Some unexpected properties of Littlewood-Richardson coefficients

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Abstract

We are interested in identities between Littlewood-Richardson coefficients, and hence in comparing different tensor product decompositions of the irreducible modules of the linear group $\mathrm{GL}_n(\mathbb{C})$. A family of partitions – called near-rectangular – is defined, and we prove a stability result which basically asserts that the decomposition of the tensor product of two representations associated to near-rectangular partitions does not depend on n. Given a partition λ, of length at most n, denote by $V_n(\lambda)$ the associated simple $\mathrm{GL}_n(\mathbb{C})$-module. We conjecture that, if λ is near-rectangular and μ any partition, the decompositions of $V_n(\lambda)\otimes V_n(\mu)$ and $V_n(\lambda)^*\otimes V_n(\mu)$ coincide modulo a mysterious bijection. We prove this conjecture if μ is also near-rectangular and report several computer-assisted computations which reinforce our conjecture.

1 Introduction

In this paper we study some properties of Littlewood-Richardson coefficients when some partitions are near-rectangular (see below for a precise definition). Let $n \geq 2$ be an integer. The irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ are parametrized by all non-increasing sequences of n integers. As is often the case, we focus on the polynomial representations among those, which correspond to the sequences containing only non-negative integers, also called partitions with at most n parts. Denote by $V_n(\lambda)$ the representation of $\mathrm{GL}_n(\mathbb{C})$ associated to such a partition λ. The Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ appears in the tensor product decomposition

$$V_n(\lambda) \otimes V_n(\mu) = \bigoplus_{\nu\in\Lambda_n} c^\nu_{\lambda\mu} V_n(\nu),$$

where Λ_n denotes the set of partitions of length at most n. Denote moreover by $V_n(\lambda)^*$ the $\mathrm{GL}_n(\mathbb{C})$-module which is dual to $V_n(\lambda)$. Then a central question in this paper is

Problem 1. Compare the two $\mathrm{GL}_n(\mathbb{C})$-modules, $V_n(\lambda) \otimes V_n(\mu)$ and $V_n(\lambda)^* \otimes V_n(\mu)$.

Obviously $V_n(\lambda)\otimes V_n(\mu)$ and $V_n(\lambda)^*\otimes V_n(\mu)$ are not isomorphic as $\mathrm{GL}_n(\mathbb{C})$-modules. Nevertheless, we may want to compare their decompositions in irreducible modules. For example, Coquereaux-Zuber [CZ11] showed that the sums of the multiplicities of these two representations coincide. For $\lambda\in\Lambda_n$, set $\lambda^*=\lambda_1-\lambda_n\geq\lambda_1-\lambda_{n-1}\geq\cdots\geq\lambda_1-\lambda_2\geq 0$. Then $V_n(\lambda)^*$ and $V_n(\lambda^*)$ only differ by a tensor power of the determinant. More precisely, $V_n(\lambda)^*$ is the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ corresponding to the sequence $-\lambda_n\geq -\lambda_{n-1}\geq\cdots\geq-\lambda_1$. As a consequence $V_n(\lambda^*)$ is isomorphic to $V_n(\lambda)^*$ tensored by the λ_1-th power of the determinant representation of $\mathrm{GL}_n(\mathbb{C})$.

Following these observations, the Coquereaux-Zuber's result can be written as

$$\sum_{\nu\in\Lambda_n} c^{\nu}_{\lambda\mu}=\sum_{\nu\in\Lambda_n} c^{\nu}_{\lambda^*\mu}. \tag{1}$$

More generally, we want to compare $\{c^{\nu}_{\lambda\mu}\,:\,\nu\in\Lambda_n\}$ and $\{c^{\nu}_{\lambda^*\mu}\,:\,\nu\in\Lambda_n\}$.

A partition $\lambda\in\Lambda_n$ is said to be *near-rectangular* if $\lambda=\lambda_1\lambda_2^{n-2}\lambda_n$ for some integers $\lambda_1\geq\lambda_2\geq\lambda_n$; that is, if $\lambda_2=\cdots=\lambda_{n-1}$. We formulate the following

Conjecture 1. *Let λ and μ in Λ_n. If λ is near-rectangular then*

$$\forall c\in\mathbb{N}\qquad \sharp\{\nu\in\Lambda_n\,:\,c^{\nu}_{\lambda\mu}=c\}=\sharp\{\nu\in\Lambda_n\,:\,c^{\nu}_{\lambda^*\mu}=c\}.$$

Equivalently, we conjecture that there exists a bijection $\varphi\,:\,\Lambda_n\longrightarrow\Lambda_n$, depending on λ and μ such that

$$\forall\nu\in\Lambda_n\qquad c^{\nu}_{\lambda\mu}=c^{\varphi(\nu)}_{\lambda^*\mu},$$

if λ is near-rectangular. In other words, the multisets $\{c^{\nu}_{\lambda\mu}\,:\,\nu\in\Lambda_n\}$ and $\{c^{\nu}_{\lambda^*\mu}\,:\,\nu\in\Lambda_n\}$ are expected to be equal, if λ is near-rectangular.

In the literature, there are a lot of equalities between Littlewood-Richardson coefficients. There are symmetries (see [BR20] and references therein), stabilities (see [BOR15]), reductions (see [CM11]). None of this numerous results seems to explain that Conjecture 1 holds even for $\mathrm{GL}_3(\mathbb{C})$. Moreover, there are various combinatorial models for the Littlewood-Richardson coefficients (see [Ful97, Lit95, Zel81, KT99, Vak06, Cos09]). None of these models seems to prove Conjecture 1 easily.

What made this Conjecture 1 even more unexpected for us is that it seems that, should the aforementioned bijection $\varphi\,:\,\Lambda_n\longrightarrow\Lambda_n$ exist, it does not seem to be expressible simply in terms of λ and μ. Indeed, even if $n=3$, we remark in Section 4 that $(\lambda,\mu,\nu)\longrightarrow(\lambda^*,\mu,\varphi(\nu))$ cannot be linear.

In Section 7.2, we give an example showing that the assumption on λ is truly necessary. Nevertheless, for $n=3$, this assumption is trivial. In this case, we sketch Conjecture 1 by computing explicitly the function

$$\begin{array}{ccl}(\Lambda_n^2)\times\mathbb{N} & \longrightarrow & \mathbb{N}\\ (\lambda,\mu,c) & \longmapsto & \mathrm{Nb}_n(c^{\bullet}_{\lambda\mu}>c):=\sharp\{\nu\in\Lambda_n\,:\,c^{\nu}_{\lambda\mu}>c\}.\end{array}$$

See Section 4 for details. In this introduction we report on this computation as follows:

Proposition 1. *The function*

$$\begin{array}{rccl} \mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c): & \Lambda_3 \times \Lambda_3 \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu, c) & \longmapsto & \sharp\{\nu \in \Lambda_n \,:\, c^{\nu}_{\lambda\mu} > c\} \end{array}$$

is piecewise polynomial of degree 2 with 7 cones. Moreover,

$$\mathrm{Nb}_3(c^{\bullet}_{\lambda\mu} > c) = \mathrm{Nb}_3(c^{\bullet}_{\lambda^*\mu} > c). \tag{2}$$

As a consequence of Proposition 1, we get

Corollary 2. *Conjecture 1 holds for $n = 3$.*

We sketched, using Sagemath, a few million examples for $\mathrm{GL}_4(\mathbb{C})$, $\mathrm{GL}_5(\mathbb{C})$, $\mathrm{GL}_6(\mathbb{C})$, and $\mathrm{GL}_{10}(\mathbb{C})$. See Section 7.2 for details.

We now consider the tensor products of representations associated to two near-rectangular partitions. Observe that a partition λ of length l parametrizes representations $V_n(\lambda)$ of $\mathrm{GL}_n(\mathbb{C})$ for any $n \geq l$. A priori, $c^{\nu}_{\lambda\mu}$ could depend on n. It is a classical result (see e.g. [Ful97]) of stability that it actually does not. Our second result is a similar stability result but for partitions of arbitrarily large length. Indeed, fix two near-rectangular partitions $\lambda = \lambda_1\, \lambda_2^{n-2} \lambda_n$ and $\mu = \mu_1\, \mu_2^{n-2} \mu_n$. We prove that the decomposition of $V_n(\lambda) \otimes V_n(\mu)$ does not depend on $n \geq 4$ but only on the six integers $\lambda_1, \lambda_2, \lambda_n, \mu_1, \mu_2$ and μ_n . More precisely, we get a simple expression of the Littlewood-Richardson coefficients that appear in this tensor product, expression independent of n.

Proposition 3. *Let $n \geq 4$. Let $\lambda = \lambda_1\, \lambda_2^{n-2}$ and $\mu = \mu_1\, \mu_2^{n-2}$ be two near-rectangular partitions. Let ν be a partition with at most n parts. Then $c^{\nu}_{\lambda\mu} = 0$ unless $\nu = \nu_1\, \nu_2\, (\lambda_2 + \mu_2)^{n-4}\, \nu_{n-1}\nu_n$ for four integers ν_1, ν_2, ν_{n-1} and ν_n such that $\nu_1 \geq \nu_2 \geq \lambda_2 + \mu_2 \geq \nu_{n-1} \geq \nu_n$. In this case*

$$c^{\nu}_{\lambda\mu} = \mathrm{Card}\big([\![\max(0, \lambda_2{+}\mu_1{-}\nu_1, -\mu_2{+}\nu_n), \min(\lambda_1{+}\mu_1{-}\nu_1, \lambda_2{+}\mu_1{-}\nu_2, -\lambda_2{-}\mu_2{+}\nu_{n-1}{+}\nu_n, -\mu_2{+}\nu_{n-1})]\!]\big).$$

In particular, this value does not depend on $n \geq 4$.

In the proposition, for simplicity we assume that $\lambda_n = \mu_n = 0$. This is not a serious assumption since $V_n(\lambda_1\lambda_2^{n-2}\lambda_n) \simeq \det^{\lambda_n} \otimes V_n((\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)^{n-2})$.

Proposition 3 positively answers [PW20, Question 2] by giving a much stronger result. It also allows to check a particular case of Conjecture 1. Indeed, with the help of a computer we compute $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c)$ for λ and μ near-rectangular. Set $\Lambda_n^{\mathrm{nr}} = \{\lambda_1\lambda_2^{n-2}\lambda_n \,:\, \lambda_1 \geq \lambda_2 \geq \lambda_n\}$ to be the set of near-rectangular partitions of length at most n.

Proposition 4. *The function*

$$\begin{array}{rccl} \mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c) : & \Lambda_4^{\mathrm{nr}} \times \Lambda_4^{\mathrm{nr}} \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu, c) & \longmapsto & \sharp\{\nu \in \Lambda_n \,:\, c^{\nu}_{\lambda\mu} > c\} \end{array}$$

is piecewise polynomial of degree 3 with 36 cones.

The 36 polynomial functions and cones are given in Section 6.3. As a consequence of Propositions 3 and 4, we get

Corollary 5. *Let $n \geq 4$. Conjecture 1 holds for $\mathrm{GL}_n(\mathbb{C})$ assuming moreover that μ is near-rectangular.*

A poor version of Conjecture 1 is

Conjecture 2. *If $\lambda \in \Lambda_n$ is near-rectangular then*

$$\sharp\{\nu \in \Lambda_n \,:\, c^{\nu}_{\lambda\mu} \neq 0\} = \sharp\{\nu \in \Lambda_n \,:\, c^{\nu}_{\lambda^*\mu} \neq 0\}.$$

Equivalently, we ask whether, for $\lambda \in \Lambda_n^{\mathrm{nr}}$,

$$\forall \mu \in \Lambda_n \qquad \mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > 0) = \mathrm{Nb}_n(c^{\bullet}_{\lambda^*\mu} > 0).$$

For $n = 4$, and λ near-rectangular, we computed $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > 0)$ and checked Conjecture 2. Here we report on this computation as follows.

Proposition 6. *The function*

$$\begin{array}{rccl} \mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > 0) : & \Lambda_4^{\mathrm{nr}} \times \Lambda_4 & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu) & \longmapsto & \sharp\{\nu \in \Lambda_n \,:\, c^{\nu}_{\lambda\mu} > 0\} \end{array}$$

is piecewise quasi-polynomial of degree 3 with 205 cones. The only congruence occurring is the parity of $\lambda_1 + |\mu|$. Moreover,

$$\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > 0) = \mathrm{Nb}_4(c^{\bullet}_{\lambda^*\mu} > 0). \tag{3}$$

This symetry with the complete duality $(\lambda, \mu) \longmapsto (\lambda^*, \mu^*)$ gives an action of $(\mathbb{Z}/2\mathbb{Z})^2$. This group acts on the 205 pairs (cone,quasi-polynomial) with 83 orbits.

This work is based on numerous computer aided computation with [VSB$^+$07, BIS, S$^+$12]. Details on these computation can be found on the webpage of the second author [Res20, Supplementary material].

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Remark. After a version of this work appeared on ArXiv, Darij Grinberg proposed a solution of the main conjecture of the paper in [Gri20]. Indeed, he defines a piecewise linear involution φ from $\mathbb{Z}^n$ to $\mathbb{Z}^n$ satisfies

$$\forall \nu \in \Lambda_n \qquad c^\nu_{\lambda\mu} = c^{\varphi(\nu)}_{\lambda^*\mu},$$

if λ is near-rectangular and solves our conjecture. An amazing fact is that this bijection does not necessarily map a partition on a partition. If $\varphi(\nu)$ is not a partition then $c^\nu_{\lambda\mu}$ vanishes, allowing φ to work.

2 Generalities on the function $\mathrm{Nb}_n(c^\bullet_{\lambda\mu} > c)$

Recall that for $\lambda, \mu \in \Lambda_n$ and $c \in \mathbb{N}$ we set

$$\mathrm{Nb}_n(c^\bullet_{\lambda\mu} > c) = \sharp\{\nu \in \Lambda_n \,:\, c^\nu_{\lambda\mu} > c\}.$$

Since $V_n(\lambda) \otimes V_n(\mu) \simeq V_n(\mu) \otimes V_n(\lambda) \simeq (V_n(\lambda^*) \otimes V_n(\mu^*))^*$ as $\mathrm{SL}_n(\mathbb{C})$-modules, the function $\mathrm{Nb}_n(c^\bullet_{\lambda\mu} > c)$ satisfies

$$\mathrm{Nb}_n(c^\bullet_{\lambda\mu} > c) = \mathrm{Nb}_n(c^\bullet_{\mu\lambda} > c) = \mathrm{Nb}_n(c^\bullet_{\lambda^*\mu^*} > c) = \mathrm{Nb}_n(c^\bullet_{\mu^*\lambda^*} > c). \tag{4}$$

Let $1^n \in \Lambda_n$ denote the partition with n parts equal to 1. Then $V_n(1^n)$ is the one dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ given by the determinant. In particular, as an $\mathrm{SL}_n(\mathbb{C})$-module $V_n(\lambda) \simeq V_n(\lambda - \lambda_n^n)$ for any k. Set $\Lambda_n^0 = \{\lambda \in \Lambda_n \,:\, \lambda_n = 0\}$. It can be seen as the set of dominant weights for the group $\mathrm{SL}_n(\mathbb{C})$. For $\lambda \in \Lambda_n$, set $\bar\lambda = \lambda - \lambda_n^n$. Then

$$\mathrm{Nb}_n(c^\bullet_{\lambda\mu} > c) = \mathrm{Nb}_n(c^\bullet_{\bar\lambda\bar\mu} > c), \qquad \text{and even more} \qquad c^\nu_{\lambda\mu} = c^{\nu-(\lambda_n+\mu_n)^n}_{\bar\lambda\bar\mu}. \tag{5}$$

Set

$$\mathrm{Horn}_n = \{(\lambda, \mu, \nu) \in (\Lambda_n)^3 \,:\, c^\nu_{\lambda\mu} \neq 0\}.$$

By a Brion-Knop's result (see [É92]), Horn_n is a finitely generated semigroup. The Knutson-Tao saturation theorem [KT99] shows that Horn_n is the set of integral points in a convex cone, the Horn cone. The Horn cone is polyhedral and the minimal list of inequalities characterizing it is known (see *e.g.* [Ful00, Bel01, KTW04]). These inequalities contain the Weyl inequalities

$$\nu_{i+j-1} \leq \lambda_i + \mu_j \quad \text{whenever } i + j - 1 \leq n; \tag{6}$$

and are all of the form

$$\sum_{k \in K} \nu_k \leq \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j, \tag{7}$$

for three subsets I, J and K in $\{1, \dots, n\}$ of the same cardinality.

Proposition 7. *Fix $n \geq 0$. The function*

$$\begin{array}{rccl} \mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > 0): & \Lambda_n \times \Lambda_n & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu) & \longmapsto & \sharp\{\nu \in \Lambda_n \,:\, c^{\nu}_{\lambda\mu} > 0\}. \end{array}$$

is piecewise quasi-polynomial.

Proof. We have

$$\mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > 0) = \sharp \mathrm{Horn}_n \cap \{(\lambda, \mu)\} \times \Lambda_n.$$

By Knutson-Tao's saturation conjecture [KT99], this is the number of integer points in the polyhedron. Moreover each inequality of this polyhedron depends linearly on $(\lambda\,\mu)$. Now the conclusion is a consequence of the general theory of Herhart polynomials (see *e.g.* [BBDL$^+$19, Theorem 1.1] or [Stu95]). $\square$

Remark. Since the function $(\lambda, \mu, \nu) \longmapsto c^{\nu}_{\lambda\mu}$ is piecewise polynomial (see [Ras04]), [BBDL$^+$19, Theorem 1.1] implies that for any $k \in \mathbb{N}$, the map $(\lambda, \mu) \mapsto N^k(\lambda, \mu) := \sum_{\nu \in \Lambda_n} (c^{\nu}_{\lambda\mu})^k$ is piecewise quasi-polynomial. Note that, since $(\lambda, \mu, \nu) \longmapsto c^{\nu}_{\lambda\mu}$ is only piecewise polynomial, one cannot apply Theorem 1.1 directly, but must apply it a certain (finite) number of times on different closed convex polyhedral subcones. Then one has to take a signed sum of the thus-obtained piecewise quasi-polynomial functions.

Moreover, Conjecture 1 could be replaced by: if λ is near-rectangular then

$$\forall k \in \mathbb{N} \qquad N^k(\lambda, \mu) = N^k(\lambda^*, \mu).$$

3 The hive model

For later use, we shortly review the hive model that expresses the Littlewood-Richardson coefficients as the number of integer points in polyhedra. Fix an integer $n \geq 2$.

Let λ, μ and ν in Λ_n such that $|\nu| = |\lambda| + |\mu|$. Otherwise $c^{\nu}_{\lambda\mu} = 0$. Place $\frac{(n+1)(n+2)}{2}$ integers on the vertices of the triangles on Figure 1 with boundaries determined by λ, μ and ν as drawn on the left. Alternatively, one can label the edges by the differences of the values on its vertices and oriented to get the picture on the right.

Neighbouring entries define three distinct types of rhombus (see Figure 2), each with its own constraint condition. For each extracted rombus we impose the following contraints

$$b + c \geq a + d \quad \Leftrightarrow \quad \beta \geq \delta \quad \Leftrightarrow \quad \alpha \geq \gamma \tag{8}$$

By definition a hive is an element of $\mathbb{Z}^{\frac{(n+1)(n+2)}{2}}$ satisfying Inequalities 8 for each one of the $3\frac{n(n-1)}{2}$ rombus. The basic Knutson-Tao's result is

Theorem 8. *(see [KT99, Appendix]) Let λ, μ and ν in Λ_n. Then $c^{\nu}_{\lambda\mu}$ is the number of hives with boundary value determined by λ, μ and ν as on the left of Figure 1.*

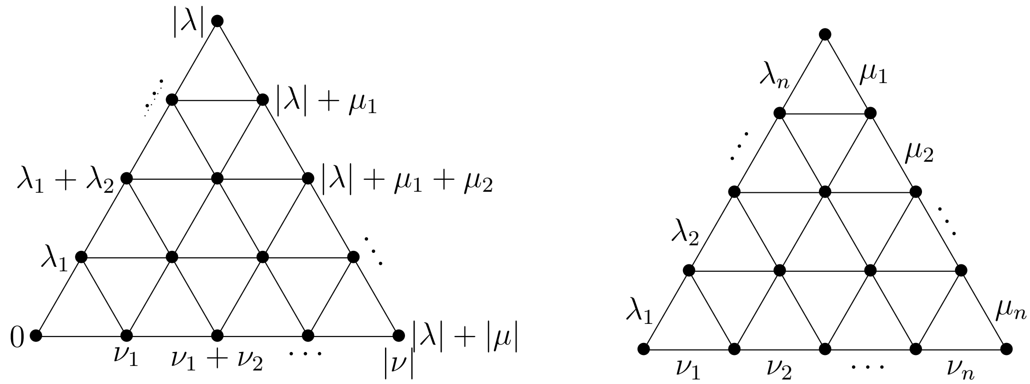


Figure 1: Hives with boundary conditions

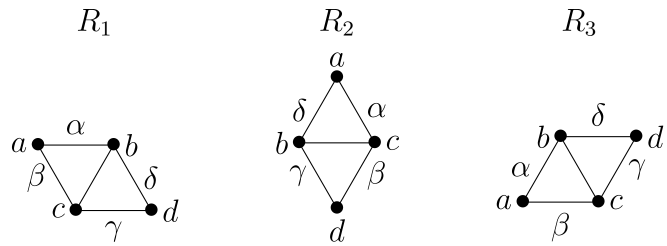


Figure 2: Rombus

Proceeding as on the right-side of Figure 1, we get an integer on the edges of each small triangle hence a point in $\mathbb{Z}^{3\frac{n(n+1)}{2}}$. Keeping the entries corresponding to the horizontal edges, one gets a point GT_h in $\mathbb{Z}^{\frac{n(n+1)}{2}}$. One can note that by Inequalities (8), GT_h is a Gelfand-Tsetlin pattern (see *e.g.* [BZ88] for a definition). One can proceed similarly, by considering parallels to one of the two other sides. One gets two more Gelfand-Tsetlin patterns. Conversely, a collection of integer entries on the vertices satisfies the rombus inequalities if and only if the three points of $\mathbb{Z}^{\frac{n(n+1)}{2}}$ are GT patterns.

4 The case of GL_3

It is known that the function $(\Lambda_n)^3 \longrightarrow \mathbb{N}$, $(\lambda, \mu, \nu) \longmapsto c^{\nu}_{\lambda\mu}$ is piecewise polynomial (see [Ras04]) of degree $\frac{n^2-3n+2}{2}$. For $n = 3$, we get precisely the following.

Proposition 9. *Let $\lambda = (\lambda_1, \lambda_2, 0)$, $\mu = (\mu_1, \mu_2, 0)$, and $\nu = (\nu_1, \nu_2, \nu_3)$ in Λ_3 such that $|\nu| = |\lambda| + |\mu|$. Then $c^{\nu}_{\lambda\mu}$ is the cardinality of*

$$\sharp\left([\![\max(\mu_1 - \lambda_2, \mu_2, \nu_1 - \lambda_1, \mu_1 - \nu_3, \nu_2 - \lambda_2, \mu_1 + \mu_2 - \nu_2), \min(\mu_1, \nu_1 - \lambda_2, \mu_1 + \mu_2 - \nu_3)]\!]\right).$$

This statement is well known and can easily be checked using the hive model. Once λ, μ and ν are fixed, there is only one interior entry x to choose in order to determine the hive.

This entry has to belong to some interval to satisfy the 9 rombus inequalities.

Proposition 9 implies that, for any nonnegative integer c, $c^\nu_{\lambda\mu} > c$ if and only if for any linear form φ and ψ appearing in the min and max respectively we have $\varphi - \psi \geq c$. Namely $c^\nu_{\lambda\mu} > c$ if and only if

$$\begin{array}{ll}
\lambda_1 - \lambda_2 - c \geq 0 & \lambda_2 - c \geq 0 \\
\mu_1 - \mu_2 - c \geq 0 & \mu_2 - c \geq 0 \\
\nu_1 - \nu_2 - c \geq 0 & \nu_2 - \nu_3 - c \geq 0 \\
\lambda_1 + \mu_1 - \nu_1 - c \geq 0 & \lambda_1 + \mu_1 - \nu_2 - \nu_3 - c \geq 0 \\
\lambda_1 + \mu_2 - \nu_2 - c \geq 0 & \lambda_1 + \lambda_2 + \mu_1 - \nu_1 - \nu_3 - c \geq 0 \\
\lambda_1 - \nu_3 - c \geq 0 & \lambda_1 + \lambda_2 + \mu_2 - \nu_2 - \nu_3 - c \geq 0 \\
\lambda_2 + \mu_1 - \nu_2 - c \geq 0 & \lambda_1 + \mu_1 + \mu_2 - \nu_1 - \nu_3 - c \geq 0 \\
\mu_1 - \nu_3 - c \geq 0 & \lambda_2 + \mu_1 + \mu_2 - \nu_2 - \nu_3 - c \geq 0 \\
\lambda_2 + \mu_2 - \nu_3 - c \geq 0 & \lambda_1 + \lambda_2 + \mu_1 + \mu_2 - \nu_1 - \nu_2 - c \geq 0
\end{array}$$

and

$$|\nu| = |\lambda| + |\mu|. \tag{9}$$

Note that, for $c = 0$, we recover the 6 inequalities saying that λ, μ and ν are dominant, the 6 Weyl inequalities and the 6 others inequalities of the Horn cone (see *e.g.* [Ful00]).

We now want to compute the function mapping (λ, μ, c) to the number of solutions of this system of inequalities in ν. The method is to put this problem in the langage of vector partition functions as in [Stu95].

Start with the matrix H of these 18 inequalities that is the transpose of

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix}.$$

Set

$$\Lambda = \{(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3, c) \in \mathbb{Z}^8 \, : \, |\nu| = |\lambda| + |\mu|\}$$

and

$$\Lambda^+ = \{(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2, \nu_3, c) \in \Lambda \, : \, \lambda, \mu, \nu \text{ dominant and } c \geq 0\}.$$

Let $\widetilde{\mathrm{Horn}}_3$ denote the set of points in Λ^+ that satisfy the 18 inequalities.

To get nonnegative variables, we make the following change of coordinates

$$\begin{array}{lll} a_1 = \lambda_1 - \lambda_2 - c & b_1 = \mu_1 - \mu_2 - c & c_1 = \nu_1 - \nu_2 - c \\ a_2 = \lambda_2 - c & b_2 = \mu_2 - c & c_2 = \nu_2 - c \end{array}$$

We get an isomorphism of abelian groups

$$\begin{array}{cccc} \varphi : & \Lambda & \longrightarrow & \mathbb{Z}^7 \\ & (\lambda, \mu, \nu, c) & \longmapsto & (\lambda_1 - \lambda_2 - c, \lambda_2 - c, \mu_1 - \mu_2 - c, \mu_2 - c, \nu_1 - \nu_2 - c, \nu_2 - c, c). \end{array}$$

Note that the last Weyl inequality is $\nu_3 - c \geq 0 (= \lambda_3 + \mu_3)$ and $\nu_2 \geq \nu_3$ imply $\nu_2 \geq c$. It follows that

$$\varphi(\widetilde{\mathrm{Horn}_3}) \subset \mathbb{N}^7.$$

Moreover, the 5 first dominancy inequalities are implied by $\varphi(\lambda, \mu, \nu, c) \in \mathbb{N}^7$. The converse of φ is given by:

$$\begin{array}{llll} \lambda_1 = a_1 + a_2 + 2c & \lambda_2 = a_2 + c & \mu_1 = b_1 + b_2 + 2c & \mu_2 = b_2 + c \\ \nu_1 = c_1 + c_2 + 2c & \nu_2 = c_2 + c & \multicolumn{2}{l}{\nu_3 = a_1 + b_1 - c_1 + 2(a_2 + b_2 - c_2) + 3c} \end{array}$$

The matrix of φ (its extension to $\mathbb{Z}^8$, more precisely) and its converse are

$$P = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & -1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The inequalities of the cone $\varphi(\widetilde{\mathrm{Horn}_3})$ are the rows of the matrix HQ. Suppressing the 5 first unuseful inequalities one gets

$$N = \begin{pmatrix} -1 & -2 & -1 & -2 & 1 & 3 & -3 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -2 & 1 & 2 & -2 \\ 0 & 1 & 1 & 1 & 0 & -1 & 1 \\ -1 & -2 & 0 & -1 & 1 & 2 & -2 \\ -1 & -1 & -1 & -1 & 1 & 2 & -2 \\ 0 & -1 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 2 & -1 & -2 & 2 \end{pmatrix}$$

Hence

$$\begin{aligned} \widetilde{\mathrm{Horn}}_3 &\simeq \{X \in \mathbb{N}^7 \,|\, NX \geq 0\} \\ &\simeq \{(X,Y) \in \mathbb{N}^7 \times \mathbb{N}^{13} \,|\, NX = Y\} \\ &\simeq \{X \in \mathbb{N}^{20} \,|\, \tilde{N}X = 0\} \end{aligned}$$

Here $\tilde{N} = (N|I_{13})$. Consider now the projection $p\,:\,\mathbb{Z}^7 \longrightarrow \mathbb{Z}^5\,:\,(a_i, b_i, c_i, c) \longmapsto (a_1, a_2, b_1, b_2, c)$.

We are interested in the map

$$\begin{array}{ccc} \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ Y & \longmapsto & \sharp\{X \in \mathbb{N}^{15}\,:\,\tilde{M}X = BY\}, \end{array}$$

where

$$\tilde{M} = \begin{pmatrix} 1 & 3 \\ -1 & -1 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \\ -1 & -2 \end{pmatrix}, \quad \tilde{M} = (M\, I_{13}), \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 1 & 2 & 3 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ -1 & -2 & -1 & -2 & -2 \end{pmatrix}.$$

Note that $\tilde{M}$ is not unimodular: the lcm of the maximal extracted determinants is not 1, but 6. There are 83 such nonzero determinants. Then [Stu95] implies that $(\lambda, \mu, c) \mapsto \mathrm{Nb}(c^{\bullet}_{\lambda\mu} > c)$ is piecewise quasi-polynomial with chambers obtained by intersecting some of these 83 simplicial cones. We used [VSB+07], an implementation of Barvinok algorithm [Bar94] to compute this function. The surprise was that we got only polynomial functions and only 7 cones. Actually, the software gave 36 cones that can be glued to give those 7.

Proposition 10. *We use the basis of fundamental weights to set* $\lambda = k_1\varpi_1 + k_2\varpi_2$ *and* $\mu = l_1\varpi_1 + l_2\varpi_2$*. Then* $\mathrm{Nb}(c^{\bullet}_{\lambda\mu} > c) = 0$ *unless*

$$c \leq \min(k_1, k_2, l_1, l_2).$$

Moreover, this cone divides into 7 cones $C_1, \dots, C_7$ *on which* $\mathrm{Nb}(c^{\bullet}_{\lambda\mu} > c)$ *is given by a polynomial function* $P_1, \dots, P_7$*. Five of these seven pairs* (C_i, P_i) *are kept unchanged by switching* k_1 *and* k_2*. The two others are exchanged by this operation.*

In particular, Conjecture 1 holds for GL_3*.*

In the basis of fundamental weights, we are interested in the function

$$\begin{array}{rccl} \psi : & \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ & (k_1,k_2,l_1,l_2,c) & \longmapsto & \sharp\{\nu \in \Lambda_3 \,|\, c^{\nu}_{k_1\varpi_1+k_2\varpi_2,l_1\varpi_1+l_2\varpi_2} > c\} \end{array} .$$

Notice moreover that switching k_1 and k_2 corresponds then to taking λ^*. Define now the following seven polynomials in k_1, k_2, l_1, l_2, c:

$$P_1 = 2\,c^2 - c(k_1+k_2+l_1+l_2+2) - \frac{1}{2}(k_1+k_2-l_1-l_2)^2 + k_1k_2 + l_1l_2 + \frac{1}{2}(k_1+k_2+l_1+l_2) + 1$$

$$P_2 = 3c^2 - 3c(k_1+k_2+1) + \frac{1}{2}(k_1+k_2)^2 + k_1k_2 + \frac{3}{2}(k_1+k_2) + 1$$

$$P_3 = 3c^2 - 3c(l_1+l_2+1) + \frac{1}{2}(l_1+l_2)^2 + l_1l_2 + \frac{3}{2}(l_1+l_2) + 1$$

$$P_4 = \frac{5}{2}\,c^2 - c\left(2\,k_1 + 2\,k_2 + l_1 + \frac{5}{2}\right) + k_1k_2 + (k_1+k_2)(l_1+1) - \frac{l_1}{2}(l_1-1) + 1$$

$$P_5 = \frac{5}{2}\,c^2 - c\left(2\,k_1 + 2\,k_2 + l_2 + \frac{5}{2}\right) + k_1k_2 + (k_1+k_2)(l_2+1) - \frac{l_2}{2}(l_2-1) + 1$$

$$P_6 = \frac{5}{2}\,c^2 - c\left(k_1 + 2l_1 + 2l_2 + \frac{5}{2}\right) + l_1l_2 + (l_1+l_2)(k_1+1) - \frac{k_1}{2}(k_1-1) + 1$$

$$P_7 = \frac{5}{2}\,c^2 - c\left(k_2 + 2l_1 + 2l_2 + \frac{5}{2}\right) + l_1l_2 + (l_1+l_2)(k_2+1) - \frac{k_2}{2}(k_2-1) + 1$$

Notice already that $P_1, \dots, P_5$ are symmetric in k_1, k_2, whereas P_6 and P_7 are exchanged when one exchanges k_1 and k_2. One might also add that P_3 is the image of P_2 under the involution corresponding to exchanging the roles of λ and μ – i.e. exchanging (k_1,k_2) and (l_1,l_2) –, as P_6 is the image of P_4 and P_7 the one of P_5 under this same involution.

Then, for $c \geq 0$, $k_1 \geq c$, $k_2 \geq c$, $l_1 \geq c$, and $l_2 \geq c$, the function ψ is given by the following piecewise polynomial:

Cones of polynomiality	Polynomial giving ψ
$C_1 : k_1+k_2 \geq \max(l_1,l_2)+c, \quad l_1+l_2 \geq \max(k_1,k_2)+c$	P_1
$C_2 : k_1+k_2 \leq \min(l_1,l_2)+c$	P_2
$C_3 : l_1+l_2 \leq \min(k_1,k_2)+c$	P_3
$C_4 : l_1+c \leq k_1+k_2 \leq l_2+c$	P_4
$C_5 : l_2+c \leq k_1+k_2 \leq l_1+c$	P_5
$C_6 : k_1+c \leq l_1+l_2 \leq k_2+c$	P_6
$C_7 : k_2+c \leq l_1+l_2 \leq k_1+c$	P_7

One can then see that the cones C_1 to C_5 are stable under permutation of k_1 and k_2 whereas the cones C_6 and C_7 are exchanged when k_1 and k_2 are. Thus, for all $k_1, k_2, l_1, l_2, c \geq 0$,

$$\psi(k_1,k_2,l_1,l_2,c) = \psi(k_2,k_1,l_1,l_2,c),$$

which proves Proposition 10.

Remark. The last part of Proposition 10 asserts that there exists a bijection $(\Lambda_3^0)^2 \times \Lambda_3 \longrightarrow (\Lambda_3^0)^2 \times \Lambda_3$, $(\lambda, \mu, \nu) \longmapsto (\lambda^*, \mu, \tilde{\nu})$ such that

$$c_{\lambda\mu}^{\nu} = c_{\lambda^*\mu}^{\tilde{\nu}}.$$

One can hope such a linear bijection. Unfortunately, it does NOT exists.

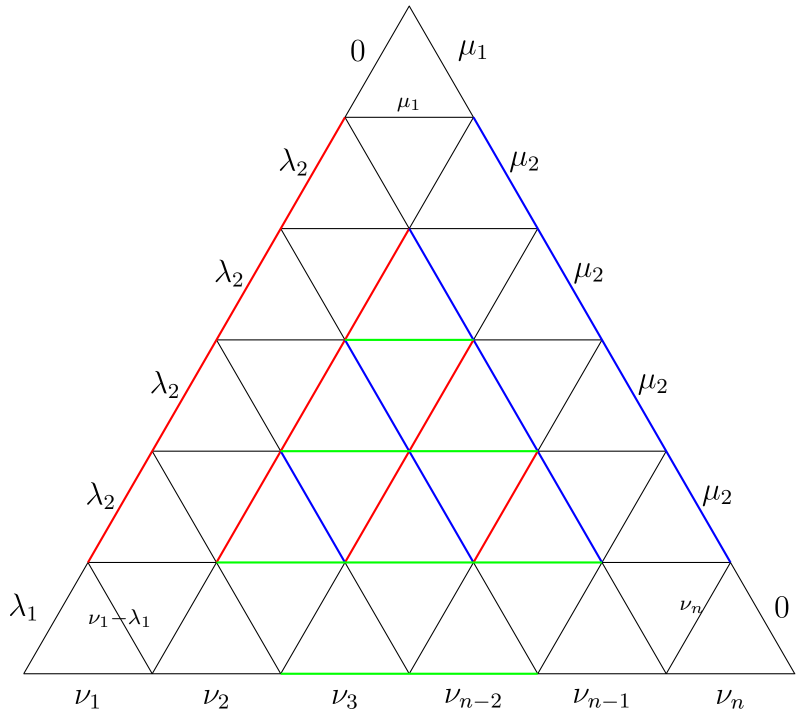
One can check this affirmation as follows. If such a map φ exists, it (or its transpose map) has to stabilize the set of the 18 inequalities of the Horn cone. These constraints are very strong and lead to a contradiction.

Note also that the linear automorphisms of $(\Lambda_3)^3$ preserving the Littlewood-Richardson coefficients are proved to form a group of cardinality 144 (so big !) in [BR20].

5 A stability result

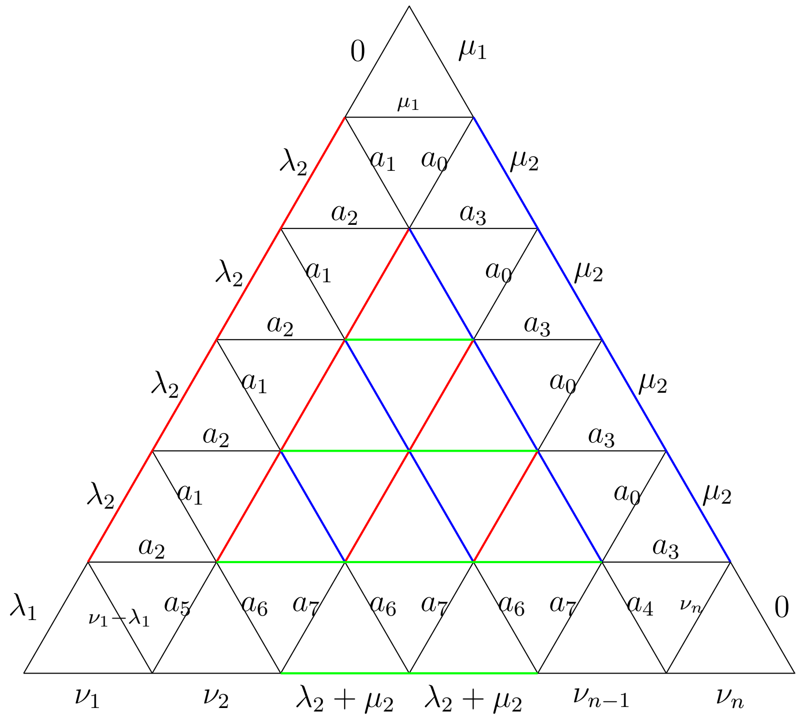
In this section, we are interested in Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$ with λ and μ near-rectangular. Using the hive model, we give a proof of the stability result stated in the introduction.

Proof of Proposition 3. We know that the Littlewood-Richardson coefficient $c_{\lambda\mu}^{\nu}$ is equal to the number of hives with exterior edges labelled by λ, μ, and ν. Let us assume that $c_{\lambda\mu}^{\nu} > 0$, meaning that such a hive exists, and consider any hive like this. In this hive, a certain number of edges have a label that is already fixed by these "boundary conditions". These edges and values are shown in the following picture (made for the sake of example with a hive of size 6, but the picture is strictly the same for all sizes at least 4):



Here we used a number of hive conditions (including the fact that, in any triangle, the values on two of the edges fix the value on the third one) and applied the following convention for colours: all the red edges correspond to the value λ_2, the blue ones to the value μ_2, and the green ones to the value $\lambda_2+\mu_2$. Thus we see that, in order for such a hive to exist, ν must be of the aforementioned form: $\nu = \nu_1\nu_2(\lambda_2+\mu_2)^{n-4}\nu_{n-1}\nu_n$. Notice moreover that, even if $n = 4$, two rombus inequalities show immediately that one must still have $\nu_2 \geq \lambda_2 + \mu_2 \geq \nu_{n-1}$.

From now on we assume that ν has this particular form. Consider once again a hive with exterior edges labelled by λ, μ, and ν. The same conditions as before apply, and the values of all the remaining edges have then to be chosen in order to determine completely the hive. Let us name these values $a_0, a_1, \ldots, a_7$ as shown in the following picture:



The fact that many of these edges must be given the same value comes everytime from the fact that, in any rombus inside a hive, if two opposite edges have the same value, then it must also be the case for the other pair of opposite edges.

Using now once again the fact that, for any triangle, the values corresponding to two edges fix the value on the third one, these 8 integers $a_0, \ldots, a_7$ are moreover related by the following equations:

$$\begin{cases} a_0 + a_1 = \mu_1 \\ a_0 + \mu_2 = a_3 \\ \lambda_2 + a_1 = a_2 \\ \nu_1 - \lambda_1 + a_5 = a_2 \\ a_4 + \nu_n = a_3 \\ a_4 + a_7 = \nu_{n-1} \\ a_5 + a_6 = \nu_2 \\ a_6 + a_7 = \lambda_2 + \mu_2 \end{cases} \iff \begin{cases} a_1 = \mu_1 - a_0 \\ a_2 = \lambda_2 + \mu_1 - a_0 \\ a_3 = \mu_2 + a_0 \\ a_4 = \mu_2 - \nu_n + a_0 \\ a_5 = \lambda_1 + \lambda_2 + \mu_1 - \nu_1 - a_0 \\ a_6 = \lambda_2 + 2\mu_2 - \nu_{n-1} - \nu_n + a_0 \\ a_7 = -\mu_2 + \nu_{n-1} + \nu_n - a_0 \end{cases}$$

(let us recall that the equality $|\lambda|+|\mu| = |\nu|$ means that $\lambda_1+2\lambda_2+\mu_1+2\mu_2 = \nu_1+\nu_2+\nu_{n-1}+\nu_n$). In particular, this means that the hive is for instance entirely determined by the value of a_0. We can now look at all the hive inequalities that must be satisfied by these a_i's:

$$\begin{array}{rcl} a_0 & \geq & 0 \\ \lambda_2 & \geq & a_0 \\ a_1 & \geq & \mu_2 \end{array} \qquad \begin{array}{rcl} \nu_1 & \geq & a_2 \\ a_1 & \geq & \nu_1 - \lambda_1 \\ a_2 & \geq & \nu_2 \\ \mu_2 & \geq & a_6 \end{array} \qquad \begin{array}{rcl} a_3 & \geq & \nu_n \\ \nu_n & \geq & a_0 \\ \nu_{n-1} & \geq & a_3 \end{array}$$

The inequalities of the first column can be obtained from the hive inequalities in the top corner of the hive, those of the second from the bottom-left corner, and those of the third from the bottom-right corner (keep in mind that some of them can of course be obtained in several ways). Thanks to the previous relations between the a_i's, all these inequalities can be expressed in terms of a_0 only, giving in the end exactly the following necessary and sufficient conditions on a_0 to obtain a hive:

$$\begin{cases} a_0 \geq \max(0, \lambda_2 + \mu_1 - \nu_1, \nu_n - \mu_2) \\ a_0 \leq \min(\lambda_2, \mu_1 - \mu_2, \lambda_1 + \mu_1 - \nu_1, \lambda_2 + \mu_1 - \nu_2, -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n, \nu_n, \nu_{n-1} - \mu_2) \end{cases}$$

As a consequence, $c_{\lambda\mu}^{\nu}$ is the cardinality of the following inteval of integers

$$[\![\max(0, \lambda_2+\mu_1-\nu_1, \nu_n-\mu_2), \min(\lambda_2, \mu_1-\mu_2, \lambda_1+\mu_1-\nu_1, \lambda_2+\mu_1-\nu_2, -\lambda_2-\mu_2+\nu_{n-1}+\nu_n, \nu_n, \nu_{n-1}-\mu_2)]\!].$$

Observe finally that $\nu_{n-1} \leq \lambda_2 + \mu_2$ and $\nu_2 \geq \lambda_2 + \mu_2$ give

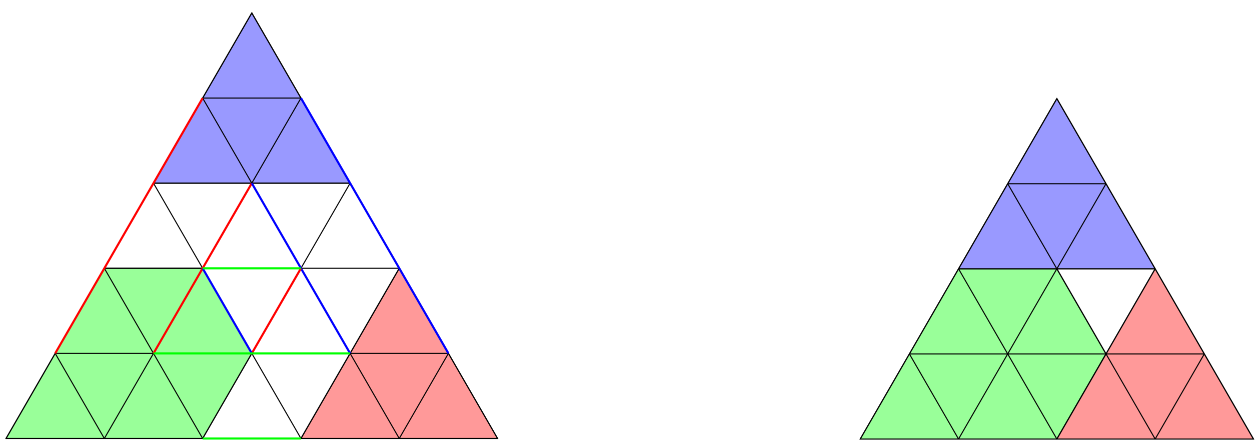
$$\nu_{n-1} - \mu_2 \leq \lambda_2, \qquad \lambda_2 + \mu_1 - \nu_2 \leq \mu_1 - \mu_2, \qquad -\lambda_2 - \mu_2 + \nu_{n-1} + \nu_n \leq \nu_n.$$

Hence $c_{\lambda\mu}^{\nu}$ is the cardinality of

$$[\![\max(0, \lambda_2+\mu_1-\nu_1, -\mu_2+\nu_n), \min(\lambda_1+\mu_1-\nu_1, \lambda_2+\mu_1-\nu_2, -\lambda_2-\mu_2+\nu_{n-1}+\nu_n, -\mu_2+\nu_{n-1})]\!].$$

$\square$

This stability can be seen as the existence of a bijection between sets of hives. Such a bijection can for instance be obtained as follows.



Starting from a hive of size n ($n \geq 4$), consider the three areas coloured in the picture above (on the left): the four triangles in the top corner, the four in the bottom-right one, and the seven in the bottom-left one. Then send this hive to the one of size 4 obtained by keeping these three coloured-areas (picture on the right). The rombus inequalities in the hive show that the values on the edges in these three particular areas determine indeed completely the hive. This means that this map is well-defined and that it is truly a bijection.

6 The case of $\mathrm{GL}_4(\mathbb{C})$

This section is written for $\mathrm{GL}_4(\mathbb{C})$. But Proposition 3 allows to extend several results to any $\mathrm{GL}_n(\mathbb{C})$ for $n \geq 4$.

6.1 The Horn cone

The set of points in Horn_n with λ and/or μ near-rectangular are those belonging on a face of this cone. Proposition 3 implies that the geometry of this face and the LR-multiplicities on it do not depend on $n \geq 4$. We denote by $\overline{\mathrm{Horn}_n}$ the set of points in Horn_n with the first two partitions λ and μ in Λ_n^0. Then $\mathrm{Horn}_n \simeq \mathbb{Z}^2 \times \overline{\mathrm{Horn}_n}$. Set

$$\mathrm{Horn}_4^{\mathrm{nr}^2} = \{(\lambda, \mu, \nu) \in \mathrm{Horn}_4 \,:\, \lambda \text{ and } \mu \text{ are near-rectangular}\}$$

and

$$\mathrm{Horn}_4^{\mathrm{nr}} = \{(\lambda, \mu, \nu) \in \mathrm{Horn}_4 \,:\, \lambda \text{ is near-rectangular}\}$$

The inequalities characterizing the cone generated by Horn_n are well known. By convex geometry and explicit computation, one can deduce the minimal lists of inequalities for $\mathrm{Horn}_4^{\mathrm{nr}^2}$ and $\mathrm{Horn}_4^{\mathrm{nr}}$. Softwares like Normaliz [BIS] allow to make the computation.

Proposition 11. *Let λ, μ in Λ_4^0 and ν in Λ_4 such that λ and μ are near-rectangular. Then $c_{\lambda\mu}^{\nu} \neq 0$ if and only if*

$$\begin{array}{ccc}
 & |\lambda| + |\mu| = |\nu| & \\
\nu_1 \geq \nu_2 & | & \nu_4 \geq 0 \\
 & \nu_3 + \nu_4 \geq \lambda_2 + \mu_2 & \\
\nu_1 + \nu_3 \geq \lambda_1 + \lambda_2 + \mu_2 & | & \nu_1 + \nu_3 \geq \lambda_2 + \mu_1 + \mu_2 \\
 & \nu_2 \geq \lambda_2 + \mu_2 \geq \nu_3 & \\
\nu_3 \geq \lambda_2 & | & \nu_3 \geq \mu_2 \\
\lambda_1 + \mu_2 \geq \nu_2 & | & \lambda_2 + \mu_1 \geq \nu_2
\end{array}$$

Remark. Proposition 3 also implies that $\nu_1 + \nu_4 \geq \lambda_2 + \mu_1$, which is a consequence of these 11 inequalities.

Proposition 12. *The cone generated by $\mathrm{Horn}_4^{\mathrm{nr}^2} \cap \overline{\mathrm{Horn}_4}$ has 8 extreamal rays generated by the inclusions*

1. *$V(1) \subset V(1) \otimes V(0)$ (twice by permuting the factors);*
2. *$V(1^3) \subset V(1^3) \otimes V(0)$ (twice by permuting the factors);*
3. *$V(1^2) \subset V(1) \otimes V(1)$;*
4. *$V(1^4) \subset V(1) \otimes V(1^3)$ (twice by permuting the factors);*
5. *$V(2^2 1^2) \subset V(1^3) \otimes V(1^3)$.*

Each such extremal ray has multiplicity one. The Hilbert basis of $\mathrm{Horn}_4^{\mathrm{nr}^2} \cap \overline{\mathrm{Horn}_4}$ *consists in these 8 elements.*

We get similar descriptions for $\mathrm{Horn}_4^{\mathrm{nr}}$.

Proposition 13. *Let λ, μ in Λ_4^0 and ν in Λ_4 such that λ is near-rectangular. Then $c_{\lambda\mu}^{\nu} \neq 0$ if and only if*

$$|\lambda| + |\mu| = |\nu|$$
$$\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4 \geq 0$$
$$\begin{array}{ll}
\lambda_1 \geq \lambda_2 \geq 0 & \mu_1 \geq \mu_2 \geq \mu_3 \geq 0 \\
\lambda_1 + \mu_1 \geq \nu_1 & \min(\lambda_2 + \mu_1, \lambda_1 + \mu_2) \geq \nu_2 \\
\min(\lambda_2 + \mu_2, \lambda_1 + \mu_3) \geq \nu_3 & \min(\lambda_1, \mu_1, \lambda_2 + \mu_3) \geq \nu_4
\end{array}$$
$$\nu_1 \geq \max(\lambda_1, \mu_1\lambda_2 + \mu_2) \qquad \nu_2 \geq \max(\mu_2, \lambda_2 + \mu_3) \qquad \nu_3 \geq \max(\lambda_2, \mu_3)$$
$$\begin{array}{ll}
\nu_1 + \nu_2 \geq \max(\lambda_1 + \lambda_2 + \mu_2, \lambda_2 + \mu_1 + \mu_2) & \nu_1 + \nu_3 \geq \max(\lambda_1 + \lambda_2 + \mu_3, \lambda_2 + \mu_1 + \mu_3) \\
\nu_2 + \nu_3 \geq \lambda_2 + \mu_2 + \mu_3 & \nu_1 + \nu_4 \geq \lambda_2 + \mu_1 \\
\nu_2 + \nu_4 \geq \lambda_2 + \mu_2 & \nu_3 + \nu_4 \geq \lambda_2 + \mu_3
\end{array}$$

We have 32 facets.

Proposition 14. *The cone generated by* $\mathrm{Horn}_4^{\mathrm{nr}} \cap \overline{\mathrm{Horn}_4}$ *has 12 extremal rays generated by the inclusions*

1. $V(1) \subset V(1) \otimes V(0)$, $V(1) \subset V(111) \otimes V(0)$, $V(1) \subset V(0) \otimes V(1)$, $V(11) \subset V(0) \otimes V(11)$ *and* $V(111) \subset V(0) \otimes V(111)$ *;*
2. $V(1^2) \subset V(1) \otimes V(1)$ *;*
3. $V(1^4)$ *is contained in* $V(1) \otimes V(1^3)$ *and* $V(1^3) \otimes V(1)$ *;*
4. $V(2211)$ *is contained in* $V(1^3) \otimes V(1^3)$ *and* $V(211) \otimes V(11)$ *;*
5. $V(1^3) \subset V(1) \otimes V(11)$ *;*
6. $V(21^3) \subset V(1^3) \otimes V(1^2)$.

Each such extremal ray has multiplicity one. The Hilbert basis of $\mathrm{Horn}_4^{\mathrm{nr}} \cap \overline{\mathrm{Horn}_4}$ *consists in these 12 elements.*

6.2 Special case of self-dual representations

Let k and l be two nonnegative integers and $n \geq 4$. The $\mathrm{SL}_n(\mathbb{C})$-modules $V_n((2k)k^{n-2})$, $V_n((2l)l^{n-2})$ and hence $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ are self dual.

In [PW20, Section 8], conjectural values (for $n = 6$) are given for the numbers of isotypical components in $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ and also self-dual such components. Here, we prove and extend these formulas.

Corollary 15. *Assume up to symmetry that $l \leq k$. The number of distinct isotypical components in $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ is given by*

$$\begin{cases} l^3 + 3l^2 + 3l + 1 & \text{if } 2l \leq k \\ \frac{1}{3}k^3 - 2k^2 l + 4kl^2 - \frac{5}{3}l^3 - k^2 + 4kl - l^2 + \frac{2}{3}k + \frac{5}{3}l + 1 & \text{if } 2l \geq k \end{cases}$$

Proof. By Proposition 3, one may assume that $n = 4$. Then, Proposition 11 shows that $\nu \in \mathbb{Z}^4$ is the highest weight of an isotypical component of $V_4((2k)k^2) \otimes V_4((2l)l^2)$ if and only if (recall that $l \leq k$)

$$\begin{array}{c} 4(k+l) = |\nu| \;|\qquad \nu_1 \geq \nu_2 \\ \nu_4 \geq 0 \qquad |\; \nu_3 + \nu_4 \geq k + l \\ \nu_1 + \nu_3 \geq 3k + l \\ 2n + m \geq \nu_2 \geq k + l \geq \nu_3 \geq k \end{array}$$

The corollary follows by explicit computations or the use of [VSB+07]. ∎

Similarly, one gets the number of self-dual representations.

Corollary 16. *Assume up to symmetry that $l \leq k$. The $\mathrm{SL}_n(\mathbb{C})$-module $V_n((2k)k^{n-2}) \otimes V_n((2l)l^{n-2})$ contains $(l+1)^2$ distinct selfdual isotypical components.*

Proof. By Proposition 3, one may assume that $n = 4$. Then, the set of self-dual isotypical components of $V_4((2k)k^2) \otimes V_4((2l)l^2)$ are obtained by adding the condition

$$\nu_1 + \nu_4 = \nu_2 + \nu_3$$

to those appearing in the proof of Corollary 15. The corollary follows by explicit computations or the use of [VSB+07]. ∎

6.3 Computation of $\mathrm{Nb}(c^{\bullet}_{\lambda\mu} > c)$ for λ and μ near-rectangular

In this subsection, we report on the computation of the function

$$\begin{array}{rccl} \mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c) : & (\Lambda_4^{\mathrm{nr}})^2 \times \mathbb{N} & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu, c) & \longmapsto & \sharp\{\nu \in \Lambda_4 \,:\, c^{\nu}_{\lambda\mu} > c\}. \end{array}$$

By Proposition 3, this function determines $\mathrm{Nb}_n(c^{\bullet}_{\lambda\mu} > c)$ for any near-rectangular partitions λ and μ of length $n \geq 4$.

Since Propositions 9 and 3 give similar expressions for the Littlewood-Richardson coefficient, we can apply the strategy of Section 4.

We get that $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c)$ is the number of points $\nu \in \Lambda_4$ such that $\lambda_1 + 2\lambda_2 + \mu_1 + 2\mu_2 = \nu_1 + \nu_2 + \nu_3 + \nu_4$ and

$$\begin{array}{ll} -\lambda_2 - \mu_2 + \nu_2 \geq 0 & \lambda_2 + \mu_2 - \nu_3 \geq 0 \\ \lambda_1 - \lambda_2 \geq c & -\lambda_2 + \nu_3 \geq c \\ \nu_1 - \nu_2 \geq c & \nu_3 - \nu_4 \geq c \\ \lambda_1 + \mu_1 - \nu_1 \geq c & \lambda_2 + \mu_1 - \nu_2 \geq c \\ -\lambda_2 - \mu_2 + \nu_3 + \nu_4 \geq c & -\mu_2 + \nu_3 \geq c \\ \lambda_1 + \mu_2 - \nu_2 \geq c & \lambda_1 + \lambda_2 + \mu_2 - \nu_2 - \nu_4 \geq c \\ \lambda_1 + \mu_1 + \mu_2 - \nu_1 - \nu_4 \geq c & \lambda_2 + \mu_1 + \mu_2 - \nu_2 - \nu_4 \geq c \end{array}$$

In particular, it is the number of integer points in some polytope depending linearly on the data (λ, μ, c). Then $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > c)$ is piecewise quasi-polynomial and can be computed using Barvinok's algorithm.

As in Section 4, from this point on we use the basis of fundamental weights to write $\lambda = k_1\varpi_1 + k_2\varpi_1^*$ and $\mu = l_1\varpi_1 + l_2\varpi_1^*$. Thus the symmetry we want to observe is once again with respect to switching k_1 and k_2. Consider the function

$$\begin{array}{rccl} \psi : & \mathbb{N}^5 & \longrightarrow & \mathbb{N} \\ & (k_1, k_2, l_1, l_2, c) & \longmapsto & \sharp\{\nu \in \Lambda_4 \,|\, c^{\nu}_{k_1\varpi_1+k_2\varpi_1^*, l_1\varpi_1+l_2\varpi_1^*} > c\} \end{array}.$$

Proposition 17. *We have $\psi(k_1, k_2, l_1, l_2, c) = 0$ unless*

$$c \leq \min(k_1, k_2, l_1, l_2).$$

Moreover, this cone divides in 36 cones $C_1, \ldots, C_{36}$ on which ψ is given by a polynomial function $P_1, \ldots, P_{36}$. 12 of these pairs (C_i, P_i) are kept unchanged by permuting k_1 and k_2 (namely (C_1, P_1) to (C_{12}, P_{12})). The 24 other such pairs are permuted two by two by this operation (for all $i \in \{7, \ldots, 18\}$, (C_{2i-1}, P_{2i-1}) and (C_{2i}, P_{2i}) are permuted).

In particular, Conjecture 1 holds for GL_4 and λ, μ near-rectangular.

Now to present as clearly as possible these cones and polynomials without writing all of them, let us use the two following involutions: s_1 corresponding to the permutation of k_1 and k_2, and s_2 corresponding to permuting (k_1, k_2) and (l_1, l_2). Then $\langle s_1, s_2\rangle$ acts on the set of all pairs (C_i, P_i) with 8 orbits. Let us give below one representative for each one of these. The labelling is the one of the complete list [Res20, pol_and_cones_SL4nr2.txt], chosen so that the stability when exchanging k_1 and k_2 is easier to see:

$$C_1: \quad l_1 + l_2 \leq k_1 + c, \quad l_1 + l_2 \leq k_2 + c,$$

$$P_1 = \left(-\frac{1}{2}\right) \cdot (-l_2 + c - 1) \cdot (-l_1 + c - 1) \cdot (-l_1 - l_2 + 2c - 2)$$

has a $\langle s_1, s_2\rangle$-orbit of size 2;

$$C_{16}: \quad l_1 + l_2 \leq k_1 + c, \quad l_1 + l_2 \geq k_2 + c, \quad k_2 \geq l_1, \quad k_2 \geq l_2,$$

$$P_{16} = P_1 - \binom{-k_2 + l_1 + l_2 - c + 2}{3}$$

has an orbit of size 4;

$$C_2: \quad l_1 + l_2 \geq k_1 + c, \quad l_1 + l_2 \geq k_2 + c, \quad k_1 \geq l_1, \quad k_1 \geq l_2, \quad k_2 \geq l_1, \quad k_2 \geq l_2,$$

$$P_2 = P_{16} - \binom{-k_1 + l_1 + l_2 - c + 2}{3}$$

has an orbit of size 2;

$$C_{19}: \quad l_1 + l_2 \geq k_1 + c, \quad k_1 \geq l_1, \quad k_2 \leq l_1, \quad k_2 \geq l_2,$$

$$P_{19} = P_2 + \binom{-k_2 + l_1 + 1}{3}$$

has an orbit of size 8;

$$C_{21}: \quad l_1 + l_2 \leq k_1 + c, \quad k_2 \leq l_1, \quad k_2 \geq l_2,$$

$$P_{21} = P_{16} + \binom{-k_2 + l_1 + 1}{3}$$

has an orbit of size 8;

$$C_{29}: \quad k_1 + k_2 \geq l_1 + l_2, \quad l_1 + l_2 \geq k_1 + c, \quad k_2 \leq l_1, \quad k_2 \leq l_2,$$

$$P_{29} = P_{19} + \binom{-k_2 + l_2 + 1}{3}$$

has an orbit of size 4;

$$C_{27}: \quad k_1 + k_2 \leq l_1 + l_2, \quad k_1 \geq l_1, \quad k_1 \geq l_2,$$

$$P_{27} = P_{29} + \binom{-k_1 - k_2 + l_1 + l_2 + 1}{3}$$

has an orbit of size 4; finally,

$$C_{36}: \quad l_1 + l_2 \leq k_1 + c, \quad k_2 \leq l_1, \quad k_2 \leq l_2,$$

$$P_{36} = P_{21} + \binom{-k_2 + l_2 + 1}{3}$$

also has an orbit of size 4.

Remark. One can observe that the polynomials P_i are exepressed using each other. We exploit here the fact that the difference of two polynomials associated to two adjacent chambers has a simple expression given by the Paradan formula [Par04, BV09].

6.4 Computation of $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > 0)$ for λ near-rectangular

In this section, we report on the computation of the function

$$\begin{array}{rccc} \mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > 0) : & \Lambda_4^{\mathrm{nr}} \times \Lambda_4 & \longrightarrow & \mathbb{N} \\ & (\lambda, \mu) & \longmapsto & \sharp\{\nu \in \Lambda_4 \,:\, c^{\nu}_{\lambda\mu} > 0\}. \end{array}$$

As we recalled in Proposition 7, $\mathrm{Nb}_4(c^{\bullet}_{\lambda\mu} > 0)$ is the number of integral points in an affine section of the Horn cone. The inequalities of this cone are explicitly given in Proposition 13. Then, one can compute explicitly the quasi-polynomial function with the program [VSB+07]. The output is too big (even using symmetries) to be collected there. The interested reader can get details from [Res20, Supplementary material].

Proposition 18. *The cone $\Lambda_4^{\mathrm{nr}} \times \Lambda_4$ divides in 205 cones of non empty interior. On 151 of them $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ is polynomial of degree 3, and on the other 54 it is quasi-polynomial. The only congruence occurring is the parity of $\lambda_1 + |\mu|$.*

Moreover, for any pair (C, P) where C is one of the 205 cones and P the corresponding function, one can see that in this list there is also a pair (C', P') obtained by replacing λ by λ^* (in 57 cases, $(C', P') = (C, P)$). In particular, Conjecture 1 holds.

Under the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ there are 61 *orbits of true polynoms and* 22 *orbits of quasipolynoms.*

Here we give three examples illustrating some of the variety of cases that one can observe. The function $\mathrm{Nb}_4(c^\bullet_{\lambda\mu} > 0)$ for $\lambda = k_1\varpi_1 + k_2\varpi_1^* \in \Lambda_4^{\mathrm{nr}} \cap \Lambda_4^0$ and $\mu = \mu_1\mu_2\mu_3 \in \Lambda_4^0$ is given:

- on the cone defined by $\mu_1 \geq k_1 + \mu_3$, $\mu_1 \geq k_2 + \mu_3$, $\mu_2 \geq \mu_3$, $k_1 + k_2 + \mu_3 \geq \mu_1 + \mu_2$, $\mu_3 \geq 0$, by the polynomial

$$\begin{aligned} P = \frac{\mu_3}{2} \cdot \Big(& \mu_2(2\mu_1 - \mu_2 + 1) + 2(\mu_1 + 1) - (\mu_3 + 1)(k_1 + k_2 + \mu_1 - \mu_2 + 2)\Big) \\ - \frac{\mu_2 + 1}{6} \cdot \Big(& 3(k_1^2 + k_2^2) - 3(k_1 + k_2)(2\mu_1 + 1) + 3\mu_1^2 + 2\mu_2^2 - 3\mu_1 + 4\mu_2 - 6\Big), \end{aligned}$$

  symmetric in k_1, k_2.

- on the cone defined by $\mu_1 + \mu_2 \geq k_1 + k_2 + \mu_3$, $k_2 + \mu_1 \geq k_1 + \mu_2 + \mu_3$, $k_2 + \mu_3 \geq \mu_2$, $k_1 + \mu_1 \geq k_2 + \mu_2 + \mu_3$, $k_1 + \mu_3 \geq \mu_2$, $k_1 + k_2 \geq \mu_1$, $\mu_3 \geq 0$ (adjacent to the previous one), by the quasi-polynomial

$$\begin{cases} \begin{aligned} & P + \frac{1}{24}(k_1 + k_2 - \mu_1 - \mu_2 + \mu_3 - 1) \\ & \cdot(k_1 + k_2 - \mu_1 - \mu_2 + \mu_3 + 1) \\ & \cdot(-2\,k_1 - 2\,k_2 + 2\,\mu_1 + 2\,\mu_2 + 4\,\mu_3 + 3) \end{aligned} & \text{if } k_1 + k_2 + \mu_1 + \mu_2 + \mu_3 \text{ is odd} \\[2ex] \begin{aligned} & P + \frac{1}{24}(k_1 + k_2 - \mu_1 - \mu_2 + \mu_3) \\ & \cdot\Big(2 + (k_1 + k_2 - \mu_1 - \mu_2 + \mu_3) \\ & \cdot(-2\,k_1 - 2\,k_2 + 2\,\mu_1 + 2\,\mu_2 + 4\,\mu_3 + 3)\Big) \end{aligned} & \text{if } k_1 + k_2 + \mu_1 + \mu_2 + \mu_3 \text{ is even,} \end{cases}$$

  also symmetric in k_1, k_2.

- on the cone defined by $\mu_1 \geq k_1$, $\mu_1 \geq k_2 + \mu_3$, $\mu_2 \geq \mu_3$, $k_2 \geq \mu_2$, $k_1 + \mu_3 \geq \mu_1$ (also adjacent to the first one), by the non-symmetric polynomial

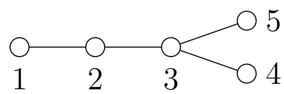
$$P + \binom{k_1 - \mu_1 + \mu_3 + 1}{3}.$$

7 Related questions

7.1 In type D_n

Apart from the type A_n, the irreducible representations of simple Lie algebras are not self-dual only in types D_n and E_6. Consider here the type D_5.

D_5



Let $(\varpi_1, \dots, \varpi_5)$ be the list of fundamental weights. Then $V(\varpi_4)^* \simeq V(\varpi_5)$ whereas $V(\varpi_1)$, $V(\varpi_2)$ and $V(\varpi_3)$ are self dual. The natural generalization of near-rectangular partition is dominant weights in $\mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$. A natural generalization of Conjecture 2 would be: for $\lambda = a\varpi_4 + b\varpi_5 \in \mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$ and μ a dominant weight of D_5, do the two tensor products

$$V_{D_5}(a\varpi_4 + b\varpi_5) \otimes V_{D_5}(\mu) \qquad \text{and} \qquad V_{D_5}(b\varpi_4 + a\varpi_5) \otimes V_{D_5}(\mu)$$

contain the same number of isotypical components?

The answer is NO, even assuming that $\mu \in \mathbb{N}\varpi_4 \oplus \mathbb{N}\varpi_5$ too. An example is $\lambda = 2\varpi_4 + \varpi_5$ and $\mu = \varpi_4 + 2\varpi_5$. The two tensor products have respectively 31 and 30 isotypical components as checked using SageMath [S+12]:

```
sage:  D5=WeylCharacterRing("D5",style="coroots")
sage:  len(D5(0,0,0,2,1)*D5(0,0,0,1,2))
31
sage:  len(D5(0,0,0,1,2)*D5(0,0,0,1,2))
30
```

7.2 In type A_n

The representations of $\mathrm{SL}_n(\mathbb{C})$ corresponding to near-rectangular partitions are of the form $V(a\varpi_1 + b\varpi_{n-1})$. Observe that (ϖ_1, ϖ_{n-1}) is a pair of mutually dual fundamental weights. One could hope that Conjecture 1 or 2 hold for any linear combinaison of a given pair of mutually dual fundamental weights. This is not true even for (ϖ_2, ϖ_3) and $n = 5$. Indeed, for $\lambda = \varpi_2 + 2\varpi_3$ and $\mu = 3\varpi_2 + \varpi_3$, the numbers of isotypical components in $V(\lambda) \otimes V(\mu)$ and $V(\lambda)^* \otimes V(\mu)$ differ:

```
sage:  len(lrcalc.mult([3,3,2],[4,4,1],5))
34
sage:  len(lrcalc.mult([3,3,1],[4,4,1],5))
33
```

Mention finally that we checked Conjecture 1 on examples, using SageMath. See [Res20, test_Conj1.sage]:

- Conjecture 1 holds for GL_4 if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 20$ and $|\mu| \leq 40$.
- Conjecture 1 holds for GL_5 if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 20$ and $|\mu| \leq 30$.
- Conjecture 1 holds for GL_6 if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 10$ and $|\mu| \leq 30$.
- Conjecture 1 holds for GL_{10} if $\max(\lambda_1 - \lambda_2, \lambda_2) \leq 10$ and $|\mu| \leq 15$.

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