AROUND THE HORN CONJECTURE

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Abstract. We discuss the problem of determining the possible spectra of a sum of Hermitian matrices each with known spectrum. We explain how the Horn conjecture, which gives a complete answer to this question, is related with algebraic geometry, symplectic geometry, and representation theory. The first lecture is an introduction to Schubert calculus, from which one direction of Horn's conjecture can be deduced. The reverse direction follows from an application of geometric invariant theory: this is treated in the second lecture. Finally, we explain in the third lecture how a version of Horn's problem for special unitary matrices is related to the quantum cohomology of Grassmannians.

Notes for Gael VIII, CIRM, March 2000.

1. Eigenvalues of hermitian matrices and Schubert calculus

The problem. Let A, B, C be complex Hermitian n by n matrices. Denote the set of eigenvalues, or spectrum of A by $\lambda(A) = (\lambda_1(A) \geq \cdots \geq \lambda_n(A))$, and similarly by $\lambda(B)$ and $\lambda(C)$ the spectra of B and C. The main theme of these notes is the following question:

Suppose that $A+B=C$. What can be their spectra $\lambda(A)$, $\lambda(B)$, $\lambda(C)$?

There are obvious relations, like $\mathrm{trace}(C) = \mathrm{trace}(A) + \mathrm{trace}(B)$ or $\lambda_1(C) \leq \lambda_1(A) + \lambda_1(B)$. But a complete answer to this longstanding question was given only recently, and combines works and ideas from representation theory, symplectic and algebraic geometry.

Weyl's inequalities. There are various characterizations of the eigenvalues of Hermitian matrices, many of which are variants of the *minimax principle*. Let $(\,|\,)$ be the standard Hermitian product on $\mathbb{C}^n$. If $s = n - r$, denote by $\mathbb{G}_{r,s}$ the Grassmannian of r-dimensional linear subspaces of $\mathbb{C}^n$. Then

$$\lambda_{j+1}(A) = \min_{L \in \mathbb{G}_{n-j,j}} \max_{\substack{x \in L \\ (x|x)=1}} (Ax|x).$$

The idea is to test the values of $(Ax|x)$ on subspaces of $\mathbb{C}^n$. That's how Hermann Weyl [24] proved in 1912 the following inequalities.

Proposition 1. $\lambda_{p+q+1}(C) \leq \lambda_{p+1}(A) + \lambda_{q+1}(B)$.

Proof. The first point is to understand what happens when you modify a Hermitian matrix by another one of small rank. To fix notations, let $e_1, \ldots, e_n$ be a basis of $\mathbb{C}^n$ made of eigenvectors of A for the eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$.

Lemma 2. *Suppose that* $\mathrm{rank}\,(B) \leq k$*. Then* $\lambda_1(C) \geq \lambda_{k+1}(A)$.

Proof. For reasons of dimensions, the vector space generated by $e_1, \ldots, e_{k+1}$ meets the kernel of B along a non-zero unitary vector x. Then $\lambda_1(C) \geq (Cx|x) = (Ax|x) \geq \lambda_{k+1}(A)$. □

After adding if necessary suitable multiples of the identity, we can suppose that the eigenvalues of A, B, C are all positive. Denote by $A^{(p)}$ the Hermitian endomorphism of $\mathbb{C}^n$ defined by $A^{(p)}(e_i) = e_i$ if $i \leq p$, zero if $i > p$ (it depends on the choice of the basis of eigenvectors when $\lambda_p(A) = \lambda_{p+1}(A)$). The largest eigenvalue of $A' = A - A^{(p)}$ is $\lambda_1(A') = \lambda_{p+1}(A)$. Let $B' = B - B^{(q)}$ and $C' = A' + B' = C - (A^{(p)} + B^{(q)})$. Since $A^{(p)} + B^{(q)}$ has rank at most $p+q$, the lemma implies that

$$\lambda_{p+q+1}(C) \leq \lambda_1(C') \leq \lambda_1(A') + \lambda_1(B') = \lambda_{p+1}(A) + \lambda_{q+1}(B). \quad \square$$

Note that the main point in the key lemma above was to find a linear subspace in special position. In general, it is intuitively clear that the eigenvalues of $C = A + B$ will depend on the relative position of the eigenspaces of A and B. That's precisely the kind of information that is encoded in Schubert varieties of Grassmannians.

Schubert varieties. Let $V_\bullet$ denote a complete flag in $\mathbb{C}^n$, that is, a sequence of linear subspaces

$$0 = V_0 \subset \cdots \subset V_i \subset \cdots \subset V_n = \mathbb{C}^n,$$

where V_i has dimension i. Let λ be a partition inscribed in a r by $s = n - r$ rectangle, that is, a sequence of integers $s \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. We define the *Schubert cell*

$$\Omega_\lambda(V_\bullet) = \{W \in \mathbb{G}_{r,s},\ \dim\,(W \cap V_j) = i \text{ for } s + i - \lambda_i \leq j \leq s + i - \lambda_{i+1}\},$$

and the *Schubert variety*

$$X_\lambda(V_\bullet) = \{W \in \mathbb{G}_{r,s},\ \dim\,(W \cap V_{s+i-\lambda_i}) \geq i,\ 1 \leq i \leq r\}.$$

For example, when λ has a unique non zero part $\lambda_1 = k$, we get a *special Schubert variety*

$$X_k(V_\bullet) = \{W \in \mathbb{G}_{r,s},\ W \cap V_{s+1-k} \neq 0\}.$$

We will use the notations Ω_λ and X_λ when the reference flag $V_\bullet$ does not matter. The following facts are well-known (see e.g. [10, 8, 19]):

1. The Schubert variety X_λ is a closed subvariety of $\mathbb{G}_{r,s}$, defined locally by the vanishing of minors of the composite maps $W \subset \mathbb{C}^n \to \mathbb{C}^n/V_i$.
2. Let $W \in \mathbb{G}_{r,s}$. The sequence $\dim\,(W \cap V_j)$ goes from 0 to r, increasing at most by one at each step. So it increases strictly at exactly r values of j, which we denote by $j = s+i-\mu_i$, with μ a partition inscribed in a r by s rectangle. This proves that

$$\mathbb{G}_{r,s} = \coprod_{\mu \in r \times s} \Omega_\mu.$$

Moreover, if $\dim\,(W \cap V_{s+i-\lambda_i}) \geq i$, we clearly have $s + i - \lambda_i \geq s + i - \mu_i$, hence

$$X_\lambda = \coprod_{\mu \supset \lambda} \Omega_\mu.$$

In particular, we have the incidence relation $X_\lambda \supset X_\mu$ if and only if $\lambda \subset \mu$.

3. Chose a basis $v_1, \ldots, v_n$ of $\mathbb{C}^n$ adapted to the reference flag, i.e. such that $V_i = \langle v_1, \ldots, v_i \rangle$. Then $W \in \Omega_\lambda$ admits a unique basis of the form

$$w_i = v_{s+i-\lambda_i} + \sum_{\substack{1 \leq j \leq s+i-\lambda_i, \\ j \neq s+k-\lambda_k,\ k \leq i}} x_{ij} v_j,$$

where $1 \leq i \leq r$. In particular, Ω_λ is affine, isomorphic to $\mathbb{C}^{rs-|\lambda|}$ where $|\lambda| = \lambda_1 + \cdots + \lambda_r$. Moreover, it is easy to check that X_λ is the Zariski closure of Ω_λ.

It follows that the Schubert cells define a complex cellular decomposition of the Grassmannian. An immediate consequence is that the fundamental classes of the Schubert varieties, the *Schubert classes* $\sigma_\lambda = [X_\lambda]$, where λ is a partition inscribed in a r by s rectangle, form a basis of the cohomology with integers coefficients:

$$H^*(\mathbb{G}_{r,s}, \mathbb{Z}) = \bigoplus_{\lambda \subset r \times s} \mathbb{Z}\sigma_\lambda.$$

Note that the Schubert classes do not depend on the reference flag. This is because $GL(n, \mathbb{C})$ acts transitively on the set of complete flags. And it is a general fact that if Y is a subvariety of some variety X on which a *connected* topological group G acts continuously, then the fundamental class of gY does not depend on $g \in G$.

Exercise. Let $P_q(\mathbb{G}_{r,s}) = \sum_{k\geq 0} q^k \mathrm{rank}_{\mathbb{Z}} H^{2k}(\mathbb{G}_{r,s}, \mathbb{Z})$ be the *Poincaré polynomial* of the Grassmannian. Prove that when q is a power of a prime, $P_q(\mathbb{G}_{r,s})$ equals the number of points of the grassmannian $\mathbb{G}_{r,s}(\mathbb{F}_q)$ over the field with q elements. Deduce from this interpretation that

$$P_q(\mathbb{G}_{r,s}) = \frac{(1-q)(1-q^2)\cdots(1-q^{r+s})}{(1-q)\cdots(1-q^r)(1-q)\cdots(1-q^s)}.$$

How to multiply Schubert classes. Now that we know the additive structure of the cohomology ring of the Grassmannian, we need to understand its multiplicative structure. It was investigated in detail by mathematicians from the last century, in particular from the German and Italian schools. Here are the main remarkable formulas.

Duality. Let again $V_\bullet$ be our reference flag, with an adapted basis $v_\bullet$. We define the *dual flag* $V'_\bullet$ by $V'_i = \langle v_{n-i+1}, \ldots, v_n \rangle$. From our explicit descriptions of a prefered basis of an element of a given Schubert cell, we easily deduce the following fact: if λ and μ are partitions such that $|\lambda| + |\mu| = rs$, then $\Omega_\lambda(V_\bullet)$ and $\Omega_\mu(V'_\bullet)$ meet transversely in a unique point if $\mu = \hat{\lambda}$, where $\hat{\lambda} = (s-\lambda_r, \ldots, s-\lambda_1)$ is the *complementary partition* of λ, and have empty intersection otherwise (see [10], p. 198 or [19], 3.2.7). This implies that the cup-product

$$\sigma_\lambda \sigma_\mu = \delta_{\mu,\hat{\lambda}} \sigma_{r\times s},$$

where $\sigma_{r\times s}$ is the class of a point. This means that the basis of the cohomology of the Grassmannian given by Schubert classes is, up to complementarity, self-dual relatively to Poincaré duality. Usually, one represents a partition λ by its *diagram* as below (this diagram has λ_i boxes of the i-th row, from top to bottom) and the complementary partition has complementary diagram (after rotation) in the r by s rectangle.

Pieri's formula. Let σ_k be the class of a *special Schubert variety.* Then

$$\sigma_\lambda \sigma_k = \sum_{\substack{\nu \subset r\times s,\\ \nu\in\lambda\otimes k}} \sigma_\nu,$$

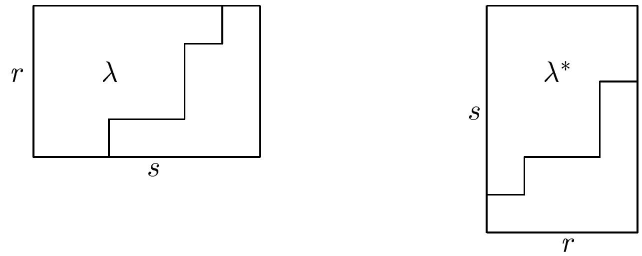
where $\lambda \otimes k$ denotes the space of partitions ν such that $|\nu| = |\lambda| + k$, and $\lambda_i \leq \nu_i \leq \lambda_{i-1}$ (see [10], p. 203 or [19], 3.2.8).

Giambelli's formula. It is a formal consequence of Pieri's formula that each Schubert class can be expressed as a determinant in special classes (see [10], p. 198 or [19], 3.2.10):

$$\sigma_\mu = \det(\sigma_{\mu_i - i + j})_{1\leq i,j\leq r}.$$

Exercise. Identifying $\mathbb{C}^n$ with its dual vector space, there is a natural isomorphism $\mathbb{G}_{r,s} \simeq \mathbb{G}_{s,r}$. Show that this isomorphism exchanges a Schubert variety $X_\lambda \subset \mathbb{G}_{r,s}$ with $X_{\lambda^*} \subset \mathbb{G}_{s,r}$, where λ^* is the *conjugate partition* of λ. Deduce from Giambelli's formula that if $\tau_k = \sigma_{(1^k)}$, then

$$\sigma_\mu = \det(\tau_{\mu^*_i - i + j})_{1\leq i,j\leq s}.$$



There are other descriptions of the cohomology ring of the Grassmannian, as a quotient of a polynomial ring. Indeed, one can deduce from Giambelli's formula that the special classes $\sigma_1, \ldots, \sigma_s$ generate $H^*(\mathbb{G}_{r,s})$. Let $\tau(z) = \sum_k z^k \tau_k = (1 + z\sigma_1 + \cdots + z^s\sigma_s)^{-1}$. One can prove that τ_k is the k-th Chern class of the dual of the tautological vector bundle on $\mathbb{G}_{r,s}$, which has rank r, so that $\tau_k = 0$ for $k > r$ ([10], p. 410); moreover, this gives a complete set of relations. More precisely:

$$H^*(\mathbb{G}_{r,s}, \mathbb{Z}) \simeq \mathbb{C}[\sigma_1, \ldots, \sigma_s]/\langle \tau_{r+1}, \ldots, \tau_n \rangle.$$

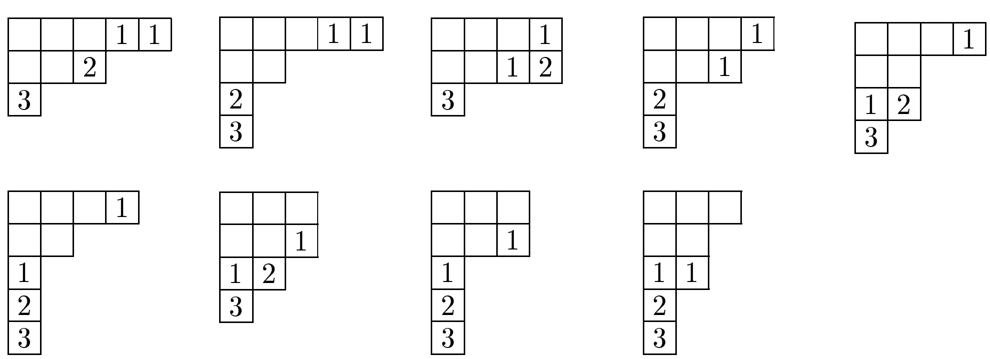
In principle, the previous formulas are enough to compute the product $\sigma_\lambda \sigma_\mu$ of any two Schubert classes: you just need to use Giambelli's formula to express σ_μ in terms of special Schubert classes, and then apply Pieri's formula r times. Of course, this is not very satisfactory. Littlewood and Richardson gave in 1934 a combinatorial algorithm for computing the multiplicities $c^\nu_{\lambda\mu}$ in the product

$$\sigma_\lambda \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu.$$

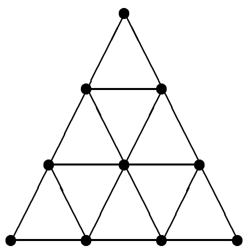
THE LITTLEWOOD-RICHARDSON RULE. First observe that $c^\nu_{\lambda\mu} = c^\nu_{\mu\lambda}$ can be non zero only when $\nu \supset \lambda, \mu$. A *skew-tableau* on μ/ν will be a way to fill the complement of the diagram of μ inside that of ν by positive integers, which increase on each column from top to bottom, and do not decrease on each line from left to right. It will have *weight* λ if each integer appears exactly λ_i times in the filling. Finally, let $w = w_1 \ldots w_{|\lambda|}$ be the associated word, obtained by reading the numbers in the skew-tableau line after line, from right to left. It is a *lattice word* if in every subword $w_1 \ldots w_i$, each integer j appears at least as often as $j+1$ (see [18], I 9).

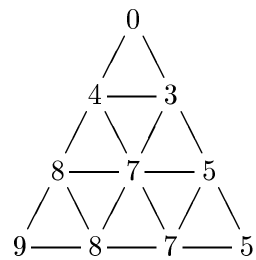
> *The Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ equals the number of skew-tableaux on μ/ν, of weight λ, whose associated word is a lattice word.*

Example. In computing the product $\sigma_{32}\sigma_{211}$, the following skew-tableaux do contribute and we get $\sigma_{32}\sigma_{211} = \sigma_{531} + \sigma_{5211} + \sigma_{441} + \sigma_{4311} + \sigma_{42111} + \sigma_{3321} + \sigma_{33111} + \sigma_{32211}$:



There exist several other combinatorial descriptions of Littlewood-Richardson coefficients. A recently discovered one is due to Knutson and Tao, and has been essential in their proof of the saturation conjecture (see below, and [5] for a very clear exposition of the main ideas). Consider a triangular array of vertices ($n+1$ on each side), and call *rhombus* a union of two small triangles with a common side.





Definition. *A* hive *is a labelling of this triangular array, such that for each rhombus, the sum of the labels at the obtuse vertices is greater than or equal to the sum of the labels at the acute vertices.*

If $\nu = (\nu_1, \dots, \nu_n)$ is any partition, define $\nu_+ = (0, \nu_1, \nu_1+\nu_2, \dots, |\nu|)$. Knutson and Tao proved that if $|\nu| = |\lambda| + |\mu|$, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ equals the number of integral hives with border labels given by $\lambda_+, |\lambda| + \mu_+, \nu_+$. (For example, on the picture above, there is an integral hive with these border labels for $\lambda = (32)$, $\mu = (211)$, $\nu = (441)$.)

Schubert calculus and Hermitian matrices. We know enough of *Schubert calculus* to generalize Weyl's inequalities. The inequalities we will obtain involve partial sums $\lambda_I(A) = \sum_{i\in I} \lambda_i(A)$ of eigenvalues of a Hermitian matrix A, where I is an increasing sequence $1 \le i_1 < \dots < i_r \le n$. To such a sequence, we associate the partition $\lambda = \lambda(I) = (i_r - r, \dots, i_1 - 1)$.

The relation between the partial sums $\lambda_I(A)$ and Schubert varieties is as follows. If $W \in \mathbb{G}_{r,s}$, let $\pi_W : \mathbb{C}^n \to W$ be the hermitian projection, and define the *Rayleigh trace*

$$R_A(W) = \operatorname{trace}(A \circ \pi_W) = \sum_{i=1}^{r} (Au_i, u_i)$$

for any orthonormal basis $(u_1, \dots, u_r)$ of W. Let $A_\bullet$ be a complete flag in $\mathbb{C}^n$, compatible with the eigenspaces of A, which means that $A_i = \langle a_1, \dots, a_i\rangle$, with $A(a_i) = \lambda_i(A)a_i$. Let $A'_\bullet$ be the opposite flag. It is an exercise to check that

$$\lambda_I(A) = \max_{W\in\Omega_{\lambda(I)}(A'_\bullet)} R_A(W).$$

Proposition 3. *Let A, B, C be Hermitian matrices such that $A + B = C$. Let $r < n$, and I, J, K be increasing subsequences of length r, with associated partitions λ, μ, ν. Suppose that the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is non zero. Then*

$$\lambda_I(A) + \lambda_J(B) \ge \lambda_K(C).$$

Proof. If $c^{\nu}_{\lambda\mu} \neq 0$, the duality properties of Schubert classes implies that the intersection of the Schubert varieties $\Omega_\lambda(A'_\bullet)$, $\Omega_\mu(B'_\bullet)$ and $\Omega_{\hat\nu}(C_\bullet)$ must be non empty. Let W be an intersection point. Denote by $\hat K$ the increasing sequence corresponding to $\hat\nu$, so that $\hat k_i = n - k_{r+1-i}$: we have $\lambda_{\hat K}(-C) = -\lambda_K(C)$. Using the linearity of the Rayleigh trace, we get

$$\lambda_I(A) + \lambda_J(B) \le R_A(W) + R_B(W) = R_C(W) = -R_{-C}(W) \le -\lambda_{\hat K}(-C) = \lambda_K(C). \quad \square$$

This proposition, due to Helmke-Rosenthal and Klyachko (see [15], Theorem 1.2), raises a number of questions. Are these inequalities sufficient for the existence of Hermitian matrices $A + B = C$ with the corresponding eigenvalues ? If it is the case, why is it enough to consider linear inequalities in the eigenvalues ? And how can we characterize the triplets (λ, μ, ν) of partitions such that the corresponding Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ is non zero ? We will answer to the first two questions in the next lecture. The answer to the last stems from a unexpected relation of Littlewood-Richardson coefficients with the representation theory of the linear group.

Representations of $GL(n,\mathbb{C})$. Let V be a complex vector space of dimension n. The linear group $GL(V)$ is (linearly) *reductive*, which means that any of its finite dimensional rational representation is a direct sum of irreducible ones (i.e. without any stable proper subspace). Moreover, these irreducible representations are classified by non-increasing sequences of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$. Among these are the partitions, corresponding to polynomial representations. For simplicity, let us suppose that λ is indeed a partition. Then the corresponding irreducible representation $S_\lambda V$, can be defined in the following way: choose any numbering N of the diagram of λ by integers from 1 to $l = |\lambda|$; denote by $(\lambda_1^*, \ldots, \lambda_m^*)$ the lenghs of the columns of this diagram. Then we have a composite map

$$p_N : \wedge^{\lambda_1^*} V \otimes \cdots \otimes \wedge^{\lambda_m^*} V \hookrightarrow V^{\otimes l} \to S^{\lambda_1} V \otimes \cdots \otimes S^{\lambda_n} V,$$

defined as follows: the inclusions $\wedge^{\lambda_i^*} V \hookrightarrow V^{\otimes \lambda_i^*}$ are the usual ones, but they involve in $V^{\otimes l}$ the factors V in the positions prescribed by the numbers in the i-th column of N; the projections $V^{\otimes \lambda_j} \to S^{\lambda_j} V$ are the usual symmetrizations, but they involve in $V^{\otimes l}$ the factors V in the positions prescribed by the numbers in the j-th line of N. These maps are obviously compatible with the diagonal action of $GL(V)$, thus the image of p_N is a $GL(V)$-module: this is $S_\lambda V$. For example, if λ has only one non zero part, say $\lambda_1 = k$, then we get the symmetric power $S^k V$; if λ has k non zero parts, all equal to one, we get the skew-symmetric power $\wedge^k V$.

The representation theory of the linear group was developped in the first decades of the century, in particular by I. Schur. Schur knew how to decompose the tensor product of any *Schur module* $S_\lambda V$ with a symmetric power. But it was only in 1947 that Lesieur realized that the answer is formally identical with Pieri's formula (up to the fact that in each case we consider partitions with slightly different restrictions: inscribed in a r by s rectangle for the Grassmannian $\mathbb{G}_{r,s}$, with at most n parts for $GL(n,\mathbb{C})$). But as we have seen, because of Giambelli's formula (which is a formal consequence of it), Pieri's formula is enough to determine the multiplication of Schubert classes, hence also of Schur modules. In particular, the multiplicity of $S_\nu V$ inside the tensor product $S_\lambda V \otimes S_\mu V$ is equal to the Littlewood-Richardson coefficient $c_{\lambda\mu}^\nu$.

There is a more geometric way to define Schur modules as spaces of sections of line bundles. Consider the variety of partial flags $0 \subset V_1 \subset \cdots \subset V_m \subset V$, where $\dim V_i = \lambda_i^*$ (repetitions are allowed when λ has several columns of the same size). Using a Plucker embedding for each V_i, then a Segre embedding, we obtain a subvariety $\mathbb{F}_\lambda(V)$ of $\mathbb{P}W$, where $W = \wedge^{\lambda_1^*} V \otimes \cdots \otimes \wedge^{\lambda_m^*} V$. The *Borel-Weil theorem* (see [4]) then asserts that

$$\Gamma(\mathbb{F}_\lambda(V), \mathcal{O}(1)) = S_\lambda V.$$

2. Principles of Geometric Invariant Theory

The general problem. Let X be an algebraic variety, defined over an algebraically closed field k, with an action of an affine algebraic group G. Can we construct an *orbit space* for this action, that is, an algebraic variety Y with a surjective morphism $f : X \longrightarrow Y$, such that the fibers of f are exactly the G-orbits in X ? The answer is scarcely yes (this would imply, for example, that all orbits are closed, i.e. the action would be closed), but we can ask for "quotients" with weaker properties. An important notion is the following:

Definition. *A* categorical quotient *of X by G is a pair (Y, f) such that f is constant on G-orbits, and such that every morphism $g : X \to Z$ with the same property factors through f.*

The affine case. Let $X = \mathrm{Spec} A$, where G acts rationally on the finitely generated k-algebra A. This is a particularly nice case when G is a reductive group (in the sense of the first lecture if char $k = 0$, in general the definition is different): indeed, it was proved by Weyl for char $k = 0$, and by Nagata in general, that the algebra A^G of G-invariants is finitely generated ([23], Theorem

3.4). One can then define the affine variety $Y = \mathrm{Spec} A^G$, and the natural morphism $f : X \longrightarrow Y$ has the following nice properties ([23], Theorem 3.5):

1. $\mathcal{O}_Y \simeq (f_* \mathcal{O}_X)^G$;
2. the image by f of any closed invariant subset is closed;
3. f separates disjoint closed invariant subsets.

In particular, two points in X have the same image iff their orbit closures meet. A consequence of these properties is that (Y, f) is a categorical quotient. Moreover, if U is an open subset of Y such that the action of G on $f^{-1}(U)$ is closed, then U is an orbit space.

The projective case. For simplicity, let us consider the case where V is a G-module, and $X \subset \mathbb{P}V$ a G-invariant closed subvariety.

Definition. *A* good quotient *of X by G is a pair (Y, f), where f is surjective, affine, constant on G-orbits, and satisfies properties 1-3 above. It is a* geometric quotient *if it's also an orbit space.*

In general, such quotients will not exist. But they will if we restrict ourselves to suitable open subsets of X. The first observation is that we need G-invariant forms to reduce the problem to the affine case. This motivates the following important definition, for which we follow [23].

Definition. *A point $x \in X$ is* semi-stable *if there exists a non constant G-invariant homogeneous form P on V such that $P(x) \neq 0$. It is* stable *if, moreover, its stabilizer G_x is finite, and one can find P as above such that the action of G on $X_P = \{y \in X,\ P(y) \neq 0\}$ is closed.*

Denote by X^{ss} and X^s the open subsets of X consisting of semi-stable and stable points. (Note that they may very well be empty.)

Fundamental theorem. *There exists a good quotient (Y, f) of X^{ss} by G, and Y is* projective. *Moreover, there exists an open subset Y^s of Y such that $f^{-1}(Y^s) = X^s$, and (Y^s, f) is a geometric quotient of X^s.*

The Hilbert-Mumford criterion. Except for very simple cases, it is extremely difficult from the definition above to determine which are the stable or semi-stable points of a given action. Indeed, this would imply to compute all the G-invariant polynomials, which is intractable in general. Let $x \in X \subset \mathbb{P}V$, and $v \in V$ lying over x. One can prove ([23], Proposition 4.7):

x is semi-stable iff the closure of Gv does not contain the origin;
x is stable iff the morphism from G to V given by $g \mapsto gv$ is proper.

From this it is possible to derive a *numerical criterion* for stability. Recall that a one parameter subgroup of G is a homomorphism $\lambda : k^* \to G$. The induced action of k^* on V can be diagonalized: we can find a basis $e_1, \dots, e_n$ of V, and integers $k_1, \dots, k_n$, such that

$$\lambda(t)v = \sum_{i=1}^{n} t^{k_i} v_i e_i \qquad \text{for } t \in k^*, \quad v = \sum_{i=1}^{n} v_i e_i \in V.$$

Define $\mu(x, \lambda) = -\min\{k_i,\ v_i \neq 0\}$. Note that when t tends to zero, $\lambda(t)x$ has a limit x_0, corresponding to $v_0 = \sum_{k_i+\mu=0} v_i e_i$. Moreover, $\mu(x, \lambda) = \mu(x_0, \lambda)$. The Hilbert-Mumford criterion states that a point is (semi-)stable iff it is (semi-)stable with respect to every one parameter subgroup ([23], Theorem 4.9):

x is (semi-)stable iff $\mu(x, \lambda) > 0 (\geq 0)$ for every one parameter subgroup λ of G.

Application to flags. We will apply the Hilbert-Mumford criterion on two examples, the first being a simple version of the second one.

1. Let U, E be vector spaces. The action of $G = SL(U)$ on $W = U \otimes E$ induces, for each integer r, an action on $\wedge^r W$, and on the Grassmannian $\mathbb{G}(r, W) \subset \mathbb{P}(\wedge^r W)$.

Proposition 4. *A point $L \in \mathbb{G}(r,W)$ is semi-stable iff for every proper subspace V of U, the subspace $M = L\cap(V\otimes E)$ of W is such that*

$$\mu(M)=\frac{\dim M}{\dim V}\leq \mu(L)=\frac{\dim L}{\dim U}.$$

Proof. Let λ be a one parameter subgroup of G: we choose a basis $u_1,\dots,u_n$ of U such that $\lambda(t)u_i = t^{r_i}u_i$, with $r_1\geq\cdots\geq r_n$, and let $U_i=\langle u_1,\dots,u_i\rangle$ and $L_i = L\cap(U_i\otimes E)$. Let $L_{p_1}\subset\cdots\subset L_{p_m}=L$ be those $L_i\neq L_{i-1}$, and denote their dimensions by $l_1,\dots,l_m$. Choose an adapted basis f_j of L. If $l_{i-1}<j\leq l_i$, we have $f_j=\sum_{k\leq p_i}u_k\otimes v_{k,j}$ for some $v_{k,j}\in V$. Then $L_0=\lim_{t\to 0}\lambda(t)L$ is the space generated by the $u_{p_i}\otimes v_{p_i,j}$, and a simple computation shows that

$$\mu(L,\lambda)=\mu(L_0,\lambda)=-\sum_j r_{p_j}\dim L_{p_j}/L_{p_{j-1}}=-\sum_{i=1}^n r_i\dim L_i/L_{i-1}.$$

If the criterion given by the proposition is fulfilled, then $\dim L_i\leq im/n$, and using the identity $r_1+\cdots+r_n=0$, which follows from the fact that λ is a subgroup of $SL(U)$, we get $\mu(L,\lambda)\geq 0$. Hence L is semi-stable. The reverse statement is an exercise. □

2. Let now $V_\bullet$ be a m-filtration of V, that is a family of m filtrations $V_\bullet^{(i)}$ of V (the dimensions of each subspace is fixed). Each such filtration defines a point of some flag manifold, and through a Segre product a m-filtration thus defines a point in the projectivization of a tensor product of wedge powers of V (or of Schur modules), which is endowed with a natural $SL(V)$-action. Applying the Hilbert-Mumford criterion as in the proof of the previous proposition, we find:

Proposition 5. *A m-filtration $V_\bullet$ is semi-stable iff for every proper subspace L of V,*

$$\mu(L)=\frac{1}{\dim L}\sum_{i,j}\dim\,(L\cap V_j^{(i)})\leq\mu(V).$$

Hermitian matrices again. We have seen in our first lecture that each non zero Littlewood-Richardson coefficient gives restrictions on the set of eigenvalues of triplets of Hermitian matrices with sum zero. We now want to prove the reverse statement, and its obvious extension to a greater number of matrices. This is due to A. Klyachko [15].

Proposition 6. *There exists Hermitian matrices $A(1),\dots,A(m)$ of size n, having for spectra the weakly decreasing sequences $\lambda(1),\dots,\lambda(m)$, and such that $A(1)+\cdots+A(m)$ is scalar, iff*

$$\frac{1}{r}\sum_{k=1}^m\lambda_{I(k)}(k)\leq\frac{1}{n}\sum_{k=1}^m\sum_{i=1}^n\lambda_i(k)$$

for all $r<n$, and all m-tuples $I(1),\dots,I(m)$ of increasing sequences of r positive integers, such that the product of the corresponding Schubert classes is non zero.

Proof. The necessity of these conditions is checked as in Proposition 1.3. To prove the reverse statement, we can use a density argument to make a few additional assumptions: first, we suppose that the spectra $\lambda(i)$ are strictly decreasing rational sequences, that they are positive (after adding, if necessary, suitable scalar operators), and even, after multiplying by some integer, that they are partitions with distinct parts; second, we suppose that all the inequalities above are strict.

Then we choose generic flags $F_\bullet^{(i)}$ in $\mathbb{C}^n$, from which we construct a m-filtration by letting

$$V_p^{(i)}=F_{\alpha(i,p)}^{(i)},\qquad\text{where}\quad \alpha(i,p)=\lambda(i)_p^*.$$

We let $A(i)$ be the sum of the Hermitian projections on the $V_p^{(i)}$, which has the required spectrum. There remains to prove that $A(1)+\cdots+A(m)$ must be scalar.

For this we choose a Hermitian metric on V, and for each i, a Hermitian basis $v(i,\bullet)$ of V adapted to the flag $F_\bullet^{(i)}$. We consider our m-filtration as a point in $\mathbb{P}W$, where $W = \otimes_{i,p}\wedge^{\alpha(i,p)}V$, and a point above it is

$$v = \otimes_{i,p} v(i,1) \wedge \cdots \wedge v(i,\alpha(i,p)) \in W.$$

Because of the previous proposition and the hypothesis, our m-filtration is stable, hence the $SL(V)$-orbit of v is closed and does not contain the origin. There is therefore a point in this orbit, say v itself, which minimizes the distance to the origin (for the induced norm on W). This implies that for all $X \in End(V)$ with trace $(X) = 0$, we have

$$0 = \mathrm{Re}(Xv, v) = \mathrm{Re}\sum_{i,p} R_X(F^{(i)}_{\alpha(i,p)}) = \mathrm{Re}\sum_i \text{trace } (XA(i)).$$

Since the $A(i)$ are Hermitian, this implies our claim. □

Corollary 7. *Let $\lambda(A), \lambda(B), \lambda(C)$ be weakly decreasing sequences of n real numbers, such that $\sum_i \lambda_i(A) + \sum_i \lambda_i(B) = \sum_i \lambda_i(C)$. Suppose that for every $r < n$, and every increasing sequences I, J, K of length r with associated partitions λ, μ, ν, such that $c^\nu_{\lambda\mu} \neq 0$, one has*

$$\lambda_I(A) + \lambda_J(B) \geq \lambda_K(C).$$

Then there exists Hermitian matrices $A + B = C$ of size n, with spectra $\lambda(A), \lambda(B), \lambda(C)$.

THE SATURATION CONJECTURE. By the very definition, a m-filtration is semi-stable if and only if there exists a $SL(V)$-invariant form on W which does not vanish at the corresponding point in the product X of flag varieties. This product has a natural projective embedding (use a Plucker embedding for each subspace, and compose with a Segre embedding), corresponding to a very ample line bundle $\mathcal{O}(1)$. By the Borel-Weil theorem, we have

$$\Gamma(X, \mathcal{O}(N)) = S_{N\lambda(A(1))}V \otimes \cdots \otimes S_{N\lambda(A(m))}V.$$

If the condition of Proposition 6 are fulfilled, there exists a semi-stable point in X, and this implies that for some $N > 0$, the above tensor product must contain some trivial factor (trivial as a representation of $SL(V)$).

A priori, one has no control on the integer N, but the saturation theorem of Knutson and Tao says that we can always take $N = 1$!

Theorem 8. *There is an integer $N > 0$ such that $S_{N\alpha(1)}V \otimes \cdots \otimes S_{N\alpha(m)}V$ contains a trivial factor, iff $S_{\alpha(1)}V \otimes \cdots \otimes S_{\alpha(m)}V$ itself contains a trivial factor.*

We will not explain the proof of this theorem, which follows from a careful study of the combinatorics of hives [17, 5]. Let us simply notice that for $m = 3$, $S_\alpha V \otimes S_\beta V \otimes S_\gamma V$ contains a trivial $SL(V)$-factor iff $|\alpha| \times |\beta| = v \times w$ for some integer w, where $v = \dim V$, and $c^\delta_{\alpha\beta} \neq 0$ for δ the complementary partition of γ in the v by w rectangle. A consequence of all this is that the non-vanishing of Littlewood-Richardson coefficients can be checked recursively.

HORN'S CONJECTURE. We can now state the original conjecture of Horn, which goes back to 1962 [11] and is also recursive in nature. Define the sets of triples of increasing sequences of r integers in $\{1, \ldots, n\}$:

$$\begin{aligned} U^n_r &= \{(I,J,K),\ |I| + |J| = |K| + \tfrac{r(r+1)}{2}\},\\ T^n_r &= \{(I,J,K) \in U^n_r,\ |I|_F + |J|_G \leq |K|_H + \tfrac{p(p+1)}{2} \quad \forall p < r,\ \forall (F,G,H) \in T^r_p\}. \end{aligned}$$

Theorem 9. *There exists Hermitian matrices (A,B,C) of size n, such that $A + B = C$, iff* trace (A) + trace (B) = trace (C), *and*

$$\lambda_I(A) + \lambda_J(B) \geq \lambda_K(C) \qquad \forall r < n,\ \forall (I,J,K) \in T^n_r.$$

Proof. One checks, using induction, that this is just a reformulation of Proposition 3 and Corollary 7. For more details, see [7], Theorem 12. □

For $n = 2$ we get $\gamma_1 \leq \alpha_1 + \beta_1$, $\gamma_2 \leq \alpha_2 + \beta_1, \alpha_1 + \beta_2$. For $n = 3$, we obtain twelve inequalities:

$$\begin{array}{rcccl} \alpha_1+\beta_3, \alpha_2+\beta_2, \alpha_3+\beta_1 & \leq & \gamma_1 & \leq & \alpha_1+\beta_1, \\ \alpha_2+\beta_3, \alpha_3+\beta_2 & \leq & \gamma_2 & \leq & \alpha_1+\beta_2, \alpha_2+\beta_1, \\ \alpha_3+\beta_3 & \leq & \gamma_3 & \leq & \alpha_1+\beta_3, \alpha_2+\beta_2, \alpha_3+\beta_1. \end{array}$$

Why polytopes? There seems to be no reason a priori why the set of eigenvalues of Hermitian matrices whose sum is zero, should be described by linear inequalities, that is, should be a *convex polytope*. It turns out that such polytopes do appear in the general context of torus actions on symplectic varieties, of which our problem is a special case.

Definition. *Let M be a manifold, with an action of a connected Lie group K preserving a symplectic form ω. Differentiating this action, we associate to each $X \in \mathfrak{k}$ (the Lie algebra of K), a vector field η_X on M. A map $\mu : M \to \mathfrak{k}^*$ is then a* moment map *for the action of K in M if it is K-equivariant (with respect to the coadjoint action of K on the dual $\mathfrak{k}^*$ of its Lie algebra), and for all $X \in \mathfrak{k}$, $d\mu(X) = \omega(\eta_X, \bullet)$ (equality of 1-forms on M).*

This is equivalent to the existence of a *Hamiltonian* $H : \mathfrak{k} \longrightarrow \mathcal{O}(M)$, the space of regular functions on M, which is a Lie algebra homomorphism and lifts the natural map $\eta : \mathfrak{k} \longrightarrow \mathcal{T}(M)$ induced by the action. Here $\mathcal{T}(M)$ is the space of vector fields on M; if $f \in \mathcal{O}(M)$, its differential df can be identified via the symplectic form with a vector field ξ_f. The moment map and the Hamiltonian are related by the identity $H(X)(m) = \mu(m)(X)$ for $m \in M$, $X \in \mathfrak{k}$.

Moment maps have good functorial properties: if N is a K-invariant submanifold of M, the restriction $\mu_{|N}$ is a moment map for the restricted action. Also, if we restrict the action to a subgroup L of K, we get a moment map for this new action by composing with the projection $\mathfrak{k}^* \to \mathfrak{l}^*$.

Example 1. The action of the unitary group $U(n+1)$ on $\mathbb{P}^n$ (endowed with the symplectic structure given by the Fubini-Study metric) is Hamiltonian: the moment map associates to each point of $\mathbb{P}^n$ the Hermitian projection on the corresponding line in $\mathbb{C}^{n+1}$. Therefore the action of any subgroup K of $U(n+1)$ preserving a subvariety X of $\mathbb{P}^n$ is also Hamiltonian.

Example 2. Let M be any coadjoint orbit in $\mathfrak{k}^*$. Let $\mathfrak{k}_m$ be the stabilizer of a point $m \in M$. Then $T_m M \simeq \mathfrak{k}/\mathfrak{k}_m$, and the identity $\omega(\bar{X}, \bar{Y}) = \langle [X, Y], m\rangle$ defines a symplectic form on M. One checks that the inclusion $M \hookrightarrow \mathfrak{k}^*$ is a moment map for the action of K on M.

Example 3. Take in particular $K = U(n)$, so that $\mathfrak{k}$ is the space of skew-Hermitian matrices. Then $\mathfrak{k}^*$ can be identified, in a K-equivariant way, with the space H_n of Hermitian matrices via the trace form $H \mapsto$ trace $(iH.)$. This identifies the coadjoint orbits in $\mathfrak{k}^*$ with the K-orbits in H_n, which are just the spaces $\mathcal{O}_\lambda$ of Hermitian matrices with fixed spectrum λ, a weakly decreasing sequence of real numbers.

The following theorem is due to Atiyah and Guillemin-Sternberg (see [13], 3.4):

Theorem 10. *Let M be a compact connected symplectic manifold. Suppose that a torus T acts on M with a moment map $u : M \to \mathfrak{t}^*$. Then $u(M)$ is a convex polytope. More precisely, the image under u of the fixed point set of T in M is finite, and $u(M)$ is the convex hull of this finite set.*

In the example above, we can restrict the action on $\mathcal{O}_\lambda$ to the subgroup T of K consisting of diagonal matrices, which is a torus group. The induced moment map $u : \mathcal{O}_\lambda \to \mathfrak{t}^*$ takes a Hermitian matrices to its diagonal entries. The theorem then asserts that the diagonal entries of the matrices with spectrum λ describe the convex hull of the permutations of λ. This is known as the Schur-Horn theorem.

For a Hamiltonian action of a Lie group which is not a torus group, the image of the moment map needs not be convex, but still a part of it must be convex. Suppose that K is compact, let T be a maximal torus in K, and $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ a positive Weyl chamber; when $K = U(n)$, the main case of interest to us, T is the torus group of diagonal matrices in K and $\mathfrak{t}^*_+$ can be chosen to be the cone of weakly decreasing sequences.

Theorem 11. *Let M be a compact connected symplectic manifold with a Hamiltonian action of K. Then $u(M) \cap \mathfrak{t}^*_+$ is a convex polytope.*

To state this theorem, which is due to F. Kirwan ([14], Theorem 2.1), in a slightly different way, consider the map $p : \mathfrak{k}^* \to \mathfrak{t}^*_+$ which (for $K = U(n)$) takes a Hermitian matrix to its spectrum. Then $p \circ u(M) \subset \mathfrak{t}_+$ is a convex polytope. This holds for any compact group, the map p being defined through the property that each K-orbit in $\mathfrak{k}^*$ meets $\mathfrak{t}^*_+$ at a single point.

Example 4. Consider the diagonal action of $K = U(n)$ on $\mathcal{O}_\lambda \times \mathcal{O}_\mu$. This action is Hamiltonian, its moment map takes a pair of Hermitian matrices to their sum. Composing with the map p above, we get that the spectrum of the sum of two Hermitian matricies with given spectra, describes a convex polytope. This justifies qualitatively the Horn conjecture. (See [16] for a quantitative discussion along the same lines.)

3. The quantum case

The multiplicative problem. Let $A \in SU(n)$. Its eigenvalues are complex numbers of norm one, which we can write in a unique way as $\exp(2i\pi\lambda_i)$, with $\lambda_1 \geq \cdots \geq \lambda_n \geq \lambda_1 - 1$ and $\lambda_1 + \cdots + \lambda_n = 0$. These inequalities define the *fundamental alcove* $\mathcal{U}$, and we denote by $\lambda(A) \in \mathcal{U}$ the spectrum of A. The multiplicative analogue of Horn's problem is then: how can we describe the set

$$\Delta_q(l) = \{(\lambda(A_1), \dots, \lambda(A_l)),\ \ A_1, \dots, A_l \in SU(n),\ A_1 \cdots A_l = I\} \quad ?$$

A geometric interpretation. For $\xi \in \mathcal{U}$, denote by $\mathcal{O}_\xi$ the space of special unitary matrices with spectrum ξ. Consider the open curve $\mathbb{P}^1$ minus l points $p_1, \dots, p_l$: its fundamental group is generated by l small loops $\gamma_1, \dots, \gamma_l$ around $p_1, \dots, p_l$, with the single relation $\gamma_1 \cdots \gamma_l = 1$. Thererefore, there is an identification between

$$\mathcal{N}(\xi_1, \dots, \xi_l) = \{(A_1, \dots, A_l) \in \mathcal{O}_{\xi_1} \times \cdots \times \mathcal{O}_{\xi_l},\ \ A_1 \cdots A_l = I\}/SU(n)$$

and the moduli space $\mathcal{M}(\xi_1, \dots, \xi_l)$ of unitary n-dimensional representations (up to global conjugacy) of $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_l\})$, such that the image of γ_i is in $\mathcal{O}_{\xi_i}$. In particular $(\xi_1, \dots, \xi_l) \in \Delta_q(l)$ iff $\mathcal{M}(\xi_1, \dots, \xi_l)$ is non empty.

There is another interpretation of this moduli space, due to Mehta and Seshadri. This involves the concept of *stable vector bundles*, which is of course closely related with the stability concept of Geometric Invariant Theory (see [23], Chapter 5). On a smooth complete curve C, a vector bundle $\mathcal{E}$ is (semi-)stable if, for every proper sub-bundle $\mathcal{F}$ of $\mathcal{E}$, we have

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\mathrm{rank}(\mathcal{F})} \leq \mu(\mathcal{E}).$$

Suppose that the genus of C is at least two, and that $C \simeq \mathbb{H}/\Gamma$, where $\mathbb{H}$ denotes the Poincaré half-plane and $\Gamma \simeq \pi_1(C)$ is a discrete subgroup of $\mathrm{Aut}(\mathbb{H}) \simeq PSL(2, \mathbb{R})$ acting freely on $\mathbb{H}$. Then it is a classical theorem of Narasimhan and Seshadri that the space of isomorphism classes of unitary representations of Γ is in bijection with the moduli space of semi-stable vector bundles of degree zero on C [22]. The definition of this correspondance is the following: to a representation $\pi : \Gamma \to U(n)$, one associates the vector bundle $E_\pi = \mathbb{H} \times_\pi \mathbb{C}^n$ on C.

The theorem of Mehta and Seshadri [20] is an extension of this result to non-compact Riemann surfaces with finite volume. Such a surface can be seen as the complement of a finite set in a complete curve ($\mathbb{P}^1$ in the case of interest to us).

Definition. *A* parabolic [1] vector bundle $\mathcal{E}$ *on* $\mathbb{P}^1$ *is a vector bundle of rank* n *plus additional data: a complete flag* $\mathcal{E}_{p_i,\bullet}$ *of subspaces of each fiber* $\mathcal{E}_{p_i}$*; weights* $\xi_{i,j}$*, which are real numbers such that* $\xi_{i,1} > \cdots > \xi_{i,n} \geq \xi_{i,1} - 1$*. Its* parabolic degree *is*

$$\operatorname{pardeg}(\mathcal{E}) = \deg(\mathcal{E}) + \sum_{i,j} \xi_{i,j}.$$

Actually, the weights will only play a role in the definition of semi-stability. Consider a sub-bundle $\mathcal{F}$ of $\mathcal{E}$. It can be endowed with a parabolic structure in the following way: each fiber $\mathcal{F}_{p_i}$ has an induced complete flag, given by taking the distinct terms in the sequence $\mathcal{F}_{p_i} \cap \mathcal{E}_{p_i,j}$, and we associate to each of these subspaces the maximum of the corresponding weights. Then $\mathcal{E}$ is *semi-stable* iff

$$\forall \mathcal{F} \subset \mathcal{E}, \quad \mu_p(\mathcal{F}) = \frac{\operatorname{pardeg}(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} \leq \mu_p(\mathcal{E}).$$

Theorem 12. *Suppose that* $\xi_1, \ldots, \xi_l$ *are rational sequences. Then the space* $\mathcal{M}(\xi_1, \ldots, \xi_l)$ *is homeomorphic to the moduli space of semi-stable parabolic vector bundles on* $\mathbb{P}^1$*, of degree zero and parabolic weights* $\xi_{i,j}$ *at* p_i*.*

Moreover, one can show that the generic point $\mathcal{E}$ of this moduli space must be semi-stable as an ordinary vector bundle ([1], Lemma 5.2). On $\mathbb{P}^1$, a vector bundle is a direct sum of line bundles, and is semi-stable iff all these line bundles have the same degree. Here, $\mathcal{E}$ has degree zero and is therefore trivial.

The quantum cohomology of the Grassmannian. A sub-bundle $\mathcal{F}$ of the trivial bundle $\mathcal{E}$ is simply given by a map $\phi_{\mathcal{F}} : \mathbb{P}^1 \to \mathbb{G}_{r,s}$, such that $\mathcal{F} = \phi_{\mathcal{F}}^* S$, where S is the tautological vector bundle on $\mathbb{G}_{r,s}$. Since $\det \mathcal{F} = \phi_{\mathcal{F}}^* \mathcal{O}(-1)$, we have

$$\operatorname{pardeg}(\mathcal{F}) = -\deg(\phi_{\mathcal{F}}) + \sum_{j \in I_1(\phi_{\mathcal{F}})} \xi_{1,j} + \cdots + \sum_{j \in I_l(\phi_{\mathcal{F}})} \xi_{l,j},$$

where the sequence $I_i(\phi_{\mathcal{F}})$ encodes the relative position of $\phi_{\mathcal{F}}(p_i) \subset \mathcal{E}_{p_i}$ with respect to the complete flag $\mathcal{E}_{p_i,\bullet}$. We get:

Theorem 13. *The set* $\Delta_q(l) \subset \mathcal{U}^l$ *is the polytope defined by the inequalities*

$$\sum_{j \in I_1} \lambda_j(A_1) + \cdots + \sum_{j \in I_l} \lambda_j(A_l) \leq d,$$

where $I_1, \ldots, I_l$ *are increasing sequences of* $r < n$ *elements in* $\{1, \ldots, n\}$ *such that there exists a degree* d *map* $\phi : \mathbb{P}^1 \to \mathbb{G}_{r,s}$ *sending* $p_1, \ldots, p_l$ *to Schubert cells* $\Omega_{I_1}, \ldots, \Omega_{I_l}$ *in general position.*

Now the question is to find necessary and sufficient conditions for such maps to exist. A major discovery of the last ten years has been that these maps can be used to construct certain q-deformations of cohomology rings, whose very striking properties have had spectacular applications, specially in enumerative geometry. This is the theory of quantum cohomology, for which we refer to [9].

We will be primarily interested in the *small quantum cohomology ring* $QH^*(\mathbb{G})$ of the Grassmannian $\mathbb{G} = \mathbb{G}_{r,s}$. This ring can be defined as the space $H^*(\mathbb{G}, \mathbb{C})[q]$, where q is an indeterminate, endowed with the following commutative and associative product:

$$\sigma_\lambda * \sigma_\mu = \sum_{d \geq 0} c^\nu_{\lambda\mu}(d) q^d \sigma_\nu,$$

where the *quantum Littlewood-Richardson number* $c^\nu_{\lambda\mu}(d)$ is defined as follows [2]. It is equal to the number of degree d maps $\phi : \mathbb{P}^1 \to \mathbb{G}$ sending three given points to Schubert cells $\Omega_\lambda, \Omega_\mu, \Omega_{\hat{\nu}}$

[1] The word "parabolic" comes from the uniformization theory of non-compact Riemann surfaces.

[2] In a more general setting, one can define quantum cohomology in terms of Gromov-Witten invariants, which are defined through moduli spaces of stable maps from a pointed curve to a variety with certain properties. The miracle is that this product is *associative*([9], Theorem 4).

in general position when this number is finite, and zero otherwise. A dimension count shows that $c^{\nu}_{\lambda\mu}(d)$ can be non-zero only when $|\nu| + nd = |\lambda| + |\mu|$ (as follows from [9], Theorem 2(i)).

Note that $c^{\nu}_{\lambda\mu}(0) = c^{\nu}_{\lambda\mu}$ is the usual Littlewood-Richardson coefficient, and that the quantum cohomology ring is a q-deformation of the ordinary cohomology. In the case of the Grassmannian one can give a simple description of this ring.

Theorem 14. *Let* $\tau(z) = (1 - z\sigma_1 + \cdots + (-1)^s z^s \sigma_s)^{-1} = \sum_k z^k \tau_k$. *Then*

$$QH^*(\mathbb{G}) \simeq QH^*_{r,s} = \mathbb{C}[\sigma_1, \dots, \sigma_s, q]/\langle \tau_{r+1}, \dots, \tau_{n-1}, \tau_n + (-1)^s q\rangle.$$

Proof. We first check that the relations $\tau_{r+1}, \dots, \tau_{n-1}, \tau_n + (-1)^s q$ do hold in $QH^*(\mathbb{G})$. First recall that if $c^{\nu}_{\lambda\mu}(d) \neq 0$, we must have $|\nu| + nd = |\lambda| + |\mu|$, which implies that $d = 0$ when $|\lambda| + |\mu| < n$: upto degree $n-1$, the quantum and the classical products coincide. Since the relations $\tau_{r+1}, \dots, \tau_{n-1}$ do hold in $H^*(\mathbb{G})$ (see the first lecture), they hold in $QH^*(\mathbb{G})$ as well.

For the remaining relation, note that the formal identity $\tau_n - \sigma_1\tau_{n-1} + \cdots + (-1)^s\sigma_s\tau_r = 0$ reduces in $QH^*(\mathbb{G})$ to $\tau_n + (-1)^s\sigma_s\tau_r = 0$. Therefore, all we need to check is that $c^{\nu}_{\lambda\mu}(1) = 1$ for $\lambda = (s)$, $\mu = (1^r)$ and $\nu = (s^r)$. The corresponding Schubert varieties are respectively $\{W \subset H\}$, $\{W \supset l\}$ and $\{W_0\}$, where H is some hyperplane and l a is line. Note that a line in $\mathbb{G}$ must be of the form $\{A \subset W \subset B\}$, the dimensions of A and B being $r-1$ and $r+1$. In general, there is a unique such line meeting the three Schubert varieties above, given by $A = W_0 \cap H$ and $B = \langle W_0, l\rangle$, and our claim follows.

This is enough to prove the theorem: indeed, we proved that there is a ring homomorphism $QH^*_{r,s} \to QH^*(\mathbb{G})$, which is an isomorphism modulo q. But these two $\mathbb{Z}[q]$-modules are free of the same rank, hence they must be isomorphic. □

Corollary 15. *The quantum cohomology ring of the projective space is*

$$QH^*(\mathbb{P}^n) = \mathbb{Z}[t, q]/\langle t^{n+1} - q\rangle.$$

Quantum Schubert calculus. What does remain of the formulas of Pieri and Giambelli in quantum cohomology? A quite surprising result, due to A. Bertram [2], is that Giambelli's formula holds without any quantum correction:

Proposition 16. *For every partition* λ *inscribed in the* r *by* s *rectangle, one has in* $QH^*(\mathbb{G})$

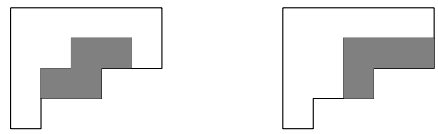
$$\sigma_\lambda = \det(\sigma_{\lambda_i - i + j})_{1 \le i,j \le r}.$$

The proof uses a generalization of Giambelli's formula, due to Kempf and Laksov, which applies to the "relative" situation where one has a morphism $u : E \to F$ between vector bundles on some variety X, and a flag of subbundles of E. One then looks at the points of X above which the kernel of u has a given relative position with respect to these subbundles. Under genericity assumptions, this defines a subvariety of X whose fundamental class is given by a Giambelli type formula in terms of Chern classes of E and F (see [19], 3.5.17). Such a description can precisely be used in the context of quantum cohomology, by defining generalized Schubert varieties on the so-called Quot-scheme, and the Kempf-Laksov formula proves the proposition.

For any partition $\lambda = (\lambda_1, \dots, \lambda_l)$, we can define the class $\sigma_\lambda = \det(\sigma_{\lambda_i - i + j})_{1 \le i,j \le l}$ in $QH^*(\mathbb{G})$. This class is obviously zero if $\lambda_1 > r$, but not necessarily when $\lambda_1^* > s$. We will give an algorithm to express such classes in terms of those corresponding to partitions contained in the r by s rectangle, and show how this leads to a formula for quantum Littlewood-Richardson coefficients.

Definition. *Choose a box on the border of the diagram of* λ, *and make* $n-1$ *steps on this border in the north-east direction to cover a* n*-rim of* λ. *This* n*-rim is* legal *if its complement in* λ *is again a partition,* illegal *otherwise. The* width *of a* n*-rim is the number of columns it occupies.*

In the picture below you see an illegal and a legal 4-rim in the partition $\lambda = (5531)$.



Lemma 17. *If λ contains an illegal n-rim, or if $\lambda_{r+1} > 0$ and λ contains no n-rim, then $\sigma_\lambda = 0$. If λ contains a legal n-rim of width w and complement μ, then $\sigma_\lambda = (-1)^{r-w} q \sigma_\mu$.*

Proof. The key observation is that the relation $\tau_n + (-1)^r q = 0$ implies, by induction, that for all $j \geq 0$, $\tau_{n+j} + (-1)^r q \tau_j = 0$. Let $\alpha = \lambda^*$: we have $\sigma_\lambda = \det(\tau_{\alpha_i - i + j})_{1 \leq i,j \leq \lambda_1}$. If λ contains no n-rim, then $\lambda_1 + \alpha_1 \leq n$, and if moreover $\lambda_{r+1} > 0$, which means that $\alpha_1 > r$, then the first line of this determinant is identically zero.

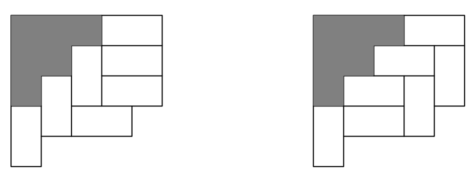
Now consider some n-rim of λ, beginning on column a and ending in column b. This n-rim is illegal precisely when $\alpha_a - a - n = \alpha_{b+1} - (b+1)$. Using the relations $\tau_{n+j} + (-1)^r q \tau_j = 0$. on the a-th line of the determinant above, we obtain a new line that is proportional to the $b+1$-th line, hence $\sigma_\lambda = 0$.

If the n-rim is legal, we pass this new line to the b-th row to obtain, up to the sign, a determinant of the same kind which is precisely σ_μ. □

This lemma gives the algorithm we were looking for: beginning with a partition λ, one can remove legal n-rims until there is no more in the remaining partition ν. (This partition does not depend on the choice of the n-rims removed: it is known as the *n-core* of λ. Nor does the sign $\varepsilon(\lambda/\nu) = (-1)^{mr-w}$, where w is the sum of the widths of the n-rims removed, and m is their number (see [12], Th. 2.7.16).) Then $\sigma_\lambda = 0$ if ν is not contained in the r by s rectangle, and otherwise

$$\sigma_\lambda = \varepsilon(\lambda/\nu) q^m \sigma_\mu.$$

Example. The partition $\lambda = (55541)$ has (321) for 2-core, and the pictures below show different ways of removing legal 2-rims.



We are now able to compute quantum Littlewood-Richardson numbers:

Proposition 18. *The quantum Littlewood-Richardson numbers can be expressed in terms of ordinary Littlewood-Richardson numbers in the following way:*

$$c^\nu_{\lambda\mu}(d) = \sum_\rho \varepsilon(\rho/\nu) c^\rho_{\lambda\mu},$$

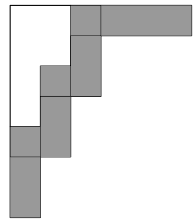
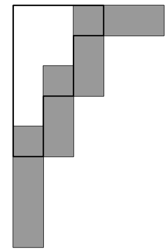
where the sum is over partitions ρ with $\rho_1 \leq r$ that can be obtained from ν by adding d n-rims (but ρ needs not be contained inside the r by s rectangle).

Proof. If we forget the relations $\tau_s, \dots, \tau_n$, the Littlewood-Richardson rule tells that

$$\sigma_\lambda \sigma_\mu = \sum_\rho c^\rho_{\lambda\mu} \sigma_\rho \qquad \text{in} \quad \mathbb{C}[\sigma_1, \dots, \sigma_s].$$

In $QH^*(\mathbb{G})$, we express the right hand side in terms of partitions inscribed in the r by s rectangle, and the algorithm above gives the claim. □

Example. Let $\lambda = (32211)$, $\mu = (432)$, $\nu = (2211)$, and let us compute $c^\nu_{\lambda\mu}(1)$ for $r = s = 6$, $n = 12$. We first add a 12-rim; there are only two such rims that can contribute, corresponding to $\rho = (6332211)$ and $\rho' = (53322111)$. Applying the Littlewood-Richardson rule one checks that $c^\rho_{\lambda\mu} = 3$ and $c^{\rho'}_{\lambda\mu} = 2$, hence $c^\nu_{\lambda\mu}(1) = 3 - 2 = 1$.

Exercise. Check that $QH^*(\mathbb{G}_{2,4})$ has the following multiplication table:

	σ_1	σ_2	σ_{11}	σ_{21}	σ_{22}
σ_1	$\sigma_2 + \sigma_{11}$	σ_{21}	σ_{21}	σ_{22}	$q\sigma_1$
σ_2	σ_{21}	σ_{22}	q	$q\sigma_1$	$q\sigma_{11}$
σ_{11}	σ_{21}	q	σ_{22}	$q\sigma_1$	$q\sigma_2$
σ_{21}	σ_{22}	$q\sigma_1$	$q\sigma_1$	$q\sigma_2 + q\sigma_{11}$	$q\sigma_{21}$
σ_{22}	$q\sigma_1$	$q\sigma_{11}$	$q\sigma_2$	$q\sigma_{21}$	q^2

Exercise. Deduce from the previous proposition the *quantum Pieri formula*:

$$\sigma_\lambda * \sigma_k = \sum_{\substack{\nu \subset r\times s, \\ \nu \in \lambda\otimes k}} \sigma_\nu + q \sum_{\substack{\rho \subset r\times s, \\ \rho\in\lambda * k}} \sigma_\rho,$$

where the quantum contribution $\lambda * k$ is the set of partitions ρ of size $|\rho| = |\lambda| + k - n$, such that $\lambda_1 - 1 \geq \rho_1 \geq \lambda_2 - 1 \geq \cdots \geq \lambda_{s-1} - 1 \geq \rho_s \geq 0$.

This is enough, in principle, to list a complete set of conditions in Theorem 13. Indeed, if their exists a map $\phi : \mathbb{P}^1 \to \mathbb{G}$ of degree d hiting a collection of Schubert cells in general position, one can show that for some $e \leq d$, there exists a finite non-zero number of maps of degree e with the same property ([1], Lemma 5.5). This means that the corresponding quantum Littlewood-Richardson coefficient is non-zero, and this can be checked with the help of Proposition 17.

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