REDUCTION FOR BRANCHING MULTIPLICITIES

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Abstract. A reduction formula for the branching coefficients of tensor products of representations and more generally restrictions of representations of a semisimple group to a semisimple subgroup is proved in [KT03, DW11]. This formula holds when the highest weights of the representations belong to a codimension 1 face of the Horn cone, which by [Res11b] corresponds to a Littlewood-Richardson coefficient equal to 1. We prove a similar reduction formula when this coefficient is equal to 2, and show some properties of the class of the branch divisor corresponding to a generically finite morphism of degree 2 naturally defined in this context.

1. Introduction

Fix an inclusion $G \subset \hat{G}$ of complex connected reductive groups. We are interested in the branching problem for the decomposition of irreducible $\hat{G}$-modules as representations of G. The appearing multiplicities are nonnegative integers parametrized by pairs of dominant weights for G and $\hat{G}$ respectively. The support of this multiplicity function is known to be a finitely generated semigroup. It generates a convex polyhedral cone $\Gamma_{\mathbb{Q}}(G \subset \hat{G})$, that we call the *Horn cone*. The set of codimension one faces of this cone is in bijection with the set of Levi-movable pairs (see Section 6.1) of Schubert classes in some projective homogeneous spaces $G/P \subset \hat{G}/\hat{P}$ such that the sum of the degrees of the classes is equal to the dimension of G/P and such that the intersection number c of the first Schubert class with the pullback of the second Schubert class in G/P is equal to one. See [Res10].

Moreover, the multiplicities on those faces satisfy reduction rules: if $L, \hat{L}$ denote the Levi subgroups of $P, \hat{P}$, the branching multiplicities for the inclusion $G \subset \hat{G}$ on such a face are equal to branching multiplicities for the inclusion $L \subset \hat{L}$. See [Res11b].

Actually, to any pair of Schubert classes such that the intersection number c is positive corresponds an inequality satisfied by $\Gamma_{\mathbb{Q}}(G \subset \hat{G})$. Our aim is to study the associated reduction rules for the multiplicities on such a face when $c = 2$. Although most of our results are general we consider in this introduction the case of the tensor product decomposition for the linear group.

Fix an n-dimensional vector space V. Let $\Lambda_n^+ = \{(\lambda_1 \geq \cdots \geq \lambda_n \geq 0 \,:\, \lambda_i \in \mathbb{N}\}$ denote the set of partitions. For $\lambda \in \Lambda_n^+$, let $S^\lambda V$ be the corresponding Schur module, that is the irreducible $\mathrm{GL}(V)$-module of highest weight $\sum \lambda_i \epsilon_i$ (notation as in [Bou02]). The Littlewood-Richardson coefficients (or LR coefficients for short) $c^\nu_{\lambda,\mu}$ are defined by

$$S^\lambda V \otimes S^\mu V \simeq \bigoplus_{\nu \in \Lambda_n^+} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes S^\nu V \tag{1}$$

(here $\mathbb{C}^{c^\nu_{\lambda,\mu}}$ is a multiplicity space).

Let $1 \leq r \leq n-1$ and $\mathrm{Gr}(r,n)$ be the Grassmannian of r-dimensional linear subspaces of V. Recall that the Schubert basis (σ^I) of the cohomology ring $H^*(\mathrm{Gr}(r,n), \mathbb{Z})$ is parametrized by the subsets I of $\{1, \dots, n\}$ with r elements. The Schubert constants $c^K_{I,J}$ are defined by

$$\sigma^I \sigma^J = \sum_K c^K_{I,J} \sigma^K. \tag{2}$$

Actually, $c^K_{I,J}$ is also a LR coefficient by [Les47], but this coincidence is specific to the type A.

Given a partition λ and a subset I, let λ_I be the partition whose parts are λ_i with $i \in I$. Let also $\overline{I}$ denote the complementary subset $\{1, \cdots, n\} \setminus I$.

Theorem 1. [KT03, DW11]

(1) Let $1 \leq rn-1$ and $I, J, K \subset \{1, \dots, n\}$ be subsets with r elements such that $c_{I,J}^K \neq 0$. For any $(\lambda, \mu, \nu) \in (\Lambda_n^+)^3$, if $c_{\lambda,\mu}^\nu \neq 0$ then

$$|\lambda_I| + |\mu_J| \geq |\nu_K| \,. \tag{3}$$

(2) Conversely, fix $(\lambda, \mu, \nu) \in (\Lambda_n^+)^3$ such that $|\lambda| + |\mu| = |\nu|$. If for any $1 \leq r \leq n-1$ and I, J, K of cardinal r such that $c_{I,J}^K = 1$, inequality (3) holds, then $c_{\lambda,\mu}^\nu \neq 0$.

The reduction result that we mentioned above, corresponding to the semisimple part $\mathrm{SL}_r \times \mathrm{SL}_{n-r} \subset \mathrm{SL}_n$ of the Levi subgroup is the following:

Theorem 2. [DW11, Res11b] *Assume that $c_{I,J}^K = 1$. Let λ, μ, ν be partitions such that*

$$|\lambda_I| + |\mu_J| = |\nu_K| \,. \tag{4}$$

Then

$$c_{\lambda,\mu}^\nu = \; c_{\lambda_I,\mu_J}^{\nu_K} \cdot c_{\lambda_{\overline{I}},\mu_{\overline{J}}}^{\nu_{\overline{K}}} \,. \tag{5}$$

Formula (5) is a multiplicativity property. Let us first report on a similar property for Belkale-Kumar [BK06] coefficients (BK coefficients for short). See Section 6.1 for a short description of these numbers which are all intersection numbers of Schubert classes or zero. Consider an inclusion $P \subset Q$ of parabolic subgroups of a reductive algebraic group G, and the corresponding fibration $G/P \to G/Q$. Richmond [Ric12] proves that any BK coefficient d of G/P is the product of two BK coefficients in G/Q and Q/P. In type A, this implies that a non-zero BK coefficient of any two steps flag manifold is a product of two LR coefficients: $d = c_1 c_2$.

If moreover $c_1 = 1$, Theorem 2 implies that c_2 itself is the product of two LR coefficients: $c_2 = c_2' c_2''$. Thus $d = c_1 c_2' c_2''$ is the product of *three* LR coefficients. This is the content of [KP11, Theorem 3], which even more generally states that on a k-step flag variety, a BK coefficient can be factorized as a product of $\frac{k(k-1)}{2}$ LR coefficients. Unfortunately, this assertion needs $c_1 = 1$ and is not correct in general, as we show in Remark 9. Our original motivation was to correct this result. We get such a correction if $c_1 = 2$.

Fix now r and $I, J, K \subset \{1, \dots, n\}$ of cardinal r such that $c_{I,J}^K = 2$, and consider the multiplicities associated to the triples of partitions satisfying equation (4). We prove (see Proposition 39) that the set of triples $(\lambda, \mu, \nu) \in (\Lambda_n^+)^3$ such that

$$|\lambda_I| + |\mu_J| = |\nu_K| \;\text{ and }\; 0 \neq c_{\lambda,\mu}^\nu < \; c_{\lambda_I,\mu_J}^{\nu_K} \cdot c_{\lambda_{\overline{I}},\mu_{\overline{J}}}^{\nu_{\overline{K}}}$$

contains a unique minimal element (α, β, γ). Theorem 8 gives an explicit expression for this triple. The interested reader can find at [CR] a program allowing to compute (α, β, γ) and its generalisations to any classical group (and under the more general assumption $c_{I,J}^K \neq 0$). Our general result, Theorem 7, holds for any inclusion $G \subset \hat{G}$. For $\hat{G} = G \times G$ and G/P a Grassmannian, it states:

Theorem 3. *Assume $c_{I,J}^K = 2$. Let λ, μ, ν be partitions such that $|\lambda_I| + |\mu_J| = |\nu_K|$. Then*

$$c_{\lambda,\mu}^\nu + c_{\lambda-\alpha,\mu-\beta}^{\nu-\gamma} = c_{\lambda_I,\mu_J}^{\nu_K} \cdot c_{\lambda_{\overline{I}},\mu_{\overline{J}}}^{\nu_{\overline{K}}} \,. \tag{6}$$

Under an assumption of Levi-movability (see Sections 6.1 and 6.2), we are able to improve this result. Indeed, in this case, the element (α, β, γ) satisfies (4), and an immediate induction expresses a multiplicity coefficient for the reduction $G \subset \hat{G}$ as an alternating sum of similar coefficients for the reduction $L \subset \hat{L}$. This is Corollary 40 below. In the setting of Theorem 3, G/P being cominuscule, the Levi-movability hypothesis is automatically satisfied, and we get:

Theorem 4. *Assume $c_{I,J}^K = 2$. Let λ, μ, ν be partitions such that $|\lambda_I| + |\mu_J| = |\nu_K|$. Then,*

$$c_{\lambda,\mu}^\nu = \sum_{k \geq 0} (-1)^k c_{\lambda_I - k\alpha_I, \mu_J - k\beta_J}^{\nu_K - k\gamma_K} \cdot c_{\lambda_{\overline{I}} - k\alpha_{\overline{I}}, \mu_{\overline{J}} - k\beta_{\overline{J}}}^{\nu_{\overline{K}} - k\gamma_{\overline{K}}} \,. \tag{7}$$

An even more particular case is in type A for a maximal parabolic subgroup P: not only (α, β, γ) belongs to the face of $\Gamma_{\mathbb{Q}}(G \subset \hat{G})$ defined by (I, J, K), but moreover we can compute the corresponding LR coefficients:

Theorem 5. *In type A, with the above notation, for all $k \geq 0$, we have*
$$c^{k\gamma}_{k\alpha,k\beta} = \frac{(k+1)(k+2)}{2},$$
and
$$c^{k\gamma_K}_{k\alpha_I,k\beta_J} = c^{k\gamma_{\overline{K}}}_{k\alpha_{\overline{I}},k\beta_{\overline{J}}} = k+1.$$

For example, Theorem 3 with $(\lambda,\mu,\nu) = (k\alpha, k\beta, k\gamma)$ is true since $\frac{(k+1)(k+2)}{2} + \frac{k(k+1)}{2} = (k+1)^2$. Theorem 5 is specific to type A. Indeed, every term can be defined in any type but the statement does not hold. See the examples in Section 7.5.

Fix a positive integer r. Any triple $(\lambda,\mu,\nu) \in (\Lambda_r^+)^3$ yields an integer N and a triple (I,J,K) of subsets of $\{1,\dots,N\}$ with r elements such that $c^\nu_{\lambda,\mu} = c^K_{IJ}$. If moreover $c^\nu_{\lambda,\mu} = 2$, Theorem 5 yields a triple $(\alpha_I,\beta_J,\gamma_K) \in (\Lambda_r^+)^3$ such that $c^{\gamma_K}_{\alpha_I,\beta_J} = 2$. It is amusing to observe that this map $(\lambda,\mu,\nu) \mapsto (\alpha_I,\beta_J,\gamma_K)$ between LR coefficients equal to two is nontrivial. See Section 7 for examples.

Given three subsets I,J,K as above, it is a difficult task to describe the face it defines in the Horn cone: to the best of our knowledge, even the dimension of such a face is not known in general, and our experience is that a computer will only give very limited information related to this problem. From a theoretical point of view as well as a computational point of view, both finding linear equations defining this face and points on it is challenging. In Section 7, we describe completely the faces corresponding to some relevant examples. Moreover, Theorem 5 provides a way to find at least one non-trivial element on this face.

The method we are using is essentially the same as in [Res11b]. We define the variety $Y \subset \mathrm{Gr}(r,n) \times (G/B)^3$ where a quadruple (V, X_1, X_2, X_3) belongs to Y if and only if $V \in \mathrm{Gr}(r,n)$ belongs to the intersection of the three Schubert varieties defined by I,J,K and the three flags X_1, X_2, X_3.

The hypothesis $c^K_{I,J} = 2$ means that the projection $Y \to (G/B)^3$ is generically finite of degree 2. We show in Subsection 3.5 that we can associate to this generically finite morphism a branch divisor in $(G/B)^3$ and that the pushforward of the structure sheaf $\mathcal{O}_Y$ is expressed in terms of this branch divisor. One half of the class of this branch divisor is the line bundle on $(G/B)^3$ whose sections are $S^\alpha V \otimes S^\beta V \otimes S^\gamma V$ for (α,β,γ) the triple of partitions defined above.

On the other hand, the morphism $Y \to \mathrm{Gr}(r,n)$ describes Y as a relative product of Schubert varieties over the Grassmannian. This shows that Y is normal and allows computing the ramification divisor as the relative canonical sheaf, adapting in this relative setting previous computations of the canonical sheaf of Schubert varieties ([Ram87, Theorem 4.2] and [Per07, Proposition 4.4]), see Section 4.5.

Using the projection formula, we are therefore able to relate the sections of line bundles on Y to those on $(G/B)^3$. Let $C \subset (G/B)^3$ be a product of flag varieties under L. As in [Res11b], taking G-invariants and restricting to C is the same as restricting to C and then taking L-invariants, which leads to the modified version of (5)

$$c^\nu_{\lambda,\mu} + c^{\nu-\gamma}_{\lambda-\alpha,\mu-\beta} = c^{\nu_K}_{\lambda_I,\mu_J} \cdot c^{\nu_{\overline{K}}}_{\lambda_{\overline{I}},\mu_{\overline{J}}}. \tag{8}$$

Equation (7) follows from (8) by an immediate induction.

Our methods do not extend easily to the case where the multiplicity $c^K_{I,J}$ is more than 2. Indeed, a key point in our arguments is that any finite degree 2 morphism is cyclic, allowing to compute the pushforward of the structural sheaf. Example 3 shows that the situation is deeply different when $c^K_{I,J} > 2$. Maybe in the case of multiplicity 3, [Mir85] could be of some help. It might also be helpful considering a deformation of Π which gives a cyclic covering. Another natural possibility would be to study directly the combinatorics as done in [KP11] with the new understanding we have of the geometry involved.

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Contents

2. Main results

2.1. **Notation.** In this section, we introduce most of our needed notation.

2.1.1. *Algebraic groups.* Come back to a general inclusion $G \subset \hat{G}$, or even more generally a finite morphism $G \to \hat{G}$ of connected reductive algebraic groups. We also assume that G and $\hat{G}$ are simply connected.

Fix a maximal torus T and a Borel subgroup B such that $T \subset B$. Let $\hat{B}$ be a Borel subgroup of $\hat{G}$ containing B and let $\hat{T}$ be a maximal torus of $\hat{B}$ containing T. We denote by $X(T)$ and $X(\hat{T})$ the groups of characters of T and $\hat{T}$ respectively, and by $X(T)^+$ and $X(\hat{T})^+$ the subsets of dominant weights.

We choose a one-parameter subgroup τ of T, which is also a one-parameter subgroup of $\hat{T}$, and thus defines parabolic subgroups $P = P(\tau) \subset G$ and $\hat{P} = \hat{P}(\tau) \subset \hat{G}$. We thus have the following inclusions:

$$
\begin{tikzcd}
T \arrow[r, hook] \arrow[d, hook] & B \arrow[r, hook] \arrow[d, hook] & P \arrow[r, hook] \arrow[d, hook] & G \arrow[d, hook] \\
\hat{T} \arrow[r, hook] & \hat{B} \arrow[r, hook] & \hat{P} \arrow[r, hook] & \hat{G}
\end{tikzcd}
\tag{9}
$$

Denote by $\iota : G/P \longrightarrow \hat{G}/\hat{P}$ the inclusion morphism. We denote by L the Levi factor of P containing T, and similarly for $\hat{L}$. We denote by S the neutral component of the center of L. Given a character γ of T or $\hat{T}$, we denote by $\gamma_{|S}$ its restriction to the torus S.

Denote by W and W_P the Weyl groups of G and P respectively, and by W^P the set of minimal length representatives of the quotient W/W_P, and similarly $\widehat{W}$, $W^{\hat{P}}$ and $W_{\hat{P}}$.

Consider the group $\mathbb{G} = G \times \hat{G}$ and use bbolded letters to refer to this group. Namely $\mathbb{T} = T \times \hat{T}$, $\mathbb{B} = B \times \hat{B}$, $\mathbb{P} = P \times \hat{P}$, $\mathbb{L} = L \times \hat{L}$ and $\mathbb{W} = W \times \widehat{W}$. The set $\mathbb{\Phi}$ of roots of $\mathbb{G}$ is

$$\mathbb{\Phi} = (\Phi \times \{0\}) \cup (\{0\} \times \hat{\Phi}),$$

where Φ and $\hat{\Phi}$ denote the sets of roots for G and $\hat{G}$ respectively.

Let Δ, $\hat{\Delta}$ and $\mathbb{\Delta}$ be the sets of simple roots for G, $\hat{G}$ and $\mathbb{G}$ respectively. For $\mathbb{\alpha} \in \mathbb{\Delta}$, we denote by $\varpi_{\mathbb{\alpha}}$ the associated fundamental weight. Let $\rho_{\mathbb{G}}$ (resp. $\rho_{\mathbb{L}}$) denote the half sum of positive roots of $\mathbb{G}$ (resp. $\mathbb{L}$). Set also $\rho^{\mathbb{L}} = \rho_{\mathbb{G}} - \rho_{\mathbb{L}}$. Similarly, we use ω_α, $\rho_{\hat{G}} \dots$

Given two dominant weights ζ and $\hat{\zeta}$ for G and $\hat{G}$ respectively, we denote by V_ζ and $V_{\hat{\zeta}}$ the highest weight modules of G and $\hat{G}$ respectively. In this paper, the following integers are what we are most interested in:

$$m_{G\subset\hat{G}}(\zeta, \hat{\zeta}) = \dim(V_\zeta \otimes V_{\hat{\zeta}})^G ,$$

where the exponent G means subset of G-invariant vectors. Indeed, they encode the branching law since

$$V_{\hat{\zeta}} = \oplus_{\zeta \in X(T)^+} \mathbb{C}^{m_{G\subset\hat{G}}(\zeta,\hat{\zeta})} \otimes V_\zeta^* ,$$

as G-modules, where V_ζ^* denotes the dual G-module of V_ζ.

2.1.2. *Schubert classes.* Let $p : G/P \to \mathrm{Spec}(\mathbb{C})$ be the structure morphism, and denote, for ξ in the Chow group $A^*(G/P)$,

$$\chi_{G/P}(\xi) = p_*\xi \in A^0(\mathrm{Spec}(\mathbb{C})) \simeq \mathbb{Z} . \tag{10}$$

Since G/P is rationally connected, $A^{\dim(G/P)} = \mathbb{Z}[pt]$ and $\chi_{G/P}(\xi)$ is the coefficient of $[pt]$ in the homogeneous part of degree $\dim(G/P)$ of ξ.

For a cycle $Z \subset G/P$, its cohomology class $[Z]$ is defined via Poincaré duality on the cohomology of G/P by the equation

$$\forall \beta \in H^*(G/P) \, , \ \chi_{G/P}([Z] \cdot \beta) = \int_Z \beta \, .$$

For $v \in W^P$, τ_v denotes the cohomology class of the Schubert variety $\overline{BvP/P}$, of degree $2(\dim(G/P) - \ell(v))$. Let w_0 be the longest element of W and $w_{0,P}$ be the longest element of W_P. Poincaré duality takes the nice form

$$\chi_{G/P}(\tau_v \cdot \tau_w) = \delta_{v^\vee, w} \text{ with } v^\vee := w_0 v w_{0,P} \, . \tag{11}$$

Similarly, $(\tau_{\hat{v}})_{\hat{v} \in W^{\hat{P}}}$ denotes the Schubert basis of $H^*(\hat{G}/\hat{P})$. When $P = B$ and $u \in W$, we denote by $\sigma_u \in H^*(G/B)$ the cohomology class of $\overline{BuB/B} \subset G/B$. Moreover, we define $\sigma^u := \sigma_{u^\vee}$ and $\tau^v := \tau_{v^\vee}$.

Let

$$\begin{array}{rccc} \delta : & G/P & \longrightarrow & \mathbb{G}/\mathbb{P} \\ & gP/P & \longmapsto & (gP/P, g\hat{P}/\hat{P}) \end{array}$$

denote the small diagonal map and consider the pullback $\delta^* : H^*(\mathbb{G}/\mathbb{P}) \longrightarrow H^*(G/P)$. For $\mathbb{v} = (v, \hat{v}) \in W^{\mathbb{P}}$, observe that

$$\delta^*(\tau_{\mathbb{v}}) = \tau_v \cdot \iota^*(\tau_{\hat{v}}) \, . \tag{12}$$

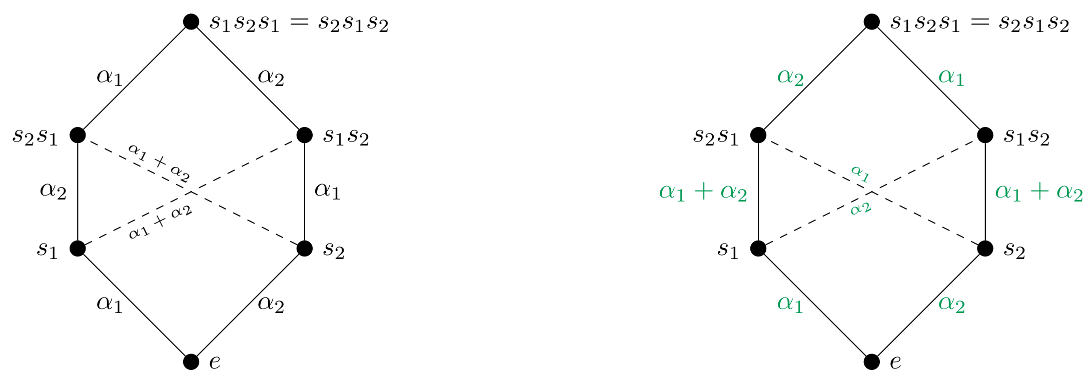


FIGURE 1. Bruhat graphs for SL_3

For $\mathbb{v} = (v, \hat{v}) \in W^{\mathbb{P}} = W^P \times W^{\hat{P}}$ we set

$$c(\mathbb{v}) = c(v,\hat{v}) := \chi_{G/P}(\delta^*(\tau_{\mathbb{v}}))\,, \text{ so that } \iota^*(\tau_{\hat{v}}) = \sum_{v\in W^P} c(v,\hat{v})\tau^v\,. \tag{13}$$

Since $\tau_{\mathbb{v}}$ has degree $\dim G/P + \dim \hat{G}/\hat{P} - \ell(\mathbb{v})$, we have $c(\mathbb{v}) = 0$ unless $\ell(\mathbb{v}) = \dim(\hat{G}/\hat{P})(= \ell(v) + \ell(\hat{v}))$.

2.1.3. *Bruhat orders.* A covering relation in the left weak Bruhat order is a pair (v, w) such that there exists a simple root α with $v = s_\alpha w$ and $\ell(v) = \ell(w) + 1$. We use the graph $\begin{array}{c} v \\ {\scriptstyle\alpha}\big| \\ w \end{array}$ to depict such a relation.

A covering relation in the strong Bruhat order is a pair (v, w) such that there exists a positive root β with $v = s_\beta w$ and $\ell(v) = \ell(w) + 1$. We use the graph $\begin{array}{c} v \\ {\scriptstyle\beta}\vdots \\ w \end{array}$ to depict such a relation. For example the graph on the left on Figure 1 is the Bruhat graph of SL_3.

We also define the (weak) twisted Bruhat graph by labelling the previous edges by $w^{-1}\alpha$ and $w^{-1}\beta$ in place of α and β. We use green color for these twisted labels to avoid any confusion. Hence $\begin{array}{c} v \\ {\scriptstyle\beta}\vdots \\ w \end{array}$ means $v = s_\beta w$ and $\begin{array}{c} v \\ \vdots{\scriptstyle\gamma} \\ w \end{array}$ means $v = ws_\gamma$, with in both cases $\ell(v) = \ell(w) + 1$. For $\mathrm{SL}_3(\mathbb{C})$ we get the right graph of Figure 1.

Moreover, we also denote a strong covering relation by $v \longrightarrow w$, or $v \longrightarrow_\gamma w$ if we want to insist on the twisted label γ, such that $v = ws_\gamma$, or also $v \xrightarrow{P} w$ if $v, w \in W^P$ for some parabolic subgroup P.

We also introduce the following notation.

Notation 1. *Let $v, \tilde{v}, u \in W$.*

- *We denote by $\tilde{v} \xrightarrow{B}_u v$ or $\tilde{v} \to_u v$ if $\tilde{v} = uv$ and $\ell(\tilde{v}) = \ell(u) + \ell(v)$.*
- *We denote by $\tilde{v} \xrightarrow{P}_u v$ if moreover $\tilde{v}$ and v belong to W^P.*

2.1.4. *Chevalley formula.* Given $\zeta \in X(T)$, we denote by $\mathcal{L}_{G/B}(\zeta)$ the G-linearized line bundle on G/B such that B acts with weight $-\zeta$ on the fiber over B/B. Similarly, define $\mathcal{L}_{\hat{G}/\hat{B}}(\hat{\zeta})$. For any v in W^P and any character ζ of P, we have the Chevalley formula, proved for example in [Pra05]:

$$c_1(\mathcal{L}_{G/P}(\zeta)) \cup \tau_v = \sum_{v \xrightarrow{P} v' = vs_\gamma} \langle \zeta, \gamma^\vee \rangle \tau_{v'}\,, \tag{14}$$

2.2. **Our reduction result for the branching problem.** In [Res11b], the following result is proved:

Theorem 6. *Let $\mathbb{v} = (v, \hat{v}) \in W^{\mathbb{P}}$ be such that $c(\mathbb{v}) = 1$. Let $\zeta \in X(T)^+$ and $\hat{\zeta} \in X(\hat{T})^+$ be such that $v^{-1}(\zeta)_{|S} + \hat{v}^{-1}(\hat{\zeta})_{|S}$ is trivial. Then $v^{-1}(\zeta)$ and $\hat{v}^{-1}(\hat{\zeta})$ are dominant weights for L and $\hat{L}$ respectively. Moreover*

$$m_{G\subset\hat{G}}(\zeta, \hat{\zeta}) = m_{L\subset\hat{L}}(v^{-1}(\zeta), \hat{v}^{-1}(\hat{\zeta}))\,.$$

The conclusion of Theorem 6 does not hold when $c(\mathbb{v}) = 2$. Our reduction result is a modification of this conclusion which holds when $c(\mathbb{v}) = 2$.

Consider the incidence variety

$$Y(\mathbb{v}) = \{(x, gB/B, \hat{g}\hat{B}/\hat{B}) \,:\, x \in g\overline{BvP/P} \text{ and } \iota(x) \in \hat{g}\overline{\hat{B}\hat{v}\hat{P}/\hat{P}}\} \subset G/P \times \mathbb{G}/\mathbb{B}\,, \tag{15}$$

endowed with its projection $\Pi : Y(\mathbb{v}) \longrightarrow \mathbb{G}/\mathbb{B}$ (in the sequel ι will be omitted and G/P will be considered as a subvariety of $\hat{G}/\hat{P}$). Consider the ramification divisor K_Π of Π (see Section 3.3 for details) and the branch divisor class $[B_\Pi] = \Pi_*(K_\Pi)$ in $\mathrm{Cl}(\mathbb{G}/\mathbb{B})$ (note that Π is proper since Y is projective). Since $\mathbb{G}/\mathbb{B}$ is smooth, $[B_\Pi]$ is an element of the Picard group and has an expression as

$$[B_\Pi] = \mathcal{L}_{G/B}(\sum_{\alpha\in\Delta} n_\alpha \varpi_\alpha) \otimes \mathcal{L}_{\hat{G}/\hat{B}}(\sum_{\hat{\alpha}\in\hat{\Delta}} n_{\hat{\alpha}} \varpi_{\hat{\alpha}})$$

for some well defined integers n_α and $n_{\hat{\alpha}}$.

Assume now that $c(\mathbb{v}) = 2$. Then, we prove in Proposition 15 that the integers n_α and $n_{\hat{\alpha}}$ are even. Define θ and $\hat{\theta}$ in $X(T)$ and $X(\hat{T})$ by setting

$$\theta = \sum_\alpha \frac{n_\alpha}{2} \varpi_\alpha \qquad \hat{\theta} = \sum_{\hat{\alpha}} \frac{n_{\hat{\alpha}}}{2} \varpi_{\hat{\alpha}}\,. \tag{16}$$

Theorem 7. *Let $\mathbb{v} = (v, \hat{v}) \in W^{\mathbb{P}}$ be such that $c(\mathbb{v}) = 2$. Let $\zeta \in X(T)^+$ and $\hat{\zeta} \in X(\hat{T})^+$ be such that $v^{-1}(\zeta)_{|S} + \hat{v}^{-1}(\hat{\zeta})_{|S}$ is trivial. Then*

$$m_{G\subset\hat{G}}(\zeta, \hat{\zeta}) + m_{G\subset\hat{G}}(\zeta - \theta, \hat{\zeta} - \hat{\theta}) = m_{L\subset\hat{L}}(v^{-1}\zeta, \hat{v}^{-1}\hat{\zeta})\,.$$

Consider now the case when $\hat{G} = G \times G$ corresponding to the tensor product decomposition. Then $\hat{P} = P \times P$. Assume moreover that P is cominuscule. Then Theorem 7 can be improved by expressing $m_{G\subset G\times G}(\zeta, \hat{\zeta})$ as an alternating sum of multiplicities for the tensor product for L as in Theorem 4 (see Corollary 40).

2.3. **A formula for the branch divisor class.** In this section, P and $\hat{P}$ are any parabolic subgroups of G and $\hat{G}$ respectively as in Diagram 9, but not necessarily associated to a given one-parameter subgroup of T. For $\beta \in \Phi$, we set:

$$\hbar(\beta) = \langle \rho_{\mathbb{G}}, \beta^\vee \rangle \tag{17}$$

Theorem 8. *Let $\mathbb{v} \in W^{\mathbb{P}}$ such that $\ell(\mathbb{v}) = \dim(\hat{G}/\hat{P})$ and $c(\mathbb{v}) \neq 0$. Consider the map*

$$\Pi : Y(\mathbb{v}) \longrightarrow \mathbb{G}/\mathbb{B}\,,$$

and its ramification divisor $K_\Pi \in \mathrm{Cl}(Y)$. Then, in $\mathrm{Cl}(\mathbb{G}/\mathbb{B})$, we have

$$\Pi_* K_\Pi = \sum_{\mathbb{v}'} \left(c_1(\mathcal{L}(2\rho^{\hat{L}}_{|T})) \cdot \delta^*(\tau_{\mathbb{v}'}) + 2c(\mathbb{v}) - \sum_{\mathbb{v}''} (\hbar(\beta) + 1) c(\mathbb{v}'') \right) \mathcal{L}_{\mathbb{G}/\mathbb{B}}(\varpi_\alpha)\,,$$

where the sums over $\mathbb{v}'$ *and* $\mathbb{v}''$ *respectively run over the covering relations* $\begin{tikzcd}[row sep=small] \mathbb{v}' \arrow[d, dash, "\mathbb{\alpha}"'] \\ \mathbb{v} \end{tikzcd}$ *and* $\begin{tikzcd}[row sep=small] \mathbb{v}' \arrow[d, dash, dashed, "\mathbb{\gamma}"] \\ \mathbb{v}'' \end{tikzcd}$ *in the Bruhat graph of* $\mathbb{G}/\mathbb{P}$*, with* $\mathbb{\alpha} \in \mathbb{\Delta}$ *and* $\mathbb{\gamma} \in \Phi^+(\mathbb{G})$ *is the twisted label. Here* c_1 *denotes the first Chern class.*

The first term $c_1(\mathcal{L}(2\rho^{\hat{L}}_{|T})) \cdot \delta^*(\tau_{\mathbb{v}'})$ is an integer as an element of $H^{2\dim(G/P)}(G/P) \simeq \mathbb{Z}$.

Notation 2. *In the case when* $G \subset \hat{G} = G^{n-1}$ *and* $\mathbb{G}/\mathbb{P} = (G/P)^n$ *for some integer* $n \geq 2$*, write* c_1^α *for the coefficient of* $\Pi_*(K_\Pi)$ *at* $\mathcal{L}_{\mathbb{G}/\mathbb{B}}(\varpi_\alpha, 0, \dots, 0)$*. We let* $(\zeta, \hat{\zeta}) = (\zeta_1, \zeta_2, \dots, \zeta_n)$*,* $\mathbb{v} = (v_1, v_2, \dots, v_n)$*,* $(\theta, \hat{\theta}) = (\theta^1, \theta^2, \dots, \theta^n)$ *and* $m_{G \subset \hat{G}}(\zeta, \hat{\zeta}) = m_G(\zeta_1, \zeta_2, \dots, \zeta_n)$*.*

Even more specifically, we call "tensor product case" the case when $n = 3$*, since then* $m_G(\zeta_1, \zeta_2, \zeta_3)$ *is the multiplicity of* $V^*_{\zeta_1}$ *in* $V_{\zeta_2} \otimes V_{\zeta_3}$*. In this case, we set* $\Gamma(G) = \Gamma(G \subset \hat{G})$*.*

Corollary 9. *Let* $n \geq 2$ *be an integer. Assume that* $\hat{G} = G^{n-1}$ *and* $\hat{P} = P^{n-1}$*. Let* $\mathbb{v} = (v_1, \dots, v_n) \in (W^P)^n$ *such that* $\ell(\mathbb{v}) = (n-1)\dim(G/P)$ *and* $c(\mathbb{v}) \neq 0$*. Consider the map*

$$\Pi : Y(\mathbb{v}) \longrightarrow \mathbb{G}/\mathbb{B}\,,$$

and its ramification divisor $K_\Pi \in \mathrm{Cl}(Y)$*. Fix* $\alpha \in \Delta$*. If* $s_\alpha v_1 \overset{P}{\to} v_1$ *then*

$$c_1^\alpha = (n-1)c_1(\mathcal{L}(2\rho^L))\tau_{v_1'} \dots \tau_{v_n'} + 2c(\mathbb{v}) - \sum_{\mathbb{v}''} (\mathbb{h}(\mathbb{\gamma}) + 1)c(\mathbb{v}'')\,,$$

where $\mathbb{v}' = (v_1', \dots, v_n') = (s_\alpha v_1, v_2, \dots, v_n)$ *and the sum runs over the covering relations* $\begin{tikzcd}[row sep=small] \mathbb{v}' \arrow[d, dash, dashed, "\mathbb{\gamma}"] \\ \mathbb{v}'' \end{tikzcd}$*. If* $s_\alpha v_1 \overset{P}{\to} v_1$ *does not hold then* $c_1^\alpha = 0$*.*

Remark 1. *In the case when moreover* G/P *is cominuscule, strong and weak Bruhat order coincide, so the covering relations* $\begin{tikzcd}[row sep=small] \mathbb{v}' \arrow[d, dash, dashed, "\mathbb{\gamma}"] \\ \mathbb{v}'' \end{tikzcd}$ *may be replaced by the covering relations* $\begin{tikzcd}[row sep=small] \mathbb{v}' \arrow[d, dash, "\mathbb{\gamma}"] \\ \mathbb{v}'' \end{tikzcd}$*.*

Proof of Corollary 9: We only need to remark that since $\hat{G}/\hat{P} = (G/P)^{n-1}$, we have $\mathcal{L}(2\rho^{\hat{L}}_{|T}) = (n-1)\mathcal{L}_{G/P}(2\rho^L)$ and $\delta^*(\tau_{\mathbb{v}'}) = \tau_{v_1'} \dots \tau_{v_n'}$. ☐

Theorem 8 is proved in Section 5.3. Here we make one comment. Let $\mathbb{v}' = (v', \hat{v}') \in W^{\mathbb{P}}$ such that $\ell(\mathbb{v}') = \dim(\hat{G}/\hat{P}) + 1$ like in the first sum of Theorem 8. Set

$$C(\mathbb{v}') = g \cdot \overline{Bv'P/P} \cap \iota^{-1}(\hat{g} \cdot \overline{B\hat{v}'P/P})\,,$$

for general elements $g \in G, \hat{g} \in \hat{G}$. Then $C(\mathbb{v}') \subset G/P$ is a curve. The degree of the pullback of the anticanonical bundle $\mathcal{L}(2\rho^{\hat{L}}_{|T})$ on $C(\mathbb{v}')$ is the first term appearing in the expression of $\Pi_* K_\Pi$. By Chevalley's formula applied to $c_1(\mathcal{L}(2\rho^{\hat{L}}_{|T})) \cdot \tau_{v_1'}$, we have an explicit formula

$$c_1(\mathcal{L}(2\rho^{\hat{L}}_{|T})) \cdot \delta^*(\tau_{\mathbb{v}'}) = \sum_{\mathbb{v}''} \langle 2\rho^{\hat{L}}_{|T}, \gamma^\vee \rangle c(\mathbb{v}'')\,, \tag{18}$$

where the sum runs over the covering relations $\begin{tikzcd}[row sep=small] \mathbb{v}' \arrow[d, dash, dashed, "\gamma"] \\ \mathbb{v}'' \end{tikzcd}$ with $\gamma \in \Phi^+(G)$.

If $c(\mathbb{v}) = 1$ then Π is birational so $\Pi_*(K_\Pi) = 0$. This gives a funny but combinatorially involved necessary condition for the equality $c(\mathbb{v}) = 1$. Let us check it on the following example.

Example 1. *Let* $G = \mathrm{SL}_3, \hat{G} = G \times G, P = B, \hat{P} = B \times B,$ *and* $\mathbb{v} = (s_2 s_1, s_2 s_1, s_1 s_2)$*. Then* $\Pi_* K_\Pi = 0$*.*

Proof. We use Corollary 9 to check this. Recall that the Bruhat graph of SL_3 is depicted in Figure 1, and that the Poincaré dual of τ_{s_i} is $\tau_{w_0 s_i} = \tau_{s_i s_{3-i}}$. Fix first $\mathbb{a} = (\alpha_1, 0, 0)$, which implies $\mathbb{v}' = (s_1 s_2 s_1, s_2 s_1, s_1 s_2)$. Then $2c_1(\mathcal{L}(2\rho))\tau_{v_1'} \dots \tau_{v_3'} + 2c(\mathbb{v}) = 2c_1(\mathcal{L}(2\rho))(\tau_{s_1} + \tau_{s_2}) + 2 \times 1 = 2\langle 2\rho, \alpha_1^\vee + \alpha_2^\vee \rangle + 2 = 10$. The possibilities for $\mathbb{b}$ with $c(\mathbb{v}'') \neq 0$ are $(0, \alpha_1 + \alpha_2, 0)$, $(0, 0, \alpha_1 + \alpha_2)$, $(\alpha_1, 0, 0)$ and $(\alpha_2, 0, 0)$. In these cases $c(\mathbb{v}'') = 1$ and $\hbar(\mathbb{b}) = 2$ twice and $\hbar(\mathbb{b}) = 1$ twice. We get $c_1^{\alpha_1} = 0$ as expected.

Fix now $\mathbb{a} = (0, 0, \alpha_2)$ with $\mathbb{v}' = (s_2 s_1, s_2 s_1, s_2 s_1 s_2)$. Then $2c_1(\mathcal{L}(2\rho))\tau_{v_1'} \dots \tau_{v_3'} + 2c(\mathbb{v}) = 2c_1(\mathcal{L}(2\rho))(\tau_{s_1}) + 2 \times 1 = 6$. The possibilities for $\mathbb{b}$ with $c(\mathbb{v}'') \neq 0$ are $(\alpha_1, 0, 0)$, $(0, \alpha_1, 0)$ and $(0, 0, \alpha_1)$ and give $3 \times 2 = 6$ for the sum.

The remaining cases are either trivial or obtained by exchanging v_1 and v_2. □

We now consider a family of examples again in the birational case:

Example 2. *Let $\hat{G} = G \times G$ and $\hat{P} = P \times P$. Let $v \in W^P$ be arbitrary and let $\mathbb{v} = (v, v^\vee, w^P)$. Then we have $\Pi_* R_\Pi = 0$.*

Proof. We check that Corollary 9 gives $\Pi_* R_\Pi = 0$. Up to exchanging v and $v^\vee$, the only case to consider is when $\mathbb{a} = (\alpha, 0, 0)$ with $\alpha \in \Delta$ and $s_\alpha v \xrightarrow{P} v$. Set $\gamma_0 = -v^{-1}\alpha$ and $\mathbb{v}' = (s_\alpha v, v^\vee, w^P)$.

Consider the Chevalley formula (14) for $c_1(\mathcal{L}(2\rho^L)) \cdot \tau_{v^\vee}$. The only summand surviving after multiplication by $\tau_{s_\alpha v}$ is $\langle 2\rho^L, \gamma \rangle \tau_{(s_\alpha v)^\vee}$. Moreover, $c(\mathbb{v}'') = 1$ for this term and $\gamma = w_{0,P}(\gamma_0)$. Finally $2c_1(\mathcal{L}(2\rho))\tau_{v_1'} \dots \tau_{v_3'} + 2c(\mathbb{v}) = \langle 4\rho^L, w_{0,P}(\gamma_0^\vee) \rangle + 2 = \langle 4\rho^L, \gamma_0^\vee \rangle + 2$.

The only case with $c(\mathbb{v}'') \neq 0$ and $\mathbb{b}$ in the first factor is $\mathbb{b} = (\gamma_0, 0, 0)$. Its contribution in the sum is $\langle \rho, \gamma_0^\vee \rangle + 1$. Similarly, with $\mathbb{b}$ in the second factor, we get $\langle w_{0,P}(\rho), \gamma_0^\vee \rangle + 1$.

Assume now that γ is in the third copy. It is a descent of w^P, hence an element of $\Delta - \Delta(L)$. We have $\tau_{w^P s_\gamma} = c_1(\mathcal{L}(\varpi_\gamma))$, so the associated coefficient $c(\mathbb{v}'')$ is $\langle \varpi_\gamma, \gamma_0^\vee \rangle$ by the Chevalley formula. The term $\hbar(\gamma) + 1$ equals $\langle \rho, \gamma^\vee \rangle + 1 = 2$. The contribution of these terms is $\sum_{\gamma \in \Delta - \Delta(L)} 2\langle \varpi_\gamma, \gamma_0^\vee \rangle = 2\langle \rho - \rho_L, \gamma_0^\vee \rangle$.

Finally,

$$c_1^\alpha = 4\langle \rho^L, \gamma_0^\vee \rangle + 2 - (\langle \rho, \gamma_0^\vee \rangle + 1) - (\langle w_{0,P}(\rho), \gamma_0^\vee \rangle + 1) - 2\langle \rho - \rho_L, \gamma_0^\vee \rangle$$

which vanishes, since $\rho^L = \rho - \rho_L$ and $w_{0,P}(\rho) = \rho^L - \rho_L$. □

In Section 7, we give several examples of Theorem 8 in the case $c(\mathbb{v}) = 2$, which is the case we are mainly interested in. Let us now give an example with $c(\mathbb{v}) = 3$:

Example 3. *In* $\mathrm{Gr}(4, 8)$, *let* $u = [35681247] = (s_2 s_3 s_5 s_4)^\vee, v = [24681357] = (s_2 s_4 s_6 s_3 s_5 s_4)^\vee$ *and* $w = [24681357] = (s_2 s_4 s_3 s_6 s_5 s_4)^\vee$. *Then* $c(u, v, w) = c_{211,321}^{4321} = 3$. *We computed with Theorem 8 and thanks to a computer that*

$$[B_\Pi] = \mathcal{L}_{(G/B)^3}(4(\varpi_2 + \varpi_4 + \varpi_6), 4(\varpi_2 + \varpi_4 + \varpi_6), 4\varpi_3 + 6\varpi_6)\,.$$

In particular, $[B_\Pi]$ *is not divisible by* 3, *which shows that the following Lemma 13 and Proposition 15 don't have obvious analogues for morphisms of degree* 3.

3. Preliminaries in Algebraic Geometry

3.1. **Geometric Invariant Theory for quotients by G and L.** In this section, X is any projective variety endowed with an action of a reductive group G. Let $\mathrm{Pic}^G(X)$ denote the group of G-linearized line bundles and $\mathrm{Pic}^G(X)_\mathbb{Q} = \mathrm{Pic}^G(X) \otimes_\mathbb{Z} \mathbb{Q}$. A line bundle $\mathcal{L}$ on X is said to be *semi-ample* if it is the pullback of an ample line bundle by some morphism from X to another projective variety or equivalently if it has a power which is globally generated. Let $\mathrm{Pic}^G(X)^+$ denote the set of semi-ample G-linearized line bundles and let $\mathrm{Pic}^G(X)^+_\mathbb{Q}$ denote the generated cone in $\mathrm{Pic}^G(X)_\mathbb{Q}$. For $\mathcal{L} \in \mathrm{Pic}^G(X)^+_\mathbb{Q}$, we define the set of semi-stable points to be

$$X^{\mathrm{ss}}(\mathcal{L}) := \{x \in X \;:\; \exists N > 0\,, \exists \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes N})^G \text{ such that } \sigma(x) \neq 0\}$$

and the GIT-cone to be

$$\Gamma(X, G) := \{\mathcal{L} \in \mathrm{Pic}^G(X)^+_\mathbb{Q} \;:\; X^{\mathrm{ss}}(\mathcal{L}) \neq \emptyset\}.$$

It is well known that $\Gamma(X, G)$ is a closed convex polyhedral cone as a subset of $\mathrm{Pic}^G(X)^+_\mathbb{Q}$.

Set also

$$X^{\mathrm{ss}}(\mathcal{L})/\!/G := \mathrm{Proj}\Big(\oplus_{k \geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes k})^G \Big).$$

It is a projective variety endowed with a G-invariant morphism $\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G$. These are the standard definitions as in [MFK94] only if $\mathcal{L}$ is ample. In particular, one needs this assumption to claim that π is a categorical quotient of $X^{\mathrm{ss}}(\mathcal{L})$.

We say that two points $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\Gamma(X, G)$ *belong to the same face* if there exist $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\Gamma(X, G)$ such that

$$\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{Q}^{+*}\mathcal{M}_1 + \mathbb{Q}^{+*}\mathcal{M}_2.$$

Proposition 10. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\Gamma(X, G)$ belong to the same face. Then*

$$\dim(X^{\mathrm{ss}}(\mathcal{L}_1)/\!/G) = \dim(X^{\mathrm{ss}}(\mathcal{L}_2)/\!/G).$$

Proof. Assume that $\mathbb{Q}^{+}\mathcal{L}_1 \neq \mathbb{Q}^{+}\mathcal{L}_2$, the statement being trivial otherwise. Up to exchanging $\mathcal{M}_1$ and $\mathcal{M}_2$, one may assume that there exist positive integers a, b and c such that $a\mathcal{L}_1 = b\mathcal{L}_2 + c\mathcal{M}_1$ holds in $\mathrm{Pic}^G(X)$. Since the line bundle $\mathcal{M}_1$ belongs to the GIT-cone, there exists a positive integer k and a nonzero G-invariant section σ_1 of $ck\mathcal{M}_1$. For any nonnegative integer n, the G-equivariant map

$$\begin{array}{ccc} \mathrm{H}^0(X, bkn\mathcal{L}_2) & \longrightarrow & \mathrm{H}^0(X, akn\mathcal{L}_1) \\ \sigma & \longmapsto & \sigma_1^{\otimes n} \otimes \sigma \end{array}$$

is injective, the variety X being irreducible. It follows that $\dim(X^{\mathrm{ss}}(\mathcal{L}_1)/\!/G) \geq \dim(X^{\mathrm{ss}}(\mathcal{L}_2)/\!/G)$. One concludes by symmetry. ☐

Let τ be a one-parameter subgroup of G and C be an irreducible component of the fixed point set X^τ. Let C^+ denote the associated Białynicki-Birula cell. Consider the fibered product $G \times^P C^+$ and the map $\eta : G \times^P C^+ \longrightarrow X$, $[g : x] \longmapsto gx$. We say that (C, τ) is *dominant* if η is (see [Res10]).

Let $\mathcal{L} \in \mathrm{Pic}^G(X)$. The action of $\tau(\mathbb{C}^*)$ on the restriction $\mathcal{L}_{|C}$ is given by a character of $\mathbb{C}^*$ which is itself given by an integer. Let $\mu^{\mathcal{L}}(C, \tau)$ denote the opposite of this integer. By [Res10, Lemma 3], if (C, τ) is dominant and $\mathcal{L} \in \Gamma(X, G)$ then $\mu^{\mathcal{L}}(C, \tau) \leq 0$.

Let G^τ denote the centralizer of $\tau(\mathbb{C}^*)$ in G: it is a Levi subgroup of G.

Proposition 11. *Let $\mathcal{L} \in \mathrm{Pic}^G(X)^+$. Assume that the pair (C, τ) is dominant and that $\mu^{\mathcal{L}}(C, \tau) = 0$. Then, there is commutative diagram*

$$\begin{tikzcd} C^{\mathrm{ss}}(\mathcal{L}) \arrow[r] \arrow[d, "\pi_C"'] & X^{\mathrm{ss}}(\mathcal{L}) \arrow[d, "\pi_X"] \\ C^{\mathrm{ss}}(\mathcal{L})/\!/G^\tau \arrow[r, "\theta"] & X^{\mathrm{ss}}(\mathcal{L})/\!/G \end{tikzcd}$$

where θ is surjective and generically finite.

Proof. This result is proved in [Res10, Proposition 9] when $\mathcal{L}$ is ample and we show here how to deduce the general case. Let $p : X \longrightarrow \bar{X}$ such that $\mathcal{L}$ is the pullback of an ample line bundle (still denoted by $\mathcal{L}$) on $\bar{X}$. Set $\bar{C}$ be the irreducible component of $\bar{X}^\tau$ containing $p(C)$. By construction $X^{\mathrm{ss}}(\mathcal{L}) = p^{-1}(X^{\mathrm{ss}}(\mathcal{L}))$. And by [Res10, Proposition 8], $C^{\mathrm{ss}}(\mathcal{L}) = C \cap X^{\mathrm{ss}}(\mathcal{L})$ (the similar equality for $\bar{C}$ and $\bar{X}$ also holds). We have the following commutative diagram

$$\begin{tikzcd} & C \arrow[rr] \arrow[dl, dashed, "\pi_C"'] \arrow[d, "p"] \arrow[drrr, dashed, "\gamma"] & & X \arrow[d, "p"] \\ C^{\mathrm{ss}}(\mathcal{L})/\!/G^\tau \arrow[dr, "\iota"'] & \bar{C} \arrow[rr] \arrow[d, dashed, "\pi_{\bar{C}}"] & & \bar{X} \arrow[d, dashed, "\pi_X"] \\ & \bar{C}^{\mathrm{ss}}(\mathcal{L})/\!/G^\tau \arrow[rr, "\bar{\theta}"] & & X^{\mathrm{ss}}(\mathcal{L})/\!/G \end{tikzcd}$$

where the dashed arrows represent rational maps defined on the semi-stable loci. The vertical maps are GIT-quotients. The map $\bar{\theta}$ is defined by factorising by the categorical quotient.

The only difficulty is that $\bar{C}$ could be bigger than $\tilde{C} := p(C)$. By construction $C^{\rm ss}(\mathcal{L})/\!/G^\tau = \tilde{C}^{\rm ss}(\mathcal{L})/\!/G^\tau$. Hence the inclusion $\tilde{C} \subset \bar{C}$ induces the morphism ι since $\mathcal{L}$ is ample on $\bar{C}$. Since ι is injective, π_C is surjective and $\bar{\theta}$ is finite and surjective by [Res10, Proposition 9], it is sufficient to prove that γ is dominant.

Since GC^+ is dense in X, it is sufficient to prove the $\pi_X \circ p(C^+) = \gamma(C)$. Let $x \in C^+ \cap X^{\rm ss}(\mathcal{L})$. Set $x_0 = \lim_{t\to 0} \tau(t)x \in C$. By [Res10, Lemma 2(ii)], x_0 belongs to $X^{\rm ss}(\mathcal{L})$. By invariance $\gamma(\tau(\mathbb{C}^*)x)$ is a point named ξ. By continuity, $\gamma(x_0) = \xi$. $\square$

3.2. **Recollections on Intersection theory.** We now recall useful notions in Intersection Theory and give some details on degree two morphisms and the associated ramification and branch divisors.

Let $f : Y \longrightarrow X$ be a dominant morphism between irreducible varieties of the same dimension. We say that f is *generically finite*. The *degree of* f is defined to be $\deg(f) = [\mathbb{C}(Y) : \mathbb{C}(X)]$.

Let X be a quasi-projective irreducible variety. We denote by $\mathrm{Cl}(X)$ the Weil divisor class group, by $\mathrm{CaCl}(X)$ the Cartier divisor class group and by $\mathrm{Pic}(X)$ the Picard group. By [Har77, Prop. II.6.15], we have a canonical isomorphism $\mathrm{CaCl}(X) \simeq \mathrm{Pic}(X)$, mapping D on $\mathcal{O}(D)$. There is also a morphism $c_1 : \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$ (see [Ful84, p. 30]) that is neither injective nor surjective. Nevertheless, if X is normal, this morphism is injective and $\mathrm{CaCl}(X) = \mathrm{Pic}(X)$ can be seen as a subgroup of $\mathrm{Cl}(X)$. Moreover if X is locally factorial (e.g. is smooth) then this morphism is an isomorphism.

We also consider the Chow ring $A^*(X)$ and identify $\mathrm{Cl}(X)$ with $A^1(X)$. Given two irreducible subvarieties Z_1 and Z_2 that intersect transversally in X, we have

$$[Z_1 \cap Z_2] = [Z_1] \cdot [Z_2] \text{ in } A^*(X). \tag{19}$$

Here $[Z_1 \cap Z_2]$ denotes the sum of the classes of the irreducible components of $Z_1 \cap Z_2$.

3.3. **Canonical, ramification and branch divisors.** Set $n = \dim(X)$. If X is smooth, the canonical divisor $K_X \in \mathrm{Pic}(X)$ on X is defined to be the line bundle $\wedge^n T^*X$. Assume that X is normal. Let $X_{\rm reg}$ denote the regular locus of X and $TX_{\rm reg}$ denote its tangent bundle. Then $\wedge^n T^*X_{\rm reg}$ is a line bundle $\mathcal{L}$ on $X_{\rm reg}$. The first Cern class $c_1(\mathcal{L}) \in \mathrm{Cl}(X_{\rm reg}) = \mathrm{Cl}(X)$ is called the canonical divisor class of X and denoted by K_X. The canonical sheaf $\mathcal{K}_X$ is defined to be the pushforward of the sheaf $U \mapsto \mathcal{L}(U)$ by the inclusion $X_{\rm reg} \subset X$.

Let $f : Y \longrightarrow X$ be a generically finite morphism between irreducible varieties. Assume moreover that Y is normal and X is smooth. The determinant of the restriction of f to the smooth locus $Y_{\rm reg}$ of Y define a Cartier divisor R_f in $Y_{\rm reg}$. By construction, the associated line bundle $\mathcal{O}(R_f)$ is $K_f := K_Y - f^*K_X$. Its closure, still denoted by R_f has a class $[R_f]$ in $\mathrm{Cl}(Y)$ called ramification divisor of f. We have

$$[R_f] = [K_Y] - f^*[K_X] \in \mathrm{Cl}(Y), \tag{20}$$

where $f^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is the pullback of line bundles. Observe that R_f is an effective Cartier divisor. We write R_f as $R_f = \sum_{i\in I} a_i R_i$, with prime divisors R_i, and we define the Cartier divisors

$$E_f = \sum_{i\in J} a_i R_i \ , \ R_f^1 = \sum_{i\notin J} a_i R_i \,, \tag{21}$$

with $J \subset I$ corresponding to the contracted divisors. We define the branch divisor $B_f := f(R_f^1) \subset X$.

3.4. **Pushforward of the structure sheaf under a finite morphisme of degree 2.** We consider a finite morphism $f : S \longrightarrow X$ and study $f_*\mathcal{O}_S$. The next lemmas are well-known to specialists, see for example [Laz80, Lemma 1.1] or [Par91]. We provide details for the singular case we are interested in.

As it is explained in [BK05, p.9], we have:

Lemma 12. *Let $f : S \longrightarrow X$ be a finite surjective morphism of degree d and assume that X is normal. Then there exists a splitting $f_*\mathcal{O}_S = \mathcal{O}_X \oplus L$ with L a coherent sheaf on X.*

We now consider the degree 2 case and exploit the fact that f is a cyclic cover.

Lemma 13. *Let $f : S \longrightarrow X$ be a finite surjective morphism of degree 2 and assume that S is normal and X is normal and locally factorial. Then, in the splitting $f_*\mathcal{O}_S = \mathcal{O}_X \oplus L$ as above, L is a line bundle, and $[L^{\otimes 2}] = -[B_f] \in \mathrm{Pic}(X) \simeq \mathrm{Cl}(X)$. Moreover, R_f is reduced and $f_*[R_f] = [B_f]$ in $\mathrm{Cl}(X)$. As Cartier divisors, we have $f^*\mathcal{O}_X(B_f) = \mathcal{O}_Y(2R_f)$.*

Proof. Let us restrict ourselves over an affine open subset $U \subset X$ and set $A = \mathbb{C}[U]$ and $B = \mathbb{C}[f^{-1}(U)]$.

Let $b \in B \setminus \mathrm{Frac}(A)$. The ring B being integral over A, there exists a monic polynomial P in $A[T]$ of minimal degree such that $P(b) = 0$. Then P is irreducible in $A[T]$. By assumption, A is unique factorization domain. Then, P is also irreducible as an element of $\mathbb{C}(X)[T]$, by [Eis95, Exercise 3.4.c]. It follows that P is the minimal polynomial of $b \in \mathbb{C}(S)$ over $\mathbb{C}(X)$. But $[\mathbb{C}(S) : \mathbb{C}(X)] = 2$, hence P has degree 2.

Write $P = T^2 + a_1T + a_2$ with a_1 and a_2 in A. Up to changing b by $b - \frac{a_1}{2}$, one may assume that $a_1 = 0$. Assume that $-a_2 = t^2c$ in A. Set $b = tb'$. Then $b'^2 - c = 0$, and $b' \in \mathbb{C}(X)$ is integral over A. By normality of S, this implies that $b' \in B$. In particular, up to changing b, one may assume that the minimal polynomial of b is $T^2 - a$ with some square free element $a \in A$.

We claim that $B = A \oplus Ab$. Let $y = \alpha + \beta b \in B$ with $\alpha, \beta \in \mathrm{Frac}(A)$. The matrix of $x \mapsto xy$ is $\begin{pmatrix} \alpha & a\beta \\ \beta & \alpha \end{pmatrix}$. Then $\mathrm{Tr}(y) = 2\alpha$ and $N(y) = \alpha^2 - a\beta^2$ are integral over A. Hence $\alpha \in A$ and $a\beta^2 \in A$. Since A is factorial and a is square free, this implies that $\beta \in A$. This proves the claim.

It follows that $f^{-1}(U) = \mathrm{Spec}(A[t]/(t^2 - a))$ and f corresponds to the inclusion $A \subset A[t]/(t^2 - a)$. Then L is the kernel of Tr, namely $tA \subset A[t]/(t^2 - a)$. This is a free A-module. This proves that L is a line bundle.

A local computation as in [BPVdV84, Lemma 16.1] shows that R_f is locally defined by t, so it is reduced. We get $f^*\mathcal{O}_X(B_f) = \mathcal{O}_S(2R_f)$, since these locally free sheaves are generated by $a = t^2$. Moreover, a being square-free, is an equation of $f(R_f) \cap U$ in $\mathbb{C}[U]$. It follows that $L^{\otimes 2}(U)$ identifies with $aA = \mathcal{O}(-B_f))(U) \subset \mathbb{C}(X)$. This proves that $L^{\otimes 2} \simeq \mathcal{O}(-B_f)$. Finally, f restricts to an isomorphism from the hypersurface defined by the equation $t = 0$ to $B_f \cap U$, proving that $f_*[R_f] = [B_f]$. □

Remark 2. *The degree 2 is a special case since any finite morphism of degree 2 is a cyclic covering. Already in degree 3, this is no longer true, and the above Lemma fails: see Example 3 below.*

3.5. **Generically finite morphisms of degree 2.** We now consider the case of a degree 2 map, but not necessarily finite: we let $f : Y \longrightarrow X$ be a generically finite morphism of degree 2, with X and Y projective, Y irreducible normal, and X normal and locally factorial. We let
$$\begin{tikzcd} Y \arrow[r, "p"] \arrow[rr, bend right, "f"'] & S \arrow[r, "h"] & X \end{tikzcd}$$
be the Stein factorization of f.

Lemma 14. *With the above notation, assume we have an irreducible divisor $C_S \subset S$. Then there is a unique irreducible component C_Y of $p^{-1}(C_S)$ which maps onto C_S, the restriction $C_Y \xrightarrow{p} C_S$ is birational, and we have the equivalence*
$$C_Y \subset R_f \Longleftrightarrow C_S \subset R_h\,.$$

Proof. Since p is surjective, there is an irreducible component C_Y of $p^{-1}(C_S)$ mapping onto C_S. This component is unique and $C_Y \xrightarrow{p} C_S$ is birational since p has connected fibers. Since Y, S and X are normal, for a generic point $y \in C_Y$, Y, S, X are normal and locally factorial at $y, p(y), f(y)$, respectively. Since $C_Y \xrightarrow{p} C_S$ is dominant, T_yp is invertible, by Zariski's Main Theorem again. Therefore T_yf will be degenerate if and only if $T_{p(y)}h$ is degenerate. In other words, $C_Y \subset R_f$ if and only if $C_S \subset R_h$. □

In the next proposition, we use the notation R_i, J introduced in Section 3.3.

Proposition 15. *Let $f : Y \longrightarrow X$ be a generically finite morphism of degree 2, with X and Y projective, Y irreducible normal, and X normal and locally factorial. Then there is a line bundle L on X such that $f_*\mathcal{O}_Y = \mathcal{O}_X \oplus L$. Moreover, in $\mathrm{Pic}(X) = \mathrm{Cl}(X)$, we have $2L + f_*(K_Y) - f_*f^*(K_X) = 0$ and $f_*[R_f] = [B_f]$. As Cartier divisors on Y, we have*

$$f^*\mathcal{O}_X(B_f) = \mathcal{O}_Y(2R_f^1 + \sum_{i \in J} b_iR_i) \tag{22}$$

for some non-negative integers b_i.

Proof. We keep the notation of the beginning of the section, and we want to prove that $p_*[R_f] = [R_h]$. By Lemma 13, R_h is reduced: $[R_h] = \sum_{C_S \subset R_h} [C_S]$. By Lemma 14, for a component C_S of R_h, there is a unique component C_Y of R_f such that $p(C_Y) = C_S$. Since $C_Y \xrightarrow{p} C_S$ is birational, we get $p_*[C_Y] = [C_S]$. Moreover, p is an isomorphism at a generic point of C_Y, so C_Y is a reduced component of R_f. For a contracted component C'_Y of R_f, we have $p_*[C'_Y] = 0$. Thus $p_*[R_f] = \sum_{C_Y \subset R_f} p_*[C_Y] = \sum_{C_S \subset R_h} [C_S] = [R_h]$. Applying h_* and using Lemma 13, we get $f_*[R_f] = [B_f]$.

By Lemma 12, let L be such that $h_*\mathcal{O}_S = \mathcal{O}_X \oplus L$. By Lemma 13, $h_*[R_h] = -2L$, so $f_*[R_f] = [B_f] = -2[L]$. It remains to prove that $f_*\mathcal{O}_Y = h_*\mathcal{O}_S$. This follows from Zariski's Main Theorem that asserts that $p_*\mathcal{O}_Y = \mathcal{O}_S$.

Let us write the Cartier divisor $f^*\mathcal{O}_X(B_f)$ as $\sum_{i \in I} b_i R_i$. Then (22) is equivalent to $b_i \geq 0$ for $i \in J$ and $b_i = 2$ for $i \notin J$. We write the Cartier divisor $\mathcal{O}_X(B_f)$ as (U_i, ξ_i), with ξ_i a local equation of B_f on U_i. Then $f^*\mathcal{O}_X(B_f)$ is defined by $(f^{-1}(U_i), \xi_i \circ f)$. Since $\xi_i \circ f$ is regular, we have $b_i \geq 0$ for all i. Moreover, if $i \notin J$, then restricting to a neighborhood of a point of R_i where p is a local isomorphism and using Lemma 13, we get $b_i = 2$. □

We now describe the branch locus as a set.

Proposition 16. *Let $f : Y \longrightarrow X$ be a generically finite morphism of degree 2, with X and Y projective, Y irreducible normal, and X normal and locally factorial. Then*

$$\mathrm{Supp}(B_f) = \{x \in X \ : \ f^{-1}(x) \textit{ is connected.}\}$$

Proof. We keep our notation. First, we claim that $B_h = B_f$. This is equivalent to saying that $h(R_h) = f(R^1_f)$. These two varieties are closed in X and of pure codimension one. Lemma 14 yields a bijection between the irreducible components of R_h and those of R^1_f, implying that $h(R_h)$ and $f(R^1_f)$ have the same irreducible components, proving the claim.

Observe now that B_h is the set of $x \in X$ such that $h^{-1}(x)$ is a point (see the proof of Lemma 13). The fibers of p being connected by Zariski's Main Theorem, the proposition follows. □

4. Preliminaries on Schubert varieties

In this section, we fix a semisimple group G with maximal torus T and Borel subgroup B containing T. Let $P \supset B$ be a standard parabolic subgroup. We are interested in the Schubert varieties in G/P, their relative versions in $G/B \times G/P$ and their Bott-Samelson resolutions. We collect both combinatorial and geometric material.

4.1. Reminder on Schubert varieties.

For $w \in W$ (resp. $w \in W^P$), denote by $\mathcal{X}^B_w = \overline{BwB/B}$ (resp. $\mathcal{X}^P_w = \overline{BwP/P}$) the associated Schubert subvariety of G/B (resp. G/P). We use notation for the Bruhat orders introduced in Section 2.1.

Lemma 17. *Let $v \in W^P$. Then*

1. *The variety $\mathcal{X}^{\mathcal{P}}_v$ is normal. In particular, $\mathrm{Pic}(\mathcal{X}^{\mathcal{P}}_v) = \mathrm{CaCl}(\mathcal{X}^{\mathcal{P}}_v)$ embeds in $\mathrm{Cl}(\mathcal{X}^{\mathcal{P}}_v)$.*
2. *We have*

$$\mathrm{Cl}(\mathcal{X}^{\mathcal{P}}_v) = \bigoplus_{v \xrightarrow{P} v'} \mathbb{Z}[\mathcal{X}^P_{v'}].$$

Proof. Good references for assertions 1 and 2 are [Mat88, XII, Lemme 75] and [BK05, §3.2]. □

4.2. Inversion sets.

4.2.1. *Inversion sets of Weyl group elements.* As before, we use notation W, W^P, W_P, Δ, Φ and Φ^+. The inversion set $\Phi(w)$ of an element w of W is defined to be $\Phi(w)=\Phi^+\cap w^{-1}(\Phi^-)$ and it satisfies $\ell(w)=\sharp\Phi(w)$. One gets a map

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{S}(\Phi^+) \\ w & \longmapsto & \Phi(w) \end{array}$$

($\mathcal{S}(\Phi^+)$ denotes the power set of Φ^+) that is injective.

The strong and left weak Bruhat orders are characterized by

$$\begin{array}{ccc} v\leq w & \iff & \mathcal{X}_v^B\subset\mathcal{X}_w^B; \\ v\leq_{\mathrm{w}} w & \iff & \Phi(v)\subset\Phi(w). \end{array} \tag{23}$$

The second point is [BB05, Proposition 1.3.1]. The direct implication is a consequence of the easy fact that $w=s_\alpha v$ and $\ell(w)=\ell(v)+1$ imply that

$$\Phi(w)=\Phi(v)\cup\{v^{-1}(\alpha)\}. \tag{24}$$

A reduced expression $w=s_{\alpha_1}\dots s_{\alpha_\ell}$ corresponds to a sequence $e\leq_{\mathrm{w}} s_{\alpha_\ell}\leq_{\mathrm{w}} s_{\alpha_{\ell-1}}s_{\alpha_\ell}\leq_{\mathrm{w}}\cdots\leq_{\mathrm{w}} w$ of weak covering relations and hence to a sequence $\emptyset\subset\mathcal{I}_1\subset\mathcal{I}_2\subset\cdots\subset\Phi(w)$ of inversion sets of Φ^+ with cardinality increasing by one at each step. In particular, given such a reduced expression we get a numbering

$$\Phi(w)=\{\gamma_1,\dots,\gamma_\ell\} \text{ with } \gamma_i=s_{\alpha_\ell}\cdots s_{\alpha_{\ell-i+2}}(\alpha_{\ell-i+1}). \tag{25}$$

Note that $w^{-1}=s_{\alpha_\ell}\dots s_{\alpha_1}$ is also a reduced expression, thus

$$\Phi(w^{-1})=\{\beta_1,\dots,\beta_\ell\} \text{ with } \beta_i=s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i). \tag{26}$$

Lemma 18. *Let $v\in W$ and $v\xrightarrow{B}v'$. Fix a reduced expression $v=s_{\alpha_1}\dots s_{\alpha_\ell}$. Then there exists a unique k such that a reduced expression of v' if obtained from $\boldsymbol{v}$ by suppressing α_k.*

Proof. This is the content of [BB05, Lemma 1.3.1]. A geometric interpretation of this result can be obtained using the Bott-Samelson resolution. Indeed, if the statement would be false, one would get nontrivial finite fibers for the Bott-Samelson resolution. By Zariski's Main Theorem, this contradicts the normality of the Schubert varieties. □

A useful result on the inversion sets is:

Lemma 19. *See e.g.* [Kum02, Corollary 1.3.22.3]. *Let $w\in W$. We have*

$$\sum_{\beta\in\Phi(w^{-1})}\beta\ =\rho-w(\rho).$$

0 We also use the following variation, see [Per07, Lemma 4.18]. The roots β_i are those defined by (26) and the function $\hbar$ by (17).

Lemma 20. *Let $v\xrightarrow{B}v'=s_\beta v$. Let $v=s_{\alpha_1}\dots s_{\alpha_\ell}$ be a reduced expression of v and $1\leq i\leq\ell$ such that v' is obtained by suppressing s_{α_i}. Then*

(1) we have $\beta_i=\beta$ and $\Phi(v'^{-1})=\{\beta_1,\dots,\beta_{i-1},s_\beta(\beta_{i+1}),\dots,s_\beta(\beta_\ell)\}$;
(2) moreover,

$$\langle\sum_{k=i}^{\ell}\beta_k,\ \beta^\vee\rangle=\hbar(-v^{-1}\beta)+1.$$

Proof. One can easily check the first assertion. We provide an argument for the second assertion simpler than that in [Per07]. By Lemma 19, we have

$$\begin{aligned} v'(\rho)-v(\rho) &= \textstyle\sum_{\theta\in\Phi(v^{-1})}\theta-\sum_{\theta\in\Phi(v'^{-1})}\theta \\ &= \beta_i+\textstyle\sum_{k=i+1}^{\ell}(\beta_k-s_\beta(\beta_k)) \\ &= \beta+\textstyle\sum_{k=i+1}^{\ell}\langle\beta_k,\beta^\vee\rangle\beta. \end{aligned}$$

Set $S=\langle\sum_{k=i}^{\ell}\beta_k,\ \beta^\vee\rangle$. Since $\beta_i=\beta$, $S=2+\langle\sum_{k=i+1}^{\ell}\beta_k,\ \beta^\vee\rangle$. We get

$$\langle v'(\rho)-v(\rho),\beta^\vee\rangle=2(S-1).$$

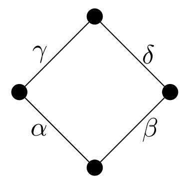
On the other hand,

$$\langle v'(\rho) - v(\rho), \beta^\vee \rangle = \langle s_\beta v(\rho) - v(\rho), \beta^\vee \rangle = -2\langle v(\rho), \beta^\vee \rangle,$$

since $s_\beta(\zeta) = \zeta - \langle \zeta, \beta^\vee \rangle \beta$ for any weight ζ. The lemma follows. □

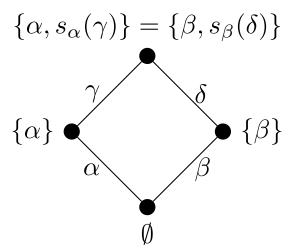
4.2.2. *Diamond lemmas.* We first state the following diamond lemma in the weak Bruhat graph.

Lemma 21. *Let $\alpha, \beta, \gamma, \delta \in \Delta$ such that*



is a subgraph of the weak Bruhat graph. Then $\alpha = \delta$, $\beta = \gamma$ and $s_\alpha s_\beta = s_\beta s_\alpha$.

Proof. Let v denote the Weyl group element of the bottom vertex. We have $s_\gamma s_\alpha v = s_\delta s_\beta v$ whose length is $\ell(v) + 2$. In particular $s_\gamma s_\alpha = s_\delta s_\beta$ has length 2: one may assume that v is trivial. Then, the four inversion sets of the vertices are



Since $\alpha \neq \beta$, we deduce that $\alpha = s_\beta(\delta) = \delta - \langle \delta, \beta^\vee \rangle \beta$. Hence α, β and δ are linearly dependant simple roots. Since $\alpha \neq \beta$ we deduce that $\alpha = \delta$. Now, the linear relation implies $\langle \delta, \beta^\vee \rangle = 0$, so s_δ and s_β commute. By symmetry this ends the proof. □

For $\alpha \in \Delta$, let $P_\alpha \subset G$ be the corresponding minimal standard parabolic subgroup. We prove a diamond lemma mixing strong and weak Bruhat orders:

Lemma 22. *Let $w \in W, \alpha \in \Delta, \gamma \in \Phi^+$ with $s_\alpha w \neq w s_\gamma$. Let P be a standard parabolic subgroup of G.*

(i) Any diagram
$$\begin{tikzcd}[column sep=small] ws_\gamma & & s_\alpha w \\ & w \arrow[ul, dashed, no head] \arrow[ur, no head] & \end{tikzcd}$$
may be completed as the diagram
$$\begin{tikzcd}[column sep=small] & s_\alpha w s_\gamma \arrow[dl, no head] \arrow[dr, dashed, no head] & \\ ws_\gamma & & s_\alpha w \\ & w \arrow[ul, dashed, no head] \arrow[ur, no head] & \end{tikzcd}.$$

(ii) Any diagram
$$\begin{tikzcd}[column sep=small] & w \arrow[dl, no head] \arrow[dr, dashed, no head] & \\ s_\alpha w & & ws_\gamma \end{tikzcd}$$
may be completed as the diagram
$$\begin{tikzcd}[column sep=small] & w \arrow[dl, no head] \arrow[dr, dashed, no head] & \\ s_\alpha w & & ws_\gamma \\ & s_\alpha w s_\gamma \arrow[ul, dashed, no head] \arrow[ur, no head] & \end{tikzcd}.$$

(iii) Given a diagram
$$\begin{tikzcd}[column sep=small] & s_\alpha w s_\gamma \arrow[dl, no head] \arrow[dr, dashed, no head] & \\ ws_\gamma & & s_\alpha w \\ & w \arrow[ul, dashed, no head] \arrow[ur, no head] & \end{tikzcd}$$
, if $ws_\gamma \in W^P$ and $s_\alpha w \in W^P$, then $w \in W^P$ and $s_\alpha w s_\gamma \in W^P$.

Proof. For (i), since $\mathcal{X}^B_{ws_\gamma}$ contains $\mathcal{X}^B_w$ and not $\mathcal{X}^B_{s_\alpha w}$, it cannot be P_α-stable. Hence $\ell(s_\alpha w s_\beta) = \ell(w) + 2$ and the first assertion follows.

In the second assertion, assume for a contradiction that $\ell(s_\alpha w s_\gamma) = \ell(ws_\gamma) + 1$. Since $\mathcal{X}^B_w$ is P_α-stable and contains $\mathcal{X}^B_{ws_\gamma}$, one must have $s_\alpha w s_\gamma = w$, a contradiction. Thus $\ell(s_\alpha w s_\gamma) = \ell(ws_\gamma) - 1$.

To prove (iii), let $s_{\alpha'} \in W_P$. Since $s_\alpha w \in W^P$, $\ell(s_\alpha w s_{\alpha'}) = \ell(w) + 2$, so $\ell(w s_{\alpha'}) = \ell(w) + 1$. Thus, $w \in W^P$. If $s_\alpha w s_\gamma \notin W^P$ then $\dim(\mathcal{X}^P_{s_\alpha w s_\gamma}) \le \dim(\mathcal{X}^P_{s_\alpha w}) = \dim(\mathcal{X}^P_{w s_\gamma})$. Then $\mathcal{X}^P_{s_\alpha w s_\gamma} = \mathcal{X}^P_{s_\alpha w} = \mathcal{X}^P_{w s_\gamma}$ and $s_\alpha w = w s_\gamma$, a contradiction. $\square$

4.3. **Relative Schubert varieties and their cohomology classes.** Fix $v \in W^P$. The usual Schubert varieties $\mathcal{X}^B_v = \overline{BvB/B} \subset G/B$ and $\mathcal{X}^P_v = \overline{BvP/P} \subset G/P$ are called *absolute* Schubert varieties. Consider the *relative* Schubert variety $\mathcal{Y}_v = G \times^B \overline{BvP/P}$. It is a closed subvariety of $G/B \times G/P$ and, more precisely, it is the G-orbit closure of $(B/B, vP/P)$ for the diagonal G-action.

The variety $\mathcal{Y}_v$ can be seen from different points of view:

(1) $\mathcal{Y}_v = G \times^B \overline{BvP/P}$ and $\mathcal{Y}_v = G \times^P \overline{Pv^{-1}B/B}$;
(2) $\mathcal{Y}_v = \overline{G.(B/B, vP/P)} = \overline{G.(v^{-1}B/B, P/P)} \subset G/B \times G/P$;
(3) $\mathcal{Y}_v = \{(gB/B, hP/P) \,:\, g^{-1}hP/P \in \overline{BvP/P}\} \subset G/B \times G/P$;
(4) $\mathcal{Y}_v = \{(gB/B, hP/P) \,:\, h^{-1}gB/B \in \overline{Pv^{-1}B/B}\} \subset G/B \times G/P$.

Recall that Schubert classes σ^w, τ_u were defined in Section 2.1.2. Think about $G/B \times G/P$ as a G^2-flag variety endowed with the action of G diagonally embedded in G^2. Then $\mathcal{Y}_v$ is an orbit closure of a spherical subgroup in a flag variety. Recalling Notation 1, by [Bri98, Theorem 1.5], we have in particular

Proposition 23. *The cohomology class of $\mathcal{Y}_v$ is given by:*

$$[\mathcal{Y}_v] = \sum_{u \,:\, uv \xrightarrow{P}_u v} \sigma^u \otimes \tau_{uv} \in H^*(G/B \times G/P, \mathbb{Z}) \ .$$

4.4. **Relative Bott-Samelson varieties.** Rational resolutions of Schubert varieties can be produced as Bott-Samelson varieties. We recall this construction in a relative context.

Definition 1. *Let $\boldsymbol{v} = (\alpha_1, \dots, \alpha_\ell)$ be a sequence of elements in Δ. We call (relative) Bott-Samelson variety the variety $\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}$ above G/B defined by*

$$\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} := \left(G \times P_{\alpha_1} \times \cdots \times P_{\alpha_\ell}\right) \Big/ B^{\ell+1} \,.$$

Here we consider the action of $B^{\ell+1}$ defined by

$$(g, p_1, \dots, p_\ell) \cdot (b_0, b_1, \dots, b_\ell) = (g{b_0}^{-1}, b_0 p_1 {b_1}^{-1}, b_1 p_2 {b_2}^{-1}, \dots, b_{\ell-1} p_\ell b_\ell^{-1}) \,.$$

Let $v = s_{\alpha_1} \cdots s_{\alpha_\ell}$ and assume that $v \in W^P$ and $\ell(v) = \ell$. As in [BK05, §2.2(6)], *we may define a morphism ω from $\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}$ to the relative Schubert variety $\mathcal{Y}_v$, by the following diagram:*

(27)
$$\begin{tikzcd}
& \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} := \left(G \times P_{\alpha_1} \times \cdots \times P_{\alpha_\ell}\right) \Big/ B^{\ell+1} \arrow[d, "\omega"] \arrow[ddl, bend right, "\boldsymbol{\pi}"'] & \\
& \mathcal{Y}_v := G \times^B \overline{BvP/P} \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & \\
G/B & & G/P
\end{tikzcd}$$

Forgetting the last term gives a chain of morphisms

$$\boldsymbol{\mathcal{Y}}_{(\alpha_1, \dots, \alpha_\ell)} \xrightarrow{\rho_\ell} \boldsymbol{\mathcal{Y}}_{(\alpha_1, \dots, \alpha_{\ell-1})} \xrightarrow{\rho_{\ell-1}} \cdots \longrightarrow \boldsymbol{\mathcal{Y}}_{(\alpha_1)} \xrightarrow{\rho_0} G/B \,,$$

which are known to be $\mathbb{P}^1$-bundles (see [BK05, p.66]). In particular, $\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}$ is smooth, and more precisely, we have:

Lemma 24. [BK05, Theorem 3.4.3] *The morphism ω is a rational resolution. Namely, $\omega_* \mathcal{O}_{\boldsymbol{\mathcal{Y}}} = \mathcal{O}_{\mathcal{Y}}$ and for $i > 0$, $R^i \omega_* \mathcal{O}_{\boldsymbol{\mathcal{Y}}} = 0$.*

Since $\mathcal{X}_v^{\mathcal{P}}$, and thus $\mathcal{Y}_v$, are smooth in codimension 1, their canonical classes are well-defined as Weil divisors, and since they have rational singularities, we may compute them using the following lemma:

Lemma 25. [BK05, Lemma 3.4.2] *Let $\omega : \boldsymbol{\mathcal{Y}} \longrightarrow \mathcal{Y}$ be a rational resolution. Then $\mathcal{Y}$ is Cohen-Macaulay with dualizing sheaf $\omega_* \mathcal{K}_{\boldsymbol{\mathcal{Y}}}$.*

Lemma 24 thus gives:

Lemma 26. *With the notation of Definition 1, we have*

$$\mathcal{K}_{\mathcal{Y}_v} = \omega_* \mathcal{K}_{\boldsymbol{\mathcal{Y}_v}} \text{ as sheaves and } K_{\mathcal{Y}_v} = \omega_* K_{\boldsymbol{\mathcal{Y}_v}} \text{ in } \mathrm{Cl}(\mathcal{Y}_v)\,.$$

Standard Weil divisors and line bundles are defined on those Bott-Samelson varieties as follows [BK05, p.67]:

Definition 2. *Let k be such that $1 \leq k \leq \ell$ and let ζ be a character of B. Then:*

- *$\boldsymbol{\mathcal{D}}_k$ is the quotient by $B^{\ell+1}$ of $G \times P_{\alpha_1} \times \cdots \times B \times \cdots \times P_{\alpha_\ell}$, where we replaced P_{α_k} by B. This is a divisor in $\boldsymbol{\mathcal{Y}_v}$, and there is an isomorphism $\sigma : \boldsymbol{\mathcal{Y}}_{(\alpha_1,\ldots,\widehat{\alpha_k},\ldots,\alpha_\ell)} \longrightarrow \boldsymbol{\mathcal{D}}_k \subset \boldsymbol{\mathcal{Y}_v}$.*
- *ζ and k define a character ζ of $B^{\ell+1}$ by $\zeta(b_0,\ldots,b_\ell) = \zeta(b_k)$. We define the line bundle $\boldsymbol{\mathcal{L}}_k(\zeta)$ by its total space $\left(G \times P_{\alpha_1} \times \cdots \times P_{\alpha_\ell} \times \mathbb{C}_{-\zeta}\right) \Big/ B^{\ell+1}$ and natural morphism to $\boldsymbol{\mathcal{Y}_v}$. Beware that as in* [BK05, §2.1(7)] *we use a minus sign here.*

In the following lemma, we use heavier notation than the notation in the previous definition: we denote by ${}^{\boldsymbol{v}}\boldsymbol{\mathcal{L}}_k(\zeta)$ the bundle on $\boldsymbol{\mathcal{Y}_v}$ denoted by $\boldsymbol{\mathcal{L}}_k(\zeta)$ above.

Given $\boldsymbol{v} = (\alpha_1,\ldots,\alpha_\ell)$ as above and $1 \leq k \leq \ell$, set $\boldsymbol{v}(\leq k) = (\alpha_1,\ldots,\alpha_k)$. The map $(g,p_1,\ldots,p_\ell) \mapsto (g,p_1,\ldots,p_k)$ induces a proper surjection

$$\rho_{\leq k} : \boldsymbol{\mathcal{Y}_v} \longrightarrow \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}(\leq k)}.$$

Note that $\rho_\ell = \rho_{\leq \ell-1}$.

Lemma 27. *Let $\boldsymbol{v}$ and k as above. Let $1 \leq i \leq k$ and $\zeta \in X(T)$. Then*

$$\rho_{\leq k}^*({}^{\boldsymbol{v}(\leq k)}\boldsymbol{\mathcal{L}}_k(\zeta)) = {}^{\boldsymbol{v}}\boldsymbol{\mathcal{L}}_k(\zeta) \qquad \text{and} \qquad \rho_{\leq k}^*(\mathcal{O}(\boldsymbol{\mathcal{D}}_i)) = \mathcal{O}(\boldsymbol{\mathcal{D}}_i).$$

This Lemma shows that there is no risk of confusion using the lighter notation $\boldsymbol{\mathcal{L}}_k(\zeta)$.

Lemma 28. *Let $\boldsymbol{v}$, k and ζ as above. Then, in $\mathrm{Pic}(\boldsymbol{\mathcal{Y}_v})$, we have*

$$\boldsymbol{\mathcal{L}}_k(\zeta) = \boldsymbol{\mathcal{L}}_{k-1}(s_{\alpha_k}(\zeta)) + \langle \zeta, \alpha_k^\vee \rangle \mathcal{O}(\boldsymbol{\mathcal{D}}_k).$$

Proof. Using Lemma 27, it is sufficient to prove the case $k = \ell$. This lemma is proved in [Dem74, Proposition 1] with some slightly different notation. Therefore we give here a quick argument. Let as above $\boldsymbol{v} = (\alpha_1,\ldots,\alpha_\ell)$ and $\boldsymbol{v}' = (\alpha_1,\ldots,\alpha_{\ell-1})$. A fiber of ρ_ℓ is $P_{\alpha_\ell}/B \simeq \mathbb{P}^1$ and the bundle $\boldsymbol{\mathcal{L}}_\ell(\zeta)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(\langle \zeta, \alpha_{\alpha_\ell}^\vee \rangle)$. Thus

$$\boldsymbol{\mathcal{L}}_\ell(\zeta) = \rho_\ell^* L \otimes \mathcal{O}(\langle \zeta, \alpha_\ell^\vee \rangle \boldsymbol{\mathcal{D}}_\ell) \tag{28}$$

for some line bundle L on $\boldsymbol{\mathcal{Y}_v}$ to be determined.

To determine L, we apply σ^* to equation (28), where $\sigma : \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}(\leq \ell-1)} \to \boldsymbol{\mathcal{D}}_\ell \subset \boldsymbol{\mathcal{Y}_v}$ is as in Definition 2. We have $\sigma^* \rho_\ell^* L = L$, and $\sigma^* \mathcal{O}(\boldsymbol{\mathcal{D}}_\ell) = \mathcal{O}_{\boldsymbol{\mathcal{Y}_v}}(\boldsymbol{\mathcal{D}}_\ell)_{|\boldsymbol{\mathcal{D}}_\ell} = \boldsymbol{\mathcal{L}}_{\ell-1}(\alpha_\ell)$ since the normal bundle of $\boldsymbol{\mathcal{D}}_\ell$ is $\boldsymbol{\mathcal{L}}_{\ell-1}(\alpha_\ell)$. Thus we get

$$L = \boldsymbol{\mathcal{L}}_{\ell-1}(\zeta - \langle \zeta, \alpha_\ell^\vee \rangle \alpha_\ell) = \boldsymbol{\mathcal{L}}_{\ell-1}(s_{\alpha_\ell}(\zeta))\,.$$

□

Since ω is proper, the pushforward $\omega_* : A^1(\boldsymbol{\mathcal{Y}_v}) \longrightarrow A^1(\mathcal{Y}_v)$ is well defined.

Lemma 29. *Let k be such that $1 \leq k \leq \ell$. Then, in $A^1(\mathcal{Y}_v)$, we have*

$$\omega_*[\boldsymbol{\mathcal{D}}_k] = \begin{cases} [\mathcal{Y}_{s_{\alpha_1}\cdots\widehat{s_{\alpha_k}}\cdots s_{\alpha_\ell}}] & \text{if } s_{\alpha_1}\cdots\widehat{s_{\alpha_k}}\cdots s_{\alpha_\ell} \text{ is reduced ;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from the fact that the restriction of ω to $\boldsymbol{\mathcal{D}}_k$ is birational onto $\mathcal{Y}_{s_{\alpha_1}\cdots\widehat{s_{\alpha_k}}\cdots s_{\alpha_\ell}}$ if the product $s_{\alpha_1}\cdots\widehat{s_{\alpha_k}}\cdots s_{\alpha_\ell}$ is reduced and contracts $\boldsymbol{\mathcal{D}}_k$ otherwise. □

4.5. **Canonical divisor of a relative Schubert variety.** We keep the notation of the previous subsection, in particular those of Definition 1 and Notation 1. We express the canonical classes of $\mathcal{Y}_v$ and $\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}$:

Proposition 30. *Assume that $\boldsymbol{v} = (\alpha_1, \ldots, \alpha_\ell)$ is a reduced expression of $v \in W^P$ and let $\gamma_1 = \alpha_\ell$, $\gamma_i = s_{\alpha_\ell} \cdots s_{\alpha_{\ell-i+2}}(\alpha_{\ell-i+1})$. We have:*

$$-K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} = \boldsymbol{\pi}^* \mathcal{L}_{G/B}\big(\rho + v(\rho)\big) + \sum_{i=1}^{\ell} (\hbar(\gamma_i) + 1)[\boldsymbol{\mathcal{D}}_i] \text{ in } \mathrm{Cl}(\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}) = \mathrm{Pic}(\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}).$$

Moreover, we have:

$$-K_{\mathcal{Y}_v} = \pi^* \mathcal{L}_{G/B}\big(\rho + v(\rho)\big) + \sum_{v'} (\hbar(\gamma) + 1)[\mathcal{Y}_{v'}] \text{ in } \mathrm{Cl}(\mathcal{Y}_v)\,,$$

where the sum runs over the covering relations $\begin{array}{c} v \\ \vdots\,\gamma \\ v' \end{array}$ *in the twisted Bruhat graph of W^P.*

Proof. An immediate induction using Lemma 28 shows that

$$\mathcal{L}_k(\alpha_k) = \mathcal{L}_{G/B}(s_{\alpha_1} \cdots s_{\alpha_k}(\alpha_k)) + \sum_{i=1}^{k} \langle s_{\alpha_{i+1}} \cdots s_{\alpha_k}(\alpha_k), \alpha_i^\vee \rangle \mathcal{O}(\boldsymbol{\mathcal{D}}_i).$$

Applying $s_{\alpha_1} \cdots s_{\alpha_i}$, we get the equality $\langle s_{\alpha_{i+1}} \cdots s_{\alpha_k}(\alpha_k), \alpha_i^\vee \rangle = \langle \beta_k, \beta_i^\vee \rangle$, since we have the relation $s_{\alpha_1} \cdots s_{\alpha_k}(\alpha_k) = -\beta_k$. Finally

$$\mathcal{L}_k(\alpha_k) = \mathcal{L}_{G/B}(-\beta_k) + \sum_{i=1}^{k} \langle \beta_k, \beta_i^\vee \rangle \mathcal{O}(\boldsymbol{\mathcal{D}}_i)\,. \tag{29}$$

The fibers of ρ_ℓ are isomorphic to $\mathbb{P}^1$ and thus the relative tangent bundle of ρ_ℓ is $\mathcal{L}_\ell(\alpha_\ell)$. Therefore, the sequence $0 \to T\rho_\ell \to T\mathcal{Y}_{\boldsymbol{v}} \to \rho_\ell^*(T\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}(\ell)}) \to 0$ reads:

$$0 \longrightarrow \mathcal{L}_\ell(\alpha_\ell) \longrightarrow T\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \longrightarrow \rho_\ell^*(T\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}(\ell)}) \longrightarrow 0\,. \tag{30}$$

Taking account that $\boldsymbol{\mathcal{Y}}_\emptyset = G/B$ and hence $K_{\boldsymbol{\mathcal{Y}}_\emptyset} = \mathcal{L}_{G/B}(-2\rho)$, an immediate induction gives

$$K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} = \mathcal{L}_\ell(-\alpha_\ell) + \rho_\ell^*(K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}(\ell)}}) = \mathcal{L}_{G/B}(-2\rho) + \sum_{k=1}^{\ell} \mathcal{L}_k(-\alpha_k).$$

Injecting (29), we get

$$\begin{aligned} K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} &= \mathcal{L}_{G/B}(-2\rho + \textstyle\sum_{k=1}^{\ell} \beta_k) + \sum_{k=1}^{\ell} \sum_{i=1}^{k} \langle \beta_k, \beta_i^\vee \rangle \mathcal{O}(-\boldsymbol{\mathcal{D}}_i) \\ &= \mathcal{L}_{G/B}(-2\rho + \textstyle\sum_{k=1}^{\ell} \beta_k) + \sum_{i=1}^{\ell} \langle \sum_{k=i}^{\ell} \beta_k, \beta_i^\vee \rangle\, \mathcal{O}(-\boldsymbol{\mathcal{D}}_i)\,. \end{aligned} \tag{31}$$

We already observed that $\{\beta_1, \ldots, \beta_\ell\} = \Phi(v^{-1})$. Then Lemma 19 shows that $\sum_{k=1}^{\ell} \beta_k = \rho - v(\rho)$. Moreover, Lemma 20 shows that $\langle \sum_{k=i}^{\ell} \beta_k, \beta_i^\vee \rangle = \hbar(\gamma_i) + 1$, and we get the given formula for $K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}}$.

The formula for $K_{\mathcal{Y}_v}$ then follows by applying ω_*, by Lemma 26. Since $\mathcal{L}_{G/B}(\zeta)$ is defined by a pullback, $\omega_*(\mathcal{L}_{G/B}(\zeta)) = \pi^*(\mathcal{L}_{G/B}(\zeta))$. Lemma 29 allows to handle the terms $\mathcal{O}(\boldsymbol{\mathcal{D}}_i)$. We can then conclude using Lemma 18 that realizes a bijection

$$\{v' \in W^P \,:\, v \xrightarrow{P} v'\} \;\longleftrightarrow\; \{1 \le i \le \ell : \omega \text{ does not contract } \boldsymbol{\mathcal{D}}_i\}.$$

The given formula for $K_{\mathcal{Y}_v}$ follows. $\square$

5. Proof of the main theorems

5.1. **Chow class of $Y(\mathbbm{v})$ for any $\mathbbm{v}$.** In this subsection $\mathbbm{v} = (v, \hat{v})$ denotes any element of $W^{\mathbb{P}}$ and we consider $Y(\mathbbm{v})$ defined by (15). Observe that the projection $\mathcal{Y}_{\hat{v}} \longrightarrow \hat{G}/\hat{P}$ is a locally trivial fibration with fiber $\overline{\hat{P}\hat{v}^{-1}\hat{B}/\hat{B}}$. Recall that G/P is embedded in $\hat{G}/\hat{P}$ and denote by $\mathcal{Y}^G_{\hat{v}}$ the preimage of G/P in $\mathcal{Y}_{\hat{v}}$. Then $\mathcal{Y}^G_{\hat{v}} \simeq G \times^P \overline{\hat{P}\hat{v}^{-1}\hat{B}/\hat{B}}$.

One can obtain $Y(\mathbbm{v})$ as a transverse intersection:

Lemma 31. *Consider $\hat{G}/\hat{B} \times \mathcal{Y}_v$ and $G/B \times \mathcal{Y}_{\hat{v}}$ as subvarieties of $G/B \times \hat{G}/\hat{B} \times \hat{G}/\hat{P}$. Then $Y(\mathbbm{v})$ is the transverse intersection*

$$Y(\mathbbm{v}) = (\hat{G}/\hat{B} \times \mathcal{Y}_v) \cap (G/B \times \mathcal{Y}_{\hat{v}}).$$

Similarly, $Y(\mathbbm{v})$ is also the transverse intersection $(\hat{G}/\hat{B} \times \mathcal{Y}_v) \cap (G/B \times \mathcal{Y}^G_{\hat{v}})$ in $G/B \times \hat{G}/\hat{B} \times G/P$.

Proof. It is plain that $Y(\mathbbm{v})$ is equal to the two intersections of the lemma; we prove that these intersections are transverse. Let $x \in G/B, \hat{x} \in \hat{G}/\hat{B}$ and $\hat{z} \in \hat{G}/\hat{P}$ such that $(x, \hat{x}, \hat{z}) \in (\hat{G}/\hat{B} \times \mathcal{Y}_v) \cap (G/B \times \mathcal{Y}_{\hat{v}})$. Let $\Big(T_{(x,\hat{x},\hat{z})}(\mathcal{Y}_v \times \hat{G}/\hat{B})\Big)^{\perp}$ denote the (conormal) space of linear forms on $T_{(x,\hat{x},\hat{z})}(G/B \times \hat{G}/\hat{B} \times \hat{G}/\hat{P})$ which vanish on the tangent space $T_{(x,\hat{x},\hat{z})}(\mathcal{Y}_v \times \hat{G}/\hat{B})$. We have

$$\begin{array}{rcl}\Big(T_{(x,\hat{x},\hat{z})}(\hat{G}/\hat{B} \times \mathcal{Y}_v)\Big)^{\perp} & \subset & T^*_x G/B \times \{0\} \times T^*_{\hat{z}}\hat{G}/\hat{P} \text{ , and} \\ \Big(T_{(x,\hat{x},\hat{z})}(G/B \times \mathcal{Y}_{\hat{v}})\Big)^{\perp} & \subset & \{0\} \times T^*_{\hat{x}}\hat{G}/\hat{B} \times T^*_{\hat{z}}\hat{G}/\hat{P} \text{ .}\end{array}$$

It follows that the intersection $\Big(T_{(x,\hat{x},\hat{z})}(\mathcal{Y}_v \times \hat{G}/\hat{B})\Big)^{\perp} \cap \Big(T_{(x,\hat{x},\hat{z})}(\mathcal{Y}_{\hat{v}} \times G/B)\Big)^{\perp}$ of these conormal spaces is included in $\{0\} \times \{0\} \times T^*_{\hat{z}}\hat{G}/\hat{P}$, and thus reduced to $\{0\}$ since the projection $G/B \times \mathcal{Y}_{\hat{v}} \to \hat{G}/\hat{P}$ is surjective. This means that the intersection is transverse. ☐

We now compute $[Y(\mathbbm{v})]$ in $A^*(\mathbb{G}/\mathbb{B} \times G/P)$ in the Schubert basis.

Proposition 32. *With the above notation, in $A^*(\mathbb{G}/\mathbb{B} \times G/P)$, we have*

$$[Y(\mathbbm{v})] = \sum_{\mathbbm{u}\mathbbm{v} \xrightarrow{\mathbb{P}}_{\mathbbm{u}} \mathbbm{v}} \sigma^{\mathbbm{u}} \otimes \delta^*(\tau_{\mathbbm{u}\mathbbm{v}}) . \tag{32}$$

Proof. By Lemma 31 and formula (19), we have

$$[Y(\mathbbm{v})] = [\hat{G}/\hat{B} \times \mathcal{Y}_v] \cdot [G/B \times \mathcal{Y}^G_{\hat{v}}] \in A^*(\mathbb{G}/\mathbb{B} \times G/P). \tag{33}$$

Proposition 23 gives a formula for $[\mathcal{Y}_v] \in H^*(G/B \times G/P)$ which we pullback in $A^*(\mathbb{G}/\mathbb{B} \times G/P)$:

$$[\mathcal{Y}_v \times \hat{G}/\hat{B}] = \sum_{uv \xrightarrow{P}_u v} \sigma^u \otimes 1 \otimes \tau_{uv} \text{ .} \tag{34}$$

Consider now the class $[\mathcal{Y}^G_{\hat{v}}] \in A^*(\hat{G}/\hat{B} \times G/P)$ and the regular embedding $i : \hat{G}/\hat{B} \times G/P \longrightarrow \hat{G}/\hat{B} \times \hat{G}/\hat{P}$. Let i^* denote the associated Gysin map. Since $\mathcal{Y}^G_{\hat{v}}$ is the transverse intersection of $\mathcal{Y}_{\hat{v}}$ and $\hat{G}/\hat{B} \times G/P$ in $\hat{G}/\hat{B} \times \hat{G}/\hat{P}$, we have

$$[\mathcal{Y}^G_{\hat{v}}] = i^*[\mathcal{Y}_{\hat{v}}] \in A^*(\hat{G}/\hat{B} \times G/P). \tag{35}$$

But $[\mathcal{Y}_{\hat{v}}]$ has an expression given by Proposition 23 as a linear combinaison of terms $\sigma^{\hat{u}} \otimes \tau_{\hat{u}\hat{v}}$. Using the relation $i^*(\sigma^{\hat{u}} \otimes \tau_{\hat{u}\hat{v}}) = \sigma^{\hat{u}} \otimes \iota^*(\tau_{\hat{u}\hat{v}})$, we get in $A^*(\hat{G}/\hat{B} \times G/P)$

$$[\mathcal{Y}^G_{\hat{v}}] = \sum_{\hat{u}\hat{v} \xrightarrow{\hat{P}}_{\hat{u}} \hat{v}} \sigma^{\hat{u}} \otimes \iota^*(\tau_{\hat{u}\hat{v}}) , \tag{36}$$

and hence in $A^*(\mathbb{G}/\mathbb{B} \times G/P)$

$$[\mathcal{Y}^G_{\hat{v}} \times G/B] = \sum_{\hat{u}\hat{v} \xrightarrow{\hat{P}}_{\hat{u}} \hat{v}} 1 \otimes \sigma^{\hat{u}} \otimes \iota^*(\tau_{\hat{u}\hat{v}}) , \tag{37}$$

Multiplying (34) with 37 and using (12), one gets the formula of the proposition. ☐

We now compute its pushforward by Π:

Corollary 33. *Assume that $\ell(\mathbb{v}) = \dim(\hat{G}/\hat{P}) - 1$ and consider the projection $\tilde{\Pi} : \mathbb{G}/\mathbb{B} \times G/P \longrightarrow \mathbb{G}/\mathbb{B}$. Then, in $A^1(\mathbb{G}/\mathbb{B})$, we have*

$$\tilde{\Pi}_*([Y(\mathbb{v})]) = \sum_{\mathbb{\alpha} \in \mathbb{\Delta}\,:\, s_{\mathbb{\alpha}}\mathbb{v} \xrightarrow{\mathbb{P}} \mathbb{v}} c(s_{\mathbb{\alpha}}\mathbb{v})\sigma^{s_{\mathbb{\alpha}}}.$$

Proof. In general, for a class $\sum a_{\mathbb{u},v}\sigma^{\mathbb{u}} \otimes \tau_v$ in $H^*(\mathbb{G}/\mathbb{B} \times G/P)$, we have

$$\tilde{\Pi}_*\left(\sum_{\mathbb{u},v} a_{\mathbb{u},v}\, \sigma^{\mathbb{u}} \otimes \tau_v\right) = \sum_{\mathbb{u}} a_{\mathbb{u},e}\, \sigma^{\mathbb{u}}. \tag{38}$$

But in (32), $\ell(\mathbb{u}\mathbb{v}) = \dim \hat{G}/\hat{P} - 1 + \ell(\mathbb{u})$ with the assumption of the corollary. So, the condition that $\tau_{\mathbb{u}\mathbb{v}} \in A^*(\mathbb{G}/\mathbb{P})$ have degree $\dim(G/P)$ is equivalent to $\ell(\mathbb{u}) = 1$. Hence the terms in the sum (32) that do not vanish after applying $\tilde{\Pi}_*$ are indexed the simple roots $\mathbb{\alpha} \in \mathbb{\Delta}$ such that $s_{\mathbb{\alpha}}\mathbb{v} \xrightarrow{\mathbb{P}} \mathbb{v}$. The corresponding term is $\sigma^{s_{\mathbb{\alpha}}} \otimes \delta^*(\tau_{s_{\mathbb{\alpha}}\mathbb{v}}) = c(s_{\mathbb{\alpha}}\mathbb{v})\sigma^{s_{\mathbb{\alpha}}} \otimes [pt]$. This implies the given formula. ☐

5.2. **Canonical class of $Y(\mathbb{v})$ when $c(\mathbb{v}) \neq 0$.** We come back to our setting with $c(\mathbb{v}) \neq 0$. In the previous subsection, we computed the canonical class of a relative Schubert variety, and we now express $Y(\mathbb{v})$ as a fibered product of two relative Schubert varieties and explain how to deduce the canonical class of Π.

Fix reduced expressions $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$ of v and $\hat{v}$ respectively, and let us use Definition 1 for v and $\hat{v}$. We define $\boldsymbol{\mathcal{Y}}_{\hat{\boldsymbol{v}}}^G$ from $\boldsymbol{\mathcal{Y}}_{\hat{\boldsymbol{v}}}$ in a way similar to what we did for $\mathcal{Y}_{\hat{v}}^G$ from $\mathcal{Y}_{\hat{v}}$. We have morphisms:

$$\begin{tikzcd}
 & \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \arrow[dl, bend right, "\boldsymbol{\pi}"'] \arrow[d, "\omega"] \arrow[dr, bend left, "\boldsymbol{\gamma}"] & & \boldsymbol{\mathcal{Y}}_{\hat{\boldsymbol{v}}}^G \arrow[dl, bend right, "\hat{\boldsymbol{\gamma}}"'] \arrow[d, "\hat{\omega}"'] \arrow[dr, bend left, "\hat{\boldsymbol{\pi}}"] & \\
 & \mathcal{Y}_v \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & & \mathcal{Y}_{\hat{v}}^G \arrow[dl, "\hat{\gamma}"'] \arrow[dr, "\hat{\pi}"] & \\
G/B & & G/P & & \hat{G}/\hat{B}
\end{tikzcd}$$

Observe that $Y(\mathbb{v})$ as in (59) is nothing else than $\mathcal{Y}_v \times_{G/P} \mathcal{Y}_{\hat{v}}^G$ and that Π is the restriction of $\tilde{\Pi} = (\pi, \hat{\pi})$ to $Y(\mathbb{v})$. We also let $\boldsymbol{\mathbb{v}} = (\boldsymbol{v}, \hat{\boldsymbol{v}})$ and $\boldsymbol{Y}(\boldsymbol{\mathbb{v}}) = \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \times_{G/P} \boldsymbol{\mathcal{Y}}_{\hat{\boldsymbol{v}}}^G$. We have the following natural morphisms:

$$\begin{tikzcd}
 & \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \arrow[d, "\omega"'] & \boldsymbol{Y}(\boldsymbol{\mathbb{v}}) \arrow[l, "\boldsymbol{\epsilon}"'] \arrow[r, "\hat{\boldsymbol{\epsilon}}"] \arrow[d, "\Omega"] & \boldsymbol{\mathcal{Y}}_{\hat{\boldsymbol{v}}}^G \arrow[d, "\hat{\omega}"] & \\
G/B \times G/P & \mathcal{Y}_v \arrow[l, hook'] & Y(\mathbb{v}) \arrow[l, "\epsilon"'] \arrow[r, "\hat{\epsilon}"] \arrow[dl, "\Gamma"'] \arrow[dr, "\Pi"] & \mathcal{Y}_{\hat{v}}^G \arrow[r, hook] & \hat{G}/\hat{B} \times G/P \\
 & G/P & & G/B \times \hat{G}/\hat{B} &
\end{tikzcd} \tag{39}$$

The first properties of the morphisms appearing in this diagram we need are:

Lemma 34. *With the above notation,*

(1) ϵ is a locally trivial fibration with fiber $\overline{\hat{P}\hat{v}^{-1}\hat{B}/\hat{B}} = \mathcal{X}^{\hat{B}}_{w_{0,\hat{P}}\hat{v}}$.

(2) $\hat{\epsilon}$ is a locally trivial fibration with fiber $\overline{Pv^{-1}B/B} = \mathcal{X}^B_{w_{0,P}v}$.
(3) $\boldsymbol{\epsilon}$ and $\hat{\boldsymbol{\epsilon}}$ are locally trivial fibrations.

In particular, these four morphisms are flat and induce pullbacks between Chow rings.

Proof. We already observed when we defined $\mathcal{Y}^G_{\hat{v}}$ at the beginning of Section 5.1 that $\mathcal{Y}^G_{\hat{v}}$ is a locally trivial fibration over G/P with fiber $\overline{\hat{P}\hat{v}^{-1}\hat{B}/\hat{B}}$. Since $Y(\mathbb{v}) = \mathcal{Y}_v \times_{G/P} \mathcal{Y}^G_{\hat{v}}$, the first assertion follows. The second one works similarly.

By the Bruhat decomposition, the morphism $G \longrightarrow G/P$ is locally trivial in Zariski topology. Hence the maps $\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}} \longrightarrow G/P$ and $\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \longrightarrow G/P$ are locally trivial fibrations. The last assertion follows. $\square$

We can now express the canonical bundle of $Y(\mathbb{v})$ in terms of the canonical bundles of $\mathcal{Y}_v$ and $\mathcal{Y}_{\hat{v}}$:

Proposition 35. *We have $K_{Y(\mathbb{v})} = \epsilon^* K_{\mathcal{Y}_v} + \hat{\epsilon}^* K_{\mathcal{Y}^G_{\hat{v}}} - \Gamma^* K_{G/P}$ in $\mathrm{Cl}(Y(\mathbb{v}))$.*

Proof. Let us first prove this result at the level of Bott-Samelson varieties. We have by definition of $\boldsymbol{Y}(\mathbf{v})$ a fibered square of smooth varieties:

$$\begin{tikzcd} \boldsymbol{Y}(\mathbf{v}) \arrow[r, "\boldsymbol{\epsilon}\times\hat{\boldsymbol{\epsilon}}"] \arrow[d, "\Gamma\circ\Omega"'] \arrow[dr, phantom, "\square"] & \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \times \boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}} \arrow[d, "\boldsymbol{\gamma}\times\hat{\boldsymbol{\gamma}}"] \\ G/P \arrow[r, "\Delta"] & G/P \times G/P \end{tikzcd}$$

Let $\mathbf{y} = (\boldsymbol{y}, \hat{\boldsymbol{y}}) \in \boldsymbol{Y}(\mathbf{v})$ and let $u = \Gamma \circ \Omega(\mathbf{y}) \in G/P$. We thus have a fibered square of tangent spaces:

$$\begin{tikzcd} T_{\mathbf{y}}\boldsymbol{Y}(\mathbf{v}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T_{\boldsymbol{y}}\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \oplus T_{\hat{\boldsymbol{y}}}\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}} \arrow[d] \\ T_uG/P \arrow[r, "d\Delta"] & T_uG/P \oplus T_uG/P \end{tikzcd}$$

The bottom horizontal map being injective, so is the top horizontal one. The right vertical map being surjective, $(T_{\boldsymbol{y}}\boldsymbol{\mathcal{Y}} \oplus T_{\hat{\boldsymbol{y}}}\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}})/T_{\mathbf{y}}\boldsymbol{Y}(\mathbf{v})$ identifies with $(T_uG/P \oplus T_uG/P)/T_uG/P$. Moreover, on $\boldsymbol{Y}(\mathbf{v})$, the bundle $(TG/P \oplus TG/P)/TG/P$ is isomorphic to $\Omega^*\Gamma^*TG/P$. We get the short exact sequence on $\boldsymbol{Y}(\mathbf{v})$

$$0 \longrightarrow T_{\boldsymbol{Y}(\mathbf{v})} \longrightarrow T_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} \oplus T_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}} \longrightarrow \Omega^*\Gamma^*T_{G/P} \longrightarrow 0\,.$$

From this we deduce that

$$\begin{aligned} K_{\boldsymbol{Y}(\mathbf{v})} &= \big(K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} + K_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}}\big)_{|\boldsymbol{Y}(\mathbf{v})} - \Omega^*\Gamma^*K_{G/P} \\ &= \boldsymbol{\epsilon}^*K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} + \hat{\boldsymbol{\epsilon}}^*K_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}} - \Omega^*\Gamma^*K_{G/P}\ . \end{aligned} \tag{40}$$

We now apply Ω_*. Lemma 24 yields $\Omega_*\mathcal{O}_{\boldsymbol{Y}(\mathbf{v})} = \mathcal{O}_{Y(\mathbb{v})}$ and $\Omega_*K_{\boldsymbol{Y}(\mathbf{v})} = K_{Y(\mathbb{v})}$. From the first point and projection formula, we deduce in $\mathrm{Cl}(Y(\mathbb{v}))$

$$\Omega_*\,\Omega^*\,\Gamma^*\,K_{G/P} \;=\; \Gamma^*\,K_{G/P}\,. \tag{41}$$

Moreover, we have a fibered product $\begin{tikzcd} \boldsymbol{Y}(\mathbf{v}) \arrow[r, "(\boldsymbol{\epsilon},\hat{\boldsymbol{\epsilon}})"] \arrow[d, "\Omega"'] \arrow[dr, phantom, "\square"] & \boldsymbol{\mathcal{Y}}_{\boldsymbol{v}} \times \boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}} \arrow[d, "\omega"] \\ Y(\mathbb{v}) \arrow[r, "(\epsilon,\hat{\epsilon})"'] & \mathcal{Y}_v \times \mathcal{Y}^G_{\hat{v}} \end{tikzcd}$. By the flat base change $(\epsilon, \hat{\epsilon})$ and [Har77, Proposition III.9.3], we deduce that

$$\Omega_*(\boldsymbol{\epsilon}^*, \hat{\boldsymbol{\epsilon}}^*)(\mathcal{K}_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} \otimes \mathcal{K}_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}}) = (\epsilon^*, \hat{\epsilon}^*)\omega_*(\mathcal{K}_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} \otimes \mathcal{K}_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}})\,. \tag{42}$$

But, by [KM98, Theorem 5.10], $\omega_*\mathcal{K}_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} = \mathcal{K}_{\mathcal{Y}_v}$ and $\omega_*\mathcal{K}_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}} = \mathcal{K}_{\mathcal{Y}^G_{\hat{v}}}$. We deduce that

$$\Omega_*(\boldsymbol{\epsilon}^*K_{\boldsymbol{\mathcal{Y}}_{\boldsymbol{v}}} + \hat{\boldsymbol{\epsilon}}^*K_{\boldsymbol{\mathcal{Y}}^G_{\hat{\boldsymbol{v}}}}) = \epsilon^*K_{\mathcal{Y}_v} + \hat{\epsilon}^*K_{\mathcal{Y}^G_{\hat{v}}}\,. \tag{43}$$

By the equalities (40), (41), (42), and (43), we get the result of the proposition. $\square$

For later use, we also compute $\Pi_* \circ \Gamma^*$ (recall from (10) that χ is the coefficient of the point class):

Lemma 36. *The map* $\Pi_* \circ \Gamma^* : A^*(G/P) \longrightarrow A^*(\mathbb{G}/\mathbb{B})$ *is determined by*

$$\Pi_* \circ \Gamma^* \xi = \sum_{\mathbb{u}\mathbb{v} \xrightarrow{\mathbb{P}} \mathbb{u} \mathbb{v}} \chi_{G/P}(\xi \cdot \delta^*(\tau_{\mathbb{u}\mathbb{v}}))\sigma^{\mathbb{u}} \quad \forall \xi \in A^*(G/P).$$

If $\xi \in A^d(G/P)$*, then all the non-vanishing terms in this summand satisfy* $\ell(\mathbb{u}) = d$*.*

Proof. Let $\xi \in A^*(G/P)$ and consider the following commutative diagram:

(44)
$$\begin{tikzcd} Y(\mathbb{v}) \arrow[r, "i"] \arrow[dr, "\Gamma"'] & \mathbb{G}/\mathbb{B} \times G/P \arrow[d, "p"] \\ & G/P. \end{tikzcd}$$

By the projection formula, we have

$$i_* \Gamma^* \xi = i_* i^* p^* \xi = p^* \xi \cup i_*[Y(\mathbb{v})]\,.$$

Proposition 32 gives $i_*[Y(\mathbb{v})]$, and since $p^*\xi = 1 \otimes \xi$, one gets

$$i_* \Gamma^* \xi = \sum_{\mathbb{u}\mathbb{v} \xrightarrow{\mathbb{P}} \mathbb{u} \mathbb{v}} \sigma^{\mathbb{u}} \otimes (\xi \cdot \delta^*(\tau_{\mathbb{u}\mathbb{v}}))\,. \tag{45}$$

Since $\Pi_* \Gamma^* \xi = \tilde{\Pi}_* i_* \Gamma^* \xi$, we deduce the formula of the proposition.

Assume now that ξ is homogeneous of degree d. As in the proof of Corollary 33, the terms to keep when applying $\tilde{\Pi}_*$ to (45) satisfy $\ell(\mathbb{u}) = d$. ☐

5.3. **Computation of the class of the branch divisor.** We compute $[B_\Pi] = \Pi_* K_\Pi$ using the already done computation of $K_{Y(\mathbb{v})}$.

Proof of Theorem 8: recall that $Y(\mathbb{v}) \subset \mathbb{G}/\mathbb{B} \times G/P$ and that Π resp. Γ is the projection of $Y(\mathbb{v})$ on $\mathbb{G}/\mathbb{B}$ resp. G/P. Consider also as before

$$\tilde{\Pi} : \mathbb{G}/\mathbb{B} \times G/P \longrightarrow \mathbb{G}/\mathbb{B}\,,$$

the restriction to $Y(\mathbb{v})$ of which is Π.

By the definition of K_Π in Section 3.3 and Proposition 35, we have

$$K_\Pi = \epsilon^* K_{\mathcal{Y}_v} + \hat{\epsilon}^* K_{\mathcal{Y}^G_{\hat{v}}} - \Gamma^* K_{G/P} - \Pi^*(K_{G/B} + K_{\hat{G}/\hat{B}}). \tag{46}$$

Consider first $K_{\mathcal{Y}^G_{\hat{v}}}$ in $A^1(\mathcal{Y}^G_{\hat{v}})$. Let $j : \mathcal{Y}^G_{\hat{v}} \longrightarrow \mathcal{Y}_{\hat{v}}$ denote the inclusion. As the pullback of the inclusion $\iota : G/P \longrightarrow \hat{G}/\hat{P}$, it is a regular embedding and we can consider the associated Gysin map j^*. At any point y of $\mathcal{Y}^G_{\hat{v}}$ with projection $\bar{y} \in G/P$ we have the exact sequence:

$$0 \longrightarrow T_y \mathcal{Y}^G_{\hat{v}} \longrightarrow T_y \mathcal{Y}_{\hat{v}} \longrightarrow T_{\bar{y}} \hat{G}/\hat{P} / T_{\bar{y}} G/P \longrightarrow 0.$$

We deduce that in $\mathrm{Cl}(\mathcal{Y}^G_{\hat{v}})$, we have

$$K_{\mathcal{Y}^G_{\hat{v}}} = j^* K_{\mathcal{Y}_{\hat{v}}} - \hat{\gamma}^*((K_{\hat{G}/\hat{P}})_{|G/P}) + \hat{\gamma}^*(K_{G/P}). \tag{47}$$

Using the formulas $-K_{\hat{G}/\hat{P}} = \mathcal{L}_{\hat{G}/\hat{P}}(2\rho^{\hat{L}})$, $-K_{\mathbb{G}/\mathbb{B}} = \mathcal{L}_{\mathbb{G}/\mathbb{B}}(2\rho_{\mathbb{G}})$, and Proposition 30, injecting (47) in (46), one gets

$$[K_\Pi] = A + B - C - D \;\text{ with}$$

$$\left\{ \begin{array}{l} A = \Pi^* c_1(\mathcal{L}_{\mathbb{G}/\mathbb{B}}(\rho_{\mathbb{G}} - \mathbb{v}(\rho_{\mathbb{G}}))) \\ B = \Gamma^* c_1(\mathcal{L}_{G/P}(2\rho^{\hat{L}}_{|T})) \\ C = \sum_\gamma (\hbar(\gamma) + 1)\epsilon^*([\mathcal{Y}_{v'}]) \\ D = \sum_{\hat{\gamma}} (\hbar(\hat{\gamma}) + 1)\hat{\epsilon}^* \circ j^*([\mathcal{Y}_{\hat{v}'}])\,, \end{array} \right. \tag{48}$$

where the sums C and D run respectively over the pictures $\begin{array}{c} v \\ \vdots\,{\scriptstyle\gamma} \\ v' \end{array}$ and $\begin{array}{c} \hat{v} \\ \vdots\,{\scriptstyle\hat{\gamma}} \\ \hat{v}' \end{array}$ in the twisted Bruhat graphs of W^P and $W^{\hat{P}}$ respectively.

We now pushforward by Π each summand of (48). First of all, Lemma 36 gives

$$\Pi_* B = \sum_{s_{\mathbb{\alpha}}\mathbb{v} \xrightarrow{\mathbb{P}} \mathbb{v}} \Big(c_1(\mathcal{L}_{G/P}(2\rho^{\hat{L}}_{|T})) \cdot \delta^*(\tau_{s_{\mathbb{\alpha}}\mathbb{v}}) \Big) \sigma^{s_{\mathbb{\alpha}}} ,$$

which is the first term of the formula of Theorem 8.

For $\mathbb{\lambda} \in X(\mathbb{T})$, the projection formula in cohomology gives $\Pi_* \circ \Pi^* c_1(\mathcal{L}_{\mathbb{G}/\mathbb{B}}(\mathbb{\lambda})) = \deg(\Pi) c_1(\mathcal{L}_{\mathbb{G}/\mathbb{B}}(\mathbb{\lambda}))$, and the Chevalley formula gives $c_1(\mathcal{L}_{\mathbb{G}/\mathbb{B}}(\mathbb{\lambda})) = \sum_{\mathbb{\alpha} \in \mathbb{\Delta}} \langle \mathbb{\lambda}, \mathbb{\alpha}^\vee \rangle \sigma^{s_{\mathbb{\alpha}}}$. Thus

$$\Pi_* A = c(\mathbb{v}) \sum_{\mathbb{\alpha} \in \mathbb{\Delta}} \langle \rho_{\mathbb{G}} - \mathbb{v}(\rho_{\mathbb{G}}), \mathbb{\alpha}^\vee \rangle \sigma^{s_{\mathbb{\alpha}}} . \tag{49}$$

Consider first a term $\mathbb{\alpha} \in \mathbb{\Delta}$ such that $s_{\mathbb{\alpha}}\mathbb{v} \in \mathbb{v}W_{\mathbb{P}}$. Then $s_{\mathbb{\alpha}}\mathbb{v} = \mathbb{v}s_{\mathbb{\gamma}}$ with $\mathbb{\gamma} = \mathbb{v}^{-1}(\mathbb{\alpha}) \in \Phi(\mathbb{L})$. Since $\mathbb{v} \in W^{\mathbb{P}}$, $\ell(\mathbb{v}) + 1 \geq \ell(s_{\mathbb{\alpha}}\mathbb{v}) = \ell(\mathbb{v}s_{\mathbb{\gamma}}) = \ell(\mathbb{v}) + \ell(s_{\mathbb{\gamma}})$. Then $\mathbb{\gamma}$ is a simple root of $\mathbb{L}$. In particular, $\langle \mathbb{v}(\rho_{\mathbb{G}}), \mathbb{\alpha}^\vee \rangle = 1$ and the corresponding term in the sum (49) vanishes.

Assume now that $\ell(s_{\mathbb{\alpha}}\mathbb{v}) = \ell(\mathbb{v}) + 1$ and $s_{\mathbb{\alpha}}\mathbb{v} \in W^{\mathbb{P}}$. Then $\langle \mathbb{v}(\rho_{\mathbb{G}}) - \rho_{\mathbb{G}}, \mathbb{\alpha}^\vee \rangle = \mathbb{h}(\mathbb{\gamma}) - 1$ for $\mathbb{\gamma} = \mathbb{v}^{-1}(\mathbb{\alpha})$. Assume finally that $\ell(s_{\mathbb{\alpha}}\mathbb{v}) = \ell(\mathbb{v}) - 1$ and $s_{\mathbb{\alpha}}\mathbb{v} \in W^{\mathbb{P}}$. Then, similarly, we have $\langle \rho_{\mathbb{G}} - \mathbb{v}(\rho_{\mathbb{G}}), \mathbb{\alpha}^\vee \rangle = \mathbb{h}(\mathbb{\gamma}) + 1$. for $\mathbb{\gamma} = -\mathbb{v}^{-1}\mathbb{\alpha}$. Summing up, we get

$$\Pi_* A = \sum_{s_{\mathbb{\alpha}}\mathbb{v} \xrightarrow{\mathbb{P}} \mathbb{v}} -c(\mathbb{v})(\mathbb{h}(\mathbb{\gamma}) - 1)\sigma^{s_{\mathbb{\alpha}}} + \sum_{\mathbb{v} \xrightarrow{\mathbb{P}} s_{\mathbb{\alpha}}\mathbb{v}} c(\mathbb{v})(\mathbb{h}(\mathbb{\gamma}) + 1)\sigma^{s_{\mathbb{\alpha}}} , \tag{50}$$

where in both summands $\mathbb{\gamma}$ is the twisted label between $\mathbb{v}$ and $s_{\mathbb{\alpha}}\mathbb{v}$.

We now compute $\Pi_* \circ \epsilon^*([\mathcal{Y}_{v'}])$, where $v \xrightarrow{P} v' = vs_\gamma$. Set $\mathbb{v}' = (v', \hat{v})$. Set $Y' = Y(\mathbb{v}') = \mathcal{Y}_{v'} \times_{G/P} \mathcal{Y}^G_{\hat{v}}$, so that we have

$$\epsilon^*[\mathcal{Y}_{v'}] = [Y'] \in \mathrm{Cl}(Y(\mathbb{v})). \tag{51}$$

Consider the class $i_*[Y']$ of Y' in $A^*(\mathbb{G}/\mathbb{B} \times G/P)$ (recall the diagram (44)). Then

$$\Pi_*(\epsilon^*[\mathcal{Y}_{v'}]) = \tilde{\Pi}_* i_*[Y']. \tag{52}$$

We apply Corollary 33 to $\mathbb{v}'$ to get

$$\tilde{\Pi}_* i_*[Y'] = \sum_{s_{\mathbb{\alpha}}\mathbb{v}' \xrightarrow{\mathbb{P}} \mathbb{v}'} c(s_{\mathbb{\alpha}}\mathbb{v}')\sigma^{s_{\mathbb{\alpha}}} . \tag{53}$$

Then

$$\Pi_* C = \sum (\mathbb{h}(\gamma) + 1) c(\mathbb{v}'')\sigma^{s_{\mathbb{\alpha}}}, \tag{54}$$

where the sum runs over the subgraphs $\begin{tikzcd}[column sep=small] \mathbb{v} \arrow[dr, dashed, "\gamma"'] & & \mathbb{v}'' \\ & \mathbb{v}' \arrow[ur, "\mathbb{\alpha}"'] & \end{tikzcd}$ in the Bruhat graph of $\mathbb{G}/\mathbb{P}$ with $\mathbb{\alpha} \in \mathbb{\Delta}$ and $\gamma \in \Phi^+(G)$.

We now compute a term $\Pi_* \circ \hat{\epsilon}^* \circ j^*(\mathcal{Y}_{\hat{v}'})$ with $\hat{v} \xrightarrow{\hat{P}} \hat{v}'$. Since the intersection $\mathcal{Y}_{\hat{v}'} \cap \mathcal{Y}^G_{\hat{v}} = \mathcal{Y}^G_{\hat{v}'}$ is transverse, we have $j^*([\mathcal{Y}_{\hat{v}'}]) = [\mathcal{Y}^G_{\hat{v}'}]$. Then

$$\Pi_* \circ \hat{\epsilon}^* \circ j^*([\mathcal{Y}_{\hat{v}'}]) = \Pi_*([\mathcal{Y}_v \times_{G/P} \mathcal{Y}_{\hat{v}'}])$$

is given by Corollary 33. One gets that

$$\Pi_* D = \sum (\mathscr{h}(\hat\gamma)+1)c(\mathbbm{v}'')\sigma^{s_{\mathbbm{\alpha}}}, \tag{55}$$

where the sum runs over the subgraphs $\begin{tikzcd}[column sep=small, row sep=small] \mathbbm{v} \arrow[dr, dash, dashed, "\hat\gamma"'] & & \mathbbm{v}'' \\ & \mathbbm{v}' \arrow[ur, dash, "\mathbbm{\alpha}"'] & \end{tikzcd}$ in the Bruhat graph of $\mathbb{G}/\mathbb{P}$ with $\mathbbm{\alpha} \in \mathbbm{\Delta}$ and $\hat\gamma \in \Phi^+(\hat G)$. Putting together (54) and (55), we get

$$\Pi_*(C+D) = \sum (\mathscr{h}(\mathbbm{\gamma})+1)c(\mathbbm{v}'')\sigma^{s_{\mathbbm{\alpha}}}, \tag{56}$$

where the sum runs over the subgraphs $\begin{tikzcd}[column sep=small, row sep=small] \mathbbm{v} \arrow[dr, dash, dashed, "\mathbbm{\gamma}"'] & & \mathbbm{v}'' \\ & \mathbbm{v}' \arrow[ur, dash, "\mathbbm{\alpha}"'] & \end{tikzcd}$ in the Bruhat graph of $\mathbb{G}/\mathbb{P}$.

By Lemma 22, the terms in (56) with $\mathbbm{v} \neq \mathbbm{v}''$ are exactly the terms in the second sum of Theorem 8. Let us now consider a term in (56) with $\mathbbm{v} = \mathbbm{v}''$, meaning that $s_{\mathbbm{\alpha}}\mathbbm{v} = \mathbbm{v}s_{\mathbbm{\gamma}}$. If $\ell(s_{\mathbbm{\alpha}}\mathbbm{v}) < \ell(\mathbbm{v})$, then the coefficient of $\sigma^{s_{\mathbbm{\alpha}}}$ in $\Pi_* A$ and in $\Pi_*(C+D)$ is $c(\mathbbm{v})(h(\mathbbm{\gamma})+1)$, by (49) and (56). So these terms cancel, and accordingly Theorem 8 states that this coefficient vanishes. If $\ell(s_{\mathbbm{\alpha}}\mathbbm{v}) > \ell(\mathbbm{v})$, then the coefficient of $\sigma^{s_{\mathbbm{\alpha}}}$ in $\Pi_* A$ is $-c(\mathbbm{v})(\mathscr{h}(\mathbbm{\gamma})-1)$ and in $\Pi_*(C+D)$ it is 0. This is also compatible with Theorem 8. $\square$

5.4. **Proof of our reduction formula.** In this section, we prove Theorem 7.

5.4.1. *The birational case.* We recall from [Res11b] the main construction in the proof of Theorem 6, because this will allow introducing useful notation. Recall that $\mathbb{G} = G \times \hat G$, $\mathbbm{v} = (v, \hat v)$ and set $\mathbb{L} = L \times \hat L$ and $\mathbbm{\zeta} = (\zeta, \hat\zeta)$. Consider the variety $X = \mathbb{G}/\mathbb{B}$ endowed with the diagonal G-action. Let $\mathcal{L} \to X$ be the line bundle defined by $\mathbbm{\zeta}$. By Borel-Weyl theorem, we have

$$m_{G\subset \hat G}(\zeta, \hat\zeta) = \dim(\mathrm{H}^0(X, \mathcal{L})^G). \tag{57}$$

Let $C \subset X$ be the irreducible component of the fixed point set X^τ of τ in X containing the point $x_0 := \mathbbm{v}^{-1}\mathbb{B}/\mathbb{B}$. Let $B_{\mathbb{L}} = \mathbb{B} \cap \mathbb{L}$. Observe that the stabilizor of x_0 in $\mathbb{L}$ is $B_{\mathbb{L}}$, so that the variety C is isomorphic to $\mathbb{L}/B_{\mathbb{L}}$. Moreover, $\mathcal{L}_{|C}$ is isomorphic to the line bundle defined by $\mathbbm{v}^{-1}\mathbbm{\zeta}$. In particular

$$m_{L\subset \hat L}(v^{-1}\zeta, \hat v^{-1}\hat\zeta) = \dim(\mathrm{H}^0(C, \mathcal{L})^L). \tag{58}$$

Recall that the Białynicki-Birula cell C^+ is

$$C^+ = \{x \in X \mid \lim_{t\to 0} \tau(t)x \in C\} = \mathbb{P}\mathbbm{v}^{-1}\mathbb{B}/\mathbb{B}\,.$$

As before consider

$$\begin{tikzcd} Y(\mathbbm{v}) := G \times_P \overline{C^+} \arrow[r, "\Pi"] \arrow[d] & X \\ G/P & \end{tikzcd} \tag{59}$$

Observe that $\Pi : Y(\mathbbm{v}) \longrightarrow X$ is nothing but the incidence variety defined in (15).

The proof in [Res11b] goes on proving the following, which does not need the hypothesis $c(\mathbbm{v}) = 1$:

Theorem 37. *Assume that $v^{-1}(\zeta)_{|S} + \hat v^{-1}(\hat\zeta)_{|S}$ is trivial. With the notation of (59), there is a natural isomorphism*

$$\mathrm{H}^0(Y(\mathbbm{v}), \Pi^*\mathcal{L})^G \simeq \mathrm{H}^0(C, \mathcal{L}_{|C})^L\,.$$

The fiber of Π over any general point in x is isomorphic to the intersection of two subvarieties of G/P of cohomology class τ_v and $\iota^*\tau_{\hat v}$: therefore the assumption $c(\mathbbm{v}) = 1$ implies that Π is birational and gives further $\mathrm{H}^0(Y(\mathbbm{v}), \Pi^*\mathcal{L}) \simeq \mathrm{H}^0(X, \mathcal{L})$, from which Theorem 6 follows.

5.4.2. *The degree 2 case.* We now deal with the case $c(\mathbb{v}) = 2$. Then Π is no longer birational, but it is generically finite of degree 2, and we apply the projection formula as in Section 3.5. We denote by $B_\Pi \subset X$ the branch divisor. Then, in the Schubert basis of $X = \mathbb{G}/\mathbb{B}$, $[B_\Pi]$ expands as

$$[B_\Pi] = \sum_\alpha n_\alpha\ \sigma^{s_\alpha} \otimes 1 + \sum_{\hat\alpha} n_{\hat\alpha}\ 1 \otimes \sigma^{s_{\hat\alpha}} \in H^*(\mathbb{G}/\mathbb{B})\,, \tag{60}$$

for some well defined integers n_α and $n_{\hat\alpha}$. These coefficients are described by Theorem 8.

We proved in Lemma 13 that $\frac{1}{2}[B_\Pi] \in \mathrm{Pic}(X)$. Recall that θ and $\hat\theta$ in $X(T)$ and $X(\hat T)$ are defined by

$$\theta = \sum_\alpha \frac{n_\alpha}{2}\varpi_\alpha \qquad \hat\theta = \sum_{\hat\alpha} \frac{n_{\hat\alpha}}{2}\varpi_{\hat\alpha}\,. \tag{61}$$

Proof of Theorem 7. Observe that C can be seen as a subvariety of $Y(\mathbb{v}) = G\times^P \overline{C^+}$ via the closed immersion $x \mapsto [e, x]$, and that $\Pi^*(\mathcal{L})_{|C}$ is equal to $\mathcal{L}_{|C}$. In particular equality (58) can be rewriten as

$$m_{L\subset\hat L}(v^{-1}\zeta, \hat v^{-1}\hat\zeta) = \dim \mathrm{H}^0(C, \Pi^*(\mathcal{L})_{|C})^L.$$

Now, by Theorem 37, there is an isomorphism

$$\mathrm{H}^0(C, \Pi^*(\mathcal{L})_{|C})^L \simeq \mathrm{H}^0(Y(\mathbb{v}), \Pi^*\mathcal{L})^G.$$

By the projection formula and Proposition 15,

$$\mathrm{H}^0(Y(\mathbb{v}), \Pi^*\mathcal{L}) = \mathrm{H}^0(X, \mathcal{L}\otimes \Pi_*\mathcal{O}_{Y(\mathbb{v})}) = \mathrm{H}^0(X, \mathcal{L}) \oplus \mathrm{H}^0\Big(X, \mathcal{L}\big(-\frac{1}{2}[B_\Pi]\big)\Big).$$

But by Borel-Weyl Theorem

$$\mathrm{H}^0\Big(X, \mathcal{L}\big(-\frac{1}{2}[B_\Pi]\big)\Big) = V_{\zeta-\theta}\otimes V_{\hat\zeta-\hat\theta}.$$

Considering the dimensions of G-invariant subspaces, we get the theorem. □

6. Properties of the branch divisor class

6.1. **The Belkale-Kumar product.** Fix a one-parameter subgroup τ of T and set $P = P(\tau)$. For $w \in W^P$, recall that $\tau^w = \tau_{w^\vee}$ is a class of degree $2\ell(w)$, see Section 2.1. For $w \in W^P$, define the BK degree of $\tau^w \in \mathrm{H}^*(G/P, \mathbb{Z})$ to be

$$\text{BK-deg}(\tau^w) := \langle w^{-1}(\rho) - \rho, \tau\rangle.$$

Let w_1, w_2 and w_3 in W^P. By [BK06, Proposition 17], if $c(w_1^\vee, w_2^\vee, w_3) \neq 0$ that is if τ^{w_3} appears in the product $\tau^{w_1}\cdot\tau^{w_2}$ then

$$\text{BK-deg}(\tau^{w_3}) \leq \text{BK-deg}(\tau^{w_1}) + \text{BK-deg}(\tau^{w_2}). \tag{62}$$

In other words the BK degree filters the cohomology ring. Let $\odot$ denotes the associated graded product on $\mathrm{H}^*(G/P, \mathbb{Z})$. For later use, note that $\odot$ coincides with the usual cup product when G/P is cominuscule (See [BK06]).

Assume now that G is embedded in $\hat G$ and set $\hat P = \hat P(\tau)$. Recall that $\iota : G/P \longrightarrow \hat G/\hat P$ denotes the inclusion. In [RR11] a graded ring morphism is defined

$$\iota^\odot : (\mathrm{H}^*(\hat G/\hat P, \mathbb{Z}), \odot) \longrightarrow (\mathrm{H}^*(G/P, \mathbb{Z}), \odot).$$

As in (13), define the integers $c^\odot(v, \hat v)$ by

$$\forall \hat v \in W^{\hat P},\ \iota^\odot(\tau_{\hat v}) = \sum_{v\in W^P} c^\odot(v, \hat v)\tau^v. \tag{63}$$

Fix $\mathbb{v} = (v, \hat v) \in W^{\mathbb{P}}$ such that $\ell(\mathbb{v}) = \dim(\hat G/\hat P)$. Consider now the incidence variety $\Pi : Y(\mathbb{v}) = G\times^P \overline{C^+} \longrightarrow \mathbb{G}/\mathbb{B}$ as above. Note that $c(\mathbb{v})$ is not zero if and only if Π is dominant. By equivariance this means that there exists a point $x \in C^+$ such the tangent map of Π at $[e : x]$ is invertible.

By [RR11, Proposition 2.3] and the definition of $\iota^\odot$, the coefficient $c^\odot(\mathbb{v})$ is not zero if and only if there exists $y_0 \in C$ such that the tangent map of Π at $[e : y_0]$ is invertible. Moreover, if $c^\odot(\mathbb{v}) \neq 0$, then $\tau(\mathbb{C}^*)$ acts trivially on the line bundle $(K_\Pi)_{|C}$, so that for $y \in C^+$ with $y_0 = \lim_{t\to 0}\tau(t)y$, the tangent map of Π at

$[e:y]$ is invertible if and only if the tangent map of Π at $[e:y_0]$ is. Writing $y_0 = (l,\hat{l}).\mathbb{v}^{-1}\mathbb{B}/\mathbb{B}$, the tangent map of Π at $[e:y_0]$ is invertible if and only if the intersection

$$\iota(lv^{-1}BvP/P) \cap (\hat{l}\hat{v}^{-1}\hat{B}\hat{v}\hat{P}/\hat{P})$$

is transverse at P/P in $\hat{G}/\hat{P}$. In [BK06], this condition is called Levi-movability. To sum up, we obtain:

Lemma 38. *In the setting of Theorem 8, assume that*

$$\tau_v \odot \iota^{\odot}(\tau_{\hat{v}}) = k[pt]\,,$$

with $k > 0$. Then, for general $(l,\hat{l})$ in $\mathbb{L}$, the intersection $\iota(lv^{-1}BvP/P) \cap (\hat{l}\hat{v}^{-1}\hat{B}\hat{v}\hat{P}/\hat{P})$ is transverse at P/P and the tangent map of Π at $[e:y_0]$ is invertible, with $y_0 = (l,\hat{l}).\mathbb{v}^{-1}\mathbb{B}/\mathbb{B}$.

6.2. **The branch divisor class in the Levi-movable case.** Consider the support

$$\Gamma(G,\hat{G}) = \{(\zeta,\hat{\zeta}) \in X(T)^+ \times X(\hat{T})^+ \;:\; m_{G\subset\hat{G}}(\zeta,\hat{\zeta}) \neq 0\}$$

of the branching multiplicities. It is known (see [Éla92]) to be a finitely generated semigroup and hence it generates a convex polyhedral cone $\Gamma_{\mathbb{Q}}(G,\hat{G}) \subset (X(T)\oplus X(\hat{T}))\otimes\mathbb{Q}$, that we call Horn cone since the introduction. Let S be the center of the Levi subgroup L of G. Given $\mathbb{v} = (v,\hat{v}) \in W^{\mathbb{P}}$ such that $c(\mathbb{v}) \neq 0$, the points $(\zeta,\hat{\zeta})$ in $\Gamma_{\mathbb{Q}}(G,\hat{G})$ satisfy

$$\langle v^{-1}(\zeta),\tau\rangle + \langle \hat{v}^{-1}(\hat{\zeta}),\tau\rangle \leq 0$$

for any dominant (for G) one-parameter subgroup τ of S. In particular, the set of pairs $(\zeta,\hat{\zeta}) \in (X(T)\times X(\hat{T}))\cap\Gamma_{\mathbb{Q}}(G,\hat{G})$ such that $v^{-1}(\zeta)_{|S} + \hat{v}^{-1}(\hat{\zeta})_{|S}$ is trivial is a face $\mathcal{F}(\mathbb{v})$ of the Horn cone.

Under the Levi-movability assumption, the branch divisor class $\frac{1}{2}[B_\Pi]$ is the minimal element of $\mathcal{F}(\mathbb{v})$ that does not satisfy the conclusion of Theorem 6:

Proposition 39. *In the setting of Theorem 8, assume that $c^{\odot}(\mathbb{v}) > 0$. Then $\frac{1}{2}[B_\Pi] \in \mathcal{F}(\mathbb{v})$, and we have, for any $(\zeta,\hat{\zeta}) \in \mathcal{F}(\mathbb{v})$,*

$$m_{G\subset\hat{G}}(\zeta,\hat{\zeta}) \leq m_{L\subset\hat{L}}(v^{-1}(\zeta),\hat{v}^{-1}(\hat{\zeta}))\,.$$

Assume more specifically that $c^{\odot}(\mathbb{v}) = 2$. Then the set of pairs $(\zeta,\hat{\zeta}) \in \mathcal{F}(\mathbb{v})$ where this inequality is strict is exactly the set

$$\frac{1}{2}[B_\Pi] + (\mathcal{F}(\mathbb{v}) \cap \Gamma(G,\hat{G}))\,.$$

Proof. First, let us prove that $[B_\Pi]$ belongs to the face $\mathcal{F}(\mathbb{v})$. If $c^{\odot}(\mathbb{v}) = 1$, then $\frac{1}{2}[B_\Pi] = 0$ and there is nothing to prove. We assume $c^{\odot}(\mathbb{v}) > 1$. Recall that $\mathcal{O}(B_\Pi) = \mathcal{L}(\theta,\hat{\theta}) \in \mathrm{Pic}(\mathbb{G}/\mathbb{B})$. We have $(\theta,\hat{\theta}) \in \mathcal{F}(\mathbb{v})$ if and only if $v^{-1}(\theta)_{|S} + \hat{v}^{-1}(\hat{\theta})_{|S}$ is trivial that is if S acts trivialy on $\mathcal{O}(B_\Pi)_{|C}$.

We pick $(l,\hat{l}) \in \mathbb{L}$ as provided by Lemma 38 and we set $y_0 = (l,\hat{l})\mathbb{v}^{-1}\mathbb{B}/\mathbb{B}$. In particular $[e:y_0] \in Y(\mathbb{v}) = G \times^P \overline{C^+}$ is an isolated point in the fiber $\Pi^{-1}(y_0)$. By transversality and degree assumption, the fiber $\Pi^{-1}(y_0)$ cannot be reduced to this point. In particular, this fiber is not connected. By Proposition 16, y_0 does not belong to the support $\mathrm{Supp}(B_\Pi)$ of B_Π.

Since $\mathcal{O}(B_\Pi)$ is the line bundle on the smooth variety X associated to some divisor with $\mathrm{Supp}(B_\Pi)$ as support, $\mathcal{O}(B_\Pi)$ has a canonical section σ that does not vanish at y_0. By unicity up to scalar multiplication of this section and G-invariance of $\mathrm{Supp}(B_\Pi)$, it has to be an eigenvector for the action of G. Since G is semi-simple this section is a G-invariant element of $H^0(X,\mathcal{O}(B_\Pi))$, meaning that σ is a G-equivariant section of $\mathcal{O}(B_\Pi)$.

But S fixes y_0 and hence acts on the fiber $\mathcal{O}(B_\Pi)_{y_0}$. This action has to fix $\sigma(y_0)$ and hence is trivial.

Let $f \in H^0(\mathbb{G}/\mathbb{B},\mathcal{L})^G$ that restricts to 0 on C. Then f vanishes on C^+ and thus $\Pi^* f$ vanishes on $Y(\mathbb{v})$. Thus, $H^0(\mathbb{G}/\mathbb{B},\mathcal{L})^G$ injects into $H^0(C,\mathcal{L}_{|C})^L$, and the inequality of the proposition follows.

Assuming that $c^{\odot}(\mathbb{v}) = 2$, Theorem 7 shows that the inequality will be strict if and only if $m_{G\subset\hat{G}}(\zeta-\theta,\hat{\zeta}-\hat{\theta}) \neq 0$, which means that $(\zeta,\hat{\zeta})$ belongs to $(\theta,\hat{\theta}) + (\mathcal{F}(\mathbb{v}) \cap \Gamma(G,\hat{G}))$. $\square$

Corollary 40. *In the setting of Theorem 7, assume that $c^{\odot}(v,\hat v)=2$. Then*
$$m_{G\subset\hat G}(\zeta,\hat\zeta)=\sum_{k\geq 0}(-1)^k m_{L\subset\hat L}(v^{-1}(\zeta-k\theta),\hat v^{-1}(\hat\zeta-k\hat\theta)).$$

Proof. Direct induction from Theorem 7. Indeed Proposition 39 allows to apply the theorem to each weight $(\zeta-k\theta,\hat\zeta-k\hat\theta)$. □

6.3. **The branch divisor class in the case of type A Grassmannians.** In this section, we prove Theorem 5. We assume that G has type A, that $P\subset G$ is a maximal parabolic subgroup (so that G/P is a Grassmannian), and that $\hat G=G\times G$ and $\hat P=P\times P$. Fix an integer $n\geq 2$ and set $G=\mathrm{SL}_n(\mathbb{C})$.

6.3.1. *About Littlewood-Richardson coefficients.* In this section, we collect some specific poperties of the LR coefficients $c^{\nu}_{\lambda,\mu}$.

In type A_{n-1}, it is more convenient to work with representations of $\mathrm{GL}_n(\mathbb{C})$ instead of $\mathrm{SL}_n(\mathbb{C})$. For $\mathrm{GL}_n(\mathbb{C})$, we have
$$\Lambda^+_{\mathrm{GL}_n}:=X(T)^+=\{\sum_{i=1}^n\lambda_i\varepsilon_i\,:\,\lambda_1\geq\cdots\geq\lambda_n\in\mathbb{Z}\},$$
with notation as in [Bou02]. Given $\lambda\in X(T)^+$, set $\bar\lambda=\sum_{i=1}^{n-1}(\lambda_i-\lambda_n)\varepsilon_i$. Then $\bar\lambda$ is a partition and the $\mathrm{GL}_n(\mathbb{C})$-representation V_λ of heighest weight λ is isomorphic to the $\mathrm{SL}_n(\mathbb{C})$-representation of heighest weight $\bar\lambda$ as an $\mathrm{SL}_n(\mathbb{C})$-representation. For $\lambda,\mu,\nu\in\Lambda^+_{\mathrm{GL}_n}$, set $m_{\mathrm{GL}_n}(\lambda,\mu,\nu)=\dim(V_\lambda\otimes V_\mu\otimes V_\nu)^{\mathrm{GL}_n(\mathbb{C})}$. Because of the action of the center of $\mathrm{GL}_n(\mathbb{C})$, we have
$$m_{\mathrm{GL}_n}(\lambda,\mu,\nu)\neq 0\Longrightarrow|\lambda|+|\mu|+|\nu|=0,$$
where $|\lambda|=\sum_i\lambda_i$ and $|\mu|$ and $|\nu|$ are defined similarly. Set $\nu^*=\sum_i-\nu_{n+1-i}\varepsilon_i$ so that V_{ν^*} is the dual representation of V_ν. If λ, μ and ν^* are partitions (that is $\lambda_n,\mu_n,\nu^*_n\geq 0$) then

$$m_{\mathrm{GL}_n}(\lambda,\mu,\nu)=c^{\nu^*}_{\lambda,\mu}\tag{64}$$
is the LR coefficient. The Horn semigroup of $\mathrm{GL}_n(\mathbb{C})$ is
$$\Gamma(\mathrm{GL}_n(\mathbb{C}))=\{(\lambda,\mu,\nu)\in(\Lambda^+_{\mathrm{GL}_n})^3\,:\,m_{\mathrm{GL}_n}(\lambda,\mu,\nu)\neq 0\}.$$
Let $1\leq r\leq n-1$ and consider $\mathrm{Gr}(r,n)=G/P$. Given a subset $I\subset\{1,\dots,n\}$ with r elements, define $v_I\in W^P$ by $v_I(\{1,\dots,r\})=I$ and set $I^\vee=\{n+1-i\,|\,i\in I\}$. Recall from (2) in the introduction that $c^K_{I,J}$ denotes the structure constants of the cohomology of the Grassmannian $\mathrm{Gr}(r,n)$ equipped with its Schubert basis. For any subsets $I,J,K\subset\{1,\dots,n\}$ with r elements, we have

$$c(v_I,v_J,v_K)=c^K_{I^\vee,J^\vee}.\tag{65}$$

Recall also that given a partition λ and a subset I, λ_I is the partition whose parts are λ_i with $i\in I$. Assume that $c(v_I,v_J,v_K)\neq 0$. The associated Horn inequality (3) expressed in terms of $\Gamma(\mathrm{GL}_n(\mathbb{C}))$ becomes

$$(\lambda,\mu,\nu)\in\Gamma(\mathrm{GL}_n(\mathbb{C}))\Longrightarrow|\lambda_I|+|\mu_J|+|\nu_K|\leq 0.\tag{66}$$
Two interpretations. Fix $0<r<n$. Let $I=\{i_1<\cdots<i_r\}$ be a subset of $\{1,\dots,n\}$ with r elements. Let
$$\lambda(I)=i_r-r\geq\cdots\geq i_1-1$$
be the associated partition.

An amazing property of the LR coefficients is that they are also the multiplicities of the tensor product decomposition for the linear groups. Namely, $c^K_{I,J}$ is also the multiplicity $c^{\lambda(K)}_{\lambda(I),\lambda(J)}$ of $V_{\mathrm{SL}_r}(\lambda(K))$ in $V_{\mathrm{SL}_r}(\lambda(I))\otimes V_{\mathrm{SL}_r}(\lambda(J))$. Hence

$$c^K_{I,J}=c^{\lambda(K)}_{\lambda(I),\lambda(J)}.\tag{67}$$

Fix now I, J and K such that $c(v_I,v_J,v_K)=:c(\mathbb{v})$ is nonzero. Consider $\Pi\,:\,Y(\mathbb{v})\longrightarrow X=(G/B)^3$.

Remark 3. *In type A, the semisimple part L^{ss} of the Levi subgroup is $L^{ss} = \mathrm{SL}_r \times \mathrm{SL}_{n-r}$ and $C = C_1 \times C_2$ with $C_1 \simeq \mathrm{Fl}(\mathbb{C}^r)^3$ and $C_2 \simeq \mathrm{Fl}(\mathbb{C}^{n-r})^3$. Using the base point of C, we fix embeddings of C_1 and C_2 in C.*

Denote by B_r and B_{n-r} the Borel subgroups of SL_r and SL_{n-r}. Observe that C is a closed subvariety of both X and $Y(\mathbb{v})$. Indeed, the map

$$\begin{array}{rccc} j_Y: & C & \longrightarrow & Y(\mathbb{v}) \\ & x & \longmapsto & [e:x] \end{array}$$

is a closed immersion. Here, $Y(\mathbb{v})$ is identified to the fibered product $G \times^P \overline{C^+}$. Similarly define the inclusions j_Y^1 and j_Y^2 of C_1 and C_2 in $Y(\mathbb{v})$ respectively. By [BKR12], we have

$$\begin{aligned} (j_Y^1)^*([R_\Pi] &= \mathcal{L}_{(SL_r/B_r)^3}(\lambda(I^\vee), \lambda(J^\vee), \lambda(K^\vee)) \in \mathrm{Pic}(C_1) \qquad \text{and} \\ (j_Y^2)^*([R_\Pi]) &= \mathcal{L}_{(SL_{n-r}/B_{n-r})^3}({}^t\lambda(I^\vee), {}^t\lambda(J^\vee), {}^t\lambda(K^\vee)) \in \mathrm{Pic}(C_2). \end{aligned} \tag{68}$$

Let λ, μ and ν be in $\Lambda^+_{\mathrm{GL}_n}$. We recall:

Polynomiality. The function $k \mapsto m_{\mathrm{GL}_n}(k\lambda, k\mu, k\nu)$ is polynomial. See [DW02]. We call it the LR-polynomial associated to (λ, μ, ν).

LR coefficients one. If $m_{\mathrm{GL}_n}(\lambda, \mu, \nu) = 1$ then for any $k \geq 0$, $m_{\mathrm{GL}_n}(k\lambda, k\mu, k\nu) = 1$. This is known as the Fulton conjecture and was first proved by Knutson-Tao-Woodward [KTW04]. See [Bel07, Ful84, BKR12] for geometric proofs.

LR coefficients two. If $m_{\mathrm{GL}_n}(\lambda, \mu, \nu) = 2$ then for any $k \geq 0$, $m_{\mathrm{GL}_n}(k\lambda, k\mu, k\nu) = k+1$. See [Ike16] and [She17, Corollary 9.4].

6.3.2. In this subsection, we prove Theorem 5. Since the Grassmannians are cominuscule, Proposition 39 holds and we can apply Theorem 7 to $\frac{k}{2}[B_\Pi]$. Then the second assertion of Theorem 5, giving the values of the polynomials $c^{k\theta^3_K}_{k\theta^1_I, k\theta^2_J}$ and $c^{k\theta^3_{\bar{K}}}_{k\theta^1_{\bar{I}}, k\theta^2_{\bar{J}}}$ implies the first one by an immediate induction on k. Nevertheless, a smaller hypothesis on those polynomials is enough to prove Theorem 5 by Lemma 41 below. Recall first that the characters $(\theta^1, \theta^2, \theta^3)$ of T satisfy

$$\mathcal{L}(\theta^1, \theta^2, \theta^3) = \frac{1}{2}[B_\Pi] \in \mathrm{Pic}((G/B)^3).$$

Lemma 41. *Assume that $c(v_I, v_J, v_K) = 2$. Let P_1, P_2 and Q be the three LR-polynomials associated to $(\theta^1_I, \theta^2_J, \theta^3_K)$, $(\theta^1_{\bar{I}}, \theta^2_{\bar{J}}, \theta^3_{\bar{K}})$ and $(\theta^1, \theta^2, \theta^3)$ respectively.*

If $\deg(P_1) = \deg(P_2) = 1$ then $P_1(X) = P_2(X) = X + 1$ and $Q(X) = \frac{1}{2}(X+1)(X+2)$.

Proof. Theorem 7 asserts that

$$Q(X) + Q(X-1) = P_1(X)P_2(X). \tag{69}$$

Thus, if P_1 and P_2 have degree 1, then Q has degree 2. Moreover, $Q(0) = P_1(0) = P_2(0) = 1$. Write the three polynomials as $P_1(X) = c_1X + 1$, $P_2(X) = c_2X + 1$ and $Q(X) = aX^2 + bX + 1$. Since P_1 and P_2 have non-negative values on $\mathbb{Z}_{\geq 0}$, $c_1 > 0$ and $c_2 > 0$.

From (69), we deduce that $b - a = 1$ and $2b - 2a = c_1 + c_2$. It follows that $c_1 + c_2 = 2$, so $c_1 = c_2 = 1$. We get $a = 1/2$ and $b = 3/2$. □

In this proof we work on the Bott-Samelson resolution $\boldsymbol{Y}(\mathbf{v})$, on which by smoothness all divisors are Cartier divisors. We first observe that $\boldsymbol{Y}(\mathbf{v})$ and $Y(\mathbb{v})$ coincide on the open subset $G \times^P C^+ \subset Y(\mathbb{v})$. Fix reduced expressions $\boldsymbol{v}$ and $\hat{\boldsymbol{v}}$ for v_I and (v_J, v_K). Let $j_Y^+ : G \times^P C^+ \to Y(\mathbb{v}) = G \times^P \overline{C^+}$ be given by the inclusion $C^+ \subset \overline{C^+}$.

Lemma 42. *There is an open immersion $j^+_{\boldsymbol{Y}}$ such that the following diagram is commutative:*

$$\begin{tikzcd} & \boldsymbol{Y}(\boldsymbol{\mathbb{v}}) \arrow[d] \\ G\times^P C^+ \arrow[ur, "j^+_{\boldsymbol{Y}}"] \arrow[r, "j^+_Y"'] & Y(\mathbb{v}) \end{tikzcd}$$

Proof. Let F be the fiber at the base point of $\boldsymbol{Y_v}\to G/P$. The projection

$$\boldsymbol{Y_v} = G\times^P F \longrightarrow Y_v = G\times^P \overline{Pv^{-1}B/B}$$

is birational and G-equivariant so it has to be an isomorphism over the open G-orbit in Y_v. Hence $\boldsymbol{Y_v}$ contains $G\times^P Pv^{-1}B/B$ as an open subset. The lemma follows since $\boldsymbol{Y}(\boldsymbol{\mathbb{v}}) = \boldsymbol{Y_v}\times_{G/P}\boldsymbol{Y_{\hat{v}}}$. $\square$

Consider the ramification divisor $R_{\boldsymbol{\Pi}}$ of $\boldsymbol{\Pi}: \boldsymbol{Y}(\boldsymbol{\mathbb{v}})\longrightarrow X$. Recall the decomposition $R_{\boldsymbol{\Pi}} = R^1_{\boldsymbol{\Pi}} + E_{\boldsymbol{\Pi}}$ as in (21). Write $E_{\boldsymbol{\Pi}} = \sum_i a_iE_i$ where a_i are positive integers and E_i are irreducible. As in Proposition 15, let b_i be non-negative integers such that

$$\boldsymbol{\Pi}^*\mathcal{O}_X(B_\Pi) = \mathcal{O}_{\boldsymbol{Y}(\boldsymbol{\mathbb{v}})}(2R^1_{\boldsymbol{\Pi}} + \sum b_iE_i)\,. \tag{70}$$

Proof of Theorem 5. Fix l in $\mathbb{N}$ such that for any i, $la_i\geq b_i$ and set

$$D = 2R^1_{\boldsymbol{\Pi}} + lE_{\boldsymbol{\Pi}}.$$

Using $j^+_{\boldsymbol{Y}}$, think about C_1 as a subvariety of $\boldsymbol{Y}(\boldsymbol{\mathbb{v}})$. Since $\boldsymbol{Y}(\boldsymbol{\mathbb{v}})$ is smooth one can consider the following line bundles on C_1: $\mathcal{O}(R^1_{\boldsymbol{\Pi}})_{|C_1}$, $\mathcal{O}(R_{\boldsymbol{\Pi}})_{|C_1}$, $\mathcal{O}(D)_{|C_1}$ and $\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1}$.

By Lemma 38, the support of $R_{\boldsymbol{\Pi}}$ does not contain C and is G-stable. Hence the four considered line bundles on $Y(\mathbb{v})$ have a G-invariant section that does not vanish identically on C. Fixing a general $x_2\in C_2$, we get a nonzero $\mathrm{SL}_r(\mathbb{C})$-invariant section of the restriction to C_1. Hence

$$\mathcal{O}(R^1_{\boldsymbol{\Pi}})_{|C_1}, \mathcal{O}(R_{\boldsymbol{\Pi}})_{|C_1}, \mathcal{O}(D)_{|C_1}, \mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1}\in\Gamma(C_1,\mathrm{SL}_r(\mathbb{C}))\,. \tag{71}$$

We claim that

$$\dim\left(C_1^{\mathrm{ss}}(\mathcal{O}(D)_{|C_1})/\!/\mathrm{SL}_r\right) = \dim\left(C_2^{\mathrm{ss}}(\mathcal{O}(D)_{|C_2})/\!/\mathrm{SL}_{n-r}\right) = 1. \tag{72}$$

By [Deb01, Lemma 7.11], for any $k\geq 0$, $\mathrm{H}^0(\boldsymbol{Y}(\boldsymbol{\mathbb{v}}),\mathcal{O}(kE_{\boldsymbol{\Pi}}))\simeq\mathbb{C}$. This implies $\dim(\boldsymbol{Y}(\boldsymbol{\mathbb{v}})^{\mathrm{ss}}(\mathcal{O}(E_{\boldsymbol{\Pi}}))/\!/\mathrm{SL}_n) = 0$. By Lemma 42, (C,τ) is dominant in $\boldsymbol{Y}(\boldsymbol{\mathbb{v}})$. Since Proposition 11 shows that

$$0 = \dim\boldsymbol{Y}(\boldsymbol{\mathbb{v}})^{\mathrm{ss}}(\mathcal{O}(E_{\boldsymbol{\Pi}}))/\!/\mathrm{SL}_n = \dim\left(C^{\mathrm{ss}}(\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C})/\!/(\mathrm{SL}_r\times\mathrm{SL}_{n-r})\right),$$

we get $\dim\left(C_1^{\mathrm{ss}}(\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1})/\!/\mathrm{SL}_r\right) + \dim\left(C_2^{\mathrm{ss}}(\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_2})/\!/\mathrm{SL}_{n-r}\right) = 0$, and these two dimensions vanish.

An open neighborhood of C in $\boldsymbol{Y}(\boldsymbol{\mathbb{v}})$ being $G\times^P C^+$, we have

$$\mathcal{O}(R_\Pi)_{|C} = \mathcal{O}(R_{\boldsymbol{\Pi}})_{|C}. \tag{73}$$

Now, (64), (65), (67) and (68) imply that

$$\dim H^0(C_1,\mathcal{O}(R_\Pi)_{|C_1})^{\mathrm{SL}_r} = \dim H^0(C_2,\mathcal{O}(R_\Pi)_{|C_2})^{\mathrm{SL}_{n-r}} = 2.$$

So, the end of Section 6.3.1 gives

$$\dim\left(C_1^{\mathrm{ss}}(\mathcal{O}(R_\Pi)_{|C_1})/\!/\mathrm{SL}_r\right) = \dim\left(C_2^{\mathrm{ss}}(\mathcal{O}(R_\Pi)_{|C_2})/\!/\mathrm{SL}_{n-r}\right) = 1. \tag{74}$$

If $\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1}$ is trivial then $\mathcal{O}(R_\Pi)_{|C_1} = \mathcal{O}(D)_{|C_1}$ and (72) follows from (74). If $\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1}$ is not trivial, the proved equalities for the dimensions of the GIT-quotients show that $\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1}$ and $\mathcal{O}(R_\Pi)_{|C_1}$ are linearly independant in $\mathrm{Pic}(C_1)_{\mathbb{Q}}$. Since $\mathcal{O}(R^1_{\boldsymbol{\Pi}})_{|C_1}$ and $\mathcal{O}(E_{\boldsymbol{\Pi}})_{|C_1}$ belong to $\Gamma(C_1,\mathrm{SL}_r)$, $\mathcal{O}(R_{\boldsymbol{\Pi}})_{|C_1}$ and $\mathcal{O}(D)_{|C_1}$ belong to the same face. Proposition 10 shows that

$$\dim\left(C_1^{\mathrm{ss}}(\mathcal{O}(D)_{|C_1})/\!/\mathrm{SL}_r\right) = \dim\left(C_1^{\mathrm{ss}}(\mathcal{O}(R_\Pi)_{|C_1})/\!/\mathrm{SL}_r\right) = 1.$$

Therefore, (72) is proved. By Proposition 11, this implies

$$\dim\left(\boldsymbol{Y}(\mathbb{v})^{\mathrm{ss}}(D)/\!/\mathrm{SL}_n\right) = 2.$$

The equation (70) and [Deb01, Lemma 7.11] imply that

$$\forall k \geq 0 \qquad \mathrm{H}^0(\boldsymbol{Y}(\mathbb{v}), \boldsymbol{\Pi}^*(\mathcal{O}_X(kB_\Pi))) \simeq \mathrm{H}^0(\boldsymbol{Y}(\mathbb{v}), \mathcal{O}_{\boldsymbol{Y}(\mathbb{v})}(kD)).$$

Hence

$$\dim\left(\boldsymbol{Y}(\mathbb{v})^{\mathrm{ss}}(\Pi^*\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_n\right) = \dim\left(\boldsymbol{Y}(\mathbb{v})^{\mathrm{ss}}(D))/\!/\mathrm{SL}_n\right) = 2.$$

Hence, by Proposition 11,

$$2 = \dim\left(\boldsymbol{Y}(\mathbb{v})^{\mathrm{ss}}(\Pi^*\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_n\right) = \dim\left(C^{\mathrm{ss}}(\boldsymbol{\Pi}^*\mathcal{O}_X(B_\Pi))/\!/(\mathrm{SL}_r \times \mathrm{SL}_{n-r})\right),$$

so $\dim\left(C_1^{\mathrm{ss}}(\boldsymbol{\Pi}^*\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_r\right) + \dim\left(C_2^{\mathrm{ss}}(\boldsymbol{\Pi}^*\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_{n-r}\right) = 2$. On the other hand, by the choice of l and (71), $\mathcal{O}_{\boldsymbol{Y}(\mathbb{v})}(D)_{|C_1} - \mathcal{O}_X(B_\Pi)_{|C_1}$ is an element of $\Gamma(C_1, \mathrm{SL}_r)$. Hence

$$\dim\left(C_1^{\mathrm{ss}}(\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_r\right) \leq \dim\left(C_1^{\mathrm{ss}}(\mathcal{O}_{\boldsymbol{Y}(\mathbb{v})}(D))/\!/\mathrm{SL}_r\right) = 1\,,$$

and similarly on C_2. We therefore conclude that

$$\dim\left(C_1^{\mathrm{ss}}(\boldsymbol{\Pi}^*\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_r\right) = \dim\left(C_2^{\mathrm{ss}}(\boldsymbol{\Pi}^*\mathcal{O}_X(B_\Pi))/\!/\mathrm{SL}_{n-r}\right) = 1.$$

Then, Lemma 41 ends the proof. □

7. Examples

7.1. **Notation.** In this section, we only consider the case $G \subset \hat{G} = G \times G$, for a given classical group G. In particular, we use Notation 2. In each case, we have a vector space V endowed with a basis $(e_1, \dots, e_n)$ and eventually a bilinear form. In types B, C, D, we choose a bilinear form β (quadratic or symplectic) such that $\beta(e_i, e_j) \neq 0$ if and only if $j = n+1-i$. With this convention, the set of diagonal resp. triangular matrices in the corresponding group is a maximal torus T resp. Borel subgroup B.

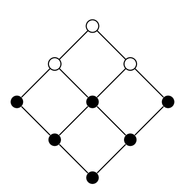
The classical Grassmannians $\mathrm{Gr} = G/P$ occur. The Schubert varieties in Gr are indexed by subsets I of $\{1, \dots, n\}$ in the following way:

$$X_I = \overline{B \cdot V_I} \text{ with } V_I := \mathrm{Span}(e_i \,:\, i \in I)\,, \quad \text{and} \quad \tau_I = [X_I] \in H^{2\mathrm{codim}(X_I)}(\mathrm{Gr}).$$

This also gives a bijection between a set of subsets of $\{1, \dots, n\}$ and W^P. Moreover, set $\bar{\imath} = n+1-i$. and $I^\vee = \{\bar{\imath} \,|\, i \in I\}$. This operation is the Poincaré duality: $\tau^I = \tau_{I^\vee}$.

In type A, we also consider the two steps flag manifolds $\mathrm{Fl}(p, q; n) = \{F_1 \subset F_2 \subset V \,:\, \dim(F_1) = p,\ \dim(F_2) = q\}$. The Schubert varieties are then indexed by "flags" $(I_p \subset I_q)$ of subsets of $\{1, \dots, n\}$: $X_{(I_p \subset I_q)} = \overline{B \cdot (V_{I_p}, V_{I_q})}$.

7.2. **A detailled example in** $\mathrm{Gr}(3,6)$**.** In this subsection, $G = \mathrm{SL}_6$, $G/P = \mathrm{Gr}(3,6)$, and $\hat{G} = G^2$. We let $I = J = K = \{2, 4, 6\} \in W^P$. Set $\mathbb{v} = (v_I, v_J, v_K)$. The inversion set $\Phi(v_I)$ of v_I is depicted with black nodes:



Since $\lambda(I^\vee) = \lambda(J^\vee) = 21$ and $\lambda(K) = 321$, we have $c(\mathbb{v}) = c^{321}_{21,\,21} = 2$, by (65) and (67). We describe geometrically the divisors R_Π and B_Π in this example, but we start with Lemma 43 about configurations of triangles in the plane.

Definition 3. *Given a 3-dimensional vector space E, a sextuple $(A_1, A_2, A_3, B_1, B_2, B_3)$ of points in the projective plane $\mathbb{P}E$ is called a bitriangle if the three sets of points $\{A_i, A_j, B_k\}$ for i, j, k distinct in $\{1, 2, 3\}$ are colinear. In the generic situation, A_1, A_2, A_3 define a triangle and the three points B_1, B_2, B_3 are on the sides of this triangle.*

The variety of all bitriangles in $\mathbb{P}E$ will be denoted by $\mathcal{T}(E)$. Given two vector spaces E and E' of dimension 3, and two bitriangles

$$T = (A_1, A_2, A_3, B_1, B_2, B_3) \in \mathcal{T}(E) \ , \ T' = (A'_1, A'_2, A'_3, B'_1, B'_2, B'_3) \in \mathcal{T}(E') \, , \tag{75}$$

a morphism $u : T \longrightarrow T'$ is a linear map $u : E \longrightarrow E'$ such that for all i, $u(A_i) \subset A'_i$ and $u(B_i) \subset B'_i$ (here A_i, B_i resp. A'_i, B'_i are considered as 1-dimensional subspaces of E resp. E'). Note that the set of morphisms of bitriangles is a subspace of the vector space of linear maps $E \to E'$.

Observe that $\mathcal{T}(E)$ is irreducible of dimension 9 and that the modality of the action of $\mathrm{PSL}(E)$ is one.

Lemma 43. *There is exactly one divisor $\mathcal{D}$ in $\mathcal{T}(E) \times \mathcal{T}(E')$ such that for all pairs (T, T') in $\mathcal{D}$, the space of morphisms from T to T' is not reduced to $\{0\}$. It may be described as the divisor of pairs of isomorphic bitriangles.*

Proof. First of all, the variety $\mathcal{D}$ of pairs of isomorphic bitriangles is a divisor, since the modality of the action of $\mathrm{PSL}(E)$ on the space of bitriangles is 1. Moreover, tautologically, if T and T' are isomorphic, then the space of morphisms $T \longrightarrow T'$ is not reduced to $\{0\}$.

On the other hand, let $u : T \longrightarrow T'$ be a morphism of bitriangles, with T and T' as in (75). If u has rank 3, then T and T' are isomorphic and $(T, T') \in \mathcal{D}$.

Assume first that T is generic. If u has rank 2, then at least 5 of the six points $A_1, A_2, A_3, B_1, B_2, B_3$ have a well defined image in $\mathbb{P}E'$ and all these images belong to the line in $\mathbb{P}E'$ defined by the image of u. It follows that 5 of the 6 points $A'_1, A'_2, A'_3, B'_1, B'_2, B'_3$ are on a line, and the set of such bitriangles has codimension 2 in $\mathcal{T}(E')$. If u has rank 1, similarly, we don't find a divisor in $\mathcal{T}(E')$ since 3 of the six points $A'_1, A'_2, A'_3, B'_1, B'_2, B'_3$ must be equal in this case. The case where T is degenerate is similar. $\square$

Proposition 44. *The hypersurface B_Π is irreducible, it is equal to the variety Δ defined below in (78), and*

$$\mathcal{O}_{(G/B)^3}(B_\Pi) = \mathcal{L}_{(G/B)^3}(2\varpi_2 + 2\varpi_4, 2\varpi_2 + 2\varpi_4, 2\varpi_2 + 2\varpi_4) \, .$$

The hypersurface R_Π is also irreducible, it is the preimage of B_Π by Π, and it also has the description given below in (79).

Proof. Let $U \subset (G/B)^3$ be the open subset of triples $({}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet)$ such that $\mathbb{C}^6 = {}_1X_2 \oplus {}_2X_2 \oplus {}_3X_2$, ${}_1X_4 \cap {}_2X_4 \cap {}_3X_4 = \{0\}$, and ${}_iX_4 \cap {}_jX_2 = \{0\}$ for $i \neq j$. We first investigate the intersection $B_\Pi \cap U$. In fact, let $({}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet) \in U$: the fiber $\Pi^{-1}({}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet)$ can be explicitly described as follows.

Let $V_3 \in \mathrm{Gr}(3, 6)$. Then $(V_3; {}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet)$ is an element of $Y(\mathtt{v})$ if and only if

$$\forall i \in \{1, 2, 3\} \ , \ \dim V_3 \cap {}_iX_2 \geq 1 \text{ and } \dim V_3 \cap {}_iX_4 \geq 2 \, . \tag{76}$$

From $\dim V_3 \cap {}_iX_2 \geq 1$, we deduce that V_3 can be written as

$$V_3 = L_1 \oplus L_2 \oplus L_3 \, , \tag{77}$$

with $L_i \subset {}_iX_2$ a subspace of dimension 1. Moreover, under our genericity assumption, we have an isomorphism ${}_iX_2 \simeq \mathbb{C}^6/{}_kX_4$ if $i \neq k$. If i, j, k are distinct, we get isomorphisms ${}_iX_2 \simeq \mathbb{C}^6/{}_kX_4$ and $\mathbb{C}^6/{}_kX_4 \simeq {}_jX_2$, and we denote $\varphi_{i,j} : {}_iX_2 \longrightarrow {}_jX_2$ the isomorphism obtained by composition.

Observe that $L_1 \oplus L_2 \oplus L_3$ will meet ${}_kX_4$ in dimension 2 if and only if the subspace of $\mathbb{C}^6$ generated by L_i, L_j and ${}_kX_4$ has dimension 5, which is equivalent to the equality $L_j = \varphi_{i,j}(L_i)$.

Thus, $V_3 = L_1 \oplus L_2 \oplus L_3$ satisfies (76) if and only if for $i \neq j$, $L_j = \varphi_{i,j}(L_i)$. In other words, $L_2 = \varphi_{1,2}(L_1), L_3 = \varphi_{2,3}(L_2)$, and $L_1 \subset {}_1X_2$ is an eigenline of $\varphi_{3,1} \circ \varphi_{2,3} \circ \varphi_{1,2}$.

By Proposition 16, $B_\Pi \cap U$ is the set of triples $({}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet)$ such that $\varphi_{3,1} \circ \varphi_{2,3} \circ \varphi_{1,2}$ has only one eigenvalue or is the identity.

We denote by Δ this divisor:

$$\Delta = \{({}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet) \,|\, \varphi_{3,1} \circ \varphi_{2,3} \circ \varphi_{1,2} \text{ has only one eigenvalue}\} \tag{78}$$

We now show that there is only one divisor in the ramification of Π. To show this, we observe that for a general element $(V_3, {}_1X_\bullet, {}_2X_\bullet, {}_3X_\bullet)$ of a divisor of $Y(\mathbb{v})$, $\dim(V_3 \cap {}_iX_2) = 1$ and $\dim(V_3 \cap {}_iX_4) = 2$ for $i \in \{1,2,3\}$. Therefore, this defines a bitriangle T in $\mathbb{P}V_3$, with vertices $A_i = V_3 \cap_i X_2$ and edges $E_i = V_3 \cap_i X_4$, and this defines also a bitriangle T' in $\mathbb{C}^6/V_3$, with vertices $A'_i = p(V_{3i}X_2)$ and edges $E'_i = p({}_iX_4)$, where $p : \mathbb{C}^6 \longrightarrow \mathbb{C}^6/V_3$ denotes the projection.

In this way, we factorize the morphism $Y(\mathbb{v}) \to \mathrm{Gr}(3,6)$ as a composition of rational maps $Y(\mathbb{v}) \overset{f}{\dashrightarrow} \mathcal{T}(E) \times_{\mathrm{Gr}(3,6)} \mathcal{T}(Q) \longrightarrow \mathrm{Gr}(3,6)$, where E and Q denote the tautological bundles on $\mathrm{Gr}(3,6)$, and $\mathcal{T}(E)$ and $\mathcal{T}(Q)$ denote the relative varieties of bitriangles therein. The rational map f is defined in codimension 1 and equidimensional. It follows that $f(R_\Pi)$ is defined and at least a divisor in $\mathcal{T}(E) \times \mathcal{T}(Q)$.

On the other hand, an element $u \in T_{V_3}\mathrm{Gr}(3,6) \simeq \mathrm{Hom}(V_3, \mathbb{C}^6/V_3)$ belongs to the tangent space to the Schubert variety defined by ${}_iX_\bullet$ if and only if $u(A_i) \subset A'_i$ and $u(E_i) \subset E'_i$. Since this must hold for all i in $\{1,2,3\}$, u defines a morphism of bitriangles (recall Definition 3). It thus follows from Lemma 43 that the image of R_Π by f is the divisor of isomorphic bitriangles in $\mathcal{T}(E) \times \mathcal{T}(Q)$:

$$R_\Pi = \{y \in Y : f(y) \in \mathcal{T}(E) \times \mathcal{T}(Q) \text{ is a pair of isomorphic bitriangles.}\} \tag{79}$$

It follows that R_Π is irreducible, equal to $\Pi^{-1}(B_\Pi)$, and that $B_\Pi = \Delta$. □

Remark 4. *One may check directly that the class of Δ is the class of B_Π as computed by Theorem 8, namely*

$$\mathcal{O}(\Delta) = \mathcal{L}_{(G/B)^3}(2\varpi_2 + 2\varpi_4, 2\varpi_2 + 2\varpi_4, 2\varpi_2 + 2\varpi_4).$$

We now check directly Theorem 7 in this case. First we give a full description of the face in the Horn cone defined by $\mathbb{v}$. Recall that $I = J = K = \{2,4,6\}$.

Lemma 45. *The face of $\Gamma(\mathrm{GL}_6)$ defined by the equation $|\lambda_I| + |\mu_J| + |\nu_K| = 0$ has equations*

$$\begin{cases} |\lambda| + |\mu| + |\nu| = 0\,; \\ \lambda_1 = \lambda_2\,, \lambda_3 = \lambda_4\,, \textit{and } \lambda_5 = \lambda_6\,; \\ \mu_1 = \mu_2\,, \mu_3 = \mu_4\,, \textit{and } \mu_5 = \mu_6\,; \\ \nu_1 = \nu_2\,, \nu_3 = \nu_4\,, \textit{and } \nu_5 = \nu_6\,. \end{cases} \tag{80}$$

Proof. The equalities (80) clearly imply $|\lambda_I| + |\mu_J| + |\nu_K| = 0$. On the other hand, if $|\lambda_I| + |\mu_J| + |\nu_K| = 0$, we get

$$0 = 2|\lambda_I| + 2|\mu_J| + 2|\nu_K| \leq |\lambda| + |\mu| + |\nu| = 0.$$

This proves that the middle inequality is an equality. So (80) hold. □

Lemma 45 shows that the face associated to (I,J,K) is contained in no face corresponding to some LR coefficient equal to one. In particular, Theorem 2 cannot be applied to the points in this face. However, applying our reduction formula and the hive model by Knutson and Tao [KT99], we compute explicitly those coefficients:

Example 4. *Let $\lambda, \mu, \nu \in \Lambda^+_{\mathrm{GL}_3}$ such that $|\lambda| + |\mu| + |\nu| = 0$. Then $m_{\mathrm{GL}_3}(\lambda,\mu,\nu)$ is equal to the number of integers in the interval*

$$[\max(\mu_1 - \lambda_2, \mu_2, -\nu_3 - \lambda_1, \mu_1 + \nu_1, -\nu_2 - \lambda_2, \mu_1 + \mu_2 + \nu_2), \min(\mu_1, -\nu_3 - \lambda_2, \mu_1 + \mu_2 + \nu_1)]\,.$$

Moreover, set $\lambda^2 = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$, and define similarly μ^2 and ν^2. Then

$$m_{\mathrm{GL}_6}(\lambda^2, \mu^2, \nu^2) = \frac{m_{\mathrm{GL}_3}(\lambda,\mu,\nu)(m_{\mathrm{GL}_3}(\lambda,\mu,\nu) + 1)}{2}\,.$$

Proof. The first equality follows from [KT99]. In fact, it is explained in [PN20, Proposition 9] that $c^\nu_{\lambda,\mu}$ is the number of integers in the interval

$$[\max(\mu_1 - \lambda_2, \mu_2, \nu_1 - \lambda_1, \mu_1 - \nu_3, \nu_2 - \lambda_2, \mu_1 + \mu_2 - \nu_2), \min(\mu_1, \nu_1 - \lambda_2, \mu_1 + \mu_2 - \nu_3)]\,.$$

Using the fact that $V^*_{(\nu_1,\nu_2,\nu_3)} \simeq V_{(-\nu_3,-\nu_2,-\nu_1)}$, we get our formula. We want to compute $m_{\mathrm{GL}_6}(\lambda^2,\mu^2,\nu^2)$ using Corollary 40 with $\mathbb{v} = (v_I, v_J, v_K)$ and $I = J = K = \{2,4,6\}$. By Proposition 44, one half of the branch divisor corresponds to the triple of partitions $((2,2,1,1,0,0),(2,2,1,1,0,0),(-1,-1,-2,-2,-3,-3))$. Thus, a term $m_L(v^{-1}(\gamma-k\theta),\hat{v}^{-1}(\hat{\gamma}-k\hat{\theta}))$ in Corollary 40 is equal in our context to

$$\Big(m_{\mathrm{GL}_3}(\lambda-k(210),\mu-k(210),\nu+k(123))\Big)^2.$$

Observe that by the first point, $m_{\mathrm{GL}_3}(\lambda-k(210),\mu-k(210),\nu+k(123))$ is $m_{\mathrm{GL}_3}(\lambda,\mu,\nu)-k$ or 0. Indeed, when λ,μ,ν are replaced respectively by $\lambda-(210),\mu-(210),\nu+(123)$, all the integers that appear in the max decrease by 1 and all the integers that appear in the min decrease by 2. Corollary 40 therefore gives

$$m_{\mathrm{GL}_6}(\lambda^2,\mu^2,\nu^2)=\sum_{k=0}^{m_{\mathrm{GL}_3}(\lambda,\mu,\nu)}(-1)^{m_{\mathrm{GL}_3}(\lambda,\mu,\nu)+k}\,k^2\,.$$

This is equal to $\frac{m_{\mathrm{GL}_3}(\lambda,\mu,\nu)(m_{\mathrm{GL}_3}(\lambda,\mu,\nu)+1)}{2}$, as stated. $\square$

Remark 5. *As we can observe here, the Littlewood-Richardson coefficients are close to being polynomials in the coefficients of the weights. This is a general phenomenon. Kostant* [Kos58, Theorem 6.2] *proves that the multiplicity of a weight in an irreducible representation is a partition function. Steinberg* [Ste61] *deduces that the multiplicity of an irreducible submodule in the tensor product of two representations is again a partition function. Rassart* [Ras04, Theorem 4.1] *deduces that the Horn cone can be subdivided into subcones where the Littlewood-Richardson coefficients are polynomial in the three weights (this holds in type A; in general the coefficients are only quasi-polynomial).*

7.3. **A bigger example.** Here $n=10$, $r=6$ and $G/P=\mathrm{Gr}(6,10)$. Set $I=\{2,4,5,6,9,10\}$, $J=\{3,4,6,7,9,10\}$ and $K=\{2,4,5,7,8,10\}$. Set $\mathbb{v}=(v_I,v_J,v_K)$ in such a way that $c(\mathbb{v})=c^{\lambda(K)}_{\lambda(I^\vee),\lambda(J^\vee)}=c^{433221}_{3222,2211}=2$. Then, applying Theorem 8, one gets:

$$\frac{1}{2}[B_\Pi]=[\mathcal{L}_{(G/B)^3}(\varpi_2+2\varpi_6,\varpi_4+\varpi_7,\varpi_2+\varpi_5+\varpi_8)].\tag{81}$$

First, we prove a statement analogous to Lemma 45:

Lemma 46. *The face defined by the equation $|\lambda_I|+|\mu_J|+|\nu_K|=0$ in $\Gamma(\mathrm{GL}_{10})$ has equations*

$$\begin{cases}|\lambda|+|\mu|+|\nu|=0\,;\\(\lambda_5+\lambda_6)+(\mu_7+\mu_{10})+(\nu_5+\nu_8)=0\,;\\\lambda_1=\lambda_2\,,\lambda_3=\lambda_4\,,\textit{and }\lambda_7=\lambda_8=\lambda_9=\lambda_{10}\,;\\\mu_1=\mu_2=\mu_3=\mu_4\,,\mu_5=\mu_6\,,\textit{and }\mu_8=\mu_9\,;\\\nu_1=\nu_2\,,\nu_3=\nu_4\,,\nu_6=\nu_7\,,\textit{and }\nu_9=\nu_{10}\,.\end{cases}\tag{82}$$

Proof. Let F be the linear space defined by these equations. We observe that the following triples (λ,μ,ν) span F and satisfy $m_{\mathrm{GL}_{10}}(\lambda,\mu,\nu)=1$:

$$\begin{array}{cccc} & (1^{10},0,(-1)^{10}) & (1^{10},0,(-1)^{10}) & \\ (1^6,(-1)^{10},1^4) & (1^5,(-1)^{10},1^5) & (1^2,(-1)^{10},1^8) & (1^6,(-1)^4(-2)^6,1^{10}) \\ (1^6,(-1)^9-2,1^5) & (1^6,(-1)^7(-2)^3,1^7) & (1^6,(-1)^6(-2)^4,1^8) & (1^5,(-1)^7(-2)^3,1^8) \\ (1^6,(-1)^7(-2)^3,2^21^3) & (1^6,(-1)^4(-2)^6,2^21^6) & (1^6,(-1)^4(-2)^3(-3)^3,2^51^3) & (2^41^2,(-2)^7(-3)^3,2^51^3)\end{array}$$

Conversely, let (λ,μ,ν) satisfying $|\lambda_I|+|\mu_J|+|\nu_K|=0$ and $m_{\mathrm{GL}_{10}}(\lambda,\mu,\nu)\neq 0$. To have nicer formulas, we write a instead of 10. We have

$$\begin{array}{rcl} 0=2|\lambda_I|+2|\mu_J|+2|\nu_K| & \leq & |\lambda_{\{12\}}|+|\lambda_{\{34\}}|+2\lambda_5+2\lambda_6+|\lambda_{\{789a\}}| \\ & + & |\mu_{\{1234\}}|+|\mu_{\{56\}}|+2\mu_7+|\mu_{\{89\}}|+2\mu_a \\ & + & |\nu_{\{12\}}|+|\nu_{\{34\}}|+2\nu_5+|\nu_{\{67\}}|+2\nu_8+|\nu_{\{9a\}}| \\ & = & |\lambda_{\{56\}}|+|\mu_{\{7a\}}|+|\nu_{\{58\}}| \\ & \leq & 0\end{array}\tag{83}$$

The last inequality is the Horn inequality associated to the LR coefficient $c(v_{\{56\}},v_{\{7a\}},v_{\{58\}})=c^{64}_{44,2}=1$. It follows that all inequalities in (83) must be equalities, which implies that (λ,μ,ν) belongs to F. $\square$

Remark 6. *We do not know about a general procedure to compute the face given by a triple I, J, K with $c(v_I, v_J, v_K) > 1$. When $c(v_I, v_J, v_K) = 1$, it is known that the corresponding face has codimension 1, but when $c(v_I, v_J, v_K) > 1$ it seems an interesting problem to determine the linear span of the corresponding face, or at least its dimension.*

Remark 7. *In this case, Lemma 46 shows that the face is contained in some regular face associated to some LR coefficient equal to one (namely $c(v_{\{56\}}, v_{\{7a\}}, v_{\{58\}}) = 1$). Then both Theorems 2 and 4 can be applied to the points of the face. These two statements are not concurrent but complementary. Indeed, by applying these results consecutively, one gets an expression of each LR coefficient on the face of Lemma 46 as an alternating sum of products of three LR coefficients. The use of a two step flag variety allows to recover this result from Corollary 40 more conceptually. Indeed, set $I_1 = \{5, 6\}$, $J_1 = \{7, 10\}$ and $K_1 = \{5, 8\}$. Set also $I_2 = I$, $J_2 = J$, and $K_2 = K$. The pairs $(I_1 \subset I_2)$, $(J_1 \subset J_2)$ and $(K_1 \subset K_2)$ define three Schubert classes in $\mathrm{Fl}(2, 6; 10)$.*

Consider the fibration $\mathrm{Fl}(2, 6; 10) \longrightarrow \mathrm{Gr}(6, 10)$ with fiber $\mathrm{Gr}(2, 6)$. The three given Schubert varieties in $\mathrm{Fl}(2, 6; 10)$ map onto X_{I_2}, X_{J_2} and X_{K_2} respectively with fibers isomorphic to $X_{\{34\}}$, $X_{\{46\}}$ and $X_{\{35\}}$ in $\mathrm{Gr}(2, 6)$. The associated Schubert coefficient in $\mathrm{Gr}(6, 10)$ is 2 and in $\mathrm{Gr}(2, 6)$ it is $c^{32}_{22,1} = 1$. Then (see e.g. [Ric12]), the associated Schubert coefficient for $\mathrm{Fl}(2, 6; 10)$ is 2.

Similarly, consider the fibration $\mathrm{Fl}(2, 6; 10) \longrightarrow \mathrm{Gr}(2, 10)$ with fiber $\mathrm{Gr}(4, 8)$. Here we find a Schubert coefficient 1 in the base and 2 in the fiber.

Recalling the BK coefficients $c^{\odot}$ from (63), these two assertions imply that

$$c^{\odot}\big((I_1 \subset I_2), (J_1 \subset J_2), (K_1 \subset K_2)\big) = 2 \tag{84}$$

and we are in position to apply Corollary 40. Since the incidence variety $\tilde{Y}$ in $\mathrm{Fl}(2, 6; 10) \times (G/B)^3$ maps birationally on that corresponding to (I_2, J_2, K_2) in $\mathrm{Gr}(6, 10)$ these two varieties have the same branch divisor given by (81):

$$\frac{1}{2}[B_\Pi] = (3^2 2^4, 2^4 1^3, 3^2 2^3 1^3).$$

Let $(\lambda, \mu, \nu) \in (\Lambda^+_{\mathrm{GL}_{10}})^3$ on the span of the considered face (see Lemma 46):

$$\begin{array}{l} \lambda = \lambda_1^2 \lambda_3^2 \lambda_5 \lambda_6 \lambda_7^4 \\ \mu = \mu_1^4 \mu_5^2 \mu_7 \mu_8^2 \mu_{10} \\ \nu = \nu_1^2 \nu_3^2 \nu_5 \nu_6^2 \nu_8 \nu_9^2 \end{array} \tag{85}$$

The restriction to $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) \otimes \mathcal{L}(\nu)$ to C is

$$\begin{array}{lll} \lambda_{\{56\}} = \lambda_5 \lambda_6 & \lambda_{\{249a\}} = \lambda_1 \lambda_3 \lambda_7^2 & \lambda_{\{1378\}} = \lambda_1 \lambda_3 \lambda_7^2 \\ \mu_{\{7a\}} = \mu_7 \mu_{10} & \mu_{\{3469\}} = \mu_1^2 \mu_5 \mu_8 & \mu_{\{1258\}} = \mu_1^2 \mu_5 \mu_8 \\ \nu_{\{58\}} = \nu_5 \nu_8 & \nu_{\{247a\}} = \nu_1 \nu_3 \nu_6 \nu_9 & \nu_{\{1369\}} = \nu_1 \nu_3 \nu_6 \nu_9. \end{array}$$

Similarly, the restriction of $\frac{1}{2}[B_\Pi]$ to C is as an $(\mathrm{SL}_2 \times \mathrm{SL}_4 \times \mathrm{SL}_4)$-linearized line bundle

$$\begin{array}{ccc} 2\ 2 & 3\ 2\ 0\ 0 & 3\ 2\ 0\ 0 \\ 1\ 0 & 2\ 2\ 1\ 0 & 2\ 2\ 1\ 0 \\ 2\ 1 & 3\ 2\ 1\ 0 & 3\ 2\ 1\ 0. \end{array}$$

It has the same invariant sections as the $(\mathrm{GL}_2 \times \mathrm{GL}_4 \times \mathrm{GL}_4)$-linearized line bundle

$$\begin{array}{ccc} 0\ 0 & 3\ 2\ 0\ 0 & 3\ 2\ 0\ 0 \\ 1\ 0 & 2\ 2\ 1\ 0 & 2\ 2\ 1\ 0 \\ 0\text{-}1 & \text{-}1\text{-}2\text{-}3\text{-}4 & \text{-}1\text{-}2\text{-}3\text{-}4. \end{array}$$

Hence Corollary 40 gives:

$$\begin{array}{rl} m_{\mathrm{GL}_{10}}(\lambda, \mu, \nu) = \sum_k (-1)^k m_{\mathrm{GL}_2}(\lambda_5 \lambda_6, \mu_{7-k}\, \mu_{10}, \nu_5\, \nu_{8+k}) & \\ \times \quad m_{\mathrm{GL}_4}(\lambda_{1-3k}\, \lambda_{3-2k}\, \lambda_7^2, (\mu_{1-2k})^2\, \mu_{5-k}\, \mu_8, \nu_{1+k}\, \nu_{3+2k}\, \nu_{6+3k}\, \nu_{9+4k})^2\,. & \end{array}$$

For $\lambda = [41, 41, 36, 36, 35, 24, 0, 0, 0, 0]$, $\mu = [-41, -41, -41, -41, -48, -48, -49, -65, -65, -72]$ and $\nu = [49, 49, 42, 42, 40, 25, 25, 22, 2, 2]$, this formula gives

$$6 = 1 \times 3^2 - 1 \times 2^2 + 1 \times 1^2.$$

Given a point as in (85), one can also apply Theorem 2 to the triple (I_1, J_1, K_1) to get

$$m_{\mathrm{GL}_{10}}(\lambda,\mu,\nu) = m_{\mathrm{GL}_2}(\lambda_5\lambda_6, \mu_7\mu_{10}, \nu_5\nu_8)\, m_{\mathrm{GL}_8}(\lambda_1^2\lambda_3^2\lambda_7^4, \mu_1^4\mu_5^2\mu_8^2, \nu_1^2\nu_3^2\nu_6^2\nu_9^2).$$

But, for $\tilde I = \{2,4,7,8\}$, $\tilde J = \{3,4,6,8\}$ and $\tilde K = \{2,4,6,8\}$ we have $c(\tilde I, \tilde J, \tilde K) = c^{4321}_{32,221} = 2$ and one can apply Theorem 4 to the second factor. Since $\frac12[B_\Pi] = (\varpi_2+\varpi_4, \varpi_4+\varpi_6, \varpi_2+\varpi_6) = (2^21^2, 2^41^2, 2^21^4)$, one gets that $m_{\mathrm{GL}_{10}}(\lambda,\mu,\nu)$ is equal to

$$m_{\mathrm{GL}_2}(\lambda_5\lambda_6, \mu_7\mu_{10}, \nu_5\nu_8) \sum_{k\ge 0} (-1)^k m_{\mathrm{GL}_4}(\lambda_1{-}2k\,\lambda_3{-}k\,\lambda_7^2, \mu_1^2{+}k\,\mu_5{+}2k\,\mu_8{+}3k, \nu_1{-}2k\,\nu_3{-}k\,\nu_6{-}k\,\nu_9)^2.$$

Applied to the explicit example above, one gets

$$6 = 1 \times (3^2 - 2^2 + 1^2).$$

7.4. **Our map between LR coefficients equal to 2.** Fix a positive integer r and consider the set

$$\mathrm{LR}_2(r) = \{(\lambda,\mu,\nu) \in (\Lambda_r^+)^3 \mid c^\nu_{\lambda,\mu} = 2\}.$$

(here Λ_r^+ denotes the set of partitions with at most r parts). Start with a triple $(\lambda,\mu,\nu) \in \mathrm{LR}_2(r)$ and let $n = r + \max(\lambda_1, \mu_1, \nu_1) = r + \nu_1$. Set $I, J, K \subset \{1, \ldots, n\}$ corresponding to λ, μ, ν. Our construction in Section 2.2 yields a variety $Y(\mathbb{v})$, a morphism $\Pi : Y(\mathbb{v}) \to \mathrm{Fl}(n)^3$, and a branch divisor $B_\Pi \subset \mathrm{Fl}(n)^3$ given by a triple (α, β, γ) of weights of SL_n. By Theorem 5,

$$(\alpha_I, \beta_J, \gamma_K) \in \mathrm{LR}_2(r),$$

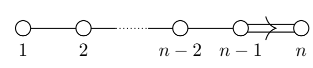
yielding a map from $\mathrm{LR}_2(r)$ to itself.

We do not understand the orbits of this map. For example, we do not know if any orbit is finite. We checked (see [CR]) that for $n \le 12$ the orbits contain at most 4 different elements of $\mathrm{LR}_2(r)$. Here is an example in $\mathrm{LR}_2(7)$:

$$\begin{tikzcd}
c^{5^24^2321}_{5421,5321^2} = 2 \arrow[d] \\
c^{16\,15\,14\,13^29^2}_{9^3874,9^27^265} = 2 \arrow[d] \\
c^{3^421}_{3^221,321} = 2 \arrow[d, bend right] \\
c^{6543^4}_{3^321,3^421} = 2 \arrow[u, bend right]
\end{tikzcd}$$

The first studied example with $I = J = K = \{2,4,6\}$, corresponding to $c^{321}_{21,21} = 2$ in $\mathrm{LR}_2(3)$, is a fixed point of this map.

7.5. **Examples in type B, C, D.** Many results in this section were obtained with the help of a computer, see [CR]. Given a weight ζ, we associate a partition λ to it with parts λ_i such that $\zeta = \sum \lambda_i \epsilon_i$, with the notation of [Bou02], (note that, given ζ all the coefficients λ_i are in $\mathbb{Z}$ or they are all in $\frac12 + \mathbb{Z}$). In type B, C, D, the vector representation has dimension N and we chose by convention that the bilinear form takes non-zero values exactly on the pairs (e_i, e_{N+1-i}) of vectors, see Section 7.1.

7.5.1. $G = \mathrm{Spin}(2n+1)$. Number the simple roots as follows: 

. Note that the Horn inequality associated to a triple (I, J, K) of subsets of $\{1, \ldots, n\} \cup \{n+2, \ldots, 2n+1\}$ such that $c(I, J, K) \ne 0$ is

$$|\lambda_{I\cap[1;n]}| - |\lambda_{\overline{I\cap[n+2;2n+1]}}| + |\mu_{J\cap[1;n]}| - |\mu_{\overline{J\cap[n+2;2n+1]}}| + |\nu_{K\cap[1;n]}| - |\nu_{\overline{K\cap[n+2;2n+1]}}| \le 0. \tag{86}$$

In the cohomology of the quadric Q^7, for $I = J = \{7\}$ and $K = \{6\}$, we have $\sigma_{\{7\}} \odot \sigma_{\{7\}} \odot \sigma_{\{6\}} = 2[pt]$ and (86) means

$$\lambda_3 + \mu_3 + \nu_4 \ge 0. \tag{87}$$

Using [CR], we get $\theta = (\varpi_2, \varpi_2, \varpi_3) = (1^2, 1^2, 1^3)$ and (87) is an equality, as predicted by Proposition 39. Here $L^{\rm ss} = {\rm Spin}(7)$. We put superscript ϖ^G and ϖ^L to make the difference between weights of G and L. We have $v_I^{-1}(\varpi_2^G) = v_I^{-1}(\epsilon_1^G + \epsilon_2^G) = \epsilon_2^G + \epsilon_3^G$, and its restriction to L is $\epsilon_1^L + \epsilon_2^L$, namely ϖ_2^L. Similarly, $(v_K^{-1}(\varpi_3^G))^L = 2\varpi_3^L$. Using the equality $m_L(\varpi_2, \varpi_2, 2\varpi_3) = 1$ and the following Remark 8, we get:

$$m_L(k{v_I}^{-1}v\theta^1, k{v_J}^{-1}\theta^2, kv_K^{-1}\theta^3) = m_L(k\varpi_2, k\varpi_2, 2k\varpi_3) = 1.$$

Corollary 40 gives $m_G(k\varpi_2, k\varpi_2, k\varpi_3) = 1$ if k is even and 0 if k is odd, consistently with the non-saturation of the tensor semigroup $\Gamma(G) := \{(\zeta_1, \zeta_2, \zeta_3) \in (\Lambda_G^+)^3 \,:\, m_G(\zeta_1, \zeta_2, \zeta_3) \neq 0\}$.

Remark 8. *The group* ${\rm Spin}_7$ *has a dense orbit in* ${\rm OG}(2,7) \times {\rm OG}(2,7) \times {\rm OG}(3,7)$.

Proof. Pierre-Emmanuel: ▶ Si tu raccourcis cette preuve, laisse la en commentaire stp, et je la mettrai dans la version longue. ◀

We consider the isotropic space $\mathbb{C}^2_a$ generated by e_1 and e_2, the isotropic space $\mathbb{C}^2_b$ generated by e_6 and e_7, and the non-isotropic space $\mathbb{C}^3$ generated by e_3, e_4, e_5. Let V_3 be the isotropic space generated by $v_1 = e_1 + e_3, v_2 = e_5 - e_7$ and $v_3 = e_2 + e_4 - e_6$. We claim that the isotropy of the triple $(\mathbb{C}^2_a, \mathbb{C}^2_b, V_3)$ in ${\rm Spin}_7$ is 1-dimensional, which proves the statement since $\dim({\rm OG}(2,7) \times {\rm OG}(2,7) \times {\rm OG}(3,7)) = 20 = \dim({\rm Spin}_7) - 1$.

Let indeed g be in the isotropy. Since g stabilizes $\mathbb{C}^2_a$ and $\mathbb{C}^2_b$, it stabilizes $\mathbb{C}^3$ as it is the subspace orthogonal to $\mathbb{C}^2_a \oplus \mathbb{C}^2_b$. The action of g on $\mathbb{C}^2_b \simeq (\mathbb{C}^2_a)^*$ is the dual of that of g on $\mathbb{C}^2_a$.

Observe that g must stabilize the line generated by v_1, since it is the intersection of V_3 and $\mathbb{C}^2_a \oplus \mathbb{C}^3$. Similarly it stabilizes the line generated by v_2. It follows that there exists a scalar λ such that $g \cdot e_1 = \lambda e_1, g \cdot e_3 = \lambda e_3, g \cdot e_7 = \lambda^{-1} e_7$ and $g \cdot e_5 = \lambda^{-1} e_5$. Then g preserves $\mathbb{C}^2_a \cap e_7^\perp$, which is the line generated by e_2. It follows that there exists a scalar μ such that $g \cdot e_2 = \mu e_2, g \cdot e_6 = \mu^{-1} e_6$ and $g \cdot e_4 = e_4$. Since g preserves the line generated by v_3, we have $\mu = 1$. Thus g is uniquely defined by λ and the isotropy has dimension 1. □

In the cohomology ring of ${\rm OG}(4,9)$, for $I = J = \{3,6,8,9\}$ and $K = \{1,3,6,8\}$, we have $\sigma_I \odot \sigma_J \odot \sigma_K = 2[pt]$. Inequality (86) means

$$\lambda_1 + \lambda_2 + \lambda_4 + \mu_1 + \mu_2 + \mu_4 + \nu_2 + \nu_4 \geq \lambda_3 + \mu_3 + \nu_1 + \nu_3. \tag{88}$$

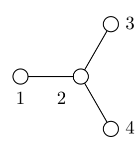
Using [CR], we get $\theta = (\varpi_3, \varpi_3, \varpi_1 + \varpi_3) = (1^3\,0, 1^3\,0, 2\,1^2\,0)$ and once again, (88) is an equality.

Here $L^{\rm ss} = {\rm SL}(4)$. We have $v_I^{-1}(\theta^1) = v_J^{-1}(\theta^2) = v_I^{-1}(\epsilon_1^G + \epsilon_2^G + \epsilon_3^G) = -\epsilon_4^G - \epsilon_3^G + \epsilon_1^G$ and this restricts to $2\epsilon_1^L + \epsilon_2^L = \varpi_1^L + \varpi_2^L$. We also have $v_K^{-1}(\theta^3)^L = \varpi_1^L + \varpi_2^L + \varpi_3^L$. We get $m_L(k{v_I}^{-1}\theta^1, k{v_J}^{-1}\theta^2, kv_K^{-1}\theta^3) = c^{3k\,2k\,k}_{2k\,k,\,2k\,k} = k+1$. Corollary 40 gives $m_G(k\varpi_3, k\varpi_3, k\varpi_1 + \varpi_3) = \lceil \frac{k}{2} \rceil$.

In the cohomology ring of ${\rm OG}(4,9)$, for $I = \{3,6,8,9\}$ and $J = K = \{2,4,7,9\}$, we have $\sigma_I\sigma_J\sigma_K = 2[pt]$ and $\sigma_I \odot \sigma_J \odot \sigma_K = 0$. Inequality (86) means

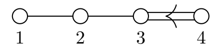
$$\lambda_1 + \lambda_2 + \lambda_4 + \mu_1 + \mu_3 + \nu_1 + \nu_3 \geq \lambda_3 + \mu_2 + \mu_4 + \nu_2 + \nu_4. \tag{89}$$

Using [CR], we get $\theta = (\varpi_3, \varpi_2 + \varpi_4, \varpi_2 + \varpi_4) = (1^3\,0, \frac{1}{2}(3^2\,1^2), \frac{1}{2}(3^2\,1^2))$, and this does not belong to the face. This shows that the Levi-movability assumption in Proposition 39 is necessary. Moreover, $\dim(V_G(\theta^1) \otimes V_G(\theta^2) \otimes V_G(\theta^3))^G = 2$ and $\dim(V_G(2\theta^1) \otimes V_G(2\theta^2) \otimes V_G(2\theta^3))^G = 6$. The restriction of $\frac{1}{2}[B_\Pi]$ to C is $(21, 321, 321)$. Corollary 40 does not apply.

7.5.2. $G = {\rm Spin}(8)$. Number the simple roots as follows: 

.

In the cohomology ring of the orthogonal Grassmannian ${\rm OG}(2,8)$, we have $\sigma_{\{6,8\}}\sigma_{\{3,7\}}\sigma_{\{3,7\}} = 2[pt]$ and $\sigma_{\{6,8\}} \odot \sigma_{\{3,7\}} \odot \sigma_{\{3,7\}} = 0$. The corresponding inequality is $\lambda_1 + \lambda_3 + \mu_2 + \nu_2 \geq \mu_3 + \nu_3$. Using [CR], we get $\frac{1}{2}[B_\Pi] = (\varpi_2, \varpi_1 + \varpi_3 + \varpi_4, \varpi_1 + \varpi_3 + \varpi_4) = (11, 211, 211)$, and this does not belong to the face. We have $\dim(V_G(\theta^1) \otimes V_G(\theta^2) \otimes V_G(\theta^3))^G = 3$ and $\dim(V_G(2\theta^1) \otimes V_G(2\theta^2) \otimes V_G(2\theta^3))^G = 7 \neq \frac{3\times 4}{2}$ (compare with Theorem 5).

7.5.3. $G = \mathrm{Sp}(8)$. Number the simple roots as follows: .

In the cohomology ring of the Lagrangian Grassmannian $\mathrm{IG}(4,8)$, we let $I = \{2,5,6,8\}$, $J = \{3,4,7,8\}$ and $K = \{2,4,6,8\}$. Then we have $\sigma_I \odot \sigma_J \odot \sigma_K = 2[pt]$ and we have $\theta = (\varpi_2, \varpi_4, \varpi_4) = (1^2\,0^2, 1^4, 1^4)$. Moreover $L^{\mathrm{ss}} = \mathrm{SL}(4)$ and $\mathcal{L}(\theta)_{|C}$ corresponds to $(211, 22, 22)$. Hence $m_L(kv_I^{-1}\theta^1, kv_I^{-1}\theta^2, kv_K^{-1}\theta^3) = 1$. Corollary 40 gives the equality $m_G(k\varpi_2, k\varpi_2, k\varpi_3) = 1$ if k is even and 0 if k is odd, consistently with the non-saturation of the tensor semigroup.

In the cohomology ring of the symplectic Grassmannian $\mathrm{IG}(2,8)$, we have $\sigma_{\{5,7\}}\sigma_{\{5,7\}}\sigma_{\{4,7\}} = 2[pt]$ and $\sigma_{\{5,7\}} \odot \sigma_{\{5,7\}} \odot \sigma_{\{4,7\}} = 0$. We get $\theta = (\varpi_1 + \varpi_3, \varpi_1 + \varpi_3, \varpi_2 + \varpi_4)$, and this does not belong to the face. We have $\dim\left(V_G(\theta^1) \otimes V_G(\theta^2) \otimes V_G(\theta^3)\right)^G = 3$ and $\dim\left(V_G(2\theta^1) \otimes V_G(2\theta^2) \otimes V_G(2\theta^3)\right)^G = 11$.

Appendix A. Appendix: puzzles

A.1. **Some computations.** In this appendix, we give an example for recent results [KP11, Res11a, Ric12] about such coefficients when G/P is a partial flag variety. Even more precisely, we consider the flag variety $\mathrm{Fl}(3,6;9)$ parametrizing flags $(V_3 \subset V_6)$ in $\mathbb{C}^9$ with $\dim(V_3) = 3$ and $\dim(V_6) = 6$. We have $\mathrm{Fl}(3,6;9) = G/P$ and $\mathrm{Gr}(6,9) = \mathrm{SL}_9/Q$ with $P \subset Q \subset G = \mathrm{SL}_9$ the corresponding parabolic subgroups.

Consider the Schubert class $\omega = (\{3,6,9\} \subset \{2,3,5,6,8,9\})$ for $\mathrm{Fl}(3,6;9)$. The projection of the Schubert variety X_ω on $\mathrm{Gr}(6,9)$ is X_I with $I = \{2,3,5,6,8,9\}$. Its intersection with the fiber $\mathrm{Gr}(3,6)$ is X_J with $J = \{2,4,6\}$.

Example 5. *We have the equalities*

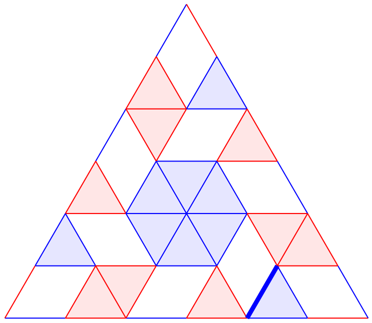
$$c_{\mathrm{Gr}(3,6)}(J,J,J) = 2\,,\ c_{\mathrm{Gr}(6,9)}(I,I,I) = 3\,,\ \textit{and}\ c_{\mathrm{Fl}(3,6;9)}(\omega,\omega,\omega) = 6\,.$$

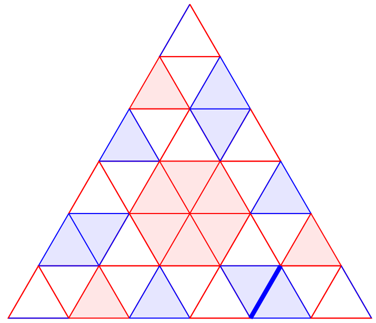
Remark 9. *Since* $6 = 3 * 2$*, this example is compatible with* [DW11, Theorem 7.14], [Ric12, Theorem 1.1], *and* [Res11a, Theorem A]. *However, it shows that* [KP11, Theorem 3] *cannot be correct, since the right hand side of* [KP11, Theorem 3] *is equal to* $2 * 2 * 2 = 8$.

Proof. Recall [KT03] that the Belkale-Kumar coefficients in type A flag varieties can be computed as a number of puzzles. The 2 puzzles corresponding to the equality $c_{\mathrm{Gr}(3,6)}(J,J,J) = 2$ are displayed in Appendix 1.2. The puzzles corresponding to the equality $c_{\mathrm{Gr}(6,9)}(I,I,I) = 3$ are displayed in Appendix 1.3, and the puzzles corresponding to the equality $c_{\mathrm{Fl}(3,6;9)}(\omega,\omega,\omega) = 6$ are displayed in Appendix 1.4. $\square$

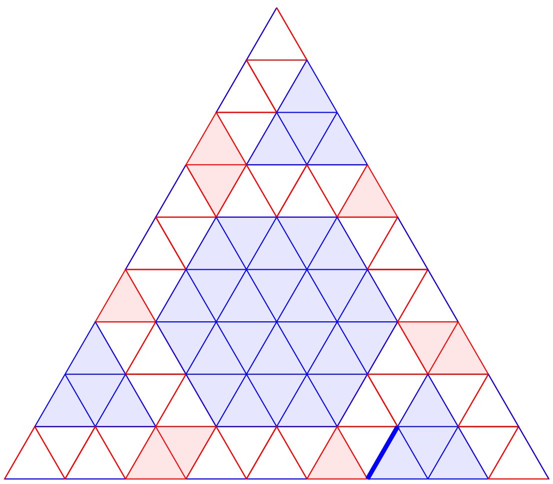
Remark 10. [KP11, Theorem 1] *is compatible with this example, but* [KP11, Theorem 2] *is not. In fact, let us denote by R the puzzle displayed on the left in Appendix 1.2 (with a Rhombus over the edge drawn with a thick line) and by T the puzzle on the right (with a Triangle). Performing the map (D_{23}, D_{13}, D_{12}) of* [KP11, Theorem 2] *on the six puzzles of Appendix 1.4, we get the six triples (R,R,R), (T,R,R), (T,T,R), (T,T,T), (R,R,T), (R,T,T). So the map in* [KP11, Theorem 2] *is injective, but its image misses the two triples (R,T,R) and (T,R,T).*

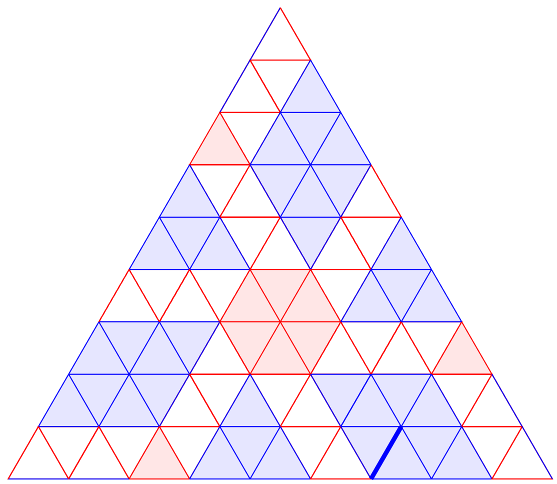
1.2. **The puzzles for** $c_{\mathrm{Gr}(3,6)}(J,J,J) = 2$ **with** $J = \{2,4,6\}$**.**

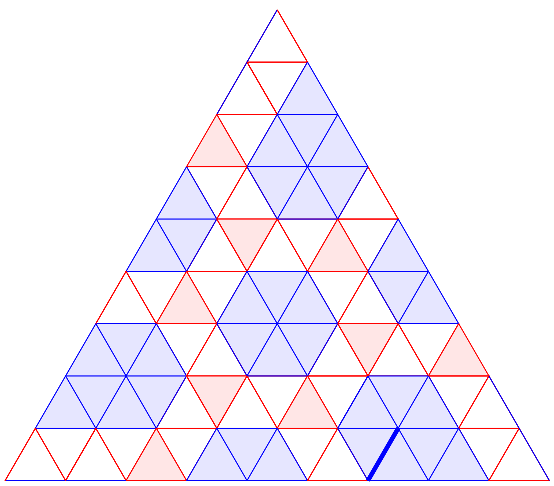




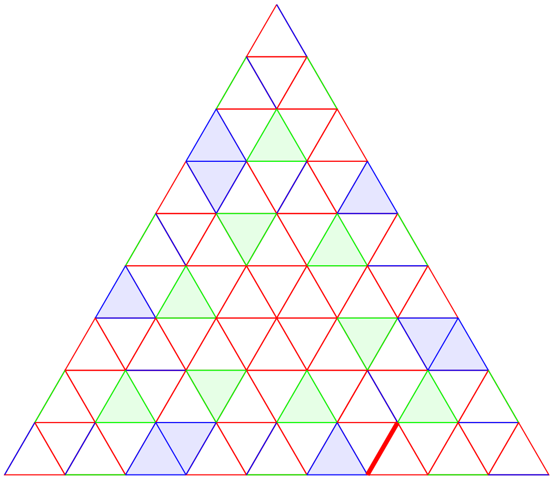
1.3. **The puzzles for** $c_{\mathrm{Gr}(6,9)}(I,I,I) = 3$ **with** $I = \{2,3,5,6,8,9\}$**.**

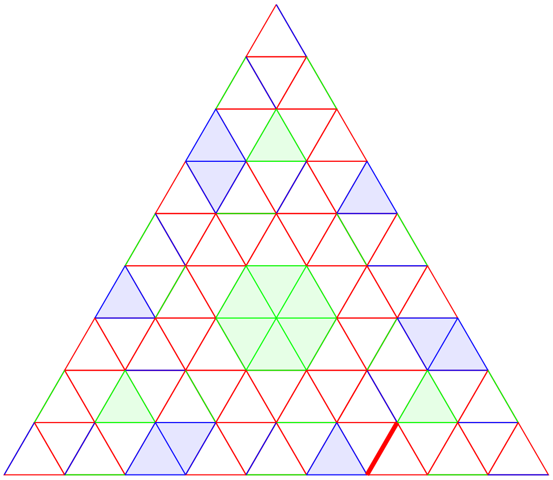


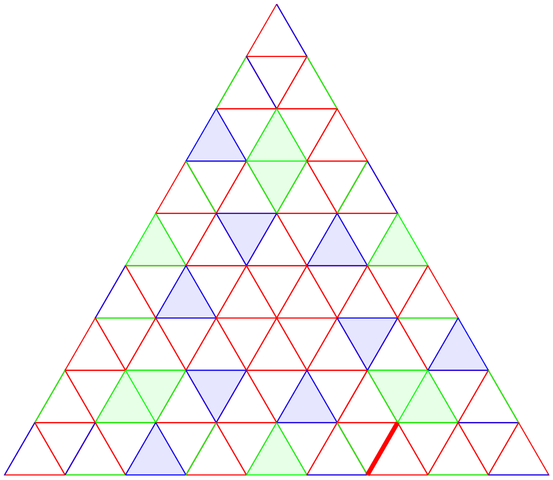


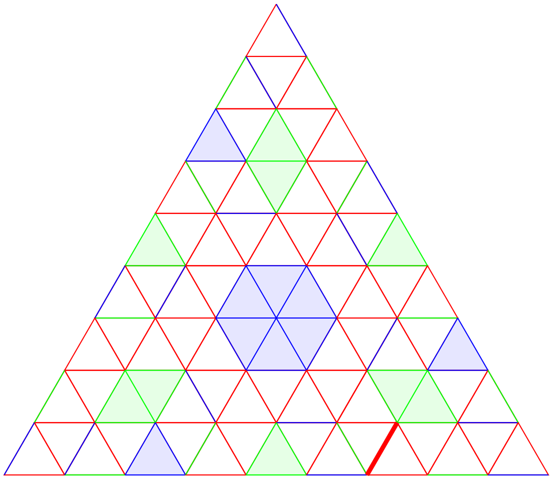


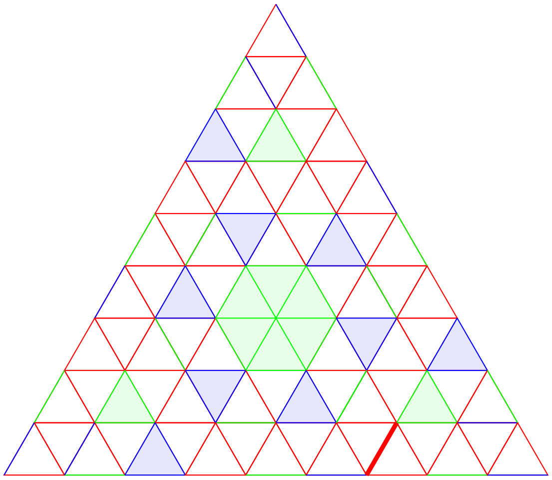
1.4. **The puzzles for** $c_{\mathcal{F}(3,6;9)}(\omega,\omega,\omega) = 6$ **with** $\omega = (\{3,6,9\} \subset \{2,3,5,6,8,9\})$**.**

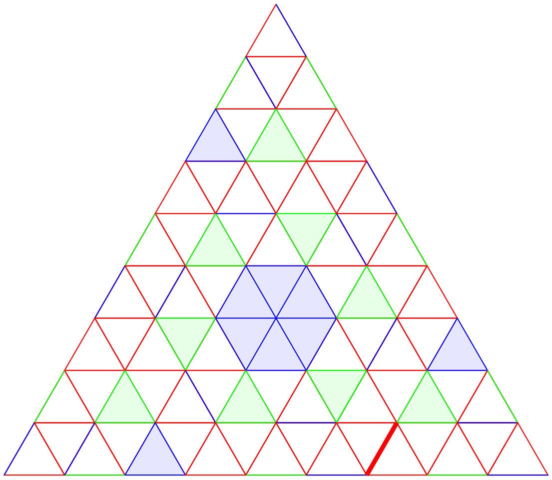












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