INVARIANT DEFORMATIONS OF ORBIT CLOSURES IN $\mathfrak{sl}(n)$

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Abstract

We study deformations of orbit closures for the action of a connected semisimple group G on its Lie algebra $\mathfrak{g}$, especially when G is the special linear group.

The tools we use are on the one hand the invariant Hilbert scheme and on the other hand the sheets of $\mathfrak{g}$. We show that when G is the special linear group, the connected components of the invariant Hilbert schemes we get are the geometric quotients of the sheets of $\mathfrak{g}$. These quotients were constructed by Katsylo for a general semisimple Lie algebra $\mathfrak{g}$; in our case, they happen to be affine spaces.

Introduction

Let G be a complex reductive group, and V be a finite dimensional G-module. A fondamental problem is to endow some sets of orbits of G in V with a structure of variety. The geometric invariant theory is the classical answer in this context: the set of closed orbits of G in V has a natural structure of affine variety. We denote by $V/\!/G$ this variety, equipped with a G-invariant quotient map $\pi : V \to V/\!/G$.

Recently, Alexeev and Brion defined in [AB] a structure of quasiprojective scheme on some sets of G-stable closed affine subscheme of V. A natural question is to wonder what happens when one applies Alexeev-Brion's construction to the orbit closures of G in V. Here, we study this construction in the case of a well known G-module, namely the adjoint representation of a semisimple group G, especially when G is the special linear group $\mathrm{SL}(n)$.

From now on, we assume that G is semisimple, and denote by $\mathfrak{g}$ its Lie algebra. Let us recall that a sheet of $\mathfrak{g}$ is an irreducible component of the set of points in $\mathfrak{g}$ whose G-orbit has a fixed dimension. Let us fix a sheet $\mathcal{S}$. We show that the G-module structure on the affine algebra $\mathbb{C}[\overline{G \cdot x}]$ of the orbit closure $\overline{G \cdot x}$ of x doesn't depend on x in $\mathcal{S}$. This allows us to define a set-theoretical application from $\mathcal{S}$ to some Alexeev-Brion's invariant Hilbert scheme of V:

$$\begin{array}{rccc} \pi_{\mathcal{S}} : & \mathcal{S} & \longrightarrow & \mathrm{Hilb}^G_{\mathcal{S}}(V) \\ & x & \longmapsto & \overline{G \cdot x}. \end{array}$$

A unique sheet is open in $\mathfrak{g}$: we call it the *regular* one, and denote it by $\mathfrak{g}_{\mathrm{reg}}$.

Firstly, we prove that $\mathrm{Hilb}^G_{\mathfrak{g}_{\mathrm{reg}}}(V)$ is canonically isomorphic to the categorical quotient $V/\!/G$. Moreover, via this isomorphism, the application $\pi_{\mathfrak{g}_{\mathrm{reg}}}$ identifies with the restriction of the quotient map $\pi : V \to V/\!/G$; in particular, it is a morphism.

In a second part, we study any sheet $\mathcal{S}$ for $G = \mathrm{SL}(n)$. Let us denote by $\pi : \mathcal{S} \to \mathcal{S}/\mathrm{SL}(n)$ the geometric quotient of $\mathcal{S}$, constructed by Katsylo in [Ka]. We show that there is a canonical morphism

$$\begin{array}{rccc} \theta : & \mathcal{S}/\mathrm{SL}(n) & \longrightarrow & \mathrm{Hilb}_{\mathcal{S}}^{\mathrm{SL}(n)}(V) \\ & \mathrm{SL}(n)\cdot x & \longmapsto & \overline{\mathrm{SL}(n)\cdot x} \end{array}$$

which is actually an isomorphism onto a connected component of $\mathrm{Hilb}_{\mathcal{S}}^{\mathrm{SL}(n)}(V)$.

Another motivation for this work is to understand examples of invariant Hilbert schemes. Indeed, the construction of Alexeev and Brion is indirect and only few examples are known (see [J], [BC]). Here, the connected components of invariant Hilbert schemes we obtain happen to be affine spaces, as in [J] and [BC]. Note that this answers in the case of $\mathrm{SL}(n)$ to a question of Katsylo who asked if the geometric quotient $\mathcal{S}/G$ is normal.

1 Hilbert's sheets

We consider schemes and affine algebraic groups over $\mathbb{C}$. Let G be a connected semisimple group. We choose a Borel subgroup B, and a maximal torus T contained in B. We denote by U the unipotent radical of B; we have $B = TU$.

We denote by Λ the character group of T. We denote by Λ^+ the set of elements of Λ that are dominant weights with respect to B. The set Λ^+ is in bijection with the set of isomorphism classes of simple rational G-modules. If λ is an element of Λ^+, we denote by $V(\lambda)$ a simple G-module associated, that is of highest weight λ.

If V is a rational G-module, we denote by $V_{(\lambda)}$ its isotypical component of type λ, that is the sum of its submodules isomorphic to $V(\lambda)$. We have the decomposition $V = \bigoplus_{\lambda\in\Lambda^+} V_{(\lambda)}$.

In any decomposition of V as a direct sum of simple modules, the multiplicity of the simple module $V(\lambda)$ is the dimension of $V_{(\lambda)}^U$. We say that V *has finite multiplicities* if these multiplicities are finite (for any dominant weight λ).

Let us recall some definitions from [AB, §1]. A *family of affine G-schemes* over some scheme S is a scheme $\mathfrak{X}$ equipped with an action of G and with a morphism $\pi : \mathfrak{X} \to S$ that is affine, of finite type and G-invariant. We have a G-equivariant morphism of $\mathcal{O}_S$-modules

$$\pi_*\mathcal{O}_{\mathfrak{X}} \simeq \bigoplus_{\lambda\in\Lambda^+} \mathcal{F}_\lambda \otimes_{\mathbb{C}} V(\lambda),$$

where each $\mathcal{F}_\lambda := (\pi_*\mathcal{O}_{\mathfrak{X}})_{(\lambda)}^U$ is equipped with the trivial action of G. Let $h : \Lambda^+ \to \mathbb{N}$ be a function. The family $\mathfrak{X}$ is said to be *of Hilbert function h* if each $\mathcal{F}_\lambda$ is an $\mathcal{O}_S$-module locally free of rank $h(\lambda)$. (Then the morphism π is flat.)

Let X be an affine G-scheme, and $h : \Lambda^+ \longrightarrow \mathbb{N}$ a function. A *family of G-stable closed subschemes of X* over some scheme S is a G-stable closed subscheme $\mathfrak{X} \subseteq S \times X$. The projection $S \times X \to S$ induces a family of affine G-schemes $\mathfrak{X} \to S$. The contravariant functor: $(\mathrm{Schemes})^\circ \longrightarrow (\mathrm{Sets})$ that associates to every scheme S the set of families $\mathfrak{X} \subseteq S \times X$ of Hilbert function h is represented by a quasiprojective scheme denoted by $Hilb_h^G(X)$ ([AB, §1.2].

The dimension of an affine G-scheme whose affine algebra has finite multiplicities can be read on its Hilbert function:

Proposition 1.1. *Let $h : \Lambda^+ \longrightarrow \mathbb{N}$ be a function. Let Y and Z be two affine schemes of Hilbert function h. Then* $\dim Y = \dim Z$.

Proof. Let us denote by A the affine ring of Y.

If Y is *horospherical*, that is ([AB, Lemma 2.4]) if for any dominant weights λ, μ, we have $A_{(\lambda)} \cdot A_{(\mu)} \subseteq A_{(\lambda+\mu)}$, it is clear that the dimension of Y can be read on its Hilbert function. Indeed, let us denote by θ_0 the linear map from $\Lambda \otimes \mathbb{Q}$ to $\mathbb{Q}$ which associates to any fundamental weight the value 1. We denote by θ the group homomorphism from Λ to $\mathbb{Z}$ that is the restriction of θ_0. We associate to θ a graduation of the algebra A by $\mathbb{N}$: its homogeneous component of degree d is

$$A_d := \bigoplus_{\lambda \in \Lambda^+,\ \theta(\lambda)=d} A_{(\lambda)}.$$

The dimension of A_d is finite, and depends only on h:

$$\dim A_d = \sum_{\lambda \in \Lambda^+,\ \theta(\lambda)=d} h(\lambda) \dim V(\lambda).$$

So the Hilbert polynomial of the graded algebra A depends only on h, and so does the dimension of Y.

We can deduce the proposition. Indeed, Y admits a flat degeneration over a connected scheme to a horospherical G-scheme Y' that admits the same Hilbert function (by [AB, Theorem 2.7]). So $\dim Y = \dim Y'$ depends only on h. $\square$

We will use the method of "asymptotic cones" of Borho and Kraft ([PV, §5.2]): let V be a finite dimensional rational G-module and F the closure of an orbit in V (or, more generally, any G-stable closed subvariety contained in a fiber of the categorical quotient $V \to V/\!/G$). We embbed V into the projective space $\mathbb{P}(\mathbb{C} \oplus V)$ of vector lines of $\mathbb{C} \oplus V$ by the inclusion $v \mapsto [1 \oplus v]$. The closure of F in $\mathbb{P}(\mathbb{C} \oplus V)$ is denoted by $\overline{F}$. The affine cone in $\mathbb{C} \oplus V$ over $\overline{F}$ is the closed cone $\mathfrak{X}$ generated by F.

The vector space $\mathbb{C} \oplus V$, equipped with its natural scheme structure, is denoted by $\mathbb{A}^1 \times V$. The cone $\mathfrak{X} \subseteq \mathbb{A}^1 \times V$, viewed as a reduced closed subscheme, is a flat family of affine G-schemes. Its fibers over non-zero elements are homothetic to F. Its fiber over 0 is a reduced cone, denoted by $\hat{F}$. It is contained in the null-cone of V (that is the fiber of the categorical quotient $V \to V/\!/G$ containing 0). Its dimension is the same as F.

We consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. If x is an element of $\mathfrak{g}$, the affine algebra of the closure of its orbit, viewed as a reduced scheme, has finite multiplicities. Let us denote by h_x its Hilbert function; we call it the Hilbert function associated to x. In this paper, we are interested in the connected component denoted Hilb_x^G of the scheme $\mathrm{Hilb}_{h_x}^G(\mathfrak{g})$ that contains

$\overline{G \cdot x}$. It gives the G-invariant deformations of $\overline{G \cdot x}$ embedded in $\mathfrak{g}$. We determine it when x is in $\mathfrak{g}_{\text{reg}}$ in §2, and for any x when G is the special linear group in §3.

Let us denote by G_x the stabilizer of x in G, and $\mathfrak{g}_x$ its Lie algebra. The coadjoint action of G_x is its natural action on the dual vector space $\mathfrak{g}_x^*$.

Proposition 1.2. *Let us assume the orbit closure $\overline{G \cdot x}$ to be normal. The tangent space $T_{\overline{G \cdot x}} \operatorname{Hilb}_x^G$ to Hilb_x^G at the point $\overline{G \cdot x}$ is canonically isomorphic to the space of invariants of the coadjoint action of G_x.*

Proof. The tangent space to $\overline{G \cdot x}$ at the point x is $\mathfrak{g}.x$; it is stable under the action of G_x. We denote by $[\mathfrak{g}/\mathfrak{g}.x]^{G_x}$ the space of invariants under the action of G_x on the quotient vector space $\mathfrak{g}/\mathfrak{g}.x$. According to [AB, Proposition 1.15 (iii)], we have a canonical isomorphism

$$T_{\overline{G \cdot x}} \operatorname{Hilb}_x^G \cong [\mathfrak{g}/\mathfrak{g}.x]^{G_x}. \tag{1}$$

Indeed, the orbit closure $\overline{G \cdot x}$ is assumed to be normal. Moreover, every orbit in $\mathfrak{g}$ has even dimension, and has a finite number of orbits in its closure ([PV, Corollary 3 page 198]), so the codimension of the boundary of $G \cdot x$ in $\overline{G \cdot x}$ is at least 2, and the proposition of [AB] can be applied.

To transform (1) into the isomorphism of the proposition, we will use the Killing form on $\mathfrak{g}$, denoted by κ. As $\mathfrak{g}$ is semisimple, its Killing form gives an isomorphism

$$\begin{array}{rccc} \phi: & \mathfrak{g} & \longrightarrow & \mathfrak{g}^* \\ & y & \longmapsto & \kappa(y, \cdot). \end{array}$$

The isomorphism ϕ is G-equivariant, thus G_x-equivariant. It sends $\mathfrak{g}.x$ onto the space $\mathfrak{g}_x^\perp$ of linear forms on $\mathfrak{g}$ that vanish on $\mathfrak{g}_x$. Indeed, the common zeros of the elements of $\phi(\mathfrak{g}.x)$ are the elements y in $\mathfrak{g}$ such that

$$\forall z \in \mathfrak{g},\ \kappa([z, x], y) = 0,$$

that is

$$\forall z \in \mathfrak{g},\ \kappa(z, [x, y]) = 0,$$

and this last condition means that y belongs to $\mathfrak{g}_x$ since κ is non-degenerate.

Thus the short exact sequence of G_x-modules

$$0 \longrightarrow \mathfrak{g}.x \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}.x \longrightarrow 0$$

identifies (thanks to ϕ) with

$$0 \longrightarrow \mathfrak{g}_x^\perp \longrightarrow \mathfrak{g}^* \longrightarrow (\mathfrak{g}_x)^* \longrightarrow 0,$$

and the proposition follows from (1). $\square$

A *sheet* of $\mathfrak{g}$ is a maximal irreducible subset of $\mathfrak{g}$ consisting of G-orbits of a fixed dimension. Every sheet of $\mathfrak{g}$ contains a unique nilpotent orbit. A *regular element* of $\mathfrak{g}$ is an element of $\mathfrak{g}$ whose orbit has maximal dimension. The open subset of $\mathfrak{g}$ whose elements are the regular elements is a sheet denoted by $\mathfrak{g}_{\text{reg}}$.

Let us call *Hilbert's sheet* a maximal irreducible subset of $\mathfrak{g}$ consisting of elements admitting a fixed associated Hilbert function.

Proposition 1.3. *The Hilbert's sheets of $\mathfrak{g}$ coincide with its sheets.*

Proof. According to Proposition 1.1, any Hilbert's sheet is contained in some sheet. It just remains to check that two points of some sheet $\mathcal{S}$ have the same associated Hilbert function.

Let F be the closure of an orbit in $\mathcal{S}$. We recalled that its asymptotic cone $\hat{F}$ is a degeneration of F. In particular, it is contained in the closure of $\mathcal{S}$. Moreover, $\hat{F}$ is contained in the null-cone of $\mathfrak{g}$, and its dimension is the same as F. So $\hat{F}$ is the closure of the nilpotent orbit of $\mathcal{S}$.

The affine algebra of $\mathfrak{g}$ is the symmetric algebra of $\mathfrak{g}^*$. Its graduation induces a G-invariant filtration on the affine algebra A of F. The affine algebra of the asymptotic cone $\hat{F}$ is isomorphic, as an algebra equipped with an action of G, to the graded algebra $\hat{A}$ associated to the filtered algebra A. In particular, A and $\hat{A}$ are isomorphic as G-modules, and their multiplicities are equal: the Hilbert function of F is equal to that of $\hat{F}$, and the proposition is proved. $\square$

2 Regular case

Let us denote by h_{reg} the Hilbert function associated to the regular elements of $\mathfrak{g}$ (Proposition 1.3). In this section, we prove that the invariant Hilbert scheme $\mathrm{H}_{\text{reg}} := \mathrm{Hilb}^G_{h_{\text{reg}}}(\mathfrak{g})$ is the categorical quotient $\mathfrak{g}/\!/G$, that is an affine space whose dimension is the rank of G.

2.1 A morphism from $\mathfrak{g}/\!/G$ to H_{reg}

Let $\mathfrak{X}_{\text{reg}}$ be the graph of the canonical projection $\mathfrak{g} \to \mathfrak{g}/\!/G$. It is a family of G-stable closed subschemes of $\mathfrak{g}$ over $\mathfrak{g}/\!/G$.

Proposition 2.1. *The closed subscheme $\mathfrak{X}_{\text{reg}}$ is a family of G-stable closed subschemes of $\mathfrak{g}$ with Hilbert function h_{reg}.*

Proof. Let us denote by $\pi : \mathfrak{X}_{\text{reg}} \to \mathfrak{g}/\!/G$ the canonical projection, and by $\mathcal{R} := \pi_* \mathcal{O}_{\mathfrak{X}_{\text{reg}}}$ the direct image by π of the structural sheaf of $\mathfrak{X}_{\text{reg}}$. We have to prove that for any dominant weight λ, we have that $\mathcal{R}^U_{(\lambda)}$ is a locally free sheaf on $\mathfrak{g}/\!/G$ of rank $h(\lambda)$.

Let us first study the case where $\lambda = 0$. The morphism $\pi/\!/G : \mathfrak{X}_{\text{reg}}/\!/G \to \mathfrak{g}/\!/G$ induced by π is clearly an isomorphism. So $\mathcal{R}^G = \mathcal{R}^U_{(0)}$ is a free module on $\mathfrak{g}/\!/G$ of rank $1 = h_{\text{reg}}(0)$.

Let λ be a dominant weight. It is known (see [AB, Lemma 1.2]) that $\mathcal{R}^U_{(\lambda)}$ is a coherent $\mathcal{R}^G$-module. Thus it is a coherent module on $\mathfrak{g}/\!/G$. To see that it is locally free, we just have to check that its rank is constant. The fibers of π are those of the canonical projection $\mathfrak{g} \to \mathfrak{g}/\!/G$, so they are the orbit closures of the regular elements, and all of them admit h_{reg} as Hilbert function. So the rank of $\mathcal{R}^U_{(\lambda)}$ at any closed point of $\mathfrak{g}/\!/G$ is $h(\lambda)$, and the proposition is proved. $\square$

This gives us a canonical morphism

$$\phi_{\text{reg}} : \mathfrak{g}/\!/G \longrightarrow \mathrm{H}_{\text{reg}}.$$

We will prove in the following of §2 that ϕ_{reg} is an isomorphism.

Lemma 2.2. *The morphism ϕ_{reg} realises a bijection from the set of closed points of $\mathfrak{g}//G$ to the set of closed points of* $\mathrm{H}_{\mathrm{reg}}$.

Proof. We remark that ϕ_{reg} is injective. Let us check it is surjective: in other words, that any G-invariant closed subscheme of $\mathfrak{g}$ of Hilbert function h_{reg} is a fiber of $\mathfrak{g} \to \mathfrak{g}//G$.

Let Y be such a subscheme. As $h_{\mathrm{reg}}(0) = 1$, it has to be contained in some fiber F of $\mathfrak{g} \to \mathfrak{g}//G$ over a reduced closed point. But F already corresponds to a closed point of $\mathrm{H}_{\mathrm{reg}}$ in the image of ϕ_{reg}. Moreover, F admits no proper closed subscheme admitting the same Hilbert function, so $F = Y$, and the lemma is proved. $\square$

Let us denote by r the rank of G. The quotient $\mathfrak{g}//G$ is an affine space of dimension r. A consequence of Lemma 2.2 is:

Corollary 2.3. *The dimension of* $\mathrm{H}_{\mathrm{reg}}$ *is r.*

2.2 Tangent space

In this section, we prove:

Proposition 2.4. *The scheme* $\mathrm{H}_{\mathrm{reg}}$ *is smooth.*

Proof. Let Z be a closed point of $\mathrm{H}_{\mathrm{reg}}$. We have to prove that the dimension of the tangent space $T_Z\mathrm{H}_{\mathrm{reg}}$ is r. We still denote by Z the closed subscheme of $\mathfrak{g}$ corresponding to Z. By Lemma 2.2, we know that Z is a fiber of the morphism $\mathfrak{g} \to \mathfrak{g}//G$, thus the closure of some regular element x. It is a normal variety. By Proposition 1.2, we have to prove that the dimension of

$$(\mathfrak{g}_x^*)^{G_x}$$

is r, or simply that it is lower or equal to r (by Corollary 2.3).

Let us prove that the dimension of the bigger space

$$(\mathfrak{g}_x^*)^{\mathfrak{g}_x}$$

is r, and the proposition will be proved.

A linear form on $\mathfrak{g}_x$ is $\mathfrak{g}_x$-invariant iff it vanishes on the derived algebra $[\mathfrak{g}_x, \mathfrak{g}_x]$, so we have to prove that

$$(\mathfrak{g}_x/[\mathfrak{g}_x, \mathfrak{g}_x])^*$$

is r-dimensional. We will prove that $\mathfrak{g}_x$ is an r-dimensional abelian algebra, and the proposition will be proved. This is true if x is semisimple, because then $\mathfrak{g}_x$ is a Cartan subalgebra of $\mathfrak{g}$. If the regular element x is not assumed to be semisimple, the dimension of $\mathfrak{g}_x$ is still r, because this doesn't depend on the regular element x, by definition. Let us check that $\mathfrak{g}_x$ is abelian.

Let us denote by $\mathrm{Grass}_r(\mathfrak{g})$ the grassmannian of r-dimensional subspaces of $\mathfrak{g}$, endowed with its projective variety structure. The subset of $\mathfrak{g}_{\mathrm{reg}} \times \mathrm{Grass}_r(\mathfrak{g})$:

$$\{(z, \mathfrak{h}) \in \mathfrak{g}_{\mathrm{reg}} \times \mathrm{Grass}_r(\mathfrak{g}) \mid \mathfrak{h} \cdot z = 0 \text{ and } [\mathfrak{h}, \mathfrak{h}] = 0\}$$

is closed, so its image by the natural projection into $\mathfrak{g}_{\mathrm{reg}}$ is closed too. As its image contains the semisimple elements of $\mathfrak{g}_{\mathrm{reg}}$, it is equal to $\mathfrak{g}_{\mathrm{reg}}$. Thus $\mathfrak{g}_x$ is abelian for any regular x, and the proposition is proved. $\square$

2.3 Conclusion

We can now conclude that the family $\mathfrak{X}_{\mathrm{reg}}$ of Proposition 2.1 is the universal family:

Theorem 2.5. *The morphism ϕ_{reg} from $\mathfrak{g}/\!/G$ to $\mathrm{H}_{\mathrm{reg}}$ is an isomorphism.*

Proof. The morphism ϕ_{reg} is bijective (Lemma 2.2) and $\mathrm{H}_{\mathrm{reg}}$ is normal. According to Zariski's main theorem, ϕ_{reg} is an isomorphism. $\square$

Remark 2.6. One knows there is a canonical morphism

$$\psi_{\mathrm{reg}} : \mathrm{H}_{\mathrm{reg}} \longrightarrow \mathfrak{g}/\!/G$$

that associates to any closed point F of $\mathrm{H}_{\mathrm{reg}}$ (viewed as a closed subscheme of $\mathfrak{g}$) its categorical quotient $F/\!/G$ (viewed as a closed point of $\mathfrak{g}/\!/G$). This morphism is a particular case of morphism

$$\eta : \mathrm{Hilb}_h^G(V) \longrightarrow \mathrm{Hilb}_{h(0)}(V/\!/G)$$

defined in [AB, §1.2], because $h_{\mathrm{reg}}(0) = 1$ and thus the punctual Hilbert scheme that parameterizes closed subschemes of length 1 in $\mathfrak{g}/\!/G$ identifies with $\mathfrak{g}/\!/G$ itself. The morphism ψ_{reg} is clearly the inverse morphism of ϕ_{reg}.

3 Case of $\mathfrak{sl}(n)$

We denote by t an indeterminate over $\mathbb{C}$, and I_n the identity matrix of size $n \times n$. If x is an element of $\mathfrak{sl}(n)$ and $i = 1 \cdots n$, we denote by $Q_i^x(t)$ the monic greatest common divisor (in the ring $\mathbb{C}[t]$) of the $(n+1-i) \times (n+1-i)$-sized minors of $x - tI_n$, and $Q_{n+1}^x(t) := 1$.

Then we put

$$q_i^x(t) := Q_i^x(t)/Q_{i+1}^x(t).$$

The polynomials $q_1^x(t), \cdots, q_n^x(t)$ are the invariant factors of the matrix $x - tI_n$ with coefficients in the euclidean ring $\mathbb{C}[t]$, ordered in such a way that $q_{i+1}^x(t)$ divides $q_i^x(t)$.

If x, y are elements of $\mathfrak{sl}(n)$, then y is in the closure of the orbit $\mathrm{SL}(n) \cdot x$ of x if and only if for any $i = 1 \ldots n$, the polynomial $Q_i^x(t)$ divides $Q_i^y(t)$. In other words, iff for any i, the polynomial $Q_i^x(t)$ divides the $(n+1-i) \times (n+1-i)$-sized minors of $y - tI_n$.

According to [W], when x is nilpotent, these conditions defines the closure of $\mathrm{SL}(n) \cdot x$ as a reduced scheme: to be more precise, when one divides a $(n+1-i) \times (n+1-i)$-sized minor of $y - tI_n$ by $Q_i^x(t)$ using Euclid algorithm, the remainder he gets is a regular function of y. All such functions generate the ideal of the closure of $\mathrm{SL}(n) \cdot x$. We will deduce easily from this difficult result that the same remains true if x is no longer assumed to be nilpotent.

One sees easily that the set of sheets of $\mathfrak{sl}(n)$ is in bijection with the set of partitions n, that is of sequences $\sigma = (b_1 \geq b_2 \geq b_3 \geq \dots)$ of nonnegative integers such that $b_1 + b_2 + b_3 + \cdots = n$. Namely, if σ is a partition of n, the elements of the correspondant sheet $\mathcal{S}_\sigma$ are those elements x

such that for any i, the polynomial $q_i^x(t)$ is of degree b_i. We denote by $\widehat{\sigma} = (c_1 \geq c_2 \geq c_3 \geq \dots)$ the conjugate partition, where c_j is the number of i such that $b_i \geq j$. We denote by h_σ the Hilbert function associated to the points of $\mathcal{S}_\sigma$ (Proposition 1.3). We denote by Z_σ the closure of the nilpotent orbit of $\mathcal{S}_\sigma$. The connected component of $\mathrm{Hilb}_{h_\sigma}^{\mathrm{SL}(n)}(\mathfrak{sl}(n))$ that contains Z_σ as a closed point is denoted H_σ. We will prove in this section that H_σ is an affine space of dimension $b_1 - 1$. The proof is similar to §2.

We recall that the sheets of $\mathfrak{sl}(n)$ are smooth ([Kr]).

3.1 A construction of the geometric quotient of $\mathcal{S}_\sigma$

Katslylo showed in [Ka] that any sheet of a semisimple Lie algebra admits a geometric quotient. Although his proof contains an explicit construction, it doesn't make clear the geometric properties of the quotient. Here we present a simple description of the quotient in the case of the Lie algebra $\mathfrak{sl}(n)$. It takes on the invariant factors theory. We get that the quotient is an affine space.

Lemma 3.1. *Given some i, the application $\mathcal{S}_\sigma \longrightarrow \mathbb{A}^{b_i}$ that associates to any x the coefficients of $q_i^x(t) = t^{b_i} + \lambda_{b_i-1}^x t^{b_i-1} + \cdots + \lambda_0^x t^0$ is regular.*

Proof. Up to scalar multiplication, the polynomial $q_i^x(t)$ is the unique nonzero polynomial of degree less or equal to b_i such that

$$\dim \ker q_i^x(x) \geq N := \sum_{j=1}^{b_i} c_j. \tag{2}$$

Thus the closed subset of $\mathcal{S}_\sigma \times \mathbb{P}^{b_i}$ consisting of elements $(x, [\mu_0 : \cdots : \mu_{b_i}])$ such that

$$\dim \ker(\sum_{j=0}^{b_i} \mu_j x^j) \geq N$$

is the graph of the application

$$\begin{array}{rccc} \psi : & \mathcal{S}_\sigma & \longrightarrow & \mathbb{P}^{b_i} \\ & x & \longmapsto & [\lambda_0^x : \cdots : \lambda_{b_i-1}^x : 1] \end{array}$$

According to [Hr, Exercise 7.8 p 76], this graph is also the graph of a rational map ϕ from $\mathcal{S}_\sigma$ to $\mathbb{P}^{b_i}$. On the open subset Ω of $\mathcal{S}_\sigma$ where ϕ is regular, ϕ coincides with ψ, so the functions $x \mapsto \lambda_j^x$ are regular functions from Ω to $\mathbb{A}^1$. As $\mathcal{S}_\sigma$ is smooth, the complementary of Ω in $\mathcal{S}_\sigma$ has codimension at least 2 ([S, Thm 3 chap II.3.1]). We conclude that the functions extend to regular functions from $\mathcal{S}_\sigma$ to $\mathbb{A}^1$. By continuity, these extensions satisfy (2), so they coincide with the functions $x \mapsto \lambda_j^x$ on $\mathcal{S}_\sigma$. $\square$

Let us define, for any x in $\mathcal{S}_\sigma$, the monic polynomial of degree $b_i - b_{i+1}$:

$$p_i^x(t) := q_i^x(t)/q_{i+1}^x(t)$$

(where $q_{n+1}^x := 1$). It follows from the previous lemma that its coefficients, viewed as functions of x, are regular functions from $\mathcal{S}_\sigma$ to $\mathbb{A}^1$.

Given an x, the family $(p_1^x(t), \dots, p_n^x(t))$ can be any family of monic polynomials of degrees $b_i - b_{i+1}$, provided the following relation is satisfied, where $S(p_i^x)$ denotes the sum of the roots of p_i^x, counted with multiplicities (given by its first nondominant coefficient):

$$\sum_{i=1}^{n} iS(p_i^x) = 0$$

(this relation simply means that the trace of x is zero).

Thus, associating to any x the coefficients of the family $(p_1^x(t), \dots, p_n^x(t))$, we get a regular map π from $\mathcal{S}_\sigma$ to a linear hyperplane of $\mathbb{C}^{b_1}$, which we will denote by $\mathbb{A}^{b_1-1}$.

Proposition 3.2. *The map $\pi : \mathcal{S}_\sigma \longrightarrow \mathbb{A}^{b_1-1}$ is the geometric quotient of $\mathcal{S}_\sigma$.*

Proof. This map is surjective, and its fibers are exactly the orbits of $\mathcal{S}_\sigma$ under the action of $\mathrm{SL}(n)$. Let us denote by $\mathcal{S}_\sigma / \mathrm{SL}(n)$ the geometric quotient of $\mathcal{S}_\sigma$ (whose existence is proved in [Ka]). The map π is the composite of the canonical projection from $\mathcal{S}_\sigma$ to $\mathcal{S}_\sigma / \mathrm{SL}(n)$ with a regular bijection

$$\mathcal{S}_\sigma / \mathrm{SL}(n) \longrightarrow \mathbb{A}^{b_1-1}.$$

This last map is bijective (thus birational), and the space $\mathbb{A}^{b_1-1}$ is normal. According to Zariski's main theorem, it is an isomorphism. $\square$

3.2 A morphism from $\mathcal{S}_\sigma / \mathrm{SL}(n)$ to H_σ

If $z = (p_1(t), \dots, p_n(t))$ is a closed point of $\mathbb{A}^{b_1-1}$ corresponding to the orbit $\mathrm{SL}(n) \cdot x$ in $\mathcal{S}_\sigma$, the polynomial

$$Q_i^x(t) = p_i(t) \cdot (p_{i+1}(t))^2 \cdot \dots \cdot (p_n(t))^{n-i+1}$$

only depends on z. Let us denote it by $Q_i^z(t)$. Its coefficients are regular functions from $\mathbb{A}^{b_1-1}$ to $\mathbb{A}^1$.

Let us consider the closed subscheme $\mathfrak{X}_\sigma$ of $\{(z, y) \in \mathbb{A}^{b_1-1} \times \mathfrak{sl}(n)\}$ defined by the vanishing, for $i = 1 \dots n$, of the remainders we get when we divide the $(n+1-i) \times (n+1-i)$-minors of $y - tI_n$ by $Q_i^z(t)$. We denote by I_σ the ideal generated by these remainders. The underlying set of $\mathfrak{X}_\sigma$ consists of all the couples (z, y) such that y is in the closure of the orbit corresponding to z.

Proposition 3.3. *The closed subscheme $\mathfrak{X}_\sigma$ is a family of $\mathrm{SL}(n)$-stable closed subschemes of $\mathfrak{sl}(n)$ with Hilbert function h_σ.*

Proof. The proof is similar to that of Proposition 2.1. The subscheme $\mathfrak{X}_\sigma$ is a family of $\mathrm{SL}(n)$-stable closed subschemes of $\mathfrak{sl}(n)$ over $\mathbb{A}^{b_1-1}$. Let us denote by π the morphism $\mathfrak{X}_\sigma \longrightarrow \mathbb{A}^{b_1-1}$.

As previouly, let us first remark that the morphism

$$\pi /\!/ \mathrm{SL}(n) : \mathfrak{X}_\sigma /\!/ \mathrm{SL}(n) \longrightarrow \mathbb{A}^{b_1-1}$$

induced by π is an isomorphism. To do this, let us verify that the comorphism

$$(\pi /\!/ \mathrm{SL}(n))^* : \mathbb{C}[\mathbb{A}^{b_1-1}] \longrightarrow \mathbb{C}[\mathbb{A}^{b_1-1}] \otimes \mathbb{C}[\mathfrak{sl}(n)]^{\mathrm{SL}(n)}/I_\sigma^{\mathrm{SL}(n)}$$

is an isomorphism. It is injective, as π is surjective. Its surjectivity comes from the relations that define $\mathfrak{X}_\sigma$: they give, for $i = 1$, that $Q_1^z(t)$ divides the determinant of $tI_n - y$, that is the characteristic polynomial of y. As their degrees are equal, $Q_1^z(t)$ and the characteristic polynomial of y are equal. This gives the surjectivity.

We go on as previously: let λ be a dominant weight. The $R^{\mathrm{SL}(n)}$-module $R^U_{(\lambda)}$ is of finite type ([AB, Lemma 1.2]). Thus $(\pi_*\mathcal{O}_{\mathfrak{X}_\sigma})^U_{(\lambda)}$ is a coherent $\mathcal{O}_{\mathbb{A}^{b_1-1}}$-module. To see that it is locally free, we just have to check that its rank is constant. Let us assume that the origin $0 \in \mathbb{A}^{b_1-1}$ corresponds to the nilpotent orbit in $\mathcal{S}_\sigma$. The fiber of π over 0 is the closure of this orbit, fitted with its structure of reduced scheme. Thus, the rank of $(\pi_*\mathcal{O}_{\mathfrak{X}_\sigma})^U_{(\lambda)}$ at 0 is $h_\sigma(\lambda)$. If z is any point of $\mathbb{A}^{b_1-1}$, the fiber of π over z is as a set the closure in $\mathfrak{sl}(n)$ of the corresponding orbit. So, by Proposition 1.3 the rank of $(\pi_*\mathcal{O}_{\mathfrak{X}_\sigma})^U_{(\lambda)}$ at z is at least $h_\sigma(\lambda)$. To conclude, we use the action of the multiplicative group on $\mathfrak{sl}(n)$ (by homotheties) and the induced action on $\mathbb{A}^{b_1-1}$, that makes π equivariant. The orbit of z goes arbitrary close to 0, and the rank of a coherent sheaf is upper semicontinuous, so the rank of $(\pi_*\mathcal{O}_{\mathfrak{X}_\sigma})^U_{(\lambda)}$ is $h_\sigma(\lambda)$ at z. □

3.3 Tangent space

In this section, we compute the dimension of the tangent space to H_σ at the point Z_σ:

Proposition 3.4. *The dimension of* $T_{\mathrm{Z}_\sigma}\mathrm{H}_\sigma$ *is* $b_1 - 1$.

Proof. Let x be an element in the open orbit in Z_σ. It is known that Z_σ is normal ([KP]). So by Proposition 1.2, we just have to prove that the dimension of

$$(\mathfrak{sl}(n)_x^*)^{\mathrm{SL}(n)_x}$$

is $b_1 - 1$. Let us consider $\mathrm{SL}(n)$ as a closed subgroup of the general linear group $\mathrm{GL}(n)$, and $\mathfrak{sl}(n)$ as a subalgebra of $\mathfrak{gl}(n)$. The stabilizer $\mathrm{GL}(n)_x$ of x in $\mathrm{GL}(n)$ is generated by $\mathrm{SL}(n)_x$ and the center of $\mathrm{GL}(n)$. It is clearly equivalent to prove that the dimension of

$$(\mathfrak{gl}(n)_x^*)^{\mathrm{GL}(n)_x}$$

is b_1. The group $\mathrm{GL}(n)_x$ is connected, so the last space is isomorphic to

$$(\mathfrak{gl}(n)_x^*)^{\mathfrak{gl}(n)_x}.$$

A linear form on $\mathfrak{gl}(n)_x$ is $\mathfrak{gl}(n)_x$-invariant iff it vanishes on the derived algebra $[\mathfrak{gl}(n)_x, \mathfrak{gl}(n)_x]$, so we have to prove that

$$(\mathfrak{gl}(n)_x/[\mathfrak{gl}(n)_x, \mathfrak{gl}(n)_x])^*$$

is b_1-dimensional. This fact is the following elementary lemma. □

Lemma 3.5. *Let $E = \bigoplus_{i=1}^{c_1} E_i$ be a graded vector space over $\mathbb{C}$, where each E_i is b_i-dimensional. We denote by $\mathfrak{h} := \mathfrak{gl}(E)$ the Lie algebra of endomorphisms of E. Let x be a nilpotent element of $\mathfrak{h}$ such that each subspace E_i is stabilized by x, and the restriction of x to each E_i is cyclic.*

Let us denote by $\mathfrak{h}_x$ the stabilizer of x in $\mathfrak{h}$. Then the codimension of the derived algebra $[\mathfrak{h}_x, \mathfrak{h}_x]$ in $\mathfrak{h}_x$ is b_1.

Proof. The graduation of E induces a graduation on the vector space $\mathfrak{h}$:

$$\mathfrak{h} = \bigoplus_{i,j} \mathrm{Hom}(E_i, E_j).$$

Let us denote by $p_i : E \longrightarrow E_i$ the natural projections. As they commute with x, the subspace $\mathfrak{h}_x$ of $\mathfrak{h}$ is homogeneous:

$$\mathfrak{h}_x = \bigoplus_{i,j} \mathrm{Hom}_x(E_i, E_j),$$

where $\mathrm{Hom}_x(E_i, E_j)$ denotes the space of homomorphisms that commute with x. Let us choose, for any i, an element e_i of E_i such that $x^{b_i - 1} e_i \neq 0$. We put $n_{ij} := b_j - b_i$ if $j < i$ and 0 otherwise. We denote by $f_{ij} : E_i \to E_j$ the unique homomophism that commutes with x and that sends e_i to $x^{n_{ij}} e_j$. Then any homomorphism from E_i to E_j that commutes with x is the composite of f_{ij} with a polynomial in x:

$$\mathrm{Hom}_x(E_i, E_j) = \mathbb{C}[x] \cdot f_{ij}.$$

We notice that if $i \neq j$, then $\mathrm{Hom}_x(E_i, E_j)$ is contained in $[\mathfrak{h}_x, \mathfrak{h}_x]$. Indeed, for any $u : E_i \to E_j$, we have $[u, p_i] = u$.

So we have to prove that the codimension in $\bigoplus_i \mathrm{Hom}_x(E_i, E_i)$ of

$$[\mathfrak{h}_x, \mathfrak{h}_x] \cap \bigoplus_i \mathrm{Hom}(E_i, E_i)$$

is b_1. The last vector space is generated by its elements of the form

$$P(x)[f_{ji}, f_{ij}] = P(x) x^{|b_i - b_j|} (\mathrm{id}_{E_i} - \mathrm{id}_{E_j}),$$

where $P(x)$ is a polynomial in x.

One checks easily that a basis of a supplementary in $\bigoplus_i \mathrm{Hom}_x(E_i, E_i)$ of this space is given by the family of elements

$$x^k \, \mathrm{id}_{E_i}$$

where $0 \leq k < b_i - b_{i+1}$, and the lemma is proved. □

3.4 Conclusion

In this section, we prove that the family $\mathfrak{X}_\sigma$ of Proposition 3.3 is the universal family:

Theorem 3.6. *The morphism ϕ_σ from $\mathcal{S}_\sigma / \mathrm{SL}(n)$ to H_σ obtained in §3.2 is an isomorphism.*

We denote by $\overline{\mathcal{S}_\sigma}$ the closure of $\mathcal{S}_\sigma$ in $\mathfrak{sl}(n)$, equipped with its reduced scheme structure. The invariant Hilbert scheme $\mathrm{H}'_\sigma := \mathrm{Hilb}_{h_\sigma}^{\mathrm{SL}(n)}(\overline{\mathcal{S}_\sigma})$ parametrizing the closed subschemes of $\overline{\mathcal{S}_\sigma}$ of Hilbert function h_σ is canonically identified with a closed subscheme of $\mathrm{Hilb}_{h_\sigma}^{\mathrm{SL}(n)}(\mathfrak{sl}(n))$. The morphism ϕ_σ factorizes by a morphism $\psi_\sigma : \mathcal{S}_\sigma/\,\mathrm{SL}(n) \to \mathrm{H}'_\sigma$.

To prove the theorem, we will get that the morphism ψ_σ is an isomorphism from $\mathcal{S}_\sigma/\,\mathrm{SL}(n)$ to H'_σ and that H'_σ is a connected component of H_σ (Corollary 3.10).

Lemma 3.7. *The morphism ψ_σ induces a bijection from the set of closed points of $\mathcal{S}_\sigma/\,\mathrm{SL}(n)$ to the set of closed points of H'_σ.*

Proof. We know that ψ_σ is injective. Let us check it is surjective: in other words, that any $\mathrm{SL}(n)$-invariant closed subscheme of $\overline{\mathcal{S}_\sigma}$ with Hilbert function h_σ is the closure of some orbit in $\mathcal{S}_\sigma$.

Let X be such a subscheme. As $h_\sigma(0) = 1$, it has to be contained in some fiber F of the categorical quotient $\overline{\mathcal{S}_\sigma} \to \overline{\mathcal{S}_\sigma}/\!/\,\mathrm{SL}(n)$ over a reduced closed point. But F already corresponds to a closed point of H'_σ in the image of ψ_σ. Moreover, F admits no proper closed subscheme admitting the same Hilbert function, so $F = X$, and the lemma is proved. $\square$

Corollary 3.8. *The dimension of H'_σ is $b_1 - 1$.*

The action of the multiplicative group $\mathbb{G}_m$ on $\mathfrak{sl}(n)$ by homotheties induces canonically an action of $\mathbb{G}_m$ on H_σ, and on H'_σ (because it stabilizes $\overline{\mathcal{S}_\sigma}$). The cone Z_σ is a $\mathbb{G}_m$-fixed point of H'_σ. In fact, it is in the closure of the $\mathbb{G}_m$-orbit of any point of H'_σ:

Proposition 3.9. *Let F be a closed point of H'_σ. The morphism $\eta : \mathbb{G}_m \longrightarrow \mathrm{H}'_\sigma$, $t \longmapsto t.X$ extends to a morphism $\mathbb{A}^1 \longrightarrow \mathrm{H}'_\sigma$, $0 \longmapsto \mathrm{Z}_\sigma$.*

Proof. The point F corresponds to a $\mathrm{SL}(n)$-invariant closed subscheme of $\overline{\mathcal{S}_\sigma}$ admitting Hilbert function h_σ. We still denote it by F. As $h_\sigma(0) = 1$, it is contained in the fiber of the categorical quotient $\mathfrak{sl}(n) \to \mathfrak{sl}(n)/\!/\mathrm{SL}(n)$ over some closed point. Thus we can apply to it the method of asymptotic cones: we obtain a flat family over $\mathbb{A}^1$ whose fiber over 0 must be Z_σ (as in the proof of Proposition 1.3). It gives a morphism from $\mathbb{A}^1$ to H'_σ whose restriction outside 0 is η. $\square$

From the proposition, we deduce that the dimension of the tangent space to H_σ at any point of H'_σ is lower or equal to that at Z_σ, that is $b_1 - 1$. As the dimension of H'_σ is $b_1 - 1$, we get:

Corollary 3.10.

- *The scheme H'_σ is reduced and smooth.*
- *It is a connected component of H_σ.*

The morphism ψ_σ is bijective (Lemma 3.7) and H'_σ is normal. According to Zariski's main theorem, ψ_σ is an isomorphism. So Theorem 3.6 is proved, thanks to the second point of Corollary 3.10.

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