Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

REGULAR LATTICE POLYTOPES AND ROOT SYSTEMS

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Abstract

Consider a lattice in a real finite dimensional vector space. Here, we are interested in the lattice polytopes, that is the convex hulls of finite subsets of the lattice. Consider the group G of the affine real transformations which map the lattice onto itself. Replacing the group of Euclidean motions by the group G one can define the notion of regular lattice polytopes. More precisely, a lattice polytope is said to be regular if the subgroup of G which preserves the polytope acts transitively on the set of its complete flags. Recently, Karpenkov obtained a classification of the regular lattice polytopes. Here we obtain this classification by using root systems.

1. *Introduction*

Let Λ be a lattice in a real finite dimensional vector space V. Here, we are interested in the lattice polytopes, that is the convex hulls of finite subsets of Λ. Consider the group G of the affine real transformations which map Λ onto itself. Replacing the group of Euclidean motions by the group G in the definition of regular polytopes, one can define the notion of regular lattice polytopes. More precisely, for a lattice polytope $\mathcal{P}$, we denote by $\mathrm{Isom}(\mathcal{P})$ the subgroup of G which preserves $\mathcal{P}$ and $\mathcal{P}$ is said to be a *regular lattice polytope* if the group $\mathrm{Isom}(\mathcal{P})$ acts transitively on the set of complete flags of $\mathcal{P}$.

The goal of this paper is to classify the lattice regular polytopes up to G and homotheties (see Section 2). Since $\mathrm{Isom}(\mathcal{P})$ is finite, there exists an invariant scalar product on V, and so $\mathcal{P}$ is an ordinary regular polytope. So, in a sense, there are less lattice regular polytopes than ordinary ones. But, the lattice equivalence relation is finer than the Euclidean one. For example, there are two non equivalent lattice regular triangles (see Figure 2). More generally, there are $\tau(n+1)$ lattice regular simplices of dimension n, where $\tau(n+1)$ denotes the number of divisors of $n+1$.

The classification in question has already been obtained by Karpenkov in [**Kar06**]: for each ordinary regular polytope he studies the possibilities to realize it as a lattice polytope. Our approach is completely different. For example, we do not use the Euclidean classification of regular polytopes but those simpler and more central in mathematics of root systems. Our method seem to be enlightening: for example, the $\tau(n+1)$ simplices are canonically associated to the $\tau(n+1)$ subgroups of $\mathbb{Z}/(n+1)\mathbb{Z}$.

Let us explain our approach in more detail. Firstly, we associate in a very natural way a reduced simply laced root system (not necessarily irreducible) to any regular lattice polytope $\mathcal{P}$. Then considering the faces of $\mathcal{P}$, we even show that the only possible root systems are of type A_n, D_n, E_6, E_7, E_8 and $(A_1)^n$ (later, we show that the exceptional root systems do not occur). Conversely, we fix such a root system Φ and seek all the regular lattice polytopes $\mathcal{P}$ with Φ as associated root system. Such a polytope is characterized up to isomorphism by a lattice between the root lattice and the weight lattice, and a dominant weight. We obtain in this way the list presented in Table 1.

A classical tool in the study of the Euclidean regular polytopes is the notion of duality. In our context, we also use this idea. Surprisingly, we show that there exist two different notions of duality for the regular lattice polytopes (see Section 4).

2000 *Mathematics Subject Classification* 00000.

For the convenience of the reader we also present below the regular lattice polytopes of dimension two in Figures 1, 2 and 3. In each figure, we have two lattices: the intersection of the gray lines and the marked points. These lattices are the weight lattices Λ_P and the root lattices Λ_R of of the root system A_2 in the two first case and $A_1 \times A_1$ in the last one. In Figure 1, we have drawn an hexagon which can be considered as a lattice polygon in Λ_P or Λ_R: this gives two classes of regular hexagons. In Figure 3, the situation is similar with squares instead hexagons. In Figure 2, we have two triangles: the dashed one in Λ_P and the other one in Λ_R. The result in dimension 2 asserts that up to evident equivalence (see Section 2) the only regular lattice polygons are these 2 hexagons, these 2 triangles and these 2 squares.

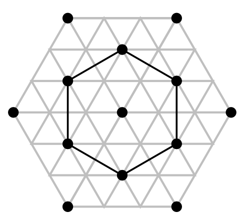


FIG. 1. *Two Hexagons*

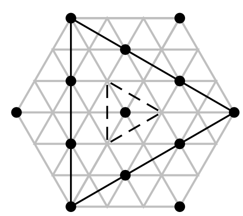


FIG. 2. *Two Triangles*

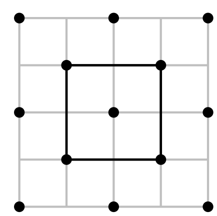


FIG. 3. *Two Squares*

Note that there is a more classical way to classify Euclidean regular polytopes by associating a root system (without the cristallographic condition) to each Euclidean regular polytope (see for example [**FR05**]); in this association the Weyl group of the root system is the isometry group of the polytope. Our construction is different; for example, the root system associated to the square is of type B_2 in the classical case, while here we associate the root system of type $A_1 \times A_1$ to the squares. Note that using the classification of root systems without the cristallographic condition (see for example [**Hum90**]) and the method of this article, one can obtain a new proof of the classification of Euclidean regular polytopes.

Finally we briefly mention the well-known link between convex polytopes and algebraic geometry. We do not use this link but our inspiration for some results are of geometric origin. To each lattice polytope $\mathcal{P}$ one can associate a toric variety $X_{\mathcal{P}}$ with T as torus, see for example [**Oda88**]. The polytope $\mathcal{P}$ is regular if and only if the group of regular toric automorphisms of $X_{\mathcal{P}}$ acts transitively on the set of the maximal chains of irreducible T-stable subvarieties of $X_{\mathcal{P}}$. In [**Pro90**], Procesi consider the toric variety X_Φ associated to the decomposition in Weyl chambers of the root system Φ. Our first results(see Proposition 3.4) can be translated in the following way: there exists an equivariant surjective morphism from X_Φ onto $X_{\mathcal{P}}$ if Φ is the root system associated to the regular lattice polytope $\mathcal{P}$.

Convention. In this paper, we only consider non degenerated polytopes that is which span the ambient real affine space.

2. *An equivalence relation*

Let Λ be a free abelian group of rank n. Let $\hat{\Lambda}$ be a set with a free transitive action of Λ denoted by: $m + z$ for any $m \in \hat{\Lambda}$ and $z \in \Lambda$. Such a set $\hat{\Lambda}$ is called a Λ-*affine space*. A map $f : \hat{\Lambda} \longrightarrow \hat{\Lambda}$ is said to be *affine* if there exists a group morphism $\overrightarrow{f} : \Lambda \longrightarrow \Lambda$ such that $f(m+z) = f(m) + \overrightarrow{f}(z)$.

Let $\mathrm{GL}(\Lambda) \simeq \mathrm{GL}_n(\mathbb{Z})$ denote the automorphism group of Λ and $\mathrm{GA}(\hat{\Lambda})$ be the group of bijective affine maps of $\hat{\Lambda}$. We have the following split exact sequence:

$$1 \longrightarrow \Lambda \longrightarrow \mathrm{GA}(\hat{\Lambda}) \longrightarrow \mathrm{GL}(\Lambda) \longrightarrow 1.$$

Consider $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$ and its affine space $\hat{\Lambda}_{\mathbb{R}} := m + \Lambda_{\mathbb{R}}$ (for any $m \in \hat{\Lambda}$). Now, $\hat{\Lambda}$ is a lattice in $\hat{\Lambda}_{\mathbb{R}}$ and $\mathrm{GA}(\hat{\Lambda})$ is the subgroup of the isomorphism group $\mathrm{GA}(\hat{\Lambda}_{\mathbb{R}})$ of $\hat{\Lambda}_{\mathbb{R}}$ of the elements which map $\hat{\Lambda}$ onto itself.

A lattice polytope $\mathcal{P}$ is the convex hull in $\hat{\Lambda}_{\mathbb{R}}$ of a finite subset of $\hat{\Lambda}$. Set

$$\mathrm{Isom}(\mathcal{P}) = \{g \in \mathrm{GA}(\hat{\Lambda}) \mid g\mathcal{P} = \mathcal{P}\}.$$

A lattice polytope $\mathcal{P}$ is said to be *regular* if $\mathrm{Isom}(\mathcal{P})$ acts transitively on the set of complete flags of $\mathcal{P}$. We want to classify the regular polytopes modulo $\mathrm{GA}(\hat{\Lambda})$ of course, but there is another reduction. We note that if h is a homothety of center in $\hat{\Lambda}$ and integer ratio then h normalize $\mathrm{GA}(\hat{\Lambda})$ and if $\mathcal{P}$ is a regular lattice polytope so is $h(\mathcal{P})$. Finally we want to classify the regular lattice polytopes up to the group generated by $\mathrm{GA}(\hat{\Lambda})$ and the homotheties of center in $\hat{\Lambda}$ and integer ratio. So we define:

DEFINITION 1. We denote $\mathcal{H}$ the subgroup of $\mathrm{GA}(\hat{\Lambda}_{\mathbb{R}})$ generated by $\mathrm{GA}(\hat{\Lambda})$ and the homotheties of center in $\hat{\Lambda}$ and integer ratio.

Actually, this group doesn't acts on the set of lattice polytopes, but on those with rational vertices. Nevertheless, this group defines a equivalence relation on the lattice polytopes. Our first reduction is to choose a common origin for the polytopes. More precisely let us fix an origin O in $\hat{\Lambda}$; now, one can identify Λ and $\hat{\Lambda}$, and embed $\mathrm{GL}(\Lambda)$ in $\mathrm{GA}(\hat{\Lambda})$). Now we can define:

DEFINITION 2. We call a lattice polytope *centered* if its barycenter is O; it is said to be *primitive* if it is not the image of another polytope by an homothety of center O and of integer ratio bigger than one.

The following proposition reduce the classification to those of primitive centered regular lattice polytope.

PROPOSITION 2.1.

(i) *Let $\mathcal{P}$ be a lattice polytope. There exits $g \in \mathcal{H}$ such that $g(\mathcal{P})$ is a primitive centered lattice polytope.*

(ii) *Conversely, if $g \in \mathcal{H}$ and $\mathcal{P}_1, \mathcal{P}_2$ are two primitive centered lattice polytopes such that $g(\mathcal{P}_1) = \mathcal{P}_2$, then $g \in \mathrm{GL}(\Lambda)$.*

Proof. The first point is obvious. For the second one, note that if $g \in \mathcal{H}$ then there exists $r \in \mathbb{Q}$ such that $r\overrightarrow{g}(\Lambda) = \Lambda$. Then let $g \in \mathcal{H}, \mathcal{P}_1, \mathcal{P}_2$ be such $g(\mathcal{P}_1) = \mathcal{P}_2$. As $\mathcal{P}_1$ and $\mathcal{P}_2$ are centered, we have $g(O) = O$ *i.e.* $g = \overrightarrow{g}$. So, we deduce there exist $r \in \mathbb{Q}$ and $h \in \mathrm{GL}(\Lambda)$ such that $r.h(\mathcal{P}_1) = \mathcal{P}_2$. But as $\mathcal{P}_1$ and $\mathcal{P}_2$ are primitive, so $r = 1$. □

From now on, we want to classify the primitive centered regular lattice polytopes up to the action of $\mathrm{GL}(\Lambda)$.

3. *Root systems*

For root systems, we will use the notation of [**Bou02**]. Let $\mathcal{P}$ be a regular lattice polytope in $\Lambda_{\mathbb{R}}$. For each edge a of $\mathcal{P}$ with vertices s_1 and s_2 we consider the subgroup $\mathbb{R}.\overrightarrow{s_1 s_2} \cap \Lambda$ of Λ and its two generators $\pm u_a$. When a runs over all the edges of $\mathcal{P}$, the $\pm u_a$ form a finite subset $\Phi(\mathcal{P})$ of Λ.

PROPOSITION 3.1. *The subset $\Phi(\mathcal{P})$ of $\Lambda_{\mathbb{R}}$ is a reduced root system.*

Proof. It is clear that $\Phi(\mathcal{P})$ is finite, does not contain zero, spans $\Lambda_{\mathbb{R}}$ and $\mathbb{Z}.\alpha \cap \Phi(\mathcal{P}) = \{\pm\alpha\}$ for any $\alpha \in \Phi(\mathcal{P})$.

Let $\alpha \in \Phi(\mathcal{P})$ and two vertices s_1 and s_2 on an edge a parallel to α. Consider a complete flag $\mathcal{D}_1$ of $\mathcal{P}$ starting with s_1 and a. Let $\mathcal{D}_2$ be the complete flag of $\mathcal{P}$ with the same faces from $\mathcal{D}_1$ except for the vertex which is s_2. Let $\sigma \in \mathrm{Isom}(\mathcal{P})$ such that $\sigma(\mathcal{D}_1) = \mathcal{D}_2$. It is clear that $\vec{\sigma}$ is a reflection which maps $\Phi(\mathcal{P})$ in $\Phi(\mathcal{P})$ and α on $-\alpha$.

Let β be another element of $\Phi(\mathcal{P})$. The vector $\vec{\sigma}(\beta) - \beta$ is an element of Λ proportional to α. Since α has been chosen primitive, $\vec{\sigma}(\beta) - \beta$ is an entire multiple of α. □

The root system $\Phi(\mathcal{P})$ is said to be *associated* to $\mathcal{P}$. We denote by Λ_P and by Λ_R, respectively the weight and root lattices of $\Phi(\mathcal{P})$. We have:

PROPOSITION 3.2. *The lattices Λ, Λ_R and Λ_P satisfy: $\Lambda_R \subset \Lambda \subset \Lambda_P$.*

Proof. The inclusion $\Lambda_R \subset \Lambda$ is true by definition of $\Phi(\mathcal{P})$. Let $\alpha \in \Phi(\mathcal{P})$ and $\lambda \in \Lambda$. We have to prove that $< \lambda, \alpha^\vee >$ belongs to $\mathbb{Z}$, where $\alpha^\vee$ is the coroot associated to α. But, $\sigma_\alpha(\lambda) = \lambda - < \lambda, \alpha^\vee > \alpha$ belongs to Λ. We can conclude since α is primitive on Λ. □

In the two following propositions, $\mathcal{P}$ is assumed to be centered in O. In this case, $\mathrm{Isom}(\mathcal{P})$ is a subgroup of $\mathrm{GL}(\Lambda)$. Let $\mathrm{Aut}(\Phi(\mathcal{P})) = \{g \in \mathrm{GL}(\Lambda_{\mathbb{R}}) \mid g.\Phi(\mathcal{P}) = \Phi(\mathcal{P})\}$ denote the automorphism group of $\Phi(\mathcal{P})$ and W denote the Weyl group of $\Phi(\mathcal{P})$. Note that $\mathrm{Aut}(\Phi(\mathcal{P}))$ is the semidirect product of W and the automorphisms of the Dynkin diagram of $\Phi(\mathcal{P})$.

PROPOSITION 3.3. *Let $\mathcal{P}$ be a centered regular lattice polytope. We have:*
(i) $W \subset \mathrm{Isom}(\mathcal{P}) \subset \mathrm{Aut}(\Phi(\mathcal{P}))$.
(ii) *The lattice Λ is stable by* $\mathrm{Isom}(\mathcal{P})$.
(iii) *The root system $\Phi(\mathcal{P})$ is homogeneous under* $\mathrm{Isom}(\mathcal{P})$.

Proof. The inclusion $W \subset \mathrm{Isom}(\mathcal{P})$ is a direct consequence of the proof of Proposition 3.1. The rest of the proposition is obvious. □

The group $\mathrm{Isom}(\mathcal{P})$ acts transitively on the set of vertices of $\mathcal{P}$; the following proposition shows a little bit more:

PROPOSITION 3.4. *The regular lattice polytope $\mathcal{P}$ is assumed to be centered. The Weyl group W acts transitively on the set of vertices of $\mathcal{P}$.*

Proof. Since any edge of $\mathcal{P}$ is parallel to a root of $\Phi(\mathcal{P})$, any maximal cone of the dual fan of $\mathcal{P}$ is an union of Weyl chambers. But, W acts transitively on the set of Weyl chambers. The proposition follows. □

A face of a regular polytope is a regular polytope. Here, one can say a little bit more:

PROPOSITION 3.5. *Let $\mathcal{F}$ be a face of a regular lattice polytope $\mathcal{P}$ and F its direction. Then, $\mathcal{F}$ is a regular lattice polytope with associated root system $\Phi(\mathcal{P}) \cap F$.*

Proof. We may assume that $\mathcal{P}$ is centered. It is clear that $\mathcal{F}$ is a regular lattice polytope with root system $\Phi(\mathcal{F})$ contained in $\Phi(\mathcal{P}) \cap F$. Let $\alpha \in \Phi(\mathcal{P}) \cap F$: we have to prove that α is parallel to an edge of $\mathcal{F}$.

We claim that the reflection σ_α of W associated to α stabilizes $\mathcal{F}$. Let A be a point of $\mathcal{F}$. The vector $\sigma_\alpha(A) - A$ is collinear to α and so belongs to F. But, $\mathcal{F} = (A + F) \cap \mathcal{P}$; and so $\sigma_\alpha(A)$ belongs to $\mathcal{F}$.

Consider the kernel H_α of $\sigma_\alpha - \mathrm{Id}$. Firstly, we assume that $H_\alpha \cap \mathcal{F}$ does not contain any vertex. Then, there exists an edge a of $\mathcal{F}$ which intersects H_α. Since a and $\sigma_\alpha(a)$ are edges of $\mathcal{F}$, we have $a = \sigma_\alpha(a)$. In particular, a is parallel to α.

We now assume that s is a vertex of $\mathcal{P}$ in $H_\alpha \cap \mathcal{F}$. Let b be an edge of $\mathcal{F}$ containing s. Let β be a root parallel to b. This root β is neither orthogonal neither collinear to α; so, Proposition 3.3 implies that $\Phi(\mathcal{P}) \cap \mathrm{Vect}(\alpha, \beta)$ is a root system of type A_2. Changing eventually α by $-\alpha$ we may assume that $\alpha + \beta$ is a root. One easily checks that $\sigma_{\alpha+\beta}(b)$ is parallel to α. □

DEFINITION 3. If Φ is a root system in the vector space E, a subsystem obtained from Φ by intersecting Φ with a linear subspace of E is called a *Levi subsystem of* Φ. Note that the Dynkin diagram of a Levi system is obtained from the first Dynkin diagram by removing vertices and the adjacent edges.

4. *Dual Polytope*

In this section, we define two notions of the dual of a centered regular lattice polytope $\mathcal{P}$. Before, we recall the situation in the Euclidean case.

4.1. *The real case*

Let E be a finite dimensional real vector space. Let $\mathcal{P}$ be a convex polytope in E containing 0 in its interior. We denote by E^* the dual of E and set:

$$\mathcal{P}^* = \{\varphi \in E^* \text{ s.t. } \varphi_{|\mathcal{P}} \geq -1\}.$$

It is known that $\mathcal{P}^*$ is a convex polytope, called dual of $\mathcal{P}$. Moreover, $\mathcal{P}^*$ contains 0 in its interior and the dual $\mathcal{P}^{**}$ of $\mathcal{P}^*$ equals $\mathcal{P}$ modulo the natural identification between E and E^{**}. There is an inclusion-reversing combinatorial correspondence between the i-dimensional faces of $\mathcal{P}$ and the $(n-1-i)$- dimensional faces of $\mathcal{P}^*$. In particular, if E is Euclidean and $\mathcal{P}$ is regular, $\mathcal{P}^*$ is regular too with an isomorphic isometry group.

Now, we assume that E is Euclidean, $\mathcal{P}$ is regular and the barycenter of the vertices of $\mathcal{P}$ is 0. Consider the convex hull $\mathcal{P}^\vee$ of the barycenters of the facets of $\mathcal{P}$. With the scalar product, one may identify E and its dual: modulo this identification and under our assumptions $\mathcal{P}^*$ are $\mathcal{P}^\vee$ are positively proportional. In particular, $\mathcal{P}^\vee$ is regular with the same group as $\mathcal{P}$ and $\mathcal{P}^{\vee\vee}$ is positively proportional to $\mathcal{P}$.

The two above constructions of the dual of a regular Euclidean polytope can be adapted to regular lattice polytopes: but the two so obtained notions differ.

4.2. *The lattice case*

The lattice $\Lambda^* := \mathrm{Hom}(\Lambda,\mathbb{Z})$ is called the dual of Λ. Let $\mathcal{P}$ be a lattice polytope in E containing 0 in its interior. Consider

$$Q = \{\varphi \in \Lambda^* \otimes \mathbb{R} \text{ s.t. } \varphi_{|\mathcal{P}} \geq -1\}.$$

It is a convex polytope in $\Lambda^* \otimes \mathbb{R}$ containing 0 in its interior. But, its vertices do not necessarily belong to Λ^* but only to $\Lambda^* \otimes \mathbb{Q}$. We denote by $\mathcal{P}^*$ the only primitive lattice polytope positively proportional to Q. This lattice polytope $\mathcal{P}^*$ is called the *$*$-dual of $\mathcal{P}$*.

Using the properties of $\mathcal{P}^*$ in the real case, one deduces that if $\mathcal{P}$ is primitive $\mathcal{P}^{**} = \mathcal{P}$ and that if $\mathcal{P}$ is centered regular so is $\mathcal{P}^*$.

Now, $\mathcal{P}$ is assumed to be a centered regular lattice polytope. There exist a unique positive rational number k such that the barycenters of the vertices of the facets of $k.\mathcal{P}$ are primitive vectors in Λ. We denote by $\mathcal{P}^\vee$ and call $\vee$-dual of $\mathcal{P}$ the convex hull of these barycenters.

Since $\mathrm{Isom}(\mathcal{P})$ is finite, there exists a scalar product on $\Lambda_\mathbb{R}$ such that $\mathcal{P}$ is an Euclidean regular polytope in $\Lambda_\mathbb{R}$. Then, using the results stated in Section 4.1, one checks that if $\mathcal{P}^\vee$ is regular with the same group as $\mathcal{P}$ and that if moreover $\mathcal{P}$ is primitive then $\mathcal{P}^{\vee\vee} = \mathcal{P}$.

The polytopes $\mathcal{P}^*$ and $\mathcal{P}^\vee$ are not equivalent. For example, in dimension two, the two triangles are their own $\vee$-dual and the $*$-dual one of the other. In Table 1, we give the $\vee$-dual and $*$-dual of each regular lattice polytope.

5. *Classification*

In this section, we will obtain the classification of the centered regular lattice polytopes.

Let us start by reducing the list of possible root systems. Let $\mathcal{P}$ be a primitive centered regular lattice polytope in $\Lambda_\mathbb{R}$ of dimension n with associated root system Φ. By Proposition 3.3 $\mathrm{Aut}(\Phi)$ acts transitively on Φ. This implies that Φ is the product of copies of an irreducible simply laced root system Φ_0. Moreover, by Proposition 3.5, there exists a Levi subsystem Φ' of Φ of rank $n-1$ which is the root system of a regular lattice polytope $\mathcal{Q}$ with $\mathrm{Isom}(\mathcal{Q})$ contained in the stabilizer of Φ' in $\mathrm{Isom}(\mathcal{P})$. We deduce that either $\Phi_0 = A_1$ or Φ is irreducible. Finally, the type of Φ is

$$A_1^n,\ A_n,\ D_n,\ E_6,\ E_7 \text{ or } E_8. \tag{5.1}$$

Conversely, let Φ be a root system in the above list. Let us choose a set of simple roots of Φ. By Proposition 3.4, the vertices of a centered primitive lattice polytope $\mathcal{P}$ with associated root system Φ are the orbit by W of a unique dominant vertex s_0 in Λ_P. Such of polytope $\mathcal{P}$ is also given with a sublattice Λ of Λ_P containing Λ_R. Moreover, the polytope $\mathcal{P}$ is completely determined by Φ, s_0 and Λ. Finally, the polytopes obtained from a pair (s_0, Λ) and its image by an automorphism of Φ are equivalent. In Sections 5.1 to 5.4, for each possible Φ we give all the possible pairs (s_0,Λ) up to the action of $\mathrm{Aut}(\Phi)$.

The last step consists to show that each given triple (Φ, s_0, Λ) gives really a regular lattice polytope. One has to check that the stabilizer of Λ in $\mathrm{Aut}(\Phi)$ acts transitively on the complete flags of the convex hull $\mathcal{P}$ of $W.s_0$ and that Φ is the root system of $\mathcal{P}$. The first verification can be made by checking the equality of the cardinality of $\mathrm{Isom}(\mathcal{P})$ and the set of complete flags of $\mathcal{P}$. Using the action of $\mathrm{Isom}(\mathcal{P})$ the second is equivalent to check that one root is primitive on one edge of $\mathcal{P}$. Thereafter, these verifications are left to the reader.

5.1. *Root System A_1^n*

Here, we assume that the root system Φ associated to the primitive centered regular lattice polytope $\mathcal{P}$ is of type A_1^n. Let $\omega_1, \ldots, \omega_n$ be a set of fundamental weights of Φ. Then, $\Lambda_P = \mathbb{Z}\omega_1 \oplus \ldots \oplus \mathbb{Z}\omega_n$ and $\Lambda_R = \mathbb{Z}.2\omega_1 \oplus \ldots \oplus \mathbb{Z}.2\omega_n$. Let $k_i \in \mathbb{Z}_{>0}$ such that the unique dominant vertex s_0 is $\sum k_i\omega_i$. By the action of W, the vertices of $\mathcal{P}$ are the $\sum_i \pm k_i\omega_i$. In particular, $\mathcal{P}$ has $n!2^n$ complete flags and $\mathrm{Isom}(\mathcal{P}) = \mathrm{Aut}(\Phi)$. This implies that all the k_i's are equal: $s_0 = k.\sum_i \omega_i$. Reciprocally, $\mathrm{Aut}(\mathcal{P})$ acts transitively on the set of flags of the convex hull of the $k.\sum_i \pm\omega_i$.

Now, we have to determine the possible lattices Λ. Necessarily, Λ/Λ_R is a subgroup of $\Lambda_P/\Lambda_R \simeq (\mathbb{Z}/2\mathbb{Z})^n$ stable by the action of $\mathrm{Aut}(\Phi)$ acting on $(\mathbb{Z}/2\mathbb{Z})^n$ by permutations. By using for example the canonical bijection between $(\mathbb{Z}/2\mathbb{Z})^n$ and the set of the subsets of $\{1, \ldots, n\}$, one easily checks that the only possibilities for Λ are:

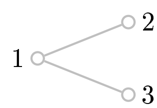
(i) $\Lambda = \Lambda_R$,
(ii) $\Lambda = \{\sum_i k_i\omega_i \mid k_i \text{ all even or all odd}\}$,
(iii) $\Lambda = \{\sum_i k_i\omega_i \mid \sum_i k_i \text{ even}\}$,
(iv) $\Lambda = \Lambda_P$.

But, the edges of the polytopes obtained with $\Lambda = \Lambda_P$ are parallel to the ω_i's; so, the root system of $\mathcal{P}$ is $\{\pm\omega_i\}$ which is a contradiction. Moreover, for $n = 2$, the second and third lattices equals. So, we obtain two primitive squares in dimension 2, and three primitive cubes for each dimension $n \geq 3$.

In Table 1, for each choice of Λ, we give a notation for the class of the corresponding cube $\mathcal{C}$, the vertex s_0, the cardinalities of $\mathcal{C} \cap \Lambda$ and of the intersection of $\mathcal{C}$ and an edge of Λ. We also give the class of the facets of $\mathcal{C}$ and of its $\vee$ and $*$ duals. All these results are obtained by direct calculations and prove that these cubes are indeed non equivalent.

5.2. *Root System D_n*

For convenience, we set $D_1 = A_1$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$. Moreover, we do not number the vertex of the Dynkin diagram of D_3 as those of A_3 but as follows:



Let $\mathcal{C}^n$ be one of the cubes obtained in the preceding section with $n \geq 2$. Its $*$-dual polytope $\mathcal{CC}^n$ is a primitive regular centered lattice polytope with $2n$ vertices and $\mathrm{Isom}(\mathcal{CC}^n)$ isomorphic to $\mathrm{Aut}(A_1^n)$. We deduce that the root system of $\mathcal{CC}^n$ is of type D_n and $\mathrm{Isom}(\mathcal{CC}^n) = \mathrm{Aut}(D_n)$, for any $n \geq 2$.

Since the facets of a cube are cubes, the stabilizer of s_0 in $\mathrm{Aut}(D_n)$ is isomorphic to $\mathrm{Aut}(D_{n-1})$; then, we may assume that $s_0 = k.\omega_1$ for a positive integer k.

The lattice Λ must be stable by the action of $\mathrm{Aut}(D_{n-1})$; there are three possibilities (for $n \geq 3$ and with notation of [**Bou02**]):

(i) $\Lambda = \Lambda_R$,
(ii) $\Lambda = \bigoplus_i \mathbb{Z}\varepsilon_i$,
(iii) $\Lambda = \Lambda_P$.

So, we obtain three cocubes for each $n \geq 3$ (see Table 1).

For $n = 2$, the two squares are $*$-dual one of the other; in particular, the cosquares are squares.

Now, let $\mathcal{P}$ be a primitive centered regular lattice polytope with root system D_n ($n \geq 4$) which is not a cocube. Using Proposition 3.3 one obtains that the root system of $\mathcal{P}^\vee$ is D_n too. The stabilizer of the dominant vertex s_0 in W is the stabilizer in W of a facet of $\mathcal{P}^\vee$. By Proposition 3.5 it is the Weyl group of a Levi subsystem of Φ which is the root system associated to a regular polytope of dimension $n-1$. We can deduce that $n \geq 5$ and $s_0 = k.\omega_n$ or $s_0 = k.\omega_{n-1}$, or $n = 4$ and s_0 is the multiple of any fundamental weight.

For $n \geq 5$, the two cases are equivalent using the action of $\mathrm{Aut}(\Phi)$. We claim that the convex hull $\mathcal{Q}$ of $W.k\omega_{n-1}$ is not a regular polytope. In an adapted base $(e_1, \dots, e_n)$ of $\Lambda_{\mathbb{R}}$, the vertices of $\mathcal{Q}$ are the $\sum_i \delta_i e_i$ with $\delta_i = \pm 1$ and $\Pi \delta_i = 1$. Let $(x_1, \dots, x_n)$ denote the dual base of $(e_1, \dots, e_n)$. Consider the two linear forms $\phi = x_1 + \ldots + x_{n-1} - x_n$ and $\psi = x_1$. The affine hyperplane $\phi = n-2$ define a facet of $\mathcal{Q}$ which is a simplex. But $\psi = 1$ is a facet with 2^{n-2} vertices. So, $\mathcal{Q}$ is not regular since $n \geq 5$.

For $n = 4$, the three fundamental weights ω_1, ω_3 and ω_4 are equivalent modulo $\mathrm{Aut}(D_4)$ and give the cocubes. Consider the case $s_0 = k.\omega_2$. Since ω_2 is the longest root, $k = 1$ and the vertices of the convex hull $\mathcal{P}$ of $W.\omega_2$ are the 24 roots of D_4 and $\mathrm{Isom}(\mathcal{P}) = \mathrm{Aut}(D_4)$.

Let $(x_1, \dots, x_4)$ be the dual basis of $(\varepsilon_1, \dots, \varepsilon_4)$ (with notation of [**Bou02**]). One easily checks that the affine hyperplane $\sum x_i = 2$ defines a facet of $\mathcal{P}$ which is a regular cocube. By the action of $\mathrm{Aut}(D_4)$, one obtains the 24 facets:

- $-2 \leq x_1 + x_2 + x_3 + x_4 \leq 2$,
- $-1 \leq x_i \leq 1$, for $i = 1, \dots, 4$,
- $-2 \leq \sum_{j \neq i} x_j - x_i \leq 2$, for $i = 1, \dots, 4$,
- $x_i + x_j - x_k + x_l \leq 2$, for $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and $i < j$ and $k < l$.

In particular, $\mathcal{P}$ is regular.

Moreover, Λ_P/Λ_R is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the only lattices Λ stable by $\mathrm{Aut}(D_4)$ such that $\Lambda_R \subset \Lambda \subset \Lambda_P$ are Λ_P and Λ_R. So, we obtain two classes of centered primitive regular lattice polytopes called 24-cells polytopes. We denote by $\mathcal{D}_1^4$ those obtained with $\Lambda = \Lambda_P$ and $\mathcal{D}_2^4$ those obtained with $\Lambda = \Lambda_R$.

The dominant weights in $\mathcal{P}$ are: ω_2, ω_1, ω_3, ω_4, $\omega_1 + \omega_3$ and 0. By acting W we deduce that $\mathcal{P} \cap \Lambda_P$ contains $24 + 6 + 6 + 6 + 32 + 1 = 81$ points and $\mathcal{P} \cap \Lambda_R$ contains $24 + 1 = 25$ points. This gives the cardinality of $\mathcal{D}_i^4 \cap \Lambda$, for $i = 1$ and 2.

Since the two 24-cells are the only lattice regular polytopes in dimension four with isomorphism group $\mathrm{Aut}(D_4)$ the $\vee$-dual of $\mathcal{D}_1^4$ is either $\mathcal{D}_2^4$ or itself. But, the barycenter of the facet $\sum_i x_i = 2 \cap \mathcal{P}$ is $\frac{1}{2} \sum_i \varepsilon_i = \omega_4$ and belongs to Λ_P. We deduce that the dual ${\mathcal{D}_1^4}^\vee$ contains strictly less points of Λ than $\mathcal{D}_1^4$. Finally, ${\mathcal{D}_1^4}^\vee = \mathcal{D}_2^4$ and ${\mathcal{D}_2^4}^\vee = \mathcal{D}_1^4$.

Remark 1. In this section, we have considered the cocubes for any $n \geq 2$. But, the others polytopes with associated root systems of type D_n have only be considered when $n \geq 4$. The case $n = 2$ has been made in Section 5.1 and those with $n = 3$ will be considered in Section 5.3.

5.3. *Root system A_n*

Consider a primitive centered regular lattice polytope $\mathcal{P}$ with root system Φ of type A_n with $n \geq 3$. Because of the orders of the Weyl groups, the root system of $\mathcal{P}^\vee$ cannot be of type E_n. So, we may assume that the root system of $\mathcal{P}^\vee$ is also of type A_n; if not, we have already meet $\mathcal{P}$.

By Proposition 3.5, the stabilizer of s_0 which is the stabilizer of a facet of $\mathcal{P}^\vee$ in $\mathrm{Isom}(\mathcal{P})$ must contains the Weyl group of a root system of type A_{n-1} or $A_1 \times A_1$ for $n = 3$. This implies that if $n \geq 4$ then s_0 equals $k.\omega_1$ or $k.\omega_n$, if $n = 3$ then s_0 is a fundamental weight and implies no restriction on s_0 if $n = 2$.

Firstly, we assume that s_0 is neither proportional to ω_1 or ω_n.

Let us fix $n = 2$. Under our assumption, $\mathcal{P}$ is an hexagon and $\mathrm{Isom}(\mathcal{P}) = \mathrm{Aut}(A_2)$. We deduce that $s_0 = k.(\omega_1 + \omega_2)$: this gives two regular hexagons obtained with Λ equal to Λ_R and Λ_P.

If $n = 3$, our assumption implies that $s_0 = k.\omega_2$. So, we obtain the three cocubes considered in Section 5.2.

We now assume that s_0 is proportional to ω_1. The case when s_0 is proportional to ω_n is equivalent up to $\mathrm{Aut}(\Phi)$. The polytope $\mathcal{P}$ is the convex hull of $W.k.\omega_1$ that is of the $k.\varepsilon_i$'s. In particular, $\mathcal{P}$ is a simplex and is regular with $\mathrm{Isom}(\mathcal{P}) = W$.

The lattice Λ can be any lattice between Λ_R and Λ_P. Since $\Lambda_P/\Lambda_R \simeq \mathbb{Z}/(n+1)\mathbb{Z}$, for each divisor d of $n+1$ we have exactly one Λ_d such that $\Lambda_P/\Lambda_d \simeq \mathbb{Z}/d\mathbb{Z}$. For Λ_d and $k = d$, $\mathcal{P}$ is a primitive simplex denoted by $\mathcal{S}_d^n$. Direct calculation shows that the edges of $\mathcal{S}_d^n$ contain $d+1$ points. In particular they are pairwise non isomorphic.

The cardinality of $\mathcal{S}_d^n \cap \Lambda$ is a little bit complicated to express. For any $\tau \in \mathbb{Z}/(n+1)\mathbb{Z}$, we denote by $c(\tau)$ the cardinality of the following set

$$\{a_1, \dots, a_{n+1} \in \tau \cap \mathbb{N} \text{ s.t. } \sum a_i = d(n+1)\}.$$

Then, one easily checks that the cardinality of $\mathcal{S}_d^n \cap \Lambda$ is

$$\sum_{\tau \in \mathbb{Z}/(n+1)\mathbb{Z}} c(\tau).$$

It would be interesting to simplify this formula !

5.4. *Root systems E_n*

By absurd, we will prove that there is no regular lattice polytope $\mathcal{P}$ with root system of type E_6. We may assume that $\mathcal{P}$ is primitive and centered. Since the Weyl group of E_6 is contained in no automorphism group of a root system of rank 6 in List 5.1, the root system of $\mathcal{P}^\vee$ is necessarily E_6. Moreover the root system of a face of $\mathcal{P}$ is either D_5 or A_5. The first case is not possible because the regular polytopes with D_5 as root systems have $\mathrm{Aut}(D_5)$ as isomorphism group. But, $\mathrm{Aut}(D_5)$ is not contained in the stabilizer of D_5 in $\mathrm{Aut}(E_6)$. In the second case, one has necessarily $s_0 = \omega_2$ that is the longest root. So, the vertices of $\mathcal{P}$ are the roots of E_6. By Proposition 3.5, $\mathcal{P}$ has a facet parallel to the Levi subsystem of type A_5 that is orthogonal to the fundamental weight ω_2. Moreover, the scalar product with ω_2 of a root is either -2, -1, 0, 1 or 2; and, the values ± 2 are reached once each one (see the table of E_6 in [**Bou02**]). This is a contradiction.

The same argument shows that there is no regular lattice polytope with root system of type E_7 and E_8.

6. *Description of the regular lattice polytopes*

In the following tabular, for each primitive centered regular lattice polytope $\mathcal{P}$, we give a notation, its root system, the group $\mathrm{Isom}(\mathcal{P})$ its lattice Λ, its dominant vertex s_0, the cardinalities of $\mathcal{P} \cap \Lambda$, the number of points in Λ on an edge of $\mathcal{P}$, primitive centered regular lattice polytope equivalent to the facets of $\mathcal{P}$ and the duals of $\mathcal{P}^\vee$ and $\mathcal{P}^*$. All these elements allow us to distinguish two non isomorphic lattice polytopes.

Proofs are given in the preceding section, the others are simple calculation left to the reader.

Remark 2. In even dimension more than four, there exist three classes of cocube. Two of these three cocubes have the same simplex as facet and the third another one. In contradiction in [**Kar06**], the three cocubes have the same simplex as facet.

References

Bou02 Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley.

FR05 Sergey Fomin and Nathan Reading, *Root systems and generalized associahedra*, 2005.

Hum90 James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

Kar06 O. Karpenkov, *Classification of lattice-regular lattice convex polytopes*, Functional Analisys and Other Mathematics **1** (2006), no. 1, 31–54, math.CO/0602193.

Oda88 Tadao Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese.

Pro90 C. Procesi, *The toric variety associated to Weyl chambers*, Mots, Lang. Raison. Calc., Hermès, Paris, 1990, pp. 153–161.

Type	Φ	Isom	Not.	Λ	s_0	Card.	Edges	Facet	$\mathcal{P}^\vee$	$\mathcal{P}^*$
Simplex	A_n $n \geq 2$	$W(\Phi) \simeq$ Σ_{n+1}	$\mathcal{S}_d^n$ for $d \mid (n+1)$.	$\Lambda_R \subset \Lambda \subset \Lambda_P$ with $\#(\Lambda_P/\Lambda) = d$.	$d\omega_1$	see Sect 5.3	$d+1$	$\mathcal{S}_n^{n-1}$	$\mathcal{S}_d^n$	$\mathcal{S}_{\frac{n+1}{d}}^n$
Cubes	A_1^n $n \geq 2$	$\mathrm{Aut}(\Phi)$ $\simeq$ $(\mathbb{Z}/2\mathbb{Z})^n \ltimes \Sigma_n$	$\mathcal{C}_1^n$	Λ_R	$2\sum \omega_i$	3^n	3	$\mathcal{C}_1^{n-1}$	$\mathcal{CC}_2^n$	$\mathcal{CC}_2^n$
			$\mathcal{C}_2^n$	$k_i \equiv k_j \bmod 2$	$\sum \omega_i$	2^n+1	2	$\mathcal{C}_1^{n-1}$	$\mathcal{CC}_3^n$	$\mathcal{CC}_3^n$
			$\mathcal{C}_3^n$ for n even	$\sum k_i$ even	$\sum \omega_i$	$\frac{3^n+1}{2}$	2	$\mathcal{C}_3^{n-1}$	$\mathcal{CC}_1^n$	$\mathcal{CC}_1^n$
			$\mathcal{C}_3^n$ for n odd		$2\sum \omega_i$	$\frac{5^n-1}{2}$	3			
Cocubes	D_n $n \geq 3$	$\mathrm{Aut}(\Phi)$ $\simeq$ $(\mathbb{Z}/2\mathbb{Z})^n \ltimes \Sigma_n$	$\mathcal{CC}_1^n$	Λ_R	$2\omega_1$	$4n^2+1$	3	$\mathcal{S}_n^{n-1}$	$\mathcal{C}_3^n$	$\mathcal{C}_3^n$
			$\mathcal{CC}_2^n$	$\bigoplus_i \mathbb{Z}\varepsilon_i$	ω_1	$2n+1$	2	$\mathcal{S}_n^{n-1}$	$\mathcal{C}_1^n$	$\mathcal{C}_1^n$
			$\mathcal{CC}_3^n$ for n even	Λ_P	ω_1	$2n+1$	2	$\mathcal{S}_{n/2}^{n-1}$	$\mathcal{C}_2^n$	$\mathcal{C}_2^n$
			$\mathcal{CC}_3^n$ for n odd					$\mathcal{S}_n^{n-1}$		
Hexagon	A_2	$\mathrm{Aut}(\Phi)$ $\simeq$ D_6	$\mathcal{H}_1^2$	Λ_R	$\omega_1 + \omega_2$	7	2		$\mathcal{H}_2^2$	$\mathcal{H}_1^2$
			$\mathcal{H}_2^2$	Λ_P		13			$\mathcal{H}_1^2$	$\mathcal{H}_2^2$
24-cell	D_4	$\mathrm{Aut}(\Phi)$ $\simeq$ $(\Sigma_4 \ltimes \mathbb{Z}/2\mathbb{Z}^3) \ltimes \Sigma_3$	$\mathcal{D}_1^4$	Λ_R	ω_2	25	2	$\mathcal{CC}_1^3$	$\mathcal{D}_2^4$	$\mathcal{D}_1^4$
			$\mathcal{D}_2^4$	Λ_P		81		$\mathcal{CC}_2^3$	$\mathcal{D}_1^4$	$\mathcal{D}_2^4$

Remark: We have the following exeptional equalities in dimension two: $\mathcal{C}_2^2 = \mathcal{C}_3^2$ and $\mathcal{C}_1^{2\vee} = \mathcal{C}_2^2$.

Table 1: List of the centered primitive regular lattice polytopes