Multiplicative formulas in Schubert calculus and quiver representation

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Abstract

Consider a flag variety X and its cohomology ring $\mathrm{H}^*(X, \mathbb{Z})$ endowed with the Schubert basis. In [Ric09], E. Richmond showed that some structure coefficients of the cup product in $\mathrm{H}^*(X, \mathbb{Z})$ are products of two such coefficients for smaller flag varieties. Consider a quiver without oriented cycle. If α and β are two dimension vectors, $\alpha \circ \beta$ denotes the number of α-dimensional subrepresentations of a general $\alpha + \beta$-dimensional representation. In [DW10], H. Derksen and J. Weyman expressed some numbers $\alpha \circ \beta$ as products of two such numbers for smaller dimension vectors. The aim of this work is to prove two generalizations of the two above results by the same method.

Keywords: Schubert calculus, Quiver representation

1. Introduction

We work over an algebraically closed field $\mathbb{K}$ of characteristic zero. Let G be a semi-simple group, let $T \subset B \subset Q \subset G$ be a maximal torus, a Borel subgroup and a parabolic subgroup respectively. In [BK06], P. Belkale and S. Kumar defined a new product $\odot_0$ (associative and commutative) on the cohomology group $\mathrm{H}^*(G/Q, \mathbb{Z})$. Any structure coefficient of $\odot_0$ in the Schubert basis is either zero or the corresponding structure coefficient for the cup product. An important motivation to study this product is its relations with the eigencone of G (see [Res10b]).

Let now $P \supset Q$ be a second parabolic subgroup of G and let L denote the Levi subgroup of P containing T.

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Theorem A. *Any structure coefficient of* $(\mathrm{H}^*(G/Q,\mathbb{Z}),\odot_0)$ *in the Schubert basis is the product of such two coefficients for* $(\mathrm{H}^*(G/P,\mathbb{Z}),\odot_0)$ *and* $(\mathrm{H}^*(L/(L\cap Q),\mathbb{Z}),\odot_0)$ *respectively.*

Actually Theorem 2, stated in Section 3, is more explicit than Theorem A. This result was already obtained in [Ric09] when $G=\mathrm{SL}_n$, Q is any parabolic subgroup and P is the maximal parabolic subgroup corresponding to the linear subspace in G/Q of minimal dimension. Note that E. Richmond obtained Theorem A in [Ric11] independently.

Let Q be a quiver. Given two dimension vectors α and β, $\alpha\circ\beta$ denotes the number of α-dimensional subrepresentations of a general $\alpha+\beta$-dimensional representation. The Ringel form (see Section 4.1) is denoted by $\langle\cdot,\cdot\rangle$.

Theorem B. *Let* α, β, *and* γ *be three dimension vectors. Assume that* $\langle\alpha,\beta\rangle=\langle\alpha,\gamma\rangle=\langle\beta,\gamma\rangle=0$. *Then*

$$(\alpha+\beta\circ\gamma).(\alpha\circ\beta)=(\alpha\circ\beta+\gamma).(\beta\circ\gamma).$$

Note that Theorem 3, stated in Section 4, is more general than Theorem B, since s dimension vectors occur. We obtain the following result as a corollary of Theorem B.

Theorem C. *Assume that* Q *has no oriented cycle. Let* α, β, *and* γ *be three dimension vectors such that* $\langle\alpha,\beta\rangle=\langle\alpha,\gamma\rangle=0$ *and* $\beta\circ\gamma=1$.
Then $\alpha\circ(\beta+\gamma)=(\alpha\circ\beta).(\alpha\circ\gamma)$.

This result is not readily stated in [DW10]. However the proof of [DW10, Theorem 7.14] implies it. Note that the proof of Theorem B is really different from that of [DW10, Theorem 7.14]. Indeed the numbers $\alpha\circ\beta$ have two nontrivially equivalent interpretations (see [DSW07]): the number of points in a general fiber of a morphism or the dimension of the subspace of invariant vectors in a representation. Here we use the first characterization while Derksen-Weyman used the second one. A consequence is that in our Theorem B, it is not useful to assume that Q has no oriented cycle.

We consider more generally a semi-simple group G acting on a variety X. Fix a one parameter subgroup λ of G. Let C be an irreducible component of the fixed point set of λ in X. In Section 2, we define and study the integers $d(G,X,C,\lambda)$. These numbers generalize both the structure coefficients of the Schubert calculus and the numbers $\alpha\circ\beta$. Theorem 1 below provides a multiplicative formula for some $d(G,X,C,\lambda)$ and then it is applied to the two situations.

2. Degree of dominant pairs

2.1. Definitions

Let G be a reductive group acting on a smooth irreducible variety X. Let λ be a one parameter subgroup of G. Let L denote the centralizer of λ in G.

Consider the usual parabolic subgroup $P(\lambda)$ associated to λ with Levi subgroup L;

$$P(\lambda) = \left\{ g \in G \,:\, \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let C be an irreducible component of the fixed point set X^λ of λ in X. Consider also the Białynicki-Birula cell C^+ associated to C:

$$C^+ = \{x \in X \mid \lim_{t \to 0} \lambda(t)x \text{ exists and belongs to } C\}.$$

Then C is stable by the action of L and C^+ is stable by the action of $P(\lambda)$. Consider over $G \times C^+$ the action of $G \times P(\lambda)$ given by the formula (with obvious notation): $(g, p).(g', y) = (gg'p^{-1}, py)$. Consider the quotient $G \times_{P(\lambda)} C^+$ of $G \times C^+$ by the action of $\{e\} \times P(\lambda)$. The class of a pair $(g, y) \in G \times C^+$ in $G \times_{P(\lambda)} C^+$ is denoted by $[g : y]$. The action of $G \times \{e\}$ induces an action of G on $G \times_{P(\lambda)} C^+$. Moreover the first projection $G \times C^+ \longrightarrow G$ induces a G-equivariant map $\pi : G \times_{P(\lambda)} C^+ \longrightarrow G/P(\lambda)$ which is a locally trivial fibration with fiber C^+. In particular

$$\dim(G \times_{P(\lambda)} C^+) = \dim(G/P(\lambda)) + \dim(C^+).$$

Consider also the G-equivariant map $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$, $[g : y] \mapsto gy$. We finally obtain

$$\begin{tikzcd} G \times_{P(\lambda)} C^+ \arrow[r, "\eta"] \arrow[d, "\pi"] & X \\ G/P(\lambda). & \end{tikzcd}$$

It is well known that the map

$$\begin{array}{rccc} (\pi, \eta) : & G \times_{P(\lambda)} C^+ & \longrightarrow & G/P(\lambda) \times X \\ & [g : y] & \longmapsto & (gP(\lambda), gy) \end{array} \tag{1}$$

is an immersion; its image is the set of the $(gP(\lambda), x) \in G/P(\lambda) \times X$ such that $g^{-1}x \in C^+$. Note that this fact can be used to endow $G \times_{P(\lambda)} C^+$ with a structure of variety.

Definition 1. Set

$$\begin{array}{rl} \delta(G, X, C, \lambda) & = \dim(X) - \dim(G/P(\lambda)) - \dim(C^+) \\ & = \operatorname{codim}(C^+, X) - \operatorname{codim}(P(\lambda), G), \end{array}$$

where $\operatorname{codim}(Z, Y)$ denotes the codimension of Z in Y. If $\delta(G, X, C, \lambda) = 0$ and η is dominant, it induces a finite field extension: $\mathbb{K}(X) \subset \mathbb{K}(G \times_{P(\lambda)} C^+)$. The degree of this extension is denoted by $d(G, X, C, \lambda)$. If $\delta(G, X, C, \lambda) \neq 0$ or η is not dominant, we set $d(G, X, C, \lambda) = 0$. More generally, we define the degree of any morphism to be the degree of the induced extension if it is finite and zero otherwise.

2.2. A product formula for $d(G, X, C, \lambda)$

Let T be a maximal torus of G and let x_0 be a T-fixed point in X. We keep notation of Section 2.1 and we assume that the image of λ is contained in T and that $x_0 \in C$. Set $P = P(\lambda)$.

Let λ_ε be another one parameter subgroup of T. Set $P_\varepsilon = P(\lambda_\varepsilon)$. Consider the irreducible component C_ε of X^{λ_ε} which contains x_0 and the set C_ε^+ of points $x \in X$ such that $\lim_{t\to 0} \lambda_\varepsilon(t)x$ exists and belongs to C_ε. Assume that

(i) $P_\varepsilon \subset P$,
(ii) $C_\varepsilon^+ \subset C^+$, and
(iii) $C_\varepsilon \subset C$.

Remark 1. Let $Y(T)$ denote the group of one parameter subgroups of T. Set $Y(T)_{\mathbb{Q}} = Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that the set of one parameter subgroups λ_ε that satisfy these three assumptions generated an open convex cone in $Y(T)_{\mathbb{Q}}$ containing λ.

To compare η and η_ε, we introduce the morphism

$$\begin{array}{rccc} \eta_L : & L \times_{P_\varepsilon \cap L} (C_\varepsilon^+ \cap C) & \longrightarrow & C, \\ & [l : x] & \longmapsto & lx. \end{array}$$

This morphism is a map η like in Section 2.1 with $G = L$, $X = C$, $C = C_\varepsilon$ and $\lambda = \lambda_\varepsilon$. In particular we have defined $\delta(L, C, C_\varepsilon, \lambda_\varepsilon)$ and $d(L, C, C_\varepsilon, \lambda_\varepsilon)$.

Theorem 1. *With above notation*

(i) $\delta(G, X, C_\varepsilon, \lambda_\varepsilon) = \delta(L, C, C_\varepsilon, \lambda_\varepsilon) + \delta(G, X, C, \lambda)$*;*
(ii) *if* $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = \delta(G, X, C, \lambda) = 0$ *then*

$$d(G, X, C_\varepsilon, \lambda_\varepsilon) = d(L, C, C_\varepsilon, \lambda_\varepsilon) \cdot d(G, X, C, \lambda).$$

2.3. Proof of Theorem 1

2.3.1— Consider the two auxiliary varieties

$$Y_L = L \times_{P_\varepsilon \cap L} (C_\varepsilon^+ \cap C) \quad \text{and} \quad Y_P = P \times_{P_\varepsilon} C_\varepsilon^+,$$

and the two auxiliary morphisms

$$\eta_P : Y_P \longrightarrow C^+, [p : x] \longmapsto px,$$

and

$$[\mathrm{Id} : \eta_P] : G \times_P Y_P \longrightarrow G \times_P C^+, \ [g : [p : x]] \longmapsto [g : px].$$

Lemma 1. *The map* $G \times_P Y_P \longrightarrow G \times_{P_\varepsilon} C_\varepsilon^+$, $[g : [p : x]] \longmapsto [gp : x]$ *is an isomorphism denoted by* ι. *Moreover* $\eta_\varepsilon \circ \iota = \eta \circ ([\mathrm{Id} : \eta_P])$.

Proof. The morphism ι commutes with the two projections on G/P. Moreover the restriction of ι over P/P is the closed immersion $P \times_{P_\varepsilon} C_\varepsilon^+ \longrightarrow G \times_{P_\varepsilon} C_\varepsilon^+$. It follows (see for example [Res04, Appendix]) that ι is an isomorphism.

The morphisms $\eta_\varepsilon \circ \iota$ and $\eta \circ ([\mathrm{Id} : \eta_P])$ are G-equivariant and extend the immersion of C_ε^+ in X. They have to be equal.

2.3.2— To study η_P, consider the two following limit morphisms:

$$\begin{array}{rccl} \Lambda_P : & P & \longrightarrow & L \\ & p & \mapsto & \lim_{t\to 0} \lambda(t) p \lambda(t^{-1}) \end{array} \quad \text{and} \quad \begin{array}{rccl} \Lambda^+ : & C^+ & \longrightarrow & C \\ & x & \mapsto & \lim_{t\to 0} \lambda(t) x. \end{array}$$

Lemma 2. *For any p in P and x in C^+, we have $\Lambda^+(px) = \Lambda_P(p)\Lambda^+(x)$.*

Proof. The lemma is obtained by taking the limit in the identity $\lambda(t)px = \lambda(t)p\lambda(t^{-1})\lambda(t)x$.

2.3.3— Recall that $\Lambda^+ : C^+ \longrightarrow C$ is an affine bundle with fibers isomorphic to affine spaces (see [BB73]). The pullback of this affine bundle by η_L is

$$\eta_L^*(C^+) = \{([l : x], y) \in Y_L \times C^+ \mid lx = \Lambda^+(y)\},$$

endowed with the first projection p_1 on Y_L. Consider the following diagram

$$\begin{tikzcd} Y_P \arrow[rr, "\Theta \,:\, {[p:x]} \mapsto ({[\Lambda_P(p) : \Lambda^+(x)]}, px)"] \arrow[dr, "{[p:x]} \mapsto {[\Lambda_P(p) : \Lambda^+(x)]}"'] & & \eta_L^*(C^+) \arrow[dl, "p_1"] \\ & Y_L \arrow[d] & \\ & L/(P_\varepsilon \cap L). & \end{tikzcd} \tag{2}$$

Lemma 3. *Diagram (2) is commutative, and the top horizontal map Θ is an isomorphism.*

Proof. Lemma 2 shows that the map $Y_P \longrightarrow Y_L$ in diagram (2) is well defined. Diagram (2) is obviously commutative.

Since all the morphisms in diagram (2) are L-equivariant, [Res04, Appendix] implies that it is sufficient to prove that Θ is an isomorphism when restricted over the class of e in $L/(P_\varepsilon \cap L)$. The fiber in Y_L over this point is $C \cap C_\varepsilon^+$. Since the unipotent radical P^u of P is contained in that P_ε^u of P_ε, the fiber in

Y_P identify with C_ε^+, by $x \in C_\varepsilon^+ \mapsto [e : x]$. Note that C_ε^+ is the set of points y in C^+ such that $\Lambda^+(y)$ belongs to $C_\varepsilon \cap C$. Then the map $C_\varepsilon^+ \longrightarrow \eta_L^*(C^+)$, $y \longmapsto ([e : \Lambda^+(y)], y)$ identifies the fiber in $\eta_L^*(C^+)$ with C_ε^+. Moreover the restriction of Θ to these fibers is the identity. It follows that Θ is an isomorphism.

2.3.4— We can now prove Theorem 1.

Proof of Theorem 1. Lemma 3 allows to consider the following commutative diagram

$$\begin{tikzcd}
Y_P \arrow[rr, "\eta_P"] \arrow[dr, "\Theta"'] & & C^+ \arrow[dd, "\Lambda^+"] \\
& \eta_L^*(C^+) \arrow[ur] \arrow[d, "p_1"'] & \\
& Y_L \arrow[r, "\eta_L"] & C.
\end{tikzcd}$$

Since Θ is an isomorphism, $\dim(C^+)-\dim(Y_P) = \delta(L, C, C_\varepsilon, \lambda_\varepsilon)$ and $d(L, C, C_\varepsilon, \lambda_\varepsilon)$ equals the degree of η_P. Moreover Lemma 1 shows that the following diagram

$$\begin{tikzcd}
G \times_{P_\varepsilon} C_\varepsilon^+ \arrow[r, "\sim"] \arrow[dr, "\eta_\varepsilon"'] & G \times_P Y_P \arrow[d, "{[id : \eta_P]}"] \\
& G \times_P C^+ \arrow[d, "\eta"] \\
& X
\end{tikzcd}$$

is commutative. The first assertion of the theorem follows immediately. Let d denote the degree of $[id : \eta_P]$ that is the degree of η_P. Since $d = d(L, C, C_\varepsilon, \lambda_\varepsilon)$, it remains to prove that $d(G, X, C_\varepsilon, \lambda_\varepsilon) = d.d(G, X, C, \lambda)$. Assume firstly that $d(G, X, C_\varepsilon, \lambda_\varepsilon) = 0$. Since $\delta(G, X, C_\varepsilon, \lambda_\varepsilon) = 0$, η_ε is not dominant. Hence η or $[id : \eta_P]$ is not dominant. It follows that either $d(G, X, C, \lambda)$ or d is zero.

Assume now that $d(G, X, C_\varepsilon, \lambda_\varepsilon) \neq 0$, and so that η_ε is dominant. Since the image of η_ε is contained in the image of η, η is dominant. Since η_ε is dominant, the dimension of the closure of the image of $[id : \eta_P]$ is at least those of X. Since $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = \delta(G, X, C, \lambda) = 0$, this implies that η_P is dominant. Now, the second assertion is a consequence of the multiplicative formula for the degree of a double extension field.

2.4. Well generically finite pairs

2.4.1— Given a smooth variety Y of dimension n, $\mathcal{T}Y$ denotes its tangent bundle. The line bundle $\bigwedge^n \mathcal{T}Y$ over Y is called the *determinant bundle* and denoted by $\mathcal{D}etY$. If $\varphi : Y \longrightarrow Y'$ is a morphism between smooth varieties, $T\varphi : \mathcal{T}Y \longrightarrow \mathcal{T}Y'$ denotes its tangent map, and $\mathcal{D}et\varphi : \mathcal{D}etY \longrightarrow \mathcal{D}etY'$ denotes its determinant. If $y \in Y$, we denote by $T_y\varphi : \mathcal{T}_yY \longrightarrow \mathcal{T}_{\varphi(y)}Y'$ the specialization over y; and similarly, $\mathcal{D}et_y\varphi$, $\mathcal{D}et_yY$, $\mathcal{D}et_{\varphi(y)}Y'$.

2.4.2— Consider the morphism $\eta : G \times_{P(\lambda)} C^+ \longrightarrow X$ like in Section 2.1.

Definition 2. The quadruplet (G, X, C, λ) is said to be *generically finite* if $d(G, X, C, \lambda) \neq 0$. The quadruplet (G, X, C, λ) is said to be *well generically finite* if it is generically finite and there exists $x \in C$ such that $T_{[e:x]}\eta$ is invertible.

2.4.3— The map $x \mapsto [e : x]$ embeds C^+ in $G \times_P C^+$. Consider the restriction of $T\eta$ and $\mathcal{D}et\eta$ to C^+:

$$T\eta_{|C^+} : \mathcal{T}(G \times_P C^+)_{|C^+} \longrightarrow \mathcal{T}(X)_{|C^+},$$

$$\mathcal{D}et\eta_{|C^+} : \mathcal{D}et(G \times_P C^+)_{|C^+} \longrightarrow \mathcal{D}et(X)_{|C^+}.$$

Since η is G-equivariant, the morphism $\mathcal{D}et\eta_{|C^+}$ is P-equivariant; it can be thought as a P-invariant section of the line bundle $\mathcal{D} := \mathcal{D}et(G \times_P C^+)^*_{|C^+} \otimes \mathcal{D}et(X)_{|C^+}$ over C^+. For any $x \in C$, $\mathbb{K}^*$ acts linearly via λ on the fiber $\mathcal{D}_x$ over x in $\mathcal{D}$: this action is given by a character of $\mathbb{K}^*$, that is an integer m. Moreover this integer does not depend on x in C: it is denoted by $\mu^{\mathcal{D}}(C, \lambda)$.

Lemma 4. *Recall that X is smooth. The following are equivalent:*

(i) *(G, X, C, λ) is well generically finite;*
(ii) *(G, X, C, λ) is generically finite and $\mu^{\mathcal{D}}(C, \lambda) = 0$.*

Proof. Assume that (G, X, C, λ) is well generically finite and choose $x \in C$ such that $T_{[e:x]}\eta$ is invertible. Then $\mathcal{D}et\eta_{[e:x]}$ is a nonzero $\mathbb{K}^*$-fixed point in $\mathcal{D}_x$: the action of $\mathbb{K}^*$ on the line $\mathcal{D}_x$ has to be trivial.

Assume conversely that (G, X, C, λ) is generically finite and that $\mu^{\mathcal{D}}(C, \lambda) = 0$. Since the base field is assumed to have characteristic zero, the exists a point y in $G \times_{P(\lambda)} C^+$ such that $T_y\eta$ is invertible. Since η is G-equivariant, one can find such a point y in C^+. In particular $\mathcal{D}et\eta_{|C^+}$ is a nonzero $P(\lambda)$-invariant section of $\mathcal{D}$. Since $\mu^{\mathcal{D}}(C, \lambda) = 0$, [Res10a, Lemma 6] implies that $\mathcal{D}et\eta_{|C}$ is not identically zero.

2.4.4— Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{p}$, and $\mathfrak{p}_\varepsilon$ denote respectively the Lie algebras of G, B, P, and P_ε. The well generically finite pairs provide a nice standing to apply Theorem 1.

Proposition 1. *With notation of Theorem 1, assume that $(G, X, C_\varepsilon, \lambda_\varepsilon)$ is well generically finite.*

Then (G, X, C, λ) and $(L, C, C_\varepsilon, \lambda_\varepsilon)$ are well generically finite.

Proof. Given a vector space V endowed with a linear action of a one parameter subgroup λ, we denote by $V^\lambda_{<0}$ the set of $v \in V$ such that $\lim_{t\to 0} \lambda(t^{-1})v = 0$.

Let x be a point in C_ε such that T_{η_ε} is invertible at $[e : x]$. Consider the subtorus S of dimension two containing the images of λ and λ_ε. It fixes x. The tangent map of η_ε at the point $[e : x]$ induces a S-equivariant linear isomorphism $\theta : \mathfrak{g}/\mathfrak{p}_\varepsilon \simeq \mathfrak{g}^{\lambda_\varepsilon}_{<0} \longrightarrow (T_xX)^{\lambda_\varepsilon}_{<0}$. By assumption $\mathfrak{g}^\lambda_{<0} \subset \mathfrak{g}^{\lambda_\varepsilon}_{<0}$ and $(T_xX)^\lambda_{<0} \subset (T_xX)^{\lambda_\varepsilon}_{<0}$. Since θ is S-equivariant, it induces an isomorphism between $\mathfrak{g}^\lambda_{<0}$ and $(T_xX)^\lambda_{<0}$. In particular $\delta(G, X, C, \lambda) = 0$.

The second assertion of Lemma 1 implies that $T_{[e:x]}\eta$ is invertible. It follows that (G, X, C, λ) is well generically finite.

Since $\delta(G, X, C_\varepsilon, \lambda_\varepsilon) = 0$, Theorem 1 implies that $\delta(L, C, C_\varepsilon, \lambda_\varepsilon) = 0$. Hence Lemma 1 implies that $T_{[e:x]}\eta_P$ is invertible. By Lemma 3, it follows that $T_{[e:x]}\eta_L$ is invertible. Then $(L, C, C_\varepsilon, \lambda_\varepsilon)$ is well generically finite.

Remark 2. Note that the converse of Proposition 1 does not hold. Indeed, it would imply that the converse of assertion (ii) of Theorem 2, stated in Section 3 holds. For $G = \mathrm{SL}_n$, we would get that any nonzero Littlewood-Richardson coefficient is a product of such coefficients for $(H^*(\mathrm{SL}_r/B, \mathbb{Z}), \odot_0)$ for some integers r. By Corollary 1 for $G = \mathrm{SL}_n$, this would imply that each nonzero Littlewood-Richardson coefficient is equal to one. Contradiction.

3. Application to the Belkale-Kumar product

3.1. An interpretation of structure coefficients

3.1.1— Let P be a parabolic subgroup of the semisimple group G. Let $T \subset B \subset P$ be a maximal torus and a Borel subgroup of G. The Weyl group of T and G is denoted by W. Given $w \in W$, we set $X(w) = \overline{BwP/P}$, $X(w)^\circ = BwP/P$ and we denote by $[X(w)] \in \mathrm{H}^*(G/P, \mathbb{Z})$ the Poincaré dual class of $X(w)$ in cohomology. Let $w_1, \cdots, w_s \in W$ be such that $\sum_i \mathrm{codim} X(w_i) = \dim G/P$. Then there exists a nonnegative integer c such that

$$[X(w_1)].\cdots.[X(w_s)] = c[\mathrm{pt}].$$

Let λ be a one parameter subgroup of T such that $P = P(\lambda)$. Consider $X = (G/B)^s$ and the T-fixed point $x = (w_1^{-1}B/B, \cdots, w_s^{-1}B/B)$ in X. Let C be the irreducible component of X^λ containing x. Then

$$C = Lw_1^{-1}B/B \times \cdots \times Lw_s^{-1}B/B$$

and

$$C^+ = Pw_1^{-1}B/B \times \cdots \times Pw_s^{-1}B/B.$$

An easy consequence of Kleiman's transversality theorem (see [Kle76]) is the following lemma which express c has a degree.

Lemma 5. *We have $\delta(G,X,C,\lambda)=0$ and $c=d(G,X,C,\lambda)$.*

Proof. See [Res10a, proof of Lemma 14].

3.1.2— The notion of Levi-movability was introduced in [BK06].

Definition 3. Recall that $\sum_i \dim(X(w_i)) = (s-1)\dim(G/P)$. We say that $(X(w_1),\cdots,X(w_s))$ is *Levi-movable* if there exist $l_1,\cdots,l_s$ in L such that the intersection $l_1w_1^{-1}X(w_1)^\circ\cap\cdots\cap l_sw_s^{-1}X(w_s)^\circ$ is transverse at P/P.

Given a point z in a locally closed subvariety Z of a variety Y, set $N_z(Z,Y)=T_zY/T_zZ$.

Lemma 6. *The following are equivalent:*

(i) $(X(w_1),\cdots,X(w_s))$ *is Levi-movable;*
(ii) (G,X,C,λ) *is well generically finite.*

Proof. Let $y\in C$ and $l_1,\cdots,l_s\in L$ such that $y=(l_1w_1^{-1}B/B,\cdots,l_sw_s^{-1}B/B)$. Since η extends the immersion of C^+ in X, the tangent map $T_{[e:y]}\eta$ induces a linear map

$$\overline{T_{[e:y]}\eta}\,:\,N_{[e:y]}(C^+,G\times_P C^+)\longrightarrow N_y(C^+,X).$$

Moreover $T_{[e:y]}\eta$ is an isomorphism if and only if $\overline{T_{[e:y]}\eta}$ is. Using π, $N_{[e:y]}(C^+,G\times_P C^+)$ identifies with T_eG/P that is with $\mathfrak{g}/\mathfrak{p}$. Moreover $N_y(C^+,X)$ is equal to $\bigoplus_i N_{w_i^{-1}B/B}(Pl_iw_i^{-1}B/B,G/B)$ which identifies with $\oplus_i\mathfrak{g}/(\mathfrak{p}+l_iw_i^{-1}\mathfrak{b}w_il_i^{-1})$. Moreover, after composing by these isomorphisms, $\overline{T_{[e:y]}\eta}$ is the canonical map $\mathfrak{g}/\mathfrak{p}\longrightarrow\oplus_i\mathfrak{g}/(\mathfrak{p}+l_iw_i^{-1}\mathfrak{b}w_il_i^{-1})$. The lemma follows.

3.2. A multiplicative formula for structure coefficients of $\odot_0$

3.2.1— Let $Q\subset P$ be two parabolic subgroups of G. Let $T\subset B\subset Q$ be a maximal torus and a Borel subgroup of G. Let L denote the Levi subgroup of P containing T and let W_P denote its Weyl group. Consider the following G-equivariant fibration

$$\begin{array}{ccc} L/(L\cap Q) & \longrightarrow & G/Q \\ & & \downarrow \\ & & G/P. \end{array}$$

There exists a natural bijection between the Schubert classes of G/Q and the pairs of Schubert classes in $L/(L\cap Q)$ and G/P. Let $w\in W$. Consider the associated Schubert varieties in G/P and G/Q:

$$X^{G/P}(w)=\overline{BwP/P} \text{ and } X^{G/Q}(w)=\overline{BwQ/Q}.$$

The intersection $w^{-1}Bw\cap L$ is a Borel subgroup of L containing T and there exists a unique $\overline{w}\in W_P$ such that

$$\overline{w}^{-1}(B\cap L)\overline{w}=w^{-1}Bw\cap L. \tag{3}$$

Consider the Schubert variety in $L/L\cap Q$ associated to $\overline{w}$:

$$X^{L/L\cap Q}(w)=\overline{(L\cap B)\overline{w}(L\cap Q/L\cap Q)}.$$

The three Schubert cells associated w are related by the following fibration

$$\begin{tikzcd}
\overline{w}^{-1}X^{L/L\cap Q}(w)^{\circ} \arrow[r] & w^{-1}X^{G/Q}(w)^{\circ} \arrow[d] \\
 & w^{-1}X^{G/P}(w)^{\circ}.
\end{tikzcd}$$

3.2.2— We can now state our main result about the Belkale-Kumar product.

Theorem 2. *Let $w_1,\cdots,w_s\in W$. Assume that $\sum_i \dim X^{G/Q}(w_i)=(s-1)\dim G/Q$ and that $(X^{G/Q}(w_1),\cdots,X^{G/Q}(w_s))$ is Levi-movable. Then*

(i) *$\sum_i \dim X^{G/P}(w_i)=(s-1)\dim G/P$ and $\sum_i \dim X^{L/L\cap Q}(w_i)=(s-1)\dim L/(L\cap Q)$;*
(ii) *$(X^{G/P}(w_1),\cdots,X^{G/P}(w_s))$ and $(X^{L/L\cap Q}(w_1),\cdots,X^{L/L\cap Q}(w_s))$ are Levi-movable.*

Assertion (i) allows to define three integers by

$$\begin{array}{rcl}
[X^{G/Q}(w_1)].\cdots.[X^{G/Q}(w_s)] &=& c^{G/Q}_{w_1,\cdots,w_s}[\mathrm{pt}],\\
[X^{G/P}(w_1)].\cdots.[X^{G/P}(w_s)] &=& c^{G/P}_{w_1,\cdots,w_s}[\mathrm{pt}], \text{ and}\\
[X^{L/L\cap Q}(w_1)].\cdots.[X^{L/L\cap Q}(w_s)] &=& c^{L/L\cap Q}_{w_1,\cdots,w_s}[\mathrm{pt}].
\end{array}$$

Then

$$c^{G/Q}_{w_1,\cdots,w_s}=c^{G/P}_{w_1,\cdots,w_s}.c^{L/L\cap Q}_{w_1,\cdots,w_s}.$$

Proof. Recall that X is the variety $(G/B)^s$ and $x = (w_1^{-1}B/B, \cdots, w_s^{-1}B/B)$. Let λ and λ_ε be two one parameter subgroups of T such that $P(\lambda)$ and $P(\lambda_\varepsilon)$ are equal to P and Q. Let C (resp. C_ε) denote the irreducible component of X^λ (resp. X^{λ_ε}) containing x. Since $Q = P(\lambda_\varepsilon) \subset P(\lambda) = P$, the assumptions of Section 2.2 are fulfilled. By Lemma 6, $(G, X, C_\varepsilon, \lambda_\varepsilon)$ is well generically finite. Now, Proposition 1 implies that (G, X, C, λ) and $(L, C, C_\varepsilon, \lambda_\varepsilon)$ are well generically finite. In particular, Lemma 6 implies Assertions (i) and (ii).

Since Assertion (i) means that $\delta(G, X, C, \lambda) = \delta(L, C, C_\varepsilon, \lambda_\varepsilon) = 0$, Theorem 1 implies that $d(G, X, C_\varepsilon, \lambda_\varepsilon) = d(G, X, C, \lambda).d(L, C, C_\varepsilon, \lambda_\varepsilon)$. Lemma 5 allows to conclude.

Remark 3. In the case when $G = \mathrm{SL}_n$, Theorem 2 was already obtained in [Ric09] for some pairs $Q \subset P$. E. Richmond also obtained Theorem 2 independently in [Ric11].

3.2.3— Assuming that one knows how to compute in $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ for any maximal P and any G, Theorem 2 allows him to compute the structure coefficients of $(\mathrm{H}^*(G/Q, \mathbb{Z}), \odot_0)$ for any parabolic subgroup Q. To illustrate this principle, we state an analogue to [Ric09, Corollary 23].

Corollary 1. *Let $G = \mathrm{Sp}_{2n}$. The nonzero structure coefficients of the ring $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$ are equal to* 1.

Proof. The proof proceeds by induction on n. Let c be a nonzero structure coefficient of $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$. Let w_1, w_2, w_3 be elements of W such that

$$[X(w_1)].[X(w_2)].[X(w_3)] = c[\mathrm{pt}].$$

Since c is nonzero, $(X(w_1), X(w_2), X(w_3))$ is Levi-movable.

Consider the stabilizer P in G of the line in $\mathbb{K}^{2n}$ fixed by B. Theorem 2 applied with $B \subset P$ shows that c is the product of a structure coefficient of $(\mathrm{H}^*(G/P, \mathbb{Z}), \odot_0)$ and one of $(\mathrm{H}^*(\mathrm{Sp}_{2n-2}/B, \mathbb{Z}), \odot_0)$. The fact that G/P is a projective space and the induction allow to conclude.

Remark 4. Since T stabilizes all the Schubert cells, Levi-movability is very easy to check for G/B. In particular, one can easily decide if a given structure coefficient of $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$ is zero or not. Now, Corollary 1 allows to compute the structure coefficients of $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$.

3.3. Some questions

3.3.1— Corollary 1 is also true (and the proof is the same) for $G = \mathrm{SL}_n$.

Is Corollary 1 true for any simple group ?

3.3.2— Let $G = \mathrm{SL}_n$ and P be a maximal parabolic subgroup of G. Then G/P is a Grassmannian variety and the structure coefficients of groups $\mathrm{H}^*(G/P, \mathbb{Z})$ are the Littlewood-Richardson coefficients (LR-coefficients for short). Let $c_{w_1 w_2 w_3}$ be a nonzero structure coefficient of $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$. By considering the projection $G/B \longrightarrow G/P$, Theorem 2 and Corollary 1 for SL_n give a LR-coefficient equals to one.

How the so obtained LR-coefficients equal to one are distributed among the LR-coefficients equal to one?

One can prove that the set of LR-coefficients equals to one is an union of some faces of Klyachko cones. Is it also true for this subset ?

4. Application to quiver representations

4.1. Definitions

Let Q be a quiver (that is, a finite oriented graph) with vertexes Q_0 and arrows Q_1. An arrow $a \in Q_1$ has initial vertex ia and terminal one ta. A representation R of Q is a family $(V(s))_{s \in Q_0}$ of finite dimensional vector spaces and a family of linear maps $u(a) \in \mathrm{Hom}(V(ia), V(ta))$ indexed by $a \in Q_1$. The dimension vector of R is $(\dim(V(s)))_{s \in Q_0} \in \mathbb{N}^{Q_0}$.

Fix $\alpha \in \mathbb{N}^{Q_0}$ and a vector space $V(s)$ of dimension $\alpha(s)$ for each $\alpha \in Q_0$. Set

$$\mathrm{Rep}(Q, \alpha) = \bigoplus_{a \in Q_1} \mathrm{Hom}(V(ia), V(ta)).$$

Consider also the group

$$\mathrm{GL}(\alpha) = \prod_{s \in Q_0} \mathrm{GL}(V(s)).$$

The group $\mathrm{GL}(\alpha)$ acts on $\mathrm{Rep}(Q, \alpha)$ in such a way the orbits are the isomorphism classes of representations of Q.

Let $\alpha, \beta \in \mathbb{Z}^{Q_0}$. The Ringel form is defined by

$$\langle \alpha, \beta \rangle = \sum_{s \in Q_0} \alpha(s)\beta(s) - \sum_{a \in Q_1} \alpha(ia)\beta(ta).$$

Assume that $\alpha, \beta \in \mathbb{N}^{Q_0}$. Following Derksen-Schofield-Weyman (see [DSW07]), we define $\alpha \circ \beta$ to be the number of α-dimensional subrepresentations of a general representation of dimension $\alpha + \beta$ if it is finite, and 0 otherwise.

4.2. Dominant pairs

4.2.1— Let λ be a one parameter subgroup of $\mathrm{GL}(\alpha)$. For any $i \in \mathbb{Z}$ and for any $s \in Q_0$, set $V_i(s) = \{v \in V(s) \,|\, \lambda(t)v = t^i v\}$ and $\alpha_i(s) = \dim V_i(s)$. Obviously almost all α_i are zero, and $\alpha = \sum_{i\in\mathbb{Z}} \alpha_i$. Moreover, λ is determined up to conjugacy by the dimension vectors α_i.

The parabolic subgroup $P(\lambda)$ of $\mathrm{GL}(\alpha)$ associated to λ is the set of $(g(s))_{s\in Q_0}$ such that for all $i \in \mathbb{Z}$, $g(s)(V_i(s)) \subset \oplus_{j\le i} V_j(s)$.

The subspace $\mathrm{Rep}(Q,\alpha)^\lambda$ is the set of tuples $(u(a))_{a\in Q_1}$ such that for any $a \in Q_1$ and for any $i \in \mathbb{Z}$, $u(a)(V_i(ia)) \subset V_i(ta)$. Hence

$$\mathrm{Rep}(Q,\alpha)^\lambda = \bigoplus_i \mathrm{Rep}(Q,\alpha_i).$$

In particular it is irreducible and denoted by C from now on. Moreover C^+ is the set of tuples $(u(a))_{a\in Q_1}$ such that for any $a \in Q_1$ and for any $i \in \mathbb{Z}$, $u(a)(V_i(ia)) \subset \oplus_{j\le i} V_j(ta)$. Consider the morphism $\eta_\lambda : G \times_{P(\lambda)} C^+ \longrightarrow \mathrm{Rep}(Q,\alpha)$.

The conjugacy class of λ is uniquely determined by the tuples $(\alpha_i)_{i\in\mathbb{Z}}$ of dimension vectors. The isomorphism class of C only depends on the underlying multiset of dimension vectors. The classes of C^+ and $P(\lambda)$ only depend on the ordered multiset of dimension vectors. This observation makes the following definition natural.

Definition 4. A *decomposition* of the dimension vector α is a family $(\beta_1, \cdots, \beta_s)$ of nonzero dimension vectors such that $\alpha = \beta_1 + \cdots + \beta_s$. The decomposition is denoted by $\alpha = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$. The tilda means that we keep the order in mind.

We can now define the *map* $\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ *associated to a decomposition of* α.

4.2.2— Consider a decomposition $\alpha = \beta_1 \tilde{+} \beta_2$ with two dimension vectors and the associated morphism $\eta = \eta_{\beta_1\tilde{+}\beta_2}$. In this section, we collect some properties of η. For each vertex s in Q_0, fix a decomposition $V(s) = V_1(s) \oplus V_2(s)$ where $\dim(V_1(s)) = \beta_1(s)$ and $\dim(V_2(s)) = \beta_2(s)$. This allows to embed $C := \mathrm{Rep}(Q,\beta_1) \oplus \mathrm{Rep}(Q,\beta_2)$ in $X := \mathrm{Rep}(Q,\alpha)$. Let $(R_1,R_2) \in \mathrm{Rep}(Q,\beta_1) \oplus \mathrm{Rep}(Q,\beta_2) \subset X$. Since η extends the immersion of C^+ in X, the tangent map $T_{[e:(R_1,R_2)]}\eta$ induces a linear map

$$\overline{T}_{[e:(R_1,R_2)]}\eta \,:\, N_{[e:(R_1,R_2)]}(C^+, G \times_P C^+) \longrightarrow N_{(R_1,R_2)}(C^+, \mathrm{Rep}(Q,\alpha)).$$

Moreover $N_{[e:(R_1,R_2)]}(C^+, G \times_P C^+)$ identifies with $\oplus_{s\in Q_0} \mathrm{Hom}(V_1(s), V_2(s))$ and $N_{(R_1,R_2)}(C^+, \mathrm{Rep}(Q,\alpha))$ identifies with $\oplus_{a\in Q_1} \mathrm{Hom}(V_1(ia), V_2(ta))$. The following lemma is the consequence of a direct computation.

Lemma 7. *For $i = 1, 2$ and $a \in Q_1$, let $u_i(a)$ denote the linear map of R_i corresponding to a. Then*

$$\overline{T}_{[e:(R_1,R_2)]}\eta\Big(\sum_{s\in Q_0} \varphi(s)\Big) = \sum_{a\in Q_1} u_2(a)\varphi(ta) - \varphi(ha)u_1(a).$$

In particular the Kernel of $\overline{T}_{[e:(R_1,R_2)]}\eta$ *is* $\mathrm{Hom}(R_1, R_2)$ *and its image is* $\mathrm{Ext}(R_1, R_2)$.

The quantities $\delta(\eta)$ and $d(\eta)$ are classic objects in the representation theory of quivers.

Lemma 8. *We have:*

(i) $\delta(\eta) = -\langle \beta_1, \beta_2 \rangle$, *and*

(ii) $d(\eta) = \beta_1 \circ \beta_2$.

PROOF. By the discussion preceding Lemma 7, $\delta(\eta)$ equals the difference between the dimension of $\oplus_{a \in Q_1}\mathrm{Hom}(V_1(ia), V_2(ta))$ and that of $\oplus_{s \in Q_0}\mathrm{Hom}(V_1(s), V_2(s))$. The first assertion follows.

Let $R \in \mathrm{Rep}(Q, \alpha)$. Using immersion (1), one identifies the fiber $\eta^{-1}(R)$ with the set of β_1-dimensional subrepresentations of R. Thus when R is general, the cardinality $|\eta^{-1}(R)| = \beta_1 \circ \beta_2 = d(\eta)$.

Consider the one parameter subgroup λ of $\mathrm{GL}(\alpha)$ defined by $\lambda(s)(t)$ stabilizes the decomposition $V_1(s) \oplus V_2(s)$, is equal to Id when restricted to $V_1(s)$ and equal to $t\mathrm{Id}$ when restricted to $V_2(s)$. Consider $P(\lambda) = P$, $\mathrm{Rep}(Q, \alpha)^\lambda = C$ and $C^+(\lambda) = C^+$. Recall that $\mathcal{D}$ denote the determinant bundle of η restricted to C^+.

Lemma 9. *Assume that* $\langle \beta_1, \beta_2 \rangle = 0$. *Then the one parameter subgroup* λ *acts trivially on* $\mathcal{D}_{|C}$.

PROOF. Since C is an affine space, λ acts by the same character on each fiber of $\mathcal{D}_{|C}$. Since η extends the identity on C^+, its character is the difference between the weights of λ acting on

$$N_0(C^+, X) \simeq \oplus_{a \in Q_1}\mathrm{Hom}(V_1(ia), V_2(ta))$$

and acting on

$$N_0(C^+, G \times_P C^+) \simeq T_e G/P \simeq \oplus_{s \in Q_0}\mathrm{Hom}(V_1(s), V_2(s)).$$

Hence this character is equal to

$$\sum_{a \in Q_1} \beta_1(ia)\beta_2(ta) - \sum_{s \in Q_0} \beta_1(s)\beta_2(s);$$

that is, it is equal to $-\langle \beta_1, \beta_2 \rangle$. The lemma follows.

Remark 5. Lemma 9 is an analogue of the fact that the Grassmannian varieties are cominuscule SL_n-homogeneous spaces.

4.3. A formula for $d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s})$

4.3.1— Applying Theorem 1 in the context of quivers, we get the following result.

Theorem 3. *Let $\alpha = \beta_1\tilde{+}\cdots\tilde{+}\beta_s$ be a decomposition of α such that for all $i<j$, $\langle\beta_i,\beta_j\rangle = 0$.*

Then $\delta(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = 0$ and

$$d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = (\beta_1\circ\alpha-\beta_1).(\beta_2\circ\alpha-\beta_1-\beta_2).\cdots.(\beta_{s-1}\circ\beta_s).$$

Proof. By Section 4.2.1, the codimension of C^+ in $G\times_P C^+$ is

$$\sum_{i<j}\sum_{s\in Q_0}\beta_i(s)\beta_j(s);$$

and the codimension of C^+ in $\mathrm{Rep}(Q,\alpha)$ is

$$\sum_{i<j}\sum_{a\in Q_1}\beta_i(ia)\beta_j(ta).$$

Since $\forall i<j$ $\langle\beta_i,\beta_j\rangle = 0$, this implies that $\delta(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = 0$.

If $s=2$, the theorem follows from Lemma 8. Assume that $s=3$. A direct application of Theorem 1 with $\eta_\varepsilon = \eta_{\beta_1\tilde{+}\beta_2\tilde{+}\beta_3}$ and $\eta = \eta_{\beta_1\tilde{+}(\alpha-\beta_1)}$ gives

$$\begin{array}{ll} d(\eta_{\beta_1\tilde{+}\beta_2\tilde{+}\beta_3}) & = (\beta_1\circ\alpha-\beta_1).d(\eta_{\beta_2\tilde{+}\beta_3}) \\ & = (\beta_1\circ\alpha-\beta_1).(\beta_2\circ\beta_3). \end{array}$$

One can easily ends the proof by an induction on s.

Remark 6. In the proof of Theorem 3, the induction was made using the bracketing $\beta_1\tilde{+}\cdots\tilde{+}\beta_s = \beta_1\tilde{+}(\beta_2(\tilde{+}\cdots\tilde{+}\beta_s))$. Any other bracketing gives a similar formula. For example using the bracketing $\beta_1\tilde{+}\cdots\tilde{+}\beta_s = (\beta_1\tilde{+}\cdots\tilde{+}\beta_{s-1})\tilde{+}\beta_s$, we get $d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = (\alpha-\beta_s\circ\beta_s).(\beta-\beta_s-\beta_{s-1}\circ\beta_{s-1}).\cdots.(\beta_1\circ\beta_2)$. It is natural to ask for a more symmetric formula.

4.3.2— The assumption "$\forall i<j$ $\langle\beta_i,\beta_j\rangle = 0$" in Theorem 3 is similar to Levi-movability. Indeed the following lemma is closed to Lemma 6.

Lemma 10. *Let $\alpha = \beta_1\tilde{+}\cdots\tilde{+}\beta_s$ be a decomposition of α such that $\delta(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) = 0$. Then the following are equivalent:*

(i) *for all $i<j$, $\langle\beta_i,\beta_j\rangle = 0$ and $d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) \neq 0$;*

(ii) *the map $\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ is well generically finite.*

PROOF. For each $s \in Q_0$, fix a decomposition $V(s) = \oplus_i V_i(s)$ of $V(s)$ such that $\dim V_i(s) = \beta_i(s)$. Consider the linear action of the torus $Z = (\mathbb{K}^*)^s$ on $\oplus_{s\in Q_0} V(s)$ given by $(t_1, \cdots, t_s).v = t_i v$ for all $t_i \in \mathbb{K}^*$ and $v \in V_i(s)$ for any $s \in Q_0$. Since Z is embedded in $\mathrm{GL}(\alpha)$, it also acts on $G \times_P C^+$.

Assuming that assertion (ii) holds, there exists a point y in C such that $T_{[e:y]}\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ is invertible. Since Z fixes $[e:y]$ and η is G-equivariant, $T_{[e:y]}\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ is Z-equivariant for the tangent action of Z. It follows that for all $i < j$, $T_{[e:y]}\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ induces an isomorphism between the eigenspaces of $T_{[e:y]}G\times_P C^+$ and $T_y\mathrm{Rep}(Q,\alpha)$ of weight $t_j t_i^{-1}$. In particular, these two eigenspaces have the same dimension. But a direct computation shows that the difference between these two dimensions is precisely $\langle \beta_i, \beta_j\rangle$. Assertion (i) follows.

Conversely, assume that assertion (i) holds. Since $d(\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}) \neq 0$, there exists a point of $G \times_P C^+$ where the tangent map of $\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ is invertible. Since η is G-equivariant, its determinant is not identically zero on C^+. Using the fact for all $i < j$ $\langle \beta_i, \beta_j\rangle = 0$, a direct computation (like in the proof of Lemma 9) shows that Z acts trivially on $\mathcal{D}_{|C}$. Lemma 4 allows to conclude.

4.3.3— The dimension of $\mathrm{Ext}(R_1, R_2)$ for general α and β dimensional representations R_1 and R_2 is denoted by $\mathrm{ext}(\alpha, \beta)$.

Corollary 2. *The quiver Q is assumed to have no oriented cycle. Let α, β, and γ be three dimension vectors. Assume that $\langle\alpha, \beta\rangle = \langle\alpha, \gamma\rangle = 0$ and $\beta \circ \gamma = 1$.*

Then $\alpha \circ (\beta + \gamma) = (\alpha \circ \beta).(\alpha \circ \gamma)$.

PROOF. Theorem 3 applied to the decomposition $\alpha\tilde{+}\beta\tilde{+}\gamma$ gives $(\alpha + \beta \circ \gamma).(\alpha \circ \beta) = (\alpha \circ \beta + \gamma).(\beta \circ \gamma)$. But $(\beta \circ \gamma) = 1$. Hence

$$(\alpha + \beta \circ \gamma).(\alpha \circ \beta) = \alpha \circ (\beta + \gamma).$$

If $\alpha\circ\beta = 0$ then the corollary follows. Assume that $\alpha\circ\beta \neq 0$. Lemma 10 implies that the determinant of $\eta_{\alpha\tilde{+}\beta}$ is not identically zero on C. But Lemma 7 implies that $\mathrm{ext}(\alpha, \beta) = 0$. Now, the corollary is a direct consequence of Lemma 11 below.

The proof of the following Lemma 11 uses Derksen-Schofield-Weyman's theorem that shows that $\alpha\circ\beta$ is the dimension of some space of invariant functions.

Lemma 11. *The quiver Q is assumed to have no oriented cycle. Let α, β, and γ be three dimension vectors. Assume that $\beta \circ \gamma = 1$ and $\mathrm{ext}(\alpha, \beta) = 0$.*

Then $(\alpha + \beta) \circ \gamma = \alpha \circ \gamma$.

PROOF. The map

$$\begin{array}{ccc} \mathbb{Z}^{\mathbb{Q}_0} & \longrightarrow & \mathrm{Hom}(\mathrm{GL}(\gamma), \mathbb{K}^*) \\ \beta & \longmapsto & \left((g(s)_{s\in Q_0}) \longmapsto \prod_{s\in Q_0} \det(g(s))^{\beta(s)} \right) \end{array}$$

identifies $\mathbb{Z}^{Q_0}$ with the group of characters of $\mathrm{GL}(\gamma)$. Moreover the pairing $(\alpha,\beta)=\sum_{s\in Q_0}\alpha(s)\beta(s)$ identifies $\mathbb{Z}^{Q_0}$ with its dual. Using these identifications, for any dimension vector θ, $\langle\theta,\cdot\rangle$ corresponds to a character of $\mathrm{GL}(\gamma)$. The corresponding eigenspace in $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]$ is denoted by $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\theta,\cdot\rangle}$.

In [DSW07], Derksen-Schofield-Weyman proved that $\alpha\circ\gamma$ is equal to the dimension of $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha,\cdot\rangle}$. Consider the multiplication morphism

$$m\,:\,\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha,\cdot\rangle}\otimes\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\beta,\cdot\rangle}\longrightarrow\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha+\beta,\cdot\rangle}.$$

We claim that m is an isomorphism. The lemma follows directly from the claim. Since $\dim(\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\beta,\cdot\rangle})=1$ and $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]$ has no zero-divisor, m is injective.

In [DW00], Derksen-Weyman proved that $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha+\beta,\cdot\rangle}$ is spanned by functions c^R associated to various $\alpha+\beta$-dimensional representations R (see also [DZ01]). This vector space is also spanned by the functions c^R for general R. Indeed, if $a=\dim(\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha+\beta,\cdot\rangle})$, the set of points $(R_i)_{1\leq i\leq a}$ in $(\mathrm{Rep}(Q,\alpha+\beta))^a$ such that $\mathbb{K}[\mathrm{Rep}(Q,\gamma)]_{\langle\alpha+\beta,\cdot\rangle}$ is spanned by the functions c^{R_i} is open.

Since $\mathrm{ext}(\alpha,\beta)=0$, $\eta_{\alpha\tilde{+}\beta}$ is dominant. In particular, for R general, there exists an α-dimensional subrepresentation R' of R. By [DW00, Lemma 1], $c^R=c^{R'}.c^{R/R'}$. It follows that m is surjective.

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