J be in $\mathcal{P}(a, a+b)$. By expanding $\sigma_I.\sigma_J$, we define the structure-coefficients c_{IJ}^K of the cup product in the Schubert basis:

$$\sigma_I.\sigma_J = \sum_K c_{IJ}^K \sigma_K.$$

The class [pt] of the point generates $H^{2ab}(\mathrm{Gr}(a,b),\mathbb{Z})$. We define $K^\vee$ by: $i \in K^\vee$ if and only if $a+b+1-i \in K$. Then, σ_K and $\sigma_{K^\vee}$ are Poincaré dual, that is $\sigma_K.\sigma_{K^\vee} = [\mathrm{pt}]$. So, if the sum of the codimensions of $\Omega_I(F_\bullet)$, $\Omega_J(F_\bullet)$ and $\Omega_I(F_\bullet)$ equals the dimension of $\mathrm{Gr}(a,b)$, we have

$$\sigma_I.\sigma_J.\sigma_K = c_{IJ}^{K^\vee}[\mathrm{pt}].$$

The following result gives a simple interpretation of the integers $c_{IJ}^{K^\vee}$ and in particular shows that they are nonnegative:

Theorem 1 (Kleiman [Kle74]) *We made the above assumption about the codimensions of Ω_I, Ω_J and Ω_K. Then for general flags $F_\bullet$, $G_\bullet$ and $H_\bullet$, we have:*

$$\Omega_I(F_\bullet) \cap \Omega_J(G_\bullet) \cap \Omega_K(H_\bullet) = \Omega_I^\circ(F_\bullet) \cap \Omega_J^\circ(G_\bullet) \cap \Omega_K^\circ(H_\bullet)$$

consists of $c_{IJ}^{K^\vee}$ points.

2.2 Producing inequalities from Schubert calculus

A spectrum or a partition $(\alpha_1 \geq \cdots \geq \alpha_n)$ is said to be *regular* if the α_i's are pairwise distinct. Let A be an $n \times n$ Hermitian matrix with a regular spectrum α. Let $I \in \mathcal{P}(r,n)$. We are going to explain how to express $\sum_{i \in I} \alpha_i$ as an extrema (see inequality (2)).

To A, we associate a complete flag $A_1 \subset \cdots \subset A_{n-1} \subset \mathbb{C}^N$, where A_i is the sum of the i eigenlines of A with the i largest eigenvalues (well defined for α regular). We also consider the following Schubert variety of the Grassmannian $\mathrm{Gr}(r,n)$ of r-dimensional subspaces of $\mathbb{C}^n$:

$$\Omega_I(A) := \{V \in \mathrm{Gr}(r,n) \ : \ \dim(V \cap A_i) \geq \#(I \cap \{1, \cdots, i\}),\, 1 \leq i \leq n\}.$$

For any linear subspace V of $\mathbb{C}^n$ the Rayleigh trace $R_A(V)$ is defined as the trace of the endomorphism $p_V \circ A_{|V}$, where p_V is the orthogonal projection onto V.

Theorem 2 *[HZ62] If the spectrum of A is regular, we have*

$$\min_{V \in \Omega_I(A)} R_A(V) = \sum_{i \in I} \alpha_i(A).$$

Moreover, the minimum is achieved when V is the sum of the corresponding eigenlines.

Let $\Delta^\circ(n)$ denote the set of triples of regular elements in $\Delta(n)$. We now state the first relation between Schubert calculus and the Horn cone.

Theorem 3 ([Tot94, HR95]) *Let I, J and K be such that $c_{IJ}^{K^\vee} \neq 0$. Then, inequality (2) holds for any point in* $\mathrm{Horn}(n)$.

Proof. We admit that $\Delta(n)$ spans $E_0(n)$. This implies that $\Delta^\circ(n)$ is dense in $\Delta(n)$; in particular, it is sufficient to prove the theorem for points in $\Delta^\circ(n)$. Let A, B and C be three Hermitian matrices such that $A+B+C=0$ that have regular spectrum. Since $\sigma_I.\sigma_J.\sigma_K \neq 0$, Theorem 1 implies that

$$\Omega_I(A) \cap \Omega_J(B) \cap \Omega_K(C)$$

is not empty. Let V_0 belong to this intersection. Theorem 2 implies that

$$\begin{aligned}
\varphi_{IJK}(A,B,C) &:= \sum_{i\in I}\alpha_i(A) + \sum_{j\in J}\beta_j(B) + \sum_{k\in K}\gamma_k(C) && (4)\\
&\leq \min_{V\in\Omega_I(A)} R_A(V) + \min_{V\in\Omega_J(B)} R_B(V) + \min_{V\in\Omega_K(C)} R_C(V) && (5)\\
&\leq R_A(V_0) + R_B(V_0) + R_C(V_0) && (6)\\
&\leq R_{A+B+C}(V_0) = 0. && (7)
\end{aligned}$$

$\square$

2.3 A complete set of inequalities from semistability

In 1998, Klyachko proved that the inequalities given by Theorem 3 are sufficient to characterize $\Delta(n)$:

Theorem 4 ([Kly98]) *The point $(\alpha_i, \beta_i, \gamma_i) \in E_0(n) \cap E(n)^+$ belongs to $\Delta(n)$ if and only if for every $r = 1, \cdots, n-1$, for every $(I,J,K) \in \mathcal{P}(r,n)^3$ such that*

$$c_{IJ}^{K^\vee} \neq 0, \tag{8}$$

the following inequality holds:

$$\sum_{i\in I}\alpha_i + \sum_{j\in J}\beta_j + \sum_{k\in K}\gamma_k \leq 0. \tag{9}$$

We are going to explain one ingredient used by Klyachko. Consider the following basic question

Given two irreducible representations V_λ and V_μ of GL_n, what are the irreducible subrepresentations of $V_\lambda \otimes V_\mu$?

Let Λ_n^+ be the set of $\lambda = (\lambda \geq \cdots \geq \lambda_n) \in \mathbb{Z}^n$. We set

$$\mathrm{LR}(\mathrm{GL}_n) = \{(\lambda, \mu, \nu) \in (\Lambda_n^+)^3 \ : \ (V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n} \neq \{0\}\}.$$

Note that $(V_\lambda \otimes V_\mu \otimes V_\nu^*)^{\mathrm{GL}_n} \neq \{0\}$ if and only if $c_{\lambda\,\mu}^{\nu} \neq 0$.

Let $\mathcal{F}l(n)$ denote the variety of complete flags in $\mathbb{C}^n$ acting on by Gl_n. The Borel-Weil Theorem shows that V_λ can be obtain as the module of regular sections of some Gl_n-linearized line bundle $\mathcal{L}_\lambda$. In particular, $(V_\lambda \otimes V_\mu \otimes V_\nu)^{\mathrm{GL}_n} \neq \{0\}$ if and only if some line bundle on $\mathcal{F}l(n)^3$ admits nonzero G-invariant sections. Now, the existence of some positive k such that $(k\lambda, k\mu, k\nu) \in \mathrm{LR}(\mathrm{GL}_n)$ can be interpreted as the existence of semistable points for some action of GL_n. This existence can be checked by linear inequalities using either slopes of vector bundles or the Hilbert-Mumford theorem. In Klyachko's paper, inequalities (9) are understood as inequalities between slopes of toric vector bundles. The Kempf-Ness Theorem ([KN79]) shows that this existence is equivalent to the fact the 0 belongs to the image of a moment map for some Hamiltonian action of U_n. Making this discussion more precise, we finally obtain the following

Theorem 5 *Let (λ, μ, ν) be a triple of nonincreasing sequences of n rational numbers. Then, $(\lambda, \mu, \nu) \in \mathrm{Horn}(n)$ if and only if $(k\lambda, k\mu, k\nu) \in \mathrm{LR}(\mathrm{GL}_n)$ for some positive integer k.*

2.4 The role of the saturation conjecture

Note that the only difference between the Horn conjecture and Theorem 4 is that condition (1) was replaced by condition (8). The inductive nature of condition (1) is mainly explained by Theorem 5 and the following classical Lesieur's result (see [Les47])

$$c_{IJ}^{K} = c_{\lambda_I\,\lambda_J}^{\lambda_K}, \tag{10}$$

where λ_I is defined in the introduction. Putting all these remarks together, the missing piece to obtain the Horn conjecture is precisely the saturation conjecture as stated in the introduction. In 1999, Knutson-Tao proved this conjecture using a new model of the Berenzentein-Zelevinski polytope (namely, the Honeycomb model) to compute the Littlewood-Richardson coefficients. Then, Belkale gave a geometric proof in [Bel06] using mainly the interpretation of the Littlewood-Richardson in terms of the cohomology of the Grassmmannians. Derksen-Weyman gave a proof in [DW00] using an interpretation of the problem in terms of representations of quivers. In [KM08], Kapovich-Millson gave a proof using the Littelmann path model to translate the problem in geometric terms in some Bruhat-Tits buildings.

3 Faces of $\Delta(n)$

3.1 Removing inequalities

For $n = 3$, 4, 5 and 6, the Horn conjecture describes $\Delta(n)$ by respectively $18(= 6+12)$, $50(= 9+41)$, $154(= 12+142)$ and $537(= 15+522)$ inequalities. In the sums, the first term corresponds to the inequalities of $E(n)^+$ and the second one to inequalities (2). Using a computer software on convex geometry, one can check that for $n = 3$, 4, 5 and 6, the cone $\Delta(n)$ by respectively 18, 50, 154 and 536 faces of codimension one. So, the Horn conjecture gives one redundant inequality for $n = 6$. It is

$$\alpha_2 + \alpha_4 + \alpha_6 + \beta_2 + \beta_4 + \beta_6 + \gamma_2 + \gamma_4 + \gamma_6 \leq 0.$$

This inequality corresponds to the coefficient $c_{II}^{I^\vee}$ with $I = \{2, 4, 6\}$ equals to 2. In 2000, Belkale improved Theorem 4 is the following way

Theorem 6 ([Bel01]) *The point $(\alpha_i, \beta_i, \gamma_i) \in E_0(n) \cap E(n)^+$ belongs to $\Delta(n)$ if and only if for every $r = 1, \cdots, n-1$, for every $(I, J, K) \in \mathcal{P}(r, n)^3$ such that*

$$c_{IJ}^{K^\vee} = 1, \tag{11}$$

the following inequality holds:

$$\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j + \sum_{k\in K} \gamma_k \leq 0. \tag{12}$$

We are now going to explain with the material already introduced why Theorem 6 should be true. Let I, J and K such that $c_{IJ}^{K^\vee} \neq 0$. In Theorem 4, we can forget inequality (9) when you saturate it you obtain no point in $\Delta^\circ(n)$. So, let us assume that there exists three Hermitian matrices A, B and C with regular spectrum such that $A + B + C = 0$ and

$$\sum_{i\in I} \alpha_i(A) + \sum_{j\in J} \beta_j(B) + \sum_{k\in K} \gamma_k(C) = 0.$$

Arguing as in the proof of Theorem 3, we obtain that any point V in the intersection $\Omega_I(A) \cap \Omega_J(B) \cap \Omega_K(C)$ satisfies $\sum_{i\in I} \alpha_i(A) = R_A(V)$. Now, Theorem 2 implies that V is the sum of the eigenlines corresponding to I. This proves that $\Omega_I(A)\cap\Omega_J(B)\cap\Omega_K(C)$ is reduced to one point. To obtain Theorem 6, it remains to prove that the intersection is transverse...

3.2 The Knutson-Tao-Woodward Theorem

In 2004, Knutson-Tao-Woodward proved that Theorem 6 is optimal, in the sense that no inequality can be removed.

Theorem 7 *The hyperplanes $\alpha_i = \alpha_{i+1}$, $\beta_i = \beta_{i+1}$ and $\gamma_i = \gamma_{i+1}$ spanned by the codimension one faces of $E(n)^+$ intersects $\Delta(n)$ along faces of codimension one.*

For any I, J and K in $\mathcal{P}(r, n)$ (for some $1 \leq r \leq n-1$) such that $c_{IJ}^{K^\vee} = 1$, the hyperplane $\sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j + \sum_{k\in K} \gamma_k = 0$ intersects $\Delta(n)$ along a face $\mathcal{F}_{IJK}$ of codimension one intersecting $\Delta^\circ(n)$.

The Knutson-Tao-Woodward's proof uses their Honeycomb model. In [Res10], we give an alternative proof using the Geometric Invariant Theory viewpoint. To prove this result, we have to produce points in $\Delta(n)$ which satisfy the equality. In [Res10], these points are interpreted as line bundles on some product of manifolds that have nonzero invariant sections (see Section 2.3). We produce such line bundles by methods of algebraic geometry.

3.3 Description of the faces of $\Delta(n)$

Let I, J and K in $\mathcal{P}(r, n)$. Define the linear isomorphism ρ_{IJK} by:

$$\begin{array}{ccc} E(n) & \longrightarrow & E(r) \oplus E(n-r) \\ (\alpha_i, \beta_i, \gamma_i) & \longmapsto & ((\alpha_i)_{i\in I}, (\beta_i)_{i\in J}, (\gamma_i)_{i\in K}) + ((\alpha_i)_{i\notin I}, (\beta_i)_{i\notin J}, (\gamma_i)_{i\notin K}). \end{array}$$

This isomorphism put together the eigenvalues $(\alpha_i)_{i\in I}$. We assume that $c_{IJ}^{K^\vee} \neq 0$. Then, by Theorem 3 inequality (2) holds for any point in $\Delta(n)$. Consider the associated face (eventually of small dimension):

$$\mathcal{F}_{IJK} = \{(\alpha,\, \beta,\, \gamma) \in \Delta(n)\, :\, \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j + \sum_{k\in K} \gamma_k = 0\}. \tag{13}$$

We can now describe $\mathcal{F}_{IJK}$ in terms of smaller Horn cones. Indeed, we will prove that the points of $\mathcal{F}_{IJK}$ correspond to simultaneously block diagonal matrices as in equation (14).

Proposition 1 *Recall that $c_{IJ}^{K^\vee} \neq 0$. Let $(\alpha, \beta, \gamma) \in E(n)^+$. Then, $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK}$ if and only if $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$.*

Proof. Assume that $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$. Let $A', B', C' \in H(r)$ and $A'', B'', C'' \in H(n-r)$ such that $A' + B' + C' = 0$ and $A'' + B'' + C'' = 0$ whose spectrums correspond to $\rho_{IJK}(\alpha, \beta, \gamma)$. Consider the three following matrices of $H(n)$

$$A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}, \quad B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \quad C = \begin{pmatrix} C' & 0 \\ 0 & C'' \end{pmatrix}. \tag{14}$$

By construction, α is the spectrum of A and $\sum_I \alpha_i = \mathrm{tr}(A')$, and similarly for B and C. We deduce that $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK}$.

Conversely, let $(\alpha, \beta, \gamma) \in \mathcal{F}_{IJK}$. It remains to prove that $\rho_{IJK}(\alpha, \beta, \gamma) \in \Delta(r) \times \Delta(n-r)$. By Theorem 7, $\mathcal{F}_{IJK}$ contains regular triples; in particular, we may assume the α, β and γ are regular.

Let now choose three Hermitian matrices A, B and C with spectrum α, β and γ and such that $A + B + C = 0$. We use notation of the proof of Theorem 3. By assumption, $\varphi_{IJK}(A, B, C) = 0$ and inequality (6) becomes an equality. Thus, $R_A(V_0) = \min_{V \in \Omega_I(A)} R_A(V_0)$. Now, Theorem 2 implies that V_0 is the sum of the eigenlines of A corresponding to I. Similarly, V_0 is stable by B and C and the spectrum of the restrictions are respectively $(\beta_j)_{j \in J}$ and $(\gamma_k)_{k \in K}$. We deduce that $((\alpha_i)_{i \in I}, (\beta_i)_{i \in J}, (\gamma_i)_{i \in K})$ belongs to $\Delta(r)$. By considering the restrictions of A, B and C to the orthogonal subspace of V_0, we obtain similarly that $((\alpha_i)_{i \notin I}, (\beta_i)_{i \notin J}, (\gamma_i)_{i \notin K})$ belongs to $\Delta(n-r)$. $\square$

Let $E(n)^{++}$ denote the open convex cone in $E(n)$ consisting of regular triples in $E(n)^+$.

Corollary 1 *Let I, J and K be as in the proposition. Then, if $\mathcal{F}_{IJK}$ contains regular triples, it has codimension one. In particular, $c_{IJ}^{K^\vee} = 1$.*

Proof. By Proposition 1, $\mathcal{F}_{IJK} \cap E(n)^{++}$ is isomorphic to an open subset of $\Delta(r) \times \Delta(n-r)$. So, $\mathcal{F}_{IJK}$ has codimension 2 in $E(n)$ and so codimension one in $\Delta(n)$. Now, Theorem 6 implies that $c_{IJ}^{K^\vee} = 1$. $\square$

Remark. Corollary 1 is proved in [Res10, Theorem 8] by purely Geometric Invariant Theoretic methods.

4 Proof of the Fulton conjecture

Let λ, μ and ν be three partitions (with at most r parts) such that $c^{\nu}_{\lambda\,\mu} = 1$ and N be a positive integer. We have to prove that $c^{N\nu}_{N\lambda\,N\mu} = 1$.

Strategy of the proof. The fact that $c^{N\nu}_{N\lambda\,N\mu} \neq 0$ is a direct consequence of the Borel-Weil Theorem. By the Lesieur Theorem (see equation (10)) and Theorem 7, the coefficient $c^{\nu}_{\lambda\,\mu}$ equal to one corresponds to a face $\mathcal{F}$ of some Horn cone. By interpreting the conclusion $c^{N\nu}_{N\lambda\,N\mu} = 1$ in similar terms, we have to prove that a certain face of some Horn cone has also codimension one. To produce points on this face becomes a game with block diagonal matrices.

In the paragraph just before Theorem 5, we already mentioned that by the Borel-Weil Theorem, if $c^{\nu}_{\lambda\,\mu} = 1$ then there exists some Gl_r-invariant section σ of some line bundle $\mathcal{L}$ on $\mathcal{F}l(r)^3$. The fact that $\sigma^{\otimes N}$ is a Gl_r-invariant section of $\mathcal{L}^{\otimes N}$ implies that $c^{N\nu}_{N\lambda\,N\mu} \neq 0$.

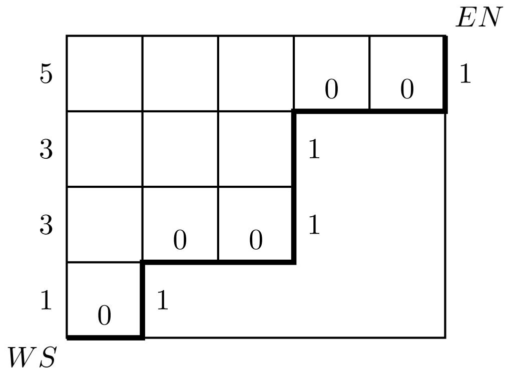
We draw the three partitions λ, μ and ν in a same rectangle: we fix an integer n such that $n - r$ is greater or equal to λ_1, μ_1 and ν_1. Set $I = \{n - r + i - \lambda_i \,:\, i = 1, \cdots, a\} \in \mathcal{P}(r, n)$ in such a way $\lambda_I = \lambda$ with the notation of the introduction. Similarly, we associate J and K to μ and ν. By equality (10), we have $c^{K}_{IJ} = 1$.

By Theorem 7, $\mathcal{F}_{IJK^\vee}$ is a face of codimension one of $\Delta(n)$. Let (A, B, C) (resp. (A', B', C')) be three Hermitian matrices of size r (resp. $n - r$) such that $A + B + C = 0$ and $A' + B' + C' = 0$. We assume that their spectrum belong to the relative interior of $\rho_{IJK^\vee}(\mathcal{F}_{IJK^\vee})$.

Let now I'', J'' and K'' be the three subsets of $r + N(n - r)$ of cardinal r corresponding to the three partitions $N\lambda$, $N\mu$ and $N\nu$ whose the Young diagram is contained is the rectangle with r lines and $N(n - r)$ columns. Since $c^{N\nu}_{N\lambda\,N\mu} \neq 0$, we can consider the face $\mathcal{F}_{I''J''K''^\vee}$ of $\Delta(r+N(n-r))$ as in Proposition 1. By Corollary 1, it remains to prove that $\mathcal{F}_{I''J''K''^\vee}$ intersects $E(r + N(n - r))^{++}$.

Consider N generic perturbations (A'_i, B'_i, C'_i) of (A', B', C') satisfying $A'_i + B'_i + C'_i = 0$. Consider now the block diagonal matrix A'' of size $r + N(n-r)$ with blocks A, $A'_1, \cdots, A'_N$; and similarly B'' and C''. We have $A'' + B'' + C'' = 0$.

It remains to prove that the point of $\Delta(r + N(n - r))$ corresponding to (A'', B'', C'') belongs to $\mathcal{F}_{I''J''K''^\vee}$. By Proposition 1, it is sufficient to prove that the spectrum of A (resp. B and C) consists of the eigenvalues of A''



$r = 4$, $n = 9$
$\lambda_I = (5 \leq 3 \leq 3 \leq 1)$
$\chi_I = 010011001$
$I = \{2, 5, 6, 9\}$.

Figure 1: From λ_I to I

(resp. B'' and C'') indexed by I'' (resp. J'' and $K''^\vee$).

Let us explain more how to recover I from λ_I. First, draw the Young diagram of λ_I. Look the path from WS to EN; its has length n. Mark each horizontal step by 0 and each vertical step by 1. We just obtained a word of length n containing r 1's: its is the characteristic function χ_I of I. We illustrate this remark by Figure 1. This description of the map $\lambda_I \mapsto I$ implies that $\chi_{I''}$ is obtained from χ_I by replacing each 0 by N ones.

Now, the spectrum of A'' is obtained from the spectrum of A by replacing each eigenvalue between two ones indexed by I by N closed eigenvalues. We deduce that $(\alpha(A'')_i)_{i\in I''} = (\alpha(A)_i)_{i\in I}$. This implies that $(\alpha(A''), \alpha(B''), \alpha(C''))$ belongs to $\mathcal{F}_{I''J''K''^\vee}$, ending the proof of Fulton's conjecture.

Remark. As pointed out by P. Belkale the construction of A'' is closed to the construction of $\mathcal{W}(N)$ in [Bel07a, p11].

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