Horn inequalities for nonzero Kronecker coefficients

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Abstract

The Kronecker coefficients $g_{\alpha\beta\gamma}$ and the Littlewood-Richardson coefficients $c_{\alpha\beta}^{\gamma}$ are nonnegative integers depending on three partitions α, β, and γ. By definition, $g_{\alpha\beta\gamma}$ (resp. $c_{\alpha\beta}^{\gamma}$) are the multiplicities of the tensor product decomposition of two irreducible representations of symmetric groups (resp. linear groups). By a classical Littlewood-Murnaghan's result the Kronecker coefficients extend the Littlewood-Richardson ones.

The nonvanishing of the Littlewood-Richardson coefficient $c_{\alpha\beta}^{\gamma}$ implies that (α, β, γ) satisfies some linear inequalities called Horn inequalities. In this paper, we extend the essential Horn inequalities to the triples of partitions corresponding to a nonzero Kronecker coefficient.

Along the way, we describe the set of tripless (α, β, γ) of partitions such that $c_{\alpha\beta}^{\gamma} \neq 0$ and $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e+f$, for some given positive integers e and f. This set is the natural analogue of the classical Horn semigroup when one thinks about $c_{\alpha\beta}^{\gamma}$ as the branching multiplicities for the subgroup $\mathrm{GL}_e \times \mathrm{GL}_f$ of GL_{e+f}.

1 Introduction

If $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_e \geq 0)$ is a partition, we set $|\alpha| = \sum_i \alpha_i$ in such a way α is a partition of $|\alpha|$. Consider the symmetric group S_n on n letters. The irreducible representations of S_n are parametrized by the partitions of n, see *e.g.* [Mac95, I. 7] . Let $[\alpha]$ denote the representation of $S_{|\alpha|}$ corresponding to α. The Kronecker coefficients $g_{\alpha\,\beta\,\gamma}$, depending on three partitions α, β, and γ of the same integer n, are defined by

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$$[\alpha] \otimes [\beta] = \sum_\gamma g_{\alpha\,\beta\,\gamma}[\gamma]. \tag{1}$$

The length $l(\alpha)$ of the partition α is the number of nonzero parts α_i. Let V be a complex vector space of dimension d. If $l(\alpha) \leq d$ then $S^\alpha V$ denotes the Schur power (see *e.g.* [FH91]): it is an irreducible polynomial representation of the linear group $\mathrm{GL}(V)$. Let β be a second partition such that $l(\beta) \leq d$. Then the Littlewood-Richardson coefficients $c_{\alpha\,\beta}^{\gamma}$ are defined by

$$S^\alpha V \otimes S^\beta V = \sum_\gamma c_{\alpha\,\beta}^{\gamma} S^\gamma V. \tag{2}$$

The partition obtained by suppressing the first part of α is denoted by $\bar{\alpha} = (\alpha_2 \geq \alpha_3 \dots)$. Observe that $\bar{\alpha}_1 = \alpha_2$. We state a classical result due to Littlewood and Murnaghan (see for example [JK81]).

Proposition 1 *Let α, β and γ be three partitons of the same integer n.*

(i) If $g_{\alpha\,\beta\,\gamma} \neq 0$ then

$$(n - \alpha_1) + (n - \beta_1) \geq n - \gamma_1. \tag{3}$$

(ii) If $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then

$$g_{\alpha\,\beta\,\gamma} = c_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}}. \tag{4}$$

In this paper, we prove many other inequalities similar to the identity (3), that are consequences of the nonvanishing of $g_{\alpha\,\beta\,\gamma}$. For the partitions (α, β, γ) satisfying equality in such an inequality, we prove a reduction rule for $g_{\alpha\,\beta\,\gamma}$ similar to the identity (4).

Observe that the formula (4) shows that the Kronecker coefficients extend the Littlewood-Richardson ones. Indeed, given $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$, one can find $\alpha = (\alpha_1, \bar{\alpha}), \beta = (\beta_1, \bar{\beta})$ and $\gamma = (\gamma_1, \bar{\gamma})$ such that $|\alpha| = |\beta| = |\gamma| =: n$, $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$. Then $c_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}} = g_{\alpha\,\beta\,\gamma}$ is a Kronecker coefficient. If $c_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}} \neq 0$ then $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ satisfy the Horn inequalities (see *e.g.* [Ful00] or below for details). If $g_{\alpha\,\beta\,\gamma} \neq 0$, our inequalities for (α, β, γ) extend some Horn inequalities. Fix such an inequality $\varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \geq 0$. We want to find an inequality $\tilde{\varphi}(\alpha, \beta, \gamma) \geq 0$ such that

(i) If $g_{\alpha\,\beta\,\gamma} \neq 0$ then $\tilde{\varphi}(\alpha, \beta, \gamma) \geq 0$;

(ii) If $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then $\tilde{\varphi}(\alpha, \beta, \gamma) = \varphi(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$.

For example, a Weyl's theorem [Wey12] asserts that if $c_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}} \neq 0$ then

$$\bar{\gamma}_{e+j-1} \leq \bar{\beta}_{j-1}, \tag{5}$$

whenever $l(\bar{\alpha}) \leq e$ and $j \geq 2$.

Before stating our extension of Weyl's theorem, we introduce some notation. Let $\mathcal{S}(r, d)$ denote the set of subsets of $\{1, \cdots, d\}$ with r elements. Given $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r, d)$ and $\alpha = (\alpha_1 \geq \cdots \geq \alpha_d)$ a partition of length at most d, we set $\alpha_I = (\alpha_{i_1} \geq \cdots \geq \alpha_{i_r})$. Observe that $\bar{\alpha}_I = (\alpha_{i_1+1} \geq \cdots \geq \alpha_{i_r+1})$.

Theorem 1 *Let e and f be two positive integers. Let α, β, and γ be three partitions of the same integer n such that*

$$l(\alpha) \leq e + 1, \quad l(\beta) \leq f + 1, \quad \textit{and} \quad l(\gamma) \leq e + f + 1. \tag{6}$$

Let $j \in \{2, \ldots, f + 1\}$.

(i) If $g_{\alpha\,\beta\,\gamma} \neq 0$ then

$$n + \gamma_1 + \gamma_{e+j} \leq \alpha_1 + \beta_1 + \beta_j \tag{7}$$

(ii) Set $J = \{1, \ldots, f\} - \{j - 1\}$ and $K = \{1, \ldots, e + f\} - \{e + j - 1\}$. If $n + \gamma_1 + \gamma_{e+j} = \alpha_1 + \beta_1 + \beta_j$ then

$$g(\alpha, \beta, \gamma) = \sum_{l(x) \leq 2e,\, l(y) \leq 2} c(x, \bar{\beta}_J; \bar{\gamma}_K) \cdot c(\gamma_1 \geq \gamma_j, y; \beta_1 \geq \beta_j) \cdot g(\bar{\alpha}, x, y).$$

Remark. In the statement of Theorem 1 (and sometimes below) we denote $c_{\alpha\,\beta}^{\gamma}$ and $g_{\alpha\,\beta\,\gamma}$ respectively by $c(\alpha, \beta; \gamma)$ and $g(\alpha, \beta, \gamma)$.

Theorem 1 extends Weyl's theorem in the sense that if $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then the inequality (7) is equivalent to $\gamma_{e+j} \leq \beta_j$, that is to the inequality (5).

For $I \in \mathcal{S}(r, d)$, consider the partition

$$\tau^I = (d - r + 1 - i_1 \geq d - r + 2 - i_2 \geq \cdots \geq d - i_r).$$

Set $|\alpha_I| := \sum_{i \in I} \alpha_i$. Observe that $|\bar{\alpha}_I| := \sum_{i \in I} \alpha_{i+1}$. We can now state our main result.

Theorem 2 *Let α, β, and γ be three partitions of the same integer n satisfying the conditions* (6).

Assume that $g_{\alpha\,\beta\,\gamma} \neq 0$. Then

$$n + |\bar{\alpha}_I| - \alpha_1 + |\bar{\beta}_J| - \beta_1 \geq |\bar{\gamma}_K| - \gamma_1, \tag{8}$$

for any $0 < r < e$, $0 < s < f$, $I \in \mathcal{S}(r, e)$, $J \in \mathcal{S}(s, f)$ and $K \in \mathcal{S}(r+s, e+f)$ such that

$$c_{\tau^I\,\tau^J}^{\tau^K} = 1. \tag{9}$$

If $(n - \alpha_1) + (n - \beta_1) = n - \gamma_1$ then the inequality (8) is equivalent to

$$|\bar{\alpha}_I| + |\bar{\beta}_J| \geq |\bar{\gamma}_K|, \tag{10}$$

which is a Horn inequality (see [Ful00] or Section 4).

Remark. Since inequalities (3), (7) and (8) are linear in (α, β, γ), the condition $g_{\alpha\,\beta\,\gamma} \neq 0$ in Proposition 1 and Theorem 1 and 2 can be replaced by the weaker condition $g_{k\alpha\,k\beta\,k\gamma} \neq 0$ for some positive k.

We get a reduction formula for the coefficients $g_{\alpha\,\beta\,\gamma}$ if the inequality (8) is saturated. If $I \in \mathcal{S}(r, d)$, we denote by $I_- \in \mathcal{S}(d - r, d)$ the complement of I in $\{1, \ldots, d\}$. By symmetry we also set $I_+ = I$.

Theorem 3 *Let α, β, and γ be three partitions of the same integer n satisfying the conditions* (6).

Let (I, J, K) be a triple that appears in Theorem 2 (in particular satisfying the condition (9)*). We assume that*

$$n + |\bar{\alpha}_I| - \alpha_1 + |\bar{\beta}_J| - \beta_1 = |\bar{\gamma}_K| - \gamma_1. \tag{11}$$

Then $g(\alpha, \beta, \gamma)$ is equal to

$$\sum_{a,b,x,y,u,v} \begin{array}{l} c(\bar{\alpha}_{I_-}, \bar{\beta}_{J_-}; y) \cdot c(x, y; \bar{\gamma}_{K_+}) \cdot c(u, v; \bar{\gamma}_{K_-}) \cdot \\ c(a, u; \bar{\alpha}_I) \cdot c(b, v; \bar{\beta}_J) \cdot g(a, b, x), \end{array} \tag{12}$$

where the sum runs over the partitions a, b, x, y, u, v satisfying

$$\begin{array}{lll} l(x) \leq (e - r)(f - s), & l(a) \leq e - r, & l(u) \leq e - r, \\ l(y) \leq r + s, & l(b) \leq f - s, & l(v) \leq f - s. \end{array} \tag{13}$$

Note that in Theorem 3, we needn't assume that $g_{\alpha\beta\gamma} \neq 0$.

Let $\mathrm{Kron}(e+1, f+1, e+f+1)$ denote the set of triples (α, β, γ) of partitions such that $|\alpha| = |\beta| = |\gamma|$, $g_{\alpha\beta\gamma} \neq 0$ and $l(\alpha) \leq e+1$, $l(\beta) \leq f+1$, $l(\gamma) \leq e+f+1$. Then $\mathrm{Kron}(e+1, f+1, e+f+1)$ is a finitely generated semigroup in ${\mathbb{Z}_{\geq 0}}^{2e+2f+3}$. In particular, the cone $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$ generated by $\mathrm{Kron}(e+1, f+1, e+f+1)$ is a closed convex polyhedral cone.

Theorem 4 *The inequalities* (7) *in Theorem 1 and the inequalities* (8) *in Theorem 2 are essential, that is correspond to codimension one faces of* $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$.

One can guess to describe the complete minimal list $\mathcal{L}$ of inequalities characterizing $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$. Such a list is known for the Littlewood-Richardson coefficients (see Theorem 7 below for details). In principle, [Res10] gives $\mathcal{L}$. Nevertheless, it is known to be untractable to make this description very explicit. Indeed, one first need to describe the so-called adapted one-parameter subgroups by describing the collection of hyperplanes spanned by subsets of a given set: a tricky combinatorial problem. And secondly one need to understand an unknown Schubert problem. In this paper we describe a natural subset of $\mathcal{L}$ related with the Horn cone.

Inequality (3) defines a codimension one face $\mathcal{F}_{LM}$ of $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$. Here "$LM$" stands for Littlewood-Murnaghan. Each Horn inequality (10) or Weyl inequality (5) define a face $\mathcal{F}$ of codimension two contained in $\mathcal{F}_{LM}$. By convex geometry $\mathcal{F}$ has to be contained in a second codimension one face $\mathcal{F}'$ of $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$. Basically, Theorem 1 and 2 describe this face $\mathcal{F}'$.

Comparaison between Theorems 1 and 2. With I, J and K respectively equal to $\{1, \dots, e\}$, $\{1, \dots, f\} - \{j-1\}$, and $\{1, \dots, e+f\} - \{e+j-1\}$ (where $j \in \{2, \dots, f+1\}$), we have $c_{\tau^I\,\tau^J}^{\tau^K} = 1$. The inequality (8) gives

$$2n + 2\gamma_1 + \gamma_j \geq 2\alpha_1 + 2\beta_1 + \beta_j. \tag{14}$$

This inequality is satisfied if $g_{\alpha\,\beta\,\gamma} \neq 0$. But the corresponding face has codimension 2 in $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$. Hence the inequality (14) is not essential. More precisely, it is a consequence of inequalities (3) and (7).

In Section 2, we define and compare several semigroups. In Section 3, we recall some results from [Res10] that allows to describe some cones generated by these semigroups. In Section 4, we describe the support of the LR-coefficients $c^{\gamma}_{\alpha\beta}$ for partitions satisfying $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e+f$, for fixed positive integers e and f. Note that these assumptions are natural if one thinks about the LR-coefficients as multiplicities for the branching from $\mathrm{GL}_e \times \mathrm{GL}_f$ to GL_{e+f}. It is a variation of the classical Horn problem. The next sections contain the proofs of the statements of the introduction.

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2 Semigroups

2.1 Definitions

2.1.1 Kronecker semigroups

We extend the definition of $g_{\alpha\beta\gamma}$ to any triple (α, β, γ) of partitions by setting $g_{\alpha\beta\gamma} = 0$ if the condition $|\alpha| = |\beta| = |\gamma|$ does not hold. Let e, f, and g be three positive integers. We define $\mathrm{Kron}(e, f, g)$ to be the set of triples $(\alpha,\ \beta,\ \gamma)$ of partitions such that $g_{\alpha\beta\gamma} \neq 0$ and $l(\alpha) \leq e$, $l(\beta) \leq f$, $l(\gamma) \leq g$. It is well known that $\mathrm{Kron}(e, f, g)$ is a finitely generated semigroup of ${\mathbb{Z}_{\geq 0}}^{e+f+g}$.

2.1.2 Littlewood-Richardson semigroups

We define $\mathrm{LR}(e, f, g)$ to be the set of triples $(\alpha,\ \beta,\ \gamma)$ of partitions such that $c^{\gamma}_{\alpha\beta} \neq 0$ and $l(\alpha) \leq e$, $l(\beta) \leq f$, $l(\gamma) \leq g$. It is well known that $\mathrm{LR}(e, f, g)$ is a finitely generated semigroup of ${\mathbb{Z}_{\geq 0}}^{e+f+g}$.

2.1.3 Branching semigroups

Let G be a connected reductive subgroup of a complex connected reductive group $\hat{G}$. Fix maximal tori $T \subset \hat{T}$ and Borel subgroups $B \supset T$ and $\hat{B} \supset \hat{T}$ of G and $\hat{G}$. Let $X(T)$ denote the group of characters of T and let $X(T)^+$ denote the set of dominant characters. The irreducible representation of highest weight $\nu \in X(T)^+$ is denoted by V_ν. Similarly, we use the notation $X(\hat{T})$, $X(\hat{T})^+$, $V_{\hat{\nu}}$ relatively to $\hat{G}$. The subspace of G-fixed vectors of the

G-module V is denoted by V^G. Set

$$c_{\nu\,\hat{\nu}} = \dim(V_\nu^* \otimes V_{\hat{\nu}})^G. \tag{15}$$

The branching problem is equivalent to the knowledge of these coefficients since

$$V_{\hat{\nu}} = \sum_{\nu \in X(T)^+} c_{\nu\,\hat{\nu}} V_\nu, \tag{16}$$

as a G-module. Consider the set

$$\mathrm{LR}(G, \hat{G}) = \{(\nu, \hat{\nu}) \in X(T)^+ \times X(\hat{T})^+ \ : \ c_{\nu\,\hat{\nu}} \neq 0\}.$$

By a result of Brion and Knop (see [É92]), $\mathrm{LR}(G, \hat{G})$ is a finitely generated semigroup.

2.1.4 GIT semigroups

Let G be a complex reductive group acting on an irreducible projective variety X. Let $\mathrm{Pic}^G(X)$ denote the group of G-linearized line bundles on X. The space $\mathrm{H}^0(X, \mathcal{L})$ of regular sections of $\mathcal{L}$ is a G-module. Consider the set

$$\mathrm{LR}(G, X) = \{\mathcal{L} \in \mathrm{Pic}^G(X) \ : \ \mathrm{H}^0(X, \mathcal{L})^G \neq \{0\}\}. \tag{17}$$

Since X is irreducible, the product of two nonzero G-invariant sections is a nonzero G-invariant section and $\mathrm{LR}(G, X)$ is a semigroup.

2.2 Relations between these semigroups

2.2.1 Kronecker semigroups as branching semigroups.

Let E and F be two complex vector spaces of dimension e and f. Consider the group $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$. Using Schur-Weyl duality, the Kronecker coefficient $g_{\alpha\,\beta\,\gamma}$ can be interpreted in terms of representations of G. Namely (see for example [Mac95, FH91]) $g_{\alpha\,\beta\,\gamma}$ is the multiplicity of $S^\alpha E \otimes S^\beta F$ in $S^\gamma(E \otimes F)$. More precisely, let γ be a partition such that $l(\gamma) \leq ef$. Then the simple $\mathrm{GL}(E \otimes F)$-module $S^\gamma(E \otimes F)$ decomposes as a sum of simple G-modules as follows

$$S^\gamma(E \otimes F) = \sum_{\substack{\text{partitions } \alpha,\ \beta \text{ s.t.} \\ l(\alpha) \leq e,\ l(\beta) \leq f}} g_{\alpha\,\beta\,\gamma}\, S^\alpha E \otimes S^\beta F. \tag{18}$$

As a consequence

$$\mathrm{Kron}(e, f, ef) = \mathrm{LR}(\mathrm{GL}(E) \times \mathrm{GL}(F), \mathrm{GL}(E \otimes F)) \cap (\mathbb{Z}^e \times \mathbb{Z}^f \times (\mathbb{Z}_{\geq 0})^{ef}). \tag{19}$$

2.2.2 Littlewood-Richardson semigroups as branching semigroups

Since the Littlewood-Richardson coefficients are multiplicities for the tensor product decomposition of GL_n, we have

$$\mathrm{LR}(e,e,e) = \mathrm{LR}(\mathrm{GL}_e, \mathrm{GL}_e \times \mathrm{GL}_e) \cap ((\mathbb{Z}_{\geq 0})^e)^3. \tag{20}$$

The Littlewood-Richardson coefficients have another interpretation in terms of representations of linear groups. Consider the embedding of $\mathrm{GL}(E) \times \mathrm{GL}(F)$ in $\mathrm{GL}(E \oplus F)$ as a Levi subgroup by its natural action on $E \oplus F$. Then (see[Mac95, Chapter I, 5.9])

$$S^\gamma(E \oplus F) = \sum_{\substack{\text{partitions } \alpha,\, \beta \text{ s.t.} \\ l(\alpha) \leq e,\, l(\beta) \leq f}} c^\gamma_{\alpha\beta} S^\alpha E \otimes S^\beta F. \tag{21}$$

In particular

$$\mathrm{LR}(e,f,e+f) = \mathrm{LR}(\mathrm{GL}_e \times \mathrm{GL}_f, \mathrm{GL}_{e+f}) \cap (\mathbb{Z}^e \times \mathbb{Z}^f \times (\mathbb{Z}_{\geq 0})^{e+f}). \tag{22}$$

2.2.3 Branching semigroups as GIT semigroups

We use notation of Section 2.1.3 and we assume that G and $\hat{G}$ are semisimple simply connected. Consider the diagonal action of G on $X = G/B \times \hat{G}/\hat{B}$. Note that $\mathrm{Pic}^G(X)$ identifies with $X(T) \times X(\hat{T})$. Then Borel-Weyl's theorem implies that $\mathrm{LR}(G, \hat{G}) = \mathrm{LR}(G, X)$.

2.2.4 Kronecker semigroups as GIT semigroups

If V is a complex finite dimensional vector space, let $\mathcal{F}l(V)$ denote the variety of complete flags of V. Given integers a_i such that $1 \leq a_1 < \cdots < a_s \leq \dim(V) - 1$, we denote by $\mathcal{F}l(a_1, \cdots, a_s; V)$ the variety of flags $V_1 \subset \cdots \subset V_s \subset V$ such that $\dim(V_i) = a_i$ for any i. If α is a partition with at most $\dim(V)$ parts then $\mathcal{L}_\alpha$ (resp. $\mathcal{L}^\alpha$) denotes the $\mathrm{GL}(V)$-linearized line bundle on $\mathcal{F}l(V)$ such that the space $\mathrm{H}^0(\mathcal{F}l(V), \mathcal{L}_\alpha)$ (resp. $\mathrm{H}^0(\mathcal{F}l(V), \mathcal{L}^\alpha)$) is isomorphic to $S^\alpha V^*$ (resp. $S^\alpha V$) as a $\mathrm{GL}(V)$-module.

Assume that E and F are two linear spaces of dimension $e+1$ and $f+1$. Set $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$. Consider the variety

$$X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(1, \cdots, e+f+1; E \otimes F)$$

endowed with its natural G-action. Let α, β, and γ be three partitions such that $l(\alpha) \leq e+1$, $l(\beta) \leq f+1$, and $l(\gamma) \leq e+f+1$. Consider the $\mathrm{GL}(E)$-linearized line bundle $\mathcal{L}^\alpha$ on $\mathcal{F}l(E)$, and respectively $\mathcal{L}^\beta$ on $\mathcal{F}l(F)$. Since $l(\gamma) \leq e+f+1$, the line bundle $\mathcal{L}_\gamma$ on $\mathcal{F}l(E\otimes F)$ is the pullback of a line bundle (still denoted by $\mathcal{L}_\gamma$) on $\mathcal{F}l(1,\cdots,e+f+1;E\otimes F)$. Consider the line bundle $\mathcal{L} = \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma$ on X endowed with its natural G-action. Then

$$\mathrm{H}^0(X,\mathcal{L}) \simeq S^\alpha E \otimes S^\beta F \otimes S^\gamma (E\otimes F)^*,$$

and, by the formula (18),

$$g_{\alpha\,\beta\,\gamma} = \dim(\mathrm{H}^0(X,\mathcal{L})^G). \tag{23}$$

The map $(\alpha,\beta,\gamma) \mapsto \mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma$ extends to a linear isomorphism from $\mathbb{Z}^{2e+2f+3}$ onto $\mathrm{Pic}^G(X)$. This isomorphism allows to identify $\mathrm{LR}(G,X)$ with a subset of $\mathbb{Z}^{2e+2f+3}$. The equality (23) implies that

$$\mathrm{Kron}(e+1,f+1,e+f+1) = (\mathbb{Z}_{\geq 0})^{2e+2f+3} \cap \mathrm{LR}(G,X).$$

3 Descriptions of branching and GIT cones

3.1 GIT cones

Assume that the connected reductive group G acts on the smooth projective variety X and that $\mathrm{Pic}^G(X)$ has finite rank. Consider the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,X)$ generated in $\mathrm{Pic}^G(X)\otimes\mathbb{Q}$ by the points of $\mathrm{LR}(G,X)$. The G-linearized ample line bundles on X generated an open convex cone $\mathrm{Pic}^G(X)^+_{\mathbb{Q}}$ in $\mathrm{Pic}^G(X)\otimes\mathbb{Q}$. In this section, we recall from [Res10] a description of the faces of $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,X)$ that intersect $\mathrm{Pic}^G(X)^+_{\mathbb{Q}}$.

Let $\mathcal{L}$ be a G-linearized line bundle on X. Consider the associated set of semistable points

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x\in X\ :\ \exists k>0 \text{ and } \sigma\in \mathrm{H}^0(X,\mathcal{L}^{\otimes k})^G \qquad \sigma(x)\neq 0\}.$$

Assume that $X^{\mathrm{ss}}(\mathcal{L})$ is nonempty. Then the projective variety $\mathrm{Proj}(\bigoplus_{k\geq 0}\mathrm{H}^0(X,\mathcal{L}^{\otimes k})^G)$ is denoted by $X^{\mathrm{ss}}(\mathcal{L})/\!/G$. For later use, observe that $\dim(\mathrm{H}^0(X,\mathcal{L}^{\otimes k})^G)$ is $O(k^{\dim(X^{\mathrm{ss}}(\mathcal{L})/\!/G)})$. If moreover $\mathcal{L}$ is ample, $X^{\mathrm{ss}}(\mathcal{L})/\!/G$ is the categorical quotient of $X^{\mathrm{ss}}(\mathcal{L})$ by G. In general, there is a canonical G-invariant regular map

$$\pi\ :\ X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G. \tag{24}$$

Let λ be a one parameter subgroup of G. The set

$$P(\lambda) = \{g \in G \,:\, \lim_{t \to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists in } G\}$$

is a parabolic subgroup of G. Consider an irreducible component C of the fixed point set X^λ of λ in X. Set

$$C^+ = \{x \in X \,:\, \lim_{t \to 0} \lambda(t) x \in C\}.$$

By Białynicki-Birula's theorem, C^+ is an irreducible smooth locally closed subvariety of X. Moreover it is stable by the action of $P(\lambda)$. Consider on $G \times C^+$ the following action of the group $P(\lambda)$:

$$p.(g, x) = (gp^{-1}, px).$$

There exists a quotient variety denoted by $G \times_{P(\lambda)} C^+$. We denote by $[g : x]$, the class of $(g, x) \in G \times C^+$. The following formula

$$h.[g : x] = [hg : x] \qquad \forall h \in G,$$

endows $G \times_{P(\lambda)} C^+$ with a G-action. Consider the G-equivariant morphism

$$\begin{array}{cccc} \eta \,: & G \times_{P(\lambda)} C^+ & \longrightarrow & X \\ & [g : x] & \longmapsto & gx. \end{array}$$

The pair (C, λ) is said to be *well covering* if there exists a $P(\lambda)$-stable open subset Ω of C^+ such that

(i) the restriction of η to $G \times_{P(\lambda)} \Omega$ is an open immersion;

(ii) Ω intersects C.

For any $\mathcal{L} \in \mathrm{Pic}^G(X)$, there exists an integer $\mu^{\mathcal{L}}(C, \lambda)$ such that

$$\lambda(t)\tilde{z} = t^{-\mu^{\mathcal{L}}(C,\lambda)} \tilde{z},$$

for any $t \in \mathbb{C}^*$, $z \in C$ and $\tilde{z}$ in the fiber $\mathcal{L}_z$ over z in $\mathcal{L}$.

Theorem 5 *(see [Res10])*

(i) For any well covering pair (C, λ) and any $\mathcal{L} \in \mathrm{LR}(G, X)$, we have $\mu^{\mathcal{L}}(C, \lambda) \leq 0$.

(ii) For any face $\mathcal{F}$ of $\mathbb{Q}_{\geq 0} \mathrm{LR}(G, X)$ intersecting $\mathrm{Pic}^G(X)^+_{\mathbb{Q}}$ there exists a well covering pair (C, λ) such that $(\mathcal{L} \otimes 1) \in \mathcal{F}$ if and only if $\mu^{\mathcal{L}}(C, \lambda) = 0$, for any ample $\mathcal{L}$ in $\mathbb{Q}_{\geq 0} \mathrm{LR}(G, X)$.

(iii) Let (C, λ) be a well covering pair and $\mathcal{L}$ be ample in $\mathrm{LR}(G, X)$. Then $\mu^{\mathcal{L}}(C, \lambda) = 0$ if and only if $X^{\mathrm{ss}}(\mathcal{L}) \cap C$ is not empty.

3.2 Branching cones

With notation of Section 2.1.3, we want to describe the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,\hat{G})$ generated by $\mathrm{LR}(G,\hat{G})$. We assume that no nonzero ideal of the Lie algebra $\mathrm{Lie}(G)$ of G is an ideal of that $\mathrm{Lie}(\hat{G})$ of $\hat{G}$: this assumption implies that the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,\hat{G})$ has nonempty interior in $(X(T)\times X(\hat{T}))\otimes\mathbb{Q}$.

Consider the natural pairing $\langle\cdot\,,\,\cdot\rangle$ between the one parameter subgroups and the characters of tori T or $\hat{T}$. Let W (resp. $\hat{W}$) denote the Weyl group of T (resp. $\hat{T}$). If λ is a one parameter subgroup of T (and thus of $\hat{T}$), we denote by W_λ (resp. $\hat{W}_\lambda$) the stabilizer of λ for the natural action of the Weyl group.

The cohomology group $\mathrm{H}^*(G/P(\lambda),\mathbb{Z})$ is freely generated by the Schubert classes σ_w parameterized by the elements $w\in W/W_\lambda$. Assume that λ is dominant. Let w_0 be the longest element of W. If $w\in W/W_\lambda$, we denote by $w^\vee\in W/W_\lambda$ the class of w_0w. By this way $\sigma_{w^\vee}$ and σ_w are Poincaré dual. We consider $\hat{G}/\hat{P}(\lambda)$, $\sigma_{\hat{w}}$ as above but with $\hat{G}$ in place of G. Consider also the canonical G-equivariant immersion $\iota\,:\,G/P(\lambda)\longrightarrow\hat{G}/\hat{P}(\lambda)$; and the corresponding morphism ι^* in cohomology.

Recall from [RR11], the definition of Levi-movability for the pair $(\sigma_w,\sigma_{\hat{w}})$. For the purpose of this paper it is only useful to known that if $(\sigma_w,\sigma_{\hat{w}})$ is Levi-movable then $\iota^*(\sigma_{\hat{w}}).\sigma_w$ is a nonzero multiple of the class $[pt]$ of the point. Moreover the converse is true if $\hat{G}/\hat{P}(\lambda)$ is minuscule.

Consider the set $\mathrm{Wt}_T(\mathrm{Lie}(\hat{G})/\mathrm{Lie}(G))$ of nontrivial weights of T in $\mathrm{Lie}(\hat{G})/\mathrm{Lie}(G)$ and the set of hyperplanes H of $X(T)\otimes\mathbb{Q}$ spanned by some elements of $\mathrm{Wt}_T(\mathrm{Lie}(\hat{G})/\mathrm{Lie}(G))$. For each such hyperplane H there exist exactly two opposite indivisible one parameter subgroups $\pm\lambda_H$ which are orthogonal (for the paring $\langle\cdot,\cdot\rangle$) to H. The so obtained one parameter subgroups are called *admissible* and form a W-stable set.

Theorem 6 *(see [Res10])*

Recall that no nonzero ideal of $\mathrm{Lie}(G)$ *is an ideal of* $\mathrm{Lie}(\hat{G})$*. Then, the cone* $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,\hat{G})$ *has nonempty interior in* $X(T\times\hat{T})\otimes\mathbb{Q}$*.*

A dominant weight $(\nu,\hat{\nu})$ *belongs to* $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,\hat{G})$ *if and only if*

$$\langle\hat{w}\lambda,\hat{\nu}\rangle\leq\langle w\lambda,\nu\rangle \tag{25}$$

for any dominant admissible one parameter subgroup λ *of* T *and for any pair* $(w,\hat{w})\in W/W_\lambda\times\hat{W}/\hat{W}_\lambda$ *such that*

(i) $\iota^*(\sigma_{\hat{w}})\cdot\sigma_{w^\vee}=[pt]\in\mathrm{H}^*(G/P(\lambda),\mathbb{Z})$*, and*

(ii) the pair $(\sigma_{w^\vee},\sigma_{\hat{w}})$ *is Levi-movable.*

Moreover, the inequalities (25) *are pairwise distinct and no one can be omitted.*

4 Description of $\mathrm{LR}(e, f, e+f)$

4.1 The statement

Theorem 7 *Let α, β, and γ be three partitions such that $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e+f$.*

Then $c_{\alpha\,\beta}^{\gamma} \neq 0$ if and only if

$$|\alpha| + |\beta| = |\gamma|, \tag{26}$$

and

$$\gamma_{f+i} \leq \alpha_i \leq \gamma_i, \quad \gamma_{e+j} \leq \beta_j \leq \gamma_j, \tag{27}$$

for any $i \in \{1, \ldots, e\}$ and $j \in \{1, \ldots, f\}$, and

$$|\gamma_K| \leq |\alpha_I| + |\beta_J|, \tag{28}$$

for any $0 < r < e$ and $0 < s < f$, for any $I \in \mathcal{S}(r, e)$, $J \in \mathcal{S}(s, f)$ and $K \in \mathcal{S}(r+s, e+f)$ such that

$$c_{\tau^I\,\tau^J}^{\tau^K} = 1. \tag{29}$$

Moreover, the inequalities (27) *or* (28) *are pairwise distinct and no one can be omitted.*

The partitions α and β in the statement of Theorem 7 are also partitions of length at most $e+f$. Hence the nonvanishing of $c_{\alpha\beta}^{\gamma}$ is equivalent to $(\alpha, \beta, \gamma) \in \mathbb{Q}_{\geq 0}\,\mathrm{LR}(e+f, e+f, e+f)$. But, by the classical Horn conjecture (see *e.g.* [Ful00]), this cone is characterized by the inequalities

$$|\gamma_{K'}| \geq |\alpha_{I'}| + |\beta_{J'}| \tag{30}$$

where $\sharp I' = \sharp J' = \sharp K'$ and

$$c_{\tau^{I'}\tau^{J'}}^{\tau^{K'}} = 1. \tag{31}$$

In some sense, Theorem 7 selects among the inequalities (30) those that remain essential when one imposes $l(\alpha) \leq e$ and $l(\beta) \leq f$.

Each inequality (28) has to be consequence of at least one Horn inequality (30). Indeed, by setting $\tilde{I} = I_- \cup \{e+s+1, \dots, e+f\}$ and $\tilde{J} = J_- \cup \{f+r+1, \dots, e+f\}$, one can check that, under the assumptions of Theorem 7 and modulo the equality (26), the inequality (28) is equivalent to

$$|\gamma_{K_-}| \geq |\alpha_{\tilde{I}}| + |\beta_{\tilde{J}}|. \tag{32}$$

But $\sharp\tilde{I} = \sharp\tilde{J} = \sharp K_- = e+f-r-s$. One can check that

$$c_{\tau^I \tau^J}^{\tau^K} = c_{\tau^{\tilde{I}} \tau^{\tilde{J}}}^{\tau^{K_-}}.$$

Hence the assumption (29) implies that the condition (32) is an Horn inequality (30) for the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(e+f, e+f, e+f)$.

For the proof of Theorem 7, we need to recall some notations and results on Schubert calculus on Grassmannians.

4.2 Schubert Calculus

Let $\mathbb{G}(r,n)$ be the Grassmann variety of r-dimensional linear subspaces of $V = \mathbb{C}^n$. Let $F_\bullet$: $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = V$ be a complete flag of V. Let $I = \{i_1 < \cdots < i_r\} \in \mathcal{S}(r,n)$. The Schubert variety $X_I(F_\bullet)$ in $\mathbb{G}(r,n)$ is defined to be

$$X_I(F_\bullet) = \{L \in \mathbb{G}(r,n) \,:\, \dim(L \cap F_{i_j}) \geq j \text{ for } 1 \leq j \leq r\}.$$

The Poincaré dual of the homology class of $X_I(F_\bullet)$ is denoted by σ_I. It does not depend on $F_\bullet$. The classes σ_I form a $\mathbb{Z}$-basis of the cohomology ring of $\mathbb{G}(r,n)$. Recall from the introduction the definition of the partition τ^I. Then σ_I has degree $2|\tau^I|$. A first cohomological interpretation of the Littlewood-Richardson coefficients is given by the formula (see *e.g.* [Man01])

$$\sigma_I.\sigma_J = \sum_{K \in \mathcal{S}(r,n)} c_{\tau^I\,\tau^J}^{\tau^K} \sigma_K, \tag{33}$$

for any I, J in $\mathcal{S}(r,n)$.

Let r and s be two integers such that $0 < r < e$ and $0 < s < f$. Fix an identification $\mathbb{C}^{e+f} = \mathbb{C}^e \oplus \mathbb{C}^f$ and consider the morphism

$$\begin{array}{cccc} \phi_{r,s} \,: & \mathbb{G}(r,e) \times \mathbb{G}(s,f) & \longrightarrow & \mathbb{G}(r+s, e+f) \\ & (F,G) & \longmapsto & F \oplus G. \end{array}$$

The associated comorphism in cohomology is

$$\phi_{r,s}^* : \mathrm{H}^*(\mathbb{G}(r+s,e+f),\mathbb{Z}) \longrightarrow \mathrm{H}^*(\mathbb{G}(r,e)\times\mathbb{G}(s,f),\mathbb{Z}).$$

By Kuneth's formula, the family $(\sigma_I \otimes \sigma_J)_{(I,J)\in\mathcal{S}(r,e)\times\mathcal{S}(s,f)}$ is a basis of $\mathrm{H}^*(\mathbb{G}(r,e)\times\mathbb{G}(s,f),\mathbb{Z})$. A second cohomological interpretation of the Littlewood-Richardson coefficients is given by the formula

$$\phi_{r,s}^*(\sigma_K) = \sum_{(I,J)\in\mathcal{S}(r,e)\times\mathcal{S}(s,f)} c_{\tau^I\,\tau^J}^{\tau^K}\ (\sigma_I\otimes\sigma_J), \tag{34}$$

for any $K\in\mathcal{S}(r+s,e+f)$.

4.3 Proof of Theorem 7

By the Knutson-Tao theorem of saturation (see [KT99]), $c_{\alpha\,\beta}^{\gamma}\neq 0$ if and only if $(\alpha,\,\beta,\,\gamma)$ belongs to the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(e,f,e+f)$. It remains to prove that the inequalities (27) and (28) characterize the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(e,f,e+f)$ in a minimal way.

Let us fix bases for the two vector spaces E and F of dimension e and f. Consider the group $\hat{G}=\mathrm{SL}(E\oplus F)$, its subgroup $G=S(\mathrm{GL}(E)\times\mathrm{GL}(F))$ and on $E\oplus F$ the basis obtained by concatenating the bases of E and F. Let $\hat{T}$ be the maximal torus of $\hat{G}$ consisting in diagonal matrices. It is contained in G; set $T=\hat{T}$. Let $\hat{B}$ be the Borel subgroup of $\hat{G}$ consisting in upper triangular matrices. Set $B=\hat{B}\cap G$. Let ε_i be the character of $\hat{T}$ mapping a matrix in $\hat{T}$ to its i^{th} diagonal entry. Since $\sum_i\varepsilon_i=0$, $(\varepsilon_1,\dots,\varepsilon_{e+f-1})$ is a $\mathbb{Z}$-basis of $X(\hat{T})$.

Let α, β, and γ be three partitions of length less or equal to e, f, and $e+f$. The highest weight of the $\hat{G}$-module $S^\gamma(E\oplus F)$ is $\tilde{\gamma}=(\gamma_1-\gamma_{e+f})\varepsilon_1+\cdots+(\gamma_{e+f-1}-\gamma_{e+f})\varepsilon_{e+f-1}$. The highest weight of the G-module $S^\alpha E\otimes S^\beta F$ is $\widetilde{(\alpha,\beta)}=(\alpha_1-\beta_f)\varepsilon_1+\cdots+(\alpha_e-\beta_f)\varepsilon_e+(\beta_1-\beta_f)\varepsilon_{e+1}+\cdots+(\beta_{f-1}-\beta_f)\varepsilon_{e+f-1}$. Then, by the formula (21)

$$\begin{aligned}
(\alpha,\beta,\gamma)\in\mathrm{LR}(e,f,e+f) &\iff (S^\alpha E^*\otimes S^\beta F^*\otimes S^\gamma(E\oplus F))^{\mathrm{GL}(E)\times\mathrm{GL}(F)}\neq 0,\\
&\iff |\alpha|+|\beta|=|\gamma|\\
&\text{and}\ \ (S^\alpha E^*\otimes S^\beta F^*\otimes S^\gamma(E\oplus F))^{G}\neq 0,\\
&\iff |\alpha|+|\beta|=|\gamma|\\
&\text{and}\ \ (\widetilde{(\alpha,\beta)},\tilde{\gamma})\in\mathrm{LR}(G,\hat{G}).
\end{aligned}$$

In particular, to determine the inequalities for the cone $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(e,f,e+f)$, it is sufficient to describe $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G,\hat{G})$. We do this using Theorem 6.

The set of weights of T acting on $\mathrm{Lie}(\hat{G})/\mathrm{Lie}(G)$ is the set of weights of T acting on $F^* \otimes E$ and their opposite. Explicitly $\mathrm{Wt}_T(\mathrm{Lie}(\hat{G})/\mathrm{Lie}(G)) = \pm\{\varepsilon_i - \varepsilon_{e+j} \,|\, 1 \leq i \leq e \quad \text{and} \quad 1 \leq j \leq f\}$. Let $(a_1, \ldots, a_e, b_1, \ldots, b_f) \in \mathbb{Z}^{e+f}$ be the exponents of a one parameter subgroup λ of T; they satisfy $\sum_i a_i + \sum_j b_j = 0$. Then $\langle \lambda, \varepsilon_i - \varepsilon_{e+j} \rangle = 0$ if and only if $a_i = b_j$. It follows that if λ is admissible then the integers a_i and b_j take at most two values. If moreover λ is dominant then there exist integers r, s, and $c > d$ such that $a_1 = \cdots = a_r = b_1 = \cdots = b_s = c$ and $a_{r+1} = \cdots = a_e = b_{s+1} = \cdots = b_f = d$. If moreover λ is indivisible, $c = \frac{e+f-r-s}{(r+s)\wedge(e+f)}$ and $d = \frac{-r-s}{(r+s)\wedge(e+f)}$, where $\wedge$ denotes the gcd. Let $\lambda_{r,s}$ denote the so obtained one-parameter subgroup of T. Conversely, one easily checks that $\lambda_{r,s}$ is an admissible dominant one-parameter subgroup of T, if $0 < r < e$ and $0 < s < f$ or if the pair (r, s) is one of the four exceptional ones $\{(1, 0), (0, 1), (e-1, f), (e, f-1)\}$.

The inclusions $G/P(\lambda_{r,s}) \subset \hat{G}/\hat{P}(\lambda_{r,s})$ associated to the four exceptional cases are $\mathbb{P}(E) \subset \mathbb{P}(E \oplus F)$, $\mathbb{P}(F) \subset \mathbb{P}(E \oplus F)$, $\mathbb{P}(E^*) \subset \mathbb{P}(E^* \oplus F^*)$ and $\mathbb{P}(F^*) \subset \mathbb{P}(E^* \oplus F^*)$. Consider $\mathbb{P}(E) \subset \mathbb{P}(E \oplus F)$. The restriction of $\sigma_{\{f+i\}} \in \mathrm{H}^*(\mathbb{P}(E \oplus F), \mathbb{Z})$ in $\mathrm{H}^*(\mathbb{P}(E), \mathbb{Z})$ is $\sigma_{\{i\}}$. Then Theorem 6 implies that

$$(e+f)\alpha_i - |\alpha| - |\beta| \geq (e+f)\gamma_{f+i} - |\gamma|.$$

Modulo the identity (26), this is equivalent to $\gamma_{f+i} \leq \alpha_i$. Similarly, we get the three other inequalities (27).

Fix now $0 < r < e$ and $0 < s < f$. The inclusion $G/P(\lambda_{r,s}) \subset \hat{G}/\hat{P}(\lambda_{r,s})$ is the morphism $\phi_{r,s}$ defined in Section 4.2. Consider $\sigma_I \otimes \sigma_J \in \mathrm{H}^*(\mathbb{G}(r, e) \times \mathbb{G}(s, f), \mathbb{Z})$ and $\sigma_K \in \mathrm{H}^*(\mathbb{G}(r+s, e+f), \mathbb{Z})$ such that $\phi_{r,s}^*(\sigma_K).(\sigma_I \otimes \sigma_J)^\vee = [pt]$. Here the Levi movability is automatic since $\hat{G}/\hat{P}(\lambda_{r,s})$ is cominuscule. Modulo (26), the inequality (25) of Theorem 6 corresponding to $\sigma_I \otimes \sigma_J$ and σ_K is the inequality (28). Then the theorem follows from Theorem 6. $\square$

4.4 Complement on stretched Littlewood-Richardson coefficients

Lemma 1 *Let α, β, and γ be three partitions such that $l(\alpha) \leq e$, $l(\beta) \leq f$ and $l(\gamma) \leq e+f$.*

Then, the map $n \longmapsto c_{n\alpha\, n\beta}^{n\gamma}$ is polynomial of degree not greater than

$$\binom{e}{2} + \binom{f}{2} + \binom{e+f}{2} - e^2 - f^2 + 1,$$

where $\binom{e}{2} = \frac{e(e-1)}{2}$.

Proof. Since $c^{\gamma}_{\alpha\beta} = c^{\gamma}_{\beta\alpha}$, we may assume that $e \leq f$. By [DW02], the function $\mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$, $n \longmapsto c^{n\gamma}_{n\alpha\, n\beta}$ is polynomial.

Recall that E and F are complex vector spaces of dimension e and f. Set $G = \mathrm{GL}(E) \times \mathrm{GL}(F)$ and $X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(E \oplus F)$. Consider on X the line bundle $\mathcal{L} = \mathcal{L}^{\alpha} \otimes \mathcal{L}^{\beta} \otimes \mathcal{L}_{\gamma}$. Since $c^{n\gamma}_{n\alpha\, n\beta} = \dim(\mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G)$, the degree of $c^{n\gamma}_{n\alpha\, n\beta}$ is equal to the dimension of $X^{\mathrm{ss}}(\mathcal{L})/\!/G$.

Consider the map π defined in (24). By Chevalley Theorem, since π is dominant, for any general $y \in X^{\mathrm{ss}}(\mathcal{L})$, one has

$$\dim \pi^{-1}(\pi(y)) = \dim(X^{\mathrm{ss}}(\mathcal{L})) - \dim(X^{\mathrm{ss}}(\mathcal{L})/\!/G).$$

But, π is G-invariant and $\pi^{-1}(\pi(y))$ contains $G.y$. Then

$$\dim \pi^{-1}(\pi(y)) \geq \dim(G.y) = \dim(G) - \dim(G_y),$$

where G_y is the stabilizer of y in G. But, for any $x \in X$, we have $\dim(G.x) \leq \dim(G.y)$ and

$$\dim(X^{\mathrm{ss}}(\mathcal{L})/\!/G) \leq \dim(X) - \dim(G) + \dim(G_x).$$

We now claim that there exists x such that $\dim(G_x) = 1$. Then the lemma follows.

We now prove the claim by constructing explicitly x, that is, defining complete flags of E, F and $E \oplus F$. Fix bases $(\eta_1, \dots, \eta_e)$ and $(\zeta_1, \dots, \zeta_f)$ of E and F. On E and F, we consider the two standard flags $F^E_\bullet$ and $F^F_\bullet$ in these bases. Consider on $E \oplus F$, the following base

$$(\eta_e + \zeta_f, \eta_e + \eta_{e-1} + \zeta_{f-1}, \dots, \eta_e + \cdots + \eta_1 + \zeta_{f-e+1}, \eta_1 + \zeta_{f-e}, \dots, \eta_1 + \zeta_1)$$

and the associated flag $F^{E\oplus F}_\bullet$. One easily checks that $x = (F^E_\bullet, F^F_\bullet, F^{E\oplus F}_\bullet)$ works. $\square$

5 Faces of $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$

5.1 Murnaghan's face

The cone $\mathbb{Q}_{\geq 0}\,\mathrm{Kron}(e+1, f+1, e+f+1)$ is contained in the linear subspace of points $(\alpha, \beta, \gamma) \in \mathbb{Q}^{e+1} \times \mathbb{Q}^{f+1} \times \mathbb{Q}^{e+f+1}$ that satisfy $|\alpha| = |\beta| = |\gamma|$. In particular its dimension is at most $2e + 2f + 1$.

Recall that $\bar{\alpha} = (\alpha_2 \geq \alpha_3 \cdots)$, if $\alpha = (\alpha_1 \geq \alpha_2 \cdots)$. By Proposition 1, the points (α, β, γ) in $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$ satisfy

$$|\bar{\alpha}| + |\bar{\beta}| \geq |\bar{\gamma}|. \tag{35}$$

The set of points of $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$ such that equality holds in the inequality (35) is a face $\mathcal{F}^M$ (M stands for Murnaghan) of the cone $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$. Consider the linear map

$$\begin{array}{rccc} \pi : & \mathbb{Q}^{2e+2f+3} & \longrightarrow & \mathbb{Q}^{2e+2f} \\ & (\alpha, \beta, \gamma) & \longmapsto & (\bar{\alpha}, \bar{\beta}, \bar{\gamma}). \end{array}$$

Lemma 2 *The face $\mathcal{F}^M$ maps by π to $\mathbb{Q}_{\geq 0} \operatorname{LR}(e, f, e+f)$. Moreover each fiber of π over $\mathbb{Q}_{\geq 0} \operatorname{LR}(e, f, e+f)$ contains an unbounded interval.*

The cone $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$ has dimension $2e+2f+1$ and the face $\mathcal{F}^M$ has dimension $2e+2f$.

Proof. Assume that equality holds in the formula (35). Assume also that the coordinates of α, β, and γ are nonnegative integers. Then

$$g_{\alpha\,\beta\,\gamma} = c_{\bar{\alpha}\,\bar{\beta}}^{\bar{\gamma}}. \tag{36}$$

Thus the face $\mathcal{F}^M$ maps by π on $\mathbb{Q}_{\geq 0} \operatorname{LR}(e, f, e+f)$. Conversely let $(\lambda, \mu, \nu) \in \operatorname{LR}(e, f, e+f)$. Let a be an integer and set $b = a + |\lambda| - |\mu|$ and $c = a + |\lambda| - |\nu|$. If a is big enough then $a \geq \lambda_1$, $b \geq \mu_1$ and $c \geq \nu_1$. Therefore $\alpha := (a, \lambda)$, $\beta = (b, \mu)$ and $\gamma = (c, \nu)$ are three partitions of the same integer such that equality holds in the inequality (35). Thus the equality (36) holds and (α, β, γ) belongs to $\mathcal{F}^M$. In particular the fiber $\pi^{-1}(\lambda, \mu, \nu)$ contains an unbounded segment.

Since $\mathbb{Q}_{\geq 0} \operatorname{LR}(e, f, e+f)$ has dimension $2e+2f-1$ and the fibers of π have dimension at least one, the cone $\mathcal{F}^M$ has dimension at least $2e+2f$. We had already noticed that $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$ has dimension at most $2e+2f+1$. These two inequalities (and the fact that $\mathcal{F}^M$ is a strict face of the cone $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$) imply the lemma. $\square$

5.2 Proof of Theorem 2

Let r, s, I, J, and K be like in Theorem 2. To such a triple (I, J, K), Theorem 7 associates a codimension one face of $\operatorname{LR}(e, f, e+f)$. Using Lemma 2, this face corresponds to a face $\mathcal{F}_{IJK}$ of $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e+1, f+1, e+f+1)$

of codimension two. Explicitly, $\mathcal{F}_{IJK}$ is the set of $(\alpha,\beta,\gamma)\in\mathbb{Q}_{\geq 0}\operatorname{Kron}(e+1,f+1,e+f+1)$ such that

$$\begin{cases} |\bar{\gamma}| = |\bar{\alpha}| + |\bar{\beta}|, \\ |\bar{\gamma}_K| = |\bar{\alpha}_I| + |\bar{\beta}_J|. \end{cases} \tag{37}$$

This face $\mathcal{F}_{IJK}$ is contained in two codimension one faces, $\mathcal{F}^M$ and another one $\mathcal{F}^M_{IJK}$ that we want to determine.

Let φ_τ denote the linear form defined by

$$\varphi_\tau(\alpha,\beta,\gamma) = \tau(|\bar{\alpha}| + |\bar{\beta}| - |\bar{\gamma}|) + (|\bar{\alpha}_I| + |\bar{\beta}_J| - |\bar{\gamma}_K|),$$

where τ is any rational number. Set also

$$\varphi_\infty(\alpha,\beta,\gamma) = |\bar{\alpha}| + |\bar{\beta}| - |\bar{\gamma}|.$$

By the theory of convex polyhedral cones, there exists τ_0 such that for any $\tau > \tau_0$, φ_τ is nonnegative on the cone and the associated face is $\mathcal{F}_{IJK}$, and, φ_{τ_0} corresponds to $\mathcal{F}^M_{IJK}$.

Here, E and F are two linear spaces of dimension $e+1$ and $f+1$ and $G = \operatorname{GL}(E)\times\operatorname{GL}(F)$. Consider the variety

$$X = \mathcal{F}l(E)\times\mathcal{F}l(F)\times\mathcal{F}l(1,\cdots,e+f+1;E\otimes F).$$

We identify $\operatorname{Pic}^G(X)$ with $\mathbb{Z}^{2e+2f+3}$ like in Section 2.2.4.

Geometric description of φ_∞. The inequality corresponding to $\mathcal{F}^M$ is $\varphi_\infty \geq 0$. By Section 2.2.4, $\mathcal{F}^M$ generates a face of $\mathbb{Q}_{\geq 0}\operatorname{LR}(G,X)$. Theorem 5 shows that there exists a well covering pair $(C_\infty,\lambda_\infty)$ of X such that $\varphi_\infty(\alpha,\beta,\gamma) = -\mu^{\mathcal{L}^\alpha\otimes\mathcal{L}^\beta\otimes\mathcal{L}_\gamma}(C_\infty,\lambda_\infty)$.

To describe such a pair $(C_\infty,\lambda_\infty)$, fix decompositions $E=\bar{E}\oplus l$ and $F=\bar{F}\oplus m$, where $\bar{E}$ and $\bar{F}$ are hyperplanes and l and m are lines. Let λ_∞ be the one-parameter subgroup of G acting with weight 1 on $\bar{E}$ and $\bar{F}$, and with weight 0 on l and m. Let C_∞ be the set of points in X such that

- the hyperplanes of the complete flags of E and F are respectively $\bar{E}$ and $\bar{F}$,
- the line of the partial flag of $E\otimes F$ is $l\otimes m$,
- the $(e+f+1)$-dimensional subspace of the partial flag of $E\otimes F$ is $(l\otimes m)\oplus(\bar{E}\otimes m)\oplus(l\otimes\bar{F})$.

One can check that $(C_\infty, \lambda_\infty)$ works (see [Res11c] for details).

Geometric description of φ_τ. Fix decompositions $\bar{E} = E_+ \oplus E_-$ and $\bar{F} = F_+ \oplus F_-$, where E_+ and F_+ have dimension r and s. Assume that $\tau > 1$ and write $\tau = \frac{p}{q}$ with two integers p and q satisfying $p \wedge q = 1$ and $q > 0$. Let λ_τ be the one parameter subgroup of G acting with weight $q + p$ on E_+ and F_+, with weight p on E_- and F_- and with weight 0 on l and m. The weight spaces of the action of λ_τ on $E \otimes F$ are

Space	$E_+ \otimes F_+$	$E_+ \otimes F_- \oplus E_- \otimes F_+$	$E_- \otimes F_-$	$E_+ \oplus F_+$	$E_- \oplus F_-$	$l \otimes m$
Weight	$2p + 2q$	$2p + q$	$2p$	$p + q$	p	0

where some "$\otimes m$" and "$l \otimes$" have been forgotten.

To $I^\vee = \{e + 1 - i \,:\, i \in I\}$ is associated an embedding $\iota_{I^\vee}$ of $\mathcal{F}l(E_+) \times \mathcal{F}l(E_-)$ in $\mathcal{F}l(\bar{E})$. Explicitly

$$\begin{array}{rccl} \iota_{I^\vee} : & \mathcal{F}l(E_+) \times \mathcal{F}l(E_-) & \longrightarrow & \mathcal{F}l(\bar{E}) \\ & ((V_i), (W_j)) & \longmapsto & (V_{\# I^\vee \cap [1,k]} \oplus W_{k - \# I^\vee \cap [1,k]})_{1 \leq k \leq e.} \end{array}$$

Similarly we consider $\iota_{J^\vee}$ and ι_K. Observe that C_∞ is canonically isomorphic to $\mathcal{F}l(\bar{E}) \times \mathcal{F}l(\bar{F}) \times \mathcal{F}l(\bar{E} \oplus \bar{F})$. Consider the embedding $(\iota_{I^\vee}, \iota_{J^\vee}, \iota_K)$ of

$$\mathcal{F}l(E_+) \times \mathcal{F}l(E_-) \times \mathcal{F}l(F_+) \times \mathcal{F}l(F_-) \times \mathcal{F}l(E_+ \oplus F_+) \times \mathcal{F}l(E_- \oplus F_-)$$

in C_∞. Denote by C_τ its image. Using for example [Res11b, Proposition 1 and Theorem 1], one can check that, for τ big enough, (C_τ, λ_τ) is a well covering pair. Moreover, $\varphi_\tau(\alpha, \beta, \gamma) = -q \mu^{\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma}(C_\tau, \lambda_\tau)$.

For any $\tau > 1$, C_τ is an irreducible component of λ_τ. Moreover, C_τ^+ and $P(\lambda_\tau)$ do not depend on $\tau > 1$. In Particular, (C_τ, λ_τ) is a well covering pair for any $\tau > 1$.

Theorem 5 shows that the face determined by the inequality φ_τ only depends on C_τ, and so does not depend on $\tau > 1$: it is $\mathcal{F}_{IJK}$. This implies that $\varphi_1 \geq 0$ on $\mathbb{Q}_{\geq 0} \operatorname{Kron}(e + 1, f + 1, e + f + 1)$. Theorem 2 follows. $\square$

Remark. Let $\mathcal{F}_1$ denote the face associated to φ_1. Up to now, we have not proved that $\mathcal{F}_1$ has codimension 1 or equivalently that $\mathcal{F}_1 = \mathcal{F}_{IJK}^M$. This is the aim of Section 7.

6 Proof of Theorem 3

Keeping the notation of Section 5.2, we give a geometric description of φ_1. The weight spaces of the action of λ_1 on $E \otimes F$ are

$E_+\otimes F_+$	$E_+\otimes F_-\oplus E_-\otimes F_+$	$E_-\otimes F_-\oplus E_+\oplus F_+$	$E_-\oplus F_-$	$l\otimes m$
4	3	2	1	0

The irreducible component C_1 of X^{λ_1} containing C_τ (for $\tau>1$) is isomorphic to

$$\begin{array}{ll} & \mathcal{F}l(E_+)\times\mathcal{F}l(E_-)\times\mathcal{F}l(F_+)\times\mathcal{F}l(F_-)\\ \times & \mathcal{F}l(1,\dots,r+s;E_-\otimes F_-\oplus E_+\oplus F_+)\times\mathcal{F}l(E_-\oplus F_-).\end{array}$$

Moreover, $C_1^+=C_\tau^+$ and $P(\lambda_1)=P(\lambda_\tau)$. In particular the pair (C_1,λ_1) is well covering.

Let G^{λ_1} denote the centralizer of λ_1 in G. Note that $G^{\lambda_1}=\mathrm{GL}(E_+)\times\mathrm{GL}(E_-)\times\mathbb{C}^*\times\mathrm{GL}(F_+)\times\mathrm{GL}(F_-)\times\mathbb{C}^*$. By [Res11c, Theorem 2], $g_{\alpha\beta\gamma}$ is the dimension of

$$\mathrm{H}^0(C_1,(\mathcal{L}^\alpha\otimes\mathcal{L}^\beta\otimes\mathcal{L}_\gamma)_{|C_1})^{G^{\lambda_1}}.$$

We have to determine the restriction $(\mathcal{L}^\alpha\otimes\mathcal{L}^\beta\otimes\mathcal{L}_\gamma)_{|C_1}$ via the identification of C_1 with a product of flag varieties. Fix a basis of $\bar{E}$ starting with a basis of E_+ followed by a basis of E_-. For the group $\mathrm{GL}(\bar{E})$ we consider standard maximal tori and Borel subgroups in this basis. Similarly, we choose subgroups of $\mathrm{GL}(\bar{F})$.

The maximal torus of $\mathrm{GL}(E)$ acts on the fiber in $\mathcal{L}^\alpha$ over the base point of $\mathcal{F}l(E)$ with weight $(\alpha_{e+1},\dots,\alpha_1)$. The maximal torus $\bar{T}$ of $\mathrm{GL}(\bar{E})$ acts on the fiber in $\mathcal{L}^\alpha$ over the base point of $\mathcal{F}l(\bar{E})$ (embedded in $\mathcal{F}l(E)$ like C_∞ is embedded in X) with weight $(\alpha_{e+1},\dots,\alpha_2)$. Let $w_{I^\vee}$ in the symmetric group S_e associated to $I^\vee$ ($w_{I^\vee}(k)$ is the k^{th} elements of $I^\vee$ and $w_{I^\vee}(r+k)$ is the k^{th} elements of $I_-^\vee$). Then $\iota_{I^\vee}$ maps the base point of $\mathcal{F}l(E_+)\times\mathcal{F}l(E_-)$ to the image by $w_{I^\vee}^{-1}$ of the base point of $\mathcal{F}l(\bar{E})$. It follows that $\bar{T}$ acts on the fiber in $\iota_{I^\vee}^*(\mathcal{L}^\alpha_{|C_\infty})$ by the weight $w_{I^\vee}^{-1}(\alpha_{e+1},\dots,\alpha_2)$. After computation this gives

$$\mathrm{H}^0(C_1,(\mathcal{L}^\alpha\otimes\mathcal{L}^0\otimes\mathcal{L}_0)_{|C_1})=S^{\bar{\alpha}_{I_+}}E_+\otimes S^{\bar{\alpha}_{I_-}}E_-.$$

Similarly

$$\mathrm{H}^0(C_1,(\mathcal{L}^0\otimes\mathcal{L}^\beta\otimes\mathcal{L}_0)_{|C_1})=S^{\bar{\beta}_{J_+}}F_+\otimes S^{\bar{\beta}_{J_-}}F_-,$$

and

$$\mathrm{H}^0(C_1,(\mathcal{L}^0\otimes\mathcal{L}^0\otimes\mathcal{L}_\gamma)_{|C_1})=S^{\bar{\gamma}_{K_+}}(E_-\otimes F_-\oplus E_+\oplus F_+)^*\otimes S^{\bar{\gamma}_{K_-}}(E_-\oplus F_-)^*.$$

We deduce that $g_{\alpha\beta\gamma}$ is the multiplicity of the $\mathrm{GL}(E_+)\times\mathrm{GL}(E_-)\times\mathrm{GL}(F_+)\times\mathrm{GL}(F_-)$-simple module

$$S^{\bar{\alpha}_{I+}}E_+\otimes S^{\bar{\alpha}_{I-}}E_-\otimes S^{\bar{\beta}_{J+}}F_+\otimes S^{\bar{\beta}_{J-}}F_-$$

in the module

$$S^{\bar{\gamma}_{K+}}(E_-\otimes F_-\oplus E_+\oplus F_+)\otimes S^{\bar{\gamma}_{K-}}(E_-\oplus F_-). \tag{38}$$

Now the theorem is obtained by using repeatedly the formulas (2), (18) and (21) to decompose the module (38).

7 Proof of Theorem 4

Recall that the aim is to prove that $\mathcal{F}_1$ has codimension one. Since $\mathcal{F}_{IJK}$ has codimension two and it is contained in $\mathcal{F}_1$, it remains to prove that $\mathcal{F}_{IJK}\neq\mathcal{F}_1$.

Assume now that (α,β,γ) belongs $\mathcal{F}_{IJK}$. Since $\mathcal{F}_{IJK}$ is contained in $\mathcal{F}^M$, $g_{\alpha\beta\gamma}=c_{\bar\alpha\bar\beta}^{\bar\gamma}$. Then (see *eg* [DW11, Theorem 7.4]) $g_{\alpha\beta\gamma}=c_{\bar\alpha_I\bar\beta_J}^{\bar\gamma_K}.c_{\bar\alpha_{I_-}\bar\beta_{J_-}}^{\bar\gamma_{K_-}}$. In particular, Lemma 1 shows that $g_{n\alpha\, n\beta\, n\gamma}$ is a polynomial function of n of degree at most

$$\begin{aligned} d_{max}=&\binom{r}{2}+\binom{s}{2}+\binom{r+s}{2}+\binom{a}{2}+\binom{b}{2}+\binom{a+b}{2}\\ &-r^2-s^2-a^2-b^2+2, \end{aligned} \tag{39}$$

where $a=e-r$ and $b=f-s$.

Given an algebraic group Γ acting on an irreducible variety Y, we denote by $\mathrm{mod}(\Gamma,Y)$ the minimal codimension of the Γ-orbits. By [Res11a, Lemma 2], for any $\mathcal{L}$ in the relative interior of $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G^{\lambda_1},C_1)$, the dimension of $C_1^{\mathrm{ss}}(\mathcal{L})/\!/G^{\lambda_1}$ is equal to $\mathrm{mod}(G^{\lambda_1},C_1)$.

By [Res10, Theorem 4], there exists a line bundle $\mathcal{M}$ on X such that $\mathcal{M}_{|C_1}$ belongs to the relative interior of $\mathbb{Q}_{\geq 0}\,\mathrm{LR}(G^{\lambda_1},C_1)$. We may assume that $\mathcal{M}=\mathcal{L}^\alpha\otimes\mathcal{L}^\beta\otimes\mathcal{L}_\gamma$ for three partitions α, β and γ. But, by [Res10, Theorem 8], $X^{\mathrm{ss}}(\mathcal{M})/\!/G\simeq C_1^{\mathrm{ss}}(\mathcal{M}_{|C_1})/\!/G^{\lambda_1}$. It follows that $\mathcal{M}$ is a point on $\mathcal{F}_1$ satisfying $\dim(X^{\mathrm{ss}}(\mathcal{M})/\!/G)=\mathrm{mod}(G^{\lambda_1},C_1)$. In particular,

$$n\mapsto g_{n\alpha\, n\beta\, n\gamma}\quad \begin{array}{l}\text{cannot be a polynomial function}\\ \text{of degree less than } \mathrm{mod}(G^{\lambda_1},C_1).\end{array} \tag{40}$$

Regarding the assertions (39) and (40), to prove that $\mathcal{F}_{IJK} \neq \mathcal{F}_1$ it is sufficient to prove the claim: $\mathrm{mod}(G^{\lambda_1}, C_1) > d_{max}$. The center of G^{λ_1} contains a dimension 3 torus acting trivially on C_1. Hence

$$\mathrm{mod}(G^{\lambda_1}, C_1) \geq \dim(C_1) - \dim(G^{\lambda_1}) + 3.$$

After simplification, we get

$$\mathrm{mod}(G^{\lambda_1}, C_1) - d_{max} \geq ab(r + s) - 1.$$

Since a, b, r, and s are positive integers, the claim follows.

8 Proof of Theorem 1

We keep notation of Section 5.2, but now $r = e$ and $I = \{1, \dots, e\}$. In particular E_- is trivial and the weight spaces of the action of λ_τ of $E \otimes F$ are

$\bar{E} \otimes F_+$	$\bar{E} \otimes F_-$	$\bar{E} \oplus F_+$	F_-	$l \otimes m$
$2p + 2q$	$2p + q$	$p + q$	p	0

Hence $C_\tau \simeq \mathcal{F}l(\bar{E}) \times \mathcal{F}l(F_+) \times \mathcal{F}l(\bar{E} \oplus F_+)$ for τ big enough. Then, for any $\tau > 0$, (C_τ, λ_τ) is a well covering pair. We conclude like in Section 5.2 that φ_0 is nonnegative on the Kronecker cone. This proves the first assertion of the theorem.

Consider now the limit case $\tau = 0$. The weight spaces are

$\bar{E} \otimes F_+$	$\bar{E} \otimes (F_- \oplus m) \oplus F_+$	$l \otimes (F_- \oplus m)$
2	1	0

Hence $C_0 \simeq \mathcal{F}l(\bar{E}) \times \mathcal{F}l(F_+) \times \mathbb{P}(F_- \oplus m) \times \mathcal{F}l(1, \dots, e+f-1; \bar{E} \otimes (F_- \oplus m) \oplus F_+) \times \mathbb{P}(F_- \oplus m)$. Moreover $G^{\lambda_0} \simeq \mathrm{GL}(\bar{E}) \times \mathbb{C}^* \times \mathrm{GL}(F_+) \times \mathrm{GL}(F_- \oplus m)$. By [Res11c, Theorem 2]

$$g_{\alpha\beta\gamma} = \dim(\mathrm{H}^0(C_0, (\mathcal{L}^\alpha \otimes \mathcal{L}^\beta \otimes \mathcal{L}_\gamma)_{|C_0})^{G^{\lambda_0}}).$$

The computation of this dimension is made using the formulas (2), (18) and (21) like in Section 6.

9 A final inequality

All but two of the inequalities of Theorem 7 had been extended to the Kronecker coefficients by Theorems 1 and 2. The two exceptions are $\alpha_i \leq \gamma_i$ and $\beta_j \leq \gamma_j$. Consider the second one, up to permuting (α, e) and (β, f). The extended inequality is

$$\alpha_1 + \beta_1 - \beta_j - n \leq \gamma_1 - \gamma_j,$$

for any $f + 1 \geq j \geq 2$.

This inequality is satisfied if $g_{\alpha\beta\gamma} \neq 0$. The proof is obtained by considering $I = \emptyset$ and $J = K = \{j - 1\}$ in Section 5.2.

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