Intersection multiplicity one for the Belkale-Kumar product in G/B

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Abstract

Consider the complete flag variety X of any complex semi-simple algebraic group G. We show that the structure coefficients of the Belkale-Kumar product $\odot_0$, on the cohomology $\mathrm{H}^*(X, \mathbb{Z})$, are all either 0 or 1. We also derive some consequences. The proof contains a geometric part and uses a combinatorial result on root systems. The geometric method is uniform whereas the combinatorial one is proved by reduction to small ranks and then, by direct checkings.

Keywords: Cohomology of Homogeneous Spaces, Root Systems.

Contents

1 Introduction

Let G be a complex semisimple group and let B be a Borel subgroup of G. In this paper, we are interested in the Belkale-Kumar product $\odot_0$ on the cohomology group of the complete flag variety G/B.

Fix a maximal torus T of B. Let W denote the Weyl group of G. For any $w \in W$, let $X_w = \overline{BwB/B}$ be the corresponding Schubert variety and let $[X_w] \in \mathrm{H}^*(G/B, \mathbb{C})$ be its cohomology cycle. Then, $([X_w])_{w\in W}$ is a basis for the cohomology group $\mathrm{H}^*(G/B, \mathbb{Z})$. The structure coefficients c_{uv}^w of the cup product are written as

$$[X_u] \cdot [X_v] = \sum_{w\in W} c_{uv}^w [X_w]. \tag{1}$$

Let Φ denote the set of roots of G, Φ^+ and Φ^- denote respectively the set of positive and negative roots corresponding to B. For $w \in W$, denote by $\Phi(w) = \Phi^+ \cap w^{-1}\Phi^-$ the set of inversions of w. For its applications to the geometry of the eigencone, Belkale-Kumar defined in [BK06] a new product $\odot_0$ on $\mathrm{H}^*(G/P, \mathbb{C})$, for any parabolic subgroup P. When $P = B$, the structure constants $\tilde{c}_{uv}^w$ of the Belkale-Kumar product,

$$[X_u]\odot_0[X_v] = \sum_{w\in W} \tilde{c}_{uv}^w [X_w] \tag{2}$$

can be defined as follows (see [BK06, Corollary 44]):

$$\tilde{c}_{uv}^w = \begin{cases} c_{uv}^w & \text{if } \Phi(u) \cap \Phi(v) = \Phi(w) \text{ and } \Phi(u) \cup \Phi(v) = \Phi^+, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The product $\odot_0$ is associative and satisfies Poincaré duality.

Our main result can be stated as follows.

Theorem 1. *Let u, v and w in W be such that $\Phi(u) \cap \Phi(v) = \Phi(w)$ and $\Phi(u) \cup \Phi(v) = \Phi^+$. Then*

$$c_{uv}^w = 1.$$

Theorem 1 was conjectured by Belkale-Kumar in oral discussions since 2006 and is stated as a question in [DR09a, Question 1]. A lot of special cases were known before. In [Ric12, Corollary 4], E. Richmond proved it in type A. As noticed in [Ric09] or [Res11, Corollary 1], Richmond's proof also works in type C. Type B is proved in [Res18, Proposition 16]. In [DR19], all the classical types are solved. The cases G_2, F_4 and E_6 can be checked using a computer. Note finally that, in [Res18, Conjecture 1], a conjecture for any homogeneous space G/P is formulated to extend Theorem 1 to any homogeneous space G/P.

Our proof of Theorem 1 has a geometric part and a combinatorial one on root systems. The first part is uniform on the type and it is based on the fact that the complete flag varieties are simply connected (see Section 4). The second one is stated as Theorem 3 in Section 2. The proof is based on reductions to root systems of small ranks.

In Section 3, we state some consequences of Theorem 1 on the Bruhat order in W, on the number of descents in W, on the geometry of the eigencone and on the cohomological components of the tensor product decomposition.

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2 On the combinatorics of root systems

In this section, Φ denotes a crystallographic root system with a fixed choice Φ^+ of positive roots and associated simple roots Δ.

A subset Φ_1 of Φ^+ is said to be *convex* if, for any $\varphi, \psi \in \Phi_1$ such that $\varphi + \psi \in \Phi$, we have $\varphi + \psi \in \Phi_1$. It is said to be *coconvex* if its complementary $\Phi^+ \backslash \Phi_1$ is convex; and it is said to be *biconvex* if it is convex and coconvex.

By [Kos61, Proposition 5.10], a subset $\Phi_1 \subset \Phi^+$ is biconvex if and only if it is equal to $\Phi(w)$ for some w in the Weyl group. In particular we have the following consequence.

Lemma 2. *If $\Phi_1 \subseteq \Phi^+$ is biconvex and $\Phi_1^c = \Phi^+ \backslash \Phi_1$, then $\mathbb{Q}_{\geq 0}\Phi_1 \cap \mathbb{Q}_{\geq 0}\Phi_1^c = \{0\}$.*

Given three subsets Φ_1, Φ_2 and Φ_3 in Φ^+, we write $\Phi_1 \sqcap \Phi_2 = \Phi_3$ if $\Phi_1 \cap \Phi_2 = \Phi_3$ and $\Phi_1 \cup \Phi_2 = \Phi^+$. Similarly, we write $\Phi_3 = \Phi_1 \sqcup \Phi_2$ if $\Phi_3 = \Phi_1 \cup \Phi_2$ and $\Phi_1 \cap \Phi_2 = \emptyset$.

For φ and ψ in Φ, we write $\varphi < \psi$ if $\psi - \varphi \in \sum_{\alpha \in \Delta} \mathbb{N}\alpha$ and $\varphi \neq \psi$. As usually, we also denote $\varphi \leq \psi$ if $\varphi = \psi$ is allowed. We set $[\varphi; \psi] = \{\gamma \in \Phi \ : \ \varphi \leq \gamma \leq \psi\}$ and $]\varphi; \psi[= \{\gamma \in \Phi \ : \ \varphi < \gamma < \psi\}$.

We can now state our combinatorial theorem.

Theorem 3. *Let Φ_1, Φ_2 and Φ_3 be three biconvex subsets of Φ^+ such that $\Phi_3 = \Phi_1 \sqcup \Phi_2$. Let β and γ be two positive roots such that*

1. *$\beta \in \Phi_1$;*
2. *$\gamma \notin \Phi_3$;*
3. *$\gamma + \beta \in \Phi_3$.*

Then $\Phi_2 \cap [\beta; \gamma]$ is empty.

The theorem is proved in Section 8.

3 Some consequences of Theorem 1

From now on, we are in the setting of the introduction. Most of the results stated in this section are proved in Section 7. We denote by w_0 the longest element of W and, for any $w \in W$, we set $w^\vee = w_0 w$ the Poincaré dual of w, so that $\Phi(w^\vee) = \Phi^+ \setminus \Phi(w)$.

Let u, v and w as in Theorem 1. To emphasize the symmetry in u, v and $w^\vee$, we set

$$w_1 = w^\vee \quad w_2 = u \quad w_3 = v.$$

The assumption $\Phi(u) \cap \Phi(v) = \Phi(w)$ and $\Phi(u) \cup \Phi(v) = \Phi^+$ can be translated as

$$\Phi^+ = \Phi(w_1^\vee) \sqcup \Phi(w_2^\vee) \sqcup \Phi(w_3^\vee). \tag{4}$$

We denote by $\leq$ the Bruhat order on W: for $v, w \in W$, $v \leq w$ if and only if $X_v \subset X_w$.

3.1 On the Bruhat order

Corollary 4. *Let w_1, w_2 and w_3 in W. If Condition (4) holds, then the only element $x \in W$ such that $w_i x \leq w_i$, for $i = 1, 2$ and 3 is the neutral element $x = e$.*

3.2 The number of descents

For $w \in W$, denote by $\ell(w)$ the cardinality of $\Phi(w)$; it is the length of w when W is thought as a Coxeter group. For $w \in W$, we consider the set of left descents:

$$D(w) = \{\alpha \in \Delta \, : \, \ell(s_\alpha u) < \ell(u)\}$$

and denote by $d(w)$ the cardinality of $D(w)$.

Corollary 5. *Let w_1, w_2 and w_3 in W satisfying (4), then*

$$d(w_1) + d(w_2) + d(w_3) = 2\mathrm{rk}(G) \quad \text{and} \quad d(w_1^\vee) + d(w_2^\vee) + d(w_3^\vee) = \mathrm{rk}(G).$$

Based on some computations with a computer we ask the following question: under the assumptions of Corollary 5, do we have

$$d(w_1^\vee) + d(w_2^\vee) = d(w_1 w_2^{-1}) \text{ or } d(w_2 w_1^{-1})?$$

This have been checked for any root system of rank at most 5 (see the source code on [Res23]).

3.3 Using a Belkale-Kumar expression of c_{uv}^w

Let B^- be the opposite Borel subgroup of B, so that $B \cap B^- = T$. Let U (also denoted by U^+) and U^- be respectively the unipotent radical of B and B^-. In [BK06, Theorem 43] Belkale and Kumar give an isomorphism of graded rings:

$$\phi\colon\ (\mathrm{H}^*(G/B, \mathbb{C}), \odot_0) \cong \left[\mathrm{H}^*(\mathfrak{u}^+) \otimes \mathrm{H}^*(\mathfrak{u}^-)\right]^{\mathfrak{t}},$$

where $\mathfrak{u}^\pm = \mathrm{Lie}(U^\pm)$, $\mathfrak{t} = \mathrm{Lie}(T)$ and $\mathrm{H}^*(\mathfrak{u}^\pm)$ denotes the Lie algebra cohomology of the nilpotent algebras $\mathfrak{u}^\pm$. They derive in [BK06, Corollary 44-(*ii*)] an expression for the coefficients $\tilde{c}_{uv}^w$. Using it, we get:

Corollary 6. *If $\Phi(w) = \Phi(w_1) \sqcup \Phi(w_2)$, then*

$$\prod_{\alpha \in \Phi(w^{-1})} \langle \rho, \alpha \rangle = \left(\prod_{\alpha \in \Phi(w_1^{-1})} \langle \rho, \alpha \rangle \right) \left(\prod_{\alpha \in \Phi(w_2^{-1})} \langle \rho, \alpha \rangle \right),$$

where ρ is one-half the sum of the positive roots and $\langle \cdot, \cdot \rangle$ the Killing form.

3.4 Minimal regular faces of the eigencone

Let $X(T)^+$ (resp. $X(T)^{++}$) denote the set of dominant (resp. strictly dominant) characters of T (relatively to B). For $\lambda \in X(T)^+$, we denote by $V(\lambda)$ the irreducible G-module of highest weight λ. If V is any G-module, we denote by V^G the set of G-invariant vectors. Set

$$\mathrm{LR}(G) = \{(\lambda_1, \lambda_2, \lambda_3) \in (X(T)^+)^3 \ :\ (V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3))^G \neq \{0\}\ \}.$$

This set is known to be a finitely generated semigroup (see e.g. [Kum15]). The convex cone $\mathcal{LR}(G)$ generated by $\mathrm{LR}(G)$ in $(X(T) \otimes \mathbb{Q})^3$ is closed and polyhedral. A face of $\mathcal{LR}(G)$ is said to be regular if it intersects $(X(T)^{++})^3$. By [Res10], the regular faces are controlled by the Belkale-Kumar product on $H^*(G/P, \mathbb{Z})$ for various standard parabolic subgroups P of G.

Corollary 7. *Let w_1, w_2 and w_3 in W satisfying Condition* (4)*. Then*

$$\mathcal{F}_{(w_1, w_2, w_3)} = \{(\lambda_1, \lambda_2, \lambda_3) \in (X(T)^+)^3 \ :\ {w_1}^{-1}\lambda_1 + {w_2}^{-1}\lambda_2 + {w_2}^{-1}\lambda_3 = 0\}$$

is the set of points in $\mathrm{LR}(G)$ *that belong to a regular face of $\mathcal{LR}(G)$. Moreover, any codimension* $\mathrm{rk}(G)$ *regular face is obtained in such a way. As a semigroup, $\mathcal{F}_{(w_1, w_2, w_3)}$ is freely generated by* $2\mathrm{rk}(G)$ *elements.*

Remarks.

1. A significant part of Corollary 7 (which is even equivalent to Theorem 1) is that there exists regular weights λ_1, λ_2 and λ_3 in $X(T)^+$ such that ${w_1}^{-1}\lambda_1 + {w_2}^{-1}\lambda_2 + {w_2}^{-1}\lambda_3 = 0$.

2. The $2\mathrm{rk}(G)$ generators of the semigroup $\mathcal{F}_{(w_1, w_2, w_3)}$ are described geometrically in the proof of the corollary (see Section 7), to be the line bundle $\mathcal{O}(D_i)$ associated to some explicit divisors D_i. The triple of weights $(\lambda_1, \lambda_2, \lambda_3)$ corresponding to $\mathcal{O}(D_i)$ can be derived from [BK20, Theorem 8].

3.5 Cohomological components of tensor products

For $\lambda \in X(T)$, we denote by $\mathcal{L}(\lambda)$ the G-linearized line bundle on G/B such that B acts on the fiber over B/B by the character $-\lambda$. If λ is dominant, the Borel-Weil theorem asserts that the space of sections $\mathrm{H}^0(G/B, \mathcal{L}(\lambda))$ is isomorphic to $V(\lambda)^*$, as a representation of G. We also set

$$\lambda^* = -w_0\lambda.$$

The points $(\lambda_1,\, \lambda_2, \lambda_1^* + \lambda_2^*)$ (for $\lambda_1, \lambda_2 \in X(T)^+$) of $\mathrm{LR}(G)$ have the following geometric property: the morphism

$$\mathrm{H}^0(G/B, \mathcal{L}(\lambda_1)) \otimes \mathrm{H}^0(G/B, \mathcal{L}(\lambda_2)) \longrightarrow \mathrm{H}^0(G/B, \mathcal{L}(\lambda_1 + \lambda_2)), \tag{5}$$

given by the product of sections is nonzero.

Following Dimitrov-Roth (see [DR09b, DR17]), we introduce a natural generalization of these points of $\mathrm{LR}(G)$ coming from the Borel-Weil-Bott theorem. For $w \in W$ and $\lambda \in X(T)$, set:

$$w \cdot \lambda = w(\lambda + \rho) - \rho. \tag{6}$$

The Borel-Weil-Bott theorem asserts that, for any dominant weight λ and any $w \in W$, $\mathrm{H}^{\ell(w)}(G/B, \mathcal{L}(w\cdot\lambda))$ is isomorphic to $V(\lambda)^*$. Let $(\lambda_1,\, \lambda_2,\, \lambda_3)$ be a triple of dominant weights. We say that $(\lambda_1,\, \lambda_2,\, \lambda_3^*)$ is a *cohomological point* of $\mathrm{LR}(G)$ if the cup product:

$$\mathrm{H}^{\ell(w_1)}(G/B, \mathcal{L}(w_1 \cdot \lambda_1)) \otimes \mathrm{H}^{\ell(w_2)}(G/B, \mathcal{L}(w_2 \cdot \lambda_2)) \longrightarrow \mathrm{H}^{\ell(w_3^\vee)}(G/B, \mathcal{L}(w_3^\vee \cdot \lambda_3)) \tag{7}$$

is nonzero for some $w_1, w_2, w_3 \in W$. This implies in particular that $\ell(w_3^\vee) = \ell(w_1) + \ell(w_2)$ and $w_1 \cdot \lambda_1 + w_2 \cdot \lambda_2 = w_3^\vee \cdot \lambda_3$.

Theorem 8 (Dimitrov-Roth)**.** *Let w_1, w_2, w_3 in W and $(\mu_1, \mu_2, \mu_3) \in (X(T)^+)^3$ such that*

1. *$\ell(w_3) = \ell(w_1) + \ell(w_2)$;*
2. *$\mu_3 = \mu_1 + \mu_2$;*
3. *$w_i \cdot \mu_i$ is dominant for $i = 1, 2, 3$.*

Then the cup product map

$$\mathrm{H}^{\ell(w_1)}(G/B, \mathcal{L}(\mu_1)) \otimes \mathrm{H}^{\ell(w_2)}(G/B, \mathcal{L}(\mu_2)) \longrightarrow \mathrm{H}^{\ell(w_3)}(G/B, \mathcal{L}(\mu_3)), \tag{8}$$

is nonzero if and only if $\Phi(w_3) = \Phi(w_1) \sqcup \Phi(w_2)$.

Under the assumption of Theorem 8 and $\Phi(w_3) = \Phi(w_1) \sqcup \Phi(w_2)$, set $\lambda_i = w_i \cdot \mu_i$ for $i = 1, 2, 3$. By the Borel-Weil-Bott theorem, Theorem 8 gives a surjective map

$$V(\lambda_1)^* \otimes V(\lambda_2)^* \longrightarrow V(\lambda_3)^*.$$

In particular the point $(\lambda_1, \lambda_2, \lambda_3^*)$ belongs to $\mathrm{LR}(G)$.

On the other hand, the condition $\mu_3 = \mu_1 + \mu_2$ is equivalent to

$$w_1{}^{-1} \cdot \lambda_1 + w_2{}^{-1} \cdot \lambda_2 = w_3{}^{-1} \cdot \lambda_3,$$

which is also equivalent to.

$$w_1{}^{-1}\lambda_1 + w_2{}^{-1}\lambda_2 = w_3{}^{-1}\lambda_3. \tag{9}$$

Indeed, using that $\Phi(w_3) = \Phi(w_1) \sqcup \Phi(w_2)$ and $\rho = \frac{1}{2}\sum_{\beta\in\Phi^+}\beta$, one easily deduces

$$\rho = w_1{}^{-1}\rho + w_2{}^{-1}\rho - w_3{}^{-1}\rho.$$

In particular, from Theorem 1, equation (9) and Corollary 7 we deduce the following Corollary.

Corollary 9. *The point $(\lambda_1, \lambda_2, \lambda_3^*) \in X(T)^3$ is a cohomological point of* $\mathrm{LR}(G)$ *if and only if it belongs to a regular face of codimension* $\mathrm{rk}(G)$ *of $\mathcal{LR}(G)$.*

4 The geometric strategy

We now start the proof of Theorem 1.

4.1 Incidence variety

Recall that for any $w \in W$, $[X_{w^\vee}]$ is the Poincaré dual of $[X_w]$.

Since $\mathrm{H}^*(G/B, \mathbb{C})$ is graded, if $c_{uv}^w \neq 0$ then

$$\ell(u) + \ell(v) = \ell(w) + \ell(w_0). \tag{10}$$

Assuming (10), by Kleiman's theorem, c_{uv}^w is the cardinality of the intersection

$$g_u X_u \cap g_v X_v \cap g_w X_{w^\vee}$$

for general $(g_u, g_v, g_w) \in G^3$.

Let $(w_1, w_2, w_3) \in W^3$ as in Section 3 and consider the incidence variety

$$Y = Y(w_1, w_2, w_3) = \{p = (z, g_1B/B, g_2B/B, g_3B/B) \in (G/B)^4 \;:\; z \in g_1X_{w_1} \cap g_2X_{w_2} \cap g_3X_{w_3}\}, \tag{11}$$

endowed with its projections $\pi : Y \longrightarrow G/B$ and $\eta : Y \longrightarrow (G/B)^3$ mapping p respectively to z and to $(g_1B/B, g_2B/B, g_3B/B)$. Then, c_{uv}^w is interpreted as the cardinality of a general fiber of η. In particular, to prove Theorem 1, it remains to prove the following proposition (recall that we are working over the complex numbers).

Proposition 10. *The map η is birational.*

4.2 About birational maps

Let $f : Y \longrightarrow X$ be a dominant morphism between irreducible varieties of the same dimension. We say that f is *generically finite*. The *degree of* f is defined to be $\deg(f) = [\mathbb{C}(Y) : \mathbb{C}(X)]$. The degree of f is one, if and only if f is birational. We use the following consequence of the main Zariski theorem (see [Mum99, Chap III, Section 9, Proposition 1]).

Proposition 11. *Assume that f is birational and that X is normal. Let D be a primitive divisor in Y. Assume that the closure of $f(D)$ is of codimension one.*

Then, the restriction of f to D is still birational. Moreover, if D and D' are two divisor as in the statement such that $\overline{f(D)} = \overline{f(D')}$ then $D = D'$.

Come back to generically finite morphism $f : Y \longrightarrow X$. Assume in addition that Y is normal and X is smooth. Let Y^{reg} denote the open set of smooth points in Y. The determinant of the tangent map of f defines a Cartier divisor R_f in Y^{reg}, called the *ramification* divisor. Taking the closure we get a Weyl divisor of Y, still denoted by R_f. Recall that $Y - Y^{\mathrm{reg}}$ has codimension at least 2. Let $\mathrm{Supp}(R_f)$ denote the reduced support of R_f.

Proposition 12. *Let $f : Y \longrightarrow X$ be a generically finite morphism. Assume, in addition, that*

1. *X is smooth and projective;*
2. *Y is normal and projective;*
3. *the closure of $f(\mathrm{Supp}(R_f))$ has codimension at least two in X;*
4. *$\pi_1(X) = \{0\}$.*

Then, f is birational.

Proof. Using the Stein factorisation [Har77, Corollary 11.5], we may assume that f is finite. Then f is a covering from $Y \backslash R_f$ onto $X \backslash f(\mathrm{Supp}(R_f))$. Since $f(\mathrm{Supp}(R_f))$ has codimension at least two in X, the fundamental groups of X and $X \backslash f(\mathrm{Supp}(R_f))$ coincide, and $X \backslash f(\mathrm{Supp}(R_f))$ is simply connected. The proposition follows. □

5 First properties of the map η

5.1 Bruhat order

For later use, we fix some notation on the Bruhat order. The Bruhat order is generated by the covering relations: $v \leq^1 w$ is equivalent to $X_v \subset X_w$ and $\dim(X_w) = \dim(X_v) + 1$.

We denote by $\leq_L$ the weak Bruhat order, which can be defined by $v \leq_L w$ if and only if $\Phi(v) \subset \Phi(w)$. It is generated by the covering relations: $v \leq_L^1 w$ is equivalent to $\Phi(v) \subset \Phi(w)$ and $\sharp\Phi(w) = \sharp\Phi(v) + 1$. Equivalently, $v \leq_L^1 w$ if and only if $v \leq w$ and there exists a simple root $\alpha \in \Delta$ such that $w = s_\alpha v$.

5.2 An open subset of the incidence variety

Fix w_1, w_2 and w_3 in W satisfying (4). Consider the incidence variety $Y = Y(w_1, w_2, w_3)$ and the two maps π and η defined in (11).

We now present an alternative description of Y. Set $X = (G/B)^3$ and

$$z_0 = ({w_1}^{-1}B/B, {w_2}^{-1}B/B, {w_3}^{-1}B/B) \in X.$$

Note that G^3 acts on X, and set $C^+ = B^3 \cdot z_0$. Denote by $\bar{C}^+$ the closure of C^+ in X. The group B acts on $G \times \bar{C}^+$ by the formula $b \cdot (g, z) = (gb^{-1}, bz)$. The quotient of $G \times \bar{C}^+$ under this action is a projective variety denoted by $G \times_B \bar{C}^+$. The class of (g, z) is denoted by $[g : z]$. The map

$$\begin{array}{rccl} \phi : & G \times_B \bar{C}^+ & \longrightarrow & Y \\ & [g : (z_1, z_2, z_3)] & \longmapsto & (gB/B, gz_1, gz_2, gz_3) \end{array}$$

is an isomorphism.

Observe that, modulo ϕ, η identifies with $[g : (z_1, z_2, z_3)] \longmapsto (gz_1, gz_2, gz_3)$, and π with $[g : (z_1, z_2, z_3)] \longmapsto gB/B$.

We now consider $G \times_B C^+$ as an open subset of $G \times_B \bar{C}^+$ and denote by η° the restriction of η to this open set. Then, $G \times_B C^+$ identifies (once more, via ϕ) with

$$Y^\circ = \{(z, g_1B/B, g_2B/B, g_3B/B) \in (G/B)^4 \, : \, z \in g_1X^\circ_{w_1} \cap g_2X^\circ_{w_2} \cap g_3X^\circ_{w_3}\},$$

where $X^\circ_w = BwB/B$, for any $w \in W$.

In W^3 (which is the Weyl group of G^3), we have $(w'_1, w'_2, w'_3) \leq^1 (w_1, w_2, w_3)$ if and only if two of the w_i are equal to the corresponding w'_i and the last one is a covering relation in the Bruhat order of W. Whenever $(w'_1, w'_2, w'_3) \leq^1 (w_1, w_2, w_3)$ is fixed, we set

$$z'_0 = ({w'_1}^{-1}B/B, {w'_2}^{-1}B/B, {w'_3}^{-1}B/B) \in (G/B)^3$$

and

$$\begin{array}{rl} D_{(w'_1, w'_2, w'_3)} & = \{p = (z, g_1B/B, g_2B/B, g_3B/B) \in (G/B)^4 \, : \, z \in g_1X_{w'_1} \cap g_2X_{w'_2} \cap g_3X_{w'_3}\}, \\ & = G \times_B (\overline{B^3 \cdot z'_0}), \end{array}$$

Then

$$G \times_B \bar{C}^+ = G \times_B C^+ \sqcup \bigcup D_{(w'_1, w'_2, w'_3)} \qquad (12)$$

where the union runs over the set of $(w'_1, w'_2, w'_3) \leq^1 (w_1, w_2, w_3)$.

5.3 The differential of η

Given $\varphi \in \Phi$, denote by $\mathfrak{g}_\varphi$ the corresponding weight space in the Lie algebra $\mathfrak{g}$ of G. For $w \in W$, set $T_w = \oplus_{\varphi \in \Phi(w)} \mathfrak{g}_{-\varphi}$. The projection $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b}$ gives an isomorphism between T_w and the tangent space $T_{B/B}w^{-1}BwB/B$ of $w^{-1}X^\circ_w$ at the point B/B. From now on we identify these two spaces.

Lemma 13. *Let* (w_1', w_2', w_3') *such that* $(w_1', w_2', w_3') \leq^1 (w_1, w_2, w_3)$ *and* $D = D_{(w_1', w_2', w_3')}$. *Then*

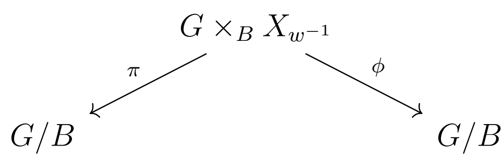
1. $\mathrm{Ker}(T_{[e:z_0]}\eta) \simeq T_{w_1} \cap T_{w_2} \cap T_{w_3}$;

2. $\mathrm{Ker}(T_{[e:(g_1,g_2,g_3)z_0']}\eta_{|D}) \simeq g_1 T_{w_1'} \cap g_2 T_{w_2'} \cap g_3 T_{w_3'}$, *for any* g_1, g_2, g_3 *in* B;

3. $\mathrm{Ker}(T_{[e:z_0']}\eta) \simeq T_{w_1} \cap T_{w_2} \cap T_{w_3}$, *if* $(w_1', w_2', w_3') \leq_L (w_1, w_2, w_3)$.

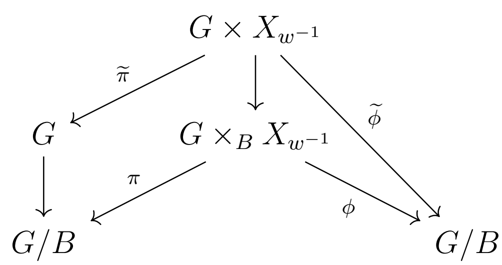
We need a preparatory lemma.

Lemma 14. *Let* $w \in W$, $h \in B$ *and* $z := w^{-1}B/B \in G/B$. *Consider the maps*



defined by $\phi([g : x]) = gx$ *and* $\pi([g : x]) = gB/B$. *The restriction of* $T_{[e:hz]}\pi$ *to* $\ker T_{[e:hz]}\phi$ *is an isomorphism with image* hT_w.

Proof. We have a commutative diagram



Where $\widetilde{\phi}((g, x)) = gx$, $\widetilde{\pi}((g, x)) = g$ and the vertical maps are the quotient maps. We claim that the restriction of $T_{(e,hz)}\widetilde{\pi}$ to $\ker T_{(e,hz)}\widetilde{\phi}$ is an isomorphism with image $hT_e(w^{-1}BwB)$. The lemma follows easily from the fact that $G \times X_{w^{-1}} \longrightarrow G \times_B X_{w^{-1}}$ and $G \longrightarrow G/B$ are locally-trivial B-bundles and that $T_e(w^{-1}BwB) = T_w \oplus \mathfrak{b}$.

We prove the claim. Note that $T_{(e,hz)}\widetilde{\pi} = p_1$, where $p_1 : \mathfrak{g} \times T_h z X_{w^{-1}} \longrightarrow \mathfrak{g}$ is the first projection. Let $\bar{w} \in N_G(T)$ be a representative of w. Multiplication for $h\bar{w}^{-1}$ is an automorphism of G/B that restrict to an isomorphism between $\overline{wBw^{-1}B/B}$ and $X_{w^{-1}}$. Since it sends B/B to hz, from now on we can identify $T_{hz}X_{w^{-1}}$ with $T_{w^{-1}}$ and $T_{hz}G/B$ with $\mathfrak{g}/\mathfrak{b}$ using the differential of these isomorphisms. For $(v, \zeta) \in \mathfrak{g} \times T_{w^{-1}}$, a standard calculation provides that

$$T_{(e,hz)}\widetilde{\phi}(v, \zeta) = \mathrm{Ad}(\bar{w}h^{-1})(v) + \zeta + \mathfrak{b}.$$

It follows immediately that the restriction of p_1 to $\mathrm{Ker}(T_{(e,hz)}\widetilde{\phi})$ is an isomorphism with image $\mathrm{Ad}(h\bar{w}^{-1})(T_{w^{-1}}+\mathfrak{b})$. But

$$\begin{array}{rcl}\mathrm{Ad}(\bar{w}^{-1})(T_{w^{-1}}+\mathfrak{b}) = & T_e w^{-1}(wBw^{-1}B)w \\ = & -T_e(Bw^{-1}Bw)^{-1} \\ = & T_e(w^{-1}BwB)\end{array}$$

Moreover, since $w^{-1}BwB$ is stable by right multiplication of B, $\mathrm{Ad}(h)(T_e(w^{-1}BwB)) = hT_e(w^{-1}BwB)$. □

Proof of Lemma 13. As a direct application of Lemma 14, the isomorphisms of the first two statements are induced by the differential of π.

In the last case, set $w_1' = s_\alpha w_1$. Then X_{w_1} is stable by the minimal parabolic subgroup associated to α. In particular $s_\alpha X_{w_1} = X_{w_1}$.

Let $w \in W$ and $\alpha \in \Delta$ such that $w' = s_\alpha w \leq_L w$. Set $\beta = w'^{-1}\alpha$. For the last assertion, we first prove the following equality.

Claim: $T_{B/B}w'^{-1}X_w = T_{B/B}w^{-1}X_w$.

Since $T_{B/B}G/B \simeq \mathfrak{u}^-$ is multiplicity free as T-module, it is sufficient to compare the sets of weights. Recall that $\Phi(w) = \Phi(w') \cup \{\beta\}$. Since $X_{w'} \subset X_w$, $-\Phi(w')$ are weights of both $T_{B/B}w'^{-1}X_w$ and $T_{B/B}w^{-1}X_w$. Since X_w is normal, $w'B/B$ is smooth in X_w and the two tangent spaces have the same dimension. Hence, to prove the claim it is sufficient to prove that $-\beta$ is a weight of $T_{B/B}w'^{-1}X_w$.

Consider now the action of the additive one parameter subgroup of U^- associated to $-\beta$ on the point B/B. The closure $\mathcal{C}$ of the orbit is isomorphic to $\mathbb{P}^1$ and contains the T-fixed points B/B and $s_\beta B/B$. Moreover, T acts with weight β on $T_{s_\beta B/B}\mathcal{C}$. Since $\beta \in \Phi(w)$, $\mathcal{C}$ is contained in $w^{-1}X_w$ and β is a weight of $T_{s_\beta B/B}w^{-1}X_w$, because $w'^{-1}X_w = s_\beta w^{-1}X_w$. Applying s_β we get that $-\beta$ is a weight of $T_{B/B}w'^{-1}X_w$.

By symmetry, assume that $w_1' = s_\alpha w_1$, $w_2' = w_2$ and $w_3' = w_3$. As for the two first assertions, one can check that

$$\mathrm{Ker}(T_{[e:z_0']}\eta) \simeq T_{B/B}{w'_1}^{-1}X_{w_1} \cap T_{w_2} \cap T_{w_3}.$$

It is clear that the claim implies the last assertion of the lemma. □

Lemma 15. *The map η° is smooth.*

Proof. Since $G \times_B C^+$ and X are smooth, it remains to prove that $T_p\eta$ is invertible for any $p \in G \times_B C^+$. Consider the set Z of such points p such that $T_p\eta$ is not invertible.

Our assumption on (w_1, w_2, w_3) and Lemma 13 imply that $T_{[e:z_0]}\eta$ is invertible.

Let τ be a dominant regular one parameter subgroup of T. It is well known that for any $z \in C^+$, $\lim_{t\to 0}\tau(t)z = z_0$. Since Z is closed and stable by the action of τ, this implies that $Z \cap C^+ = \emptyset$ (here C^+ identified to a subvariety of $G \times_B C^+$ by the map $x \mapsto [e : x]$).

The map η being G-equivariant, Z is empty. □

Lemma 15 implies that $\Omega = \eta(G \times_B C^+)$ is open in X. Moreover, we can prove the following proposition.

Proposition 16. *If Proposition 10 holds, then $\eta^\circ : G \times_B C^+ \longrightarrow \Omega$ is an isomorphism.*

Proof. Let $Z = G \times_B C^+$. Fix $q \in \Omega$ and denote by Z_q its schematic fiber for η°. Since η° is smooth of relative dimension zero and Z is of finite type, Z_q is a variety of dimension zero, hence affine. Moreover, by flatness of η°, $\dim \mathbb{C}[Z_q] = \deg \eta^\circ = 1$. Hence Z_q is a single point. The proposition follows from the Zariski main theorem. □

Lemma 17. *Set $z'_0 = (w_1'^{-1}B/B, w_2'^{-1}B/B, w_3'^{-1}B/B) \in \bar{C}^+$. If $(w'_1, w'_2, w'_3) \leq^1_L (w_1, w_2, w_3)$ then $T_{[e:z'_0]}\eta$ is an isomorphism.*

Proof. The statement follows easily from Condition (4) and Lemma 13. □

5.4 The case of Poincaré duality

Let $w \in W$. Keep the notation of the previous section assuming in addition that $(w_1, w_2, w_3) = (w, w^\vee, w_0)$. Then, by Poincaré duality η is birational. We now describe the behaviour of the divisors on the boundary of Y° using Proposition 11.

Proposition 18. *Let w'_1, w'_2 and w'_3 in W such that $(w'_1, w'_2, w'_3) \leq^1 (w, w^\vee, w_0)$ in W^3. Then*

1. *the restriction of η to $D_{(w'_1,w'_2,w'_3)}$ is birational if and only if $(w'_1, w'_2, w'_3) \leq_L (w, w^\vee, w_0)$. Moreover, there are exactly $2\mathrm{rk}(G)$ such divisors.*

2. *If $(w'_1, w'_2, w'_3) \leq_L (w, w^\vee, w_0)$ does not hold, $\overline{\eta(D_{(w'_1,w'_2,w'_3)})}$ has codimension at least two.*

Proof. By Proposition 11, the first assertion implies the second one. The proof of the first one proceeds in three steps.
Step 1. If $(w'_1, w'_2, w'_3) \leq_L (w, w^\vee, w_0)$ then $\overline{\eta(D)}$ is a divisor in $X = (G/B)^3$.

Observe that, by normality of the Schubert varieties, $[e : z'_0]$ is a smooth point. Lemma 17 shows that $T_{[e:z'_0]}\eta$ is injective. In particular, the fiber $\eta^{-1}(\eta([e : z'_0]))$ is finite. Hence $\dim(\overline{\eta(D_{(w'_1,w'_2,w'_3)})}) = \dim X - 1$. Now, Proposition 11 allows to conclude.

Step 2. Let $(w'_1, w'_2, w'_3) \in W^3$ such that $(w'_1, w'_2, w'_3) \leq^1_L (w, w^\vee, w_0)$. We claim that there are exactly $2\mathrm{rk}(G)$ triples. We have three possibilities.

1. $w'_1 = w$ and $w'_2 = w^\vee$. Then $w'_3 = s_\alpha w_0$ some $\alpha \in \Delta$ and any such w'_3 works. We get $\mathrm{rk}(G)$ such cases.

2. $w'_3 = w_0$ and $w'_2 = w^\vee$. Then $w'_1 = s_\alpha w$ for some descent α of w and any such w'_1 works.

3. $w_3' = w_0$ and $w_1' = w$. Then $w_2' = s_\alpha w^\vee$ for some descent α of $w^\vee$ and any such w_2' works.

The count is correct since, for any $\alpha \in \Delta$, either α is a descent of w or (exclusively) $-w_0\alpha$ is a descent of $w^\vee$.

Step 3. The closed subset $X - \Omega$ has $2\mathrm{rk}(G)$ irreducible components of codimesion one in X.

Note that $G \times_B C^+$ contains $U^- \times C^+$ as an open subset. The latter being an affine space, it follows that $\mathbb{C}[G \times_B C^+]^* = \mathbb{C}^*$. By Proposition 16, $\mathbb{C}[\Omega]^* = \mathbb{C}^*$. Let $E_1, \dots, E_s$ be the irreducible components of $X - \Omega$ of codimension one in X. The previous discussion and the fact that X is smooth imply that we have an exact sequence (see e.g. [Har77, Proposition 6.5]):

$$0 \longrightarrow \oplus_{i=1}^s \mathbb{Z}E_s \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\Omega) \longrightarrow 0. \qquad (13)$$

Observe first that $\mathrm{Pic}(X) \simeq X(T)^3$ is a free abelian group of rank $3\mathrm{rk}(G)$. Then, the irreducible components of the complement of $U^- \times C^+$ in $G \times_B C^+$ are the pullbacks by π of the divisors $\overline{B^- s_\alpha B/B}$. There are $\mathrm{rk}(G)$ of them. Using the exact sequence analogue to (13) and Proposition 16 we deduce that $\mathrm{Pic}(\Omega) \simeq \mathrm{Pic}(G \times_B C^+)$ is a free abelian group of rank $\mathrm{rk}(G)$.

Now, the exactness of sequence (13) implies that $s = 2\mathrm{rk}(G)$.

Step 3 and the last statment of Proposition 11 imply that $2\mathrm{rk}(G)$ irreducible divisors are not contracted by η. At Steps 1 and 2, we found $2\mathrm{rk}(G)$ of them. This ends the proof. □

6 The kernel of the differential map

Fix $(w_1, w_2, w_3) \in W^3$ satisfying Assumption (4) and consider the map $\eta : Y \longrightarrow X$ defined in Section 4.1. Lemma 15 shows that the restriction η° of η to $G \times_B C^+$ is smooth. Let D be an irreducible component of $Y \backslash (G \times_B C^+)$. By (12), there exists $(w_1', w_2', w_3') \leq^1 (w_1, w_2, w_3)$ such that $D = D_{(w_1', w_2', w_3')}$. Lemma 17 shows that D is not contained in the ramification divisor R_η if $(w_1', w_2', w_3') \leq^1_L (w_1, w_2, w_3)$ holds. Otherwise, one can apply the following proposition.

Proposition 19. *Assume that $(w_1', w_2', w_3') \leq^1_L (w_1, w_2, w_3)$ does not holds. Then $D_{(w_1', w_2', w_3')}$ is contracted by η.*

The previous proposition allows to prove Proposition 10.

Proof of Proposition 10. Let R_η be the ramification divisor of η. Lemmas 17, 15 and Proposition 19 imply that any irreducible component of R_η of codimension one in Y is contracted by η. The statement follows from Proposition 12. □

The rest of the section is a proof of Proposition 19. Observe first that it is sufficient to prove the following assertion.
Claim: For any $x \in D = D_{(w'_1, w'_2, w'_3)}$ the linear map $T_x\eta_{|D}$ is not injective.

Indeed, since we work over $\mathbb{C}$, η is separated. Moreover, by G-equivariance and semicontinuity, it is sufficient to prove the claim for $x \in B^3 z'_0 = U^3 z'_0$.

6.1 A description as the kernel of a matrix

Up to S_3-symmetry, assume that $w'_1 \neq w_1$. It is convenient to set

$$w = w_1 \quad x = w_2^\vee \quad y = w_3^\vee \quad v = w'_1.$$

Then, the assumptions are equivalent to

$$\Phi(w) = \Phi(x) \sqcup \Phi(y), \quad v \leq^1 w \quad \text{and} \quad \Phi(v) \not\subset \Phi(w). \tag{14}$$

For any $\varphi \in \Phi$, fix nonzero elements ξ_φ and ξ^φ in $\mathfrak{g}_{-\varphi}$ and $(\mathfrak{g}^*)_\varphi$ respectively.

Fix g_x and g_y in U. To this data, we attach a matrix $M = M(v, w, x, y, g_x, g_y)$ whose rows are indexed by $\Phi(w)$ and columns by $\Phi(v)$. The entry at row $\beta \in \Phi(w)$ and column $\gamma \in \Phi(v)$ is

$$M_{\beta\gamma} = \begin{cases} \xi^\beta({g_x}^{-1}\xi_\gamma) & \text{if } \beta \in \Phi(x) \\ \xi^\beta({g_y}^{-1}\xi_\gamma) & \text{if } \beta \in \Phi(y) \end{cases}$$

Lemma 20. *The Kernel of M is isomorphic to the intersection*

$$T_v \cap g_x T_{x^\vee} \cap g_y T_{y^\vee} \simeq T_p \eta_{|D},$$

where $p = [e : (B/B, g_x x^{-1} B/B, g_y y^{-1} B/B)]$.

Hence, by Lemma 13 and B-invariance, to prove Proposition 19 it is sufficient to prove the following.

Proposition 21. *The Kernel of $M = M(v, w, x, y, g_x, g_y)$ is nonzero.*

Example. In the root system D_4, consider $w = s_2 s_3 s_1 s_2 s_4 s_2$ and $v = w s_2$. Then

$$\Phi(w) = \{\alpha_2, \alpha_2 + \alpha_4, \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$$

and

$$\Phi(v) = \{\alpha_4, \alpha_2 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}.$$

Consider a generic matrix

$$\xi = \left(\begin{array}{cccc|cccc} 0 & x_0 & x_1 & x_2 & x_6 & x_7 & x_8 & 0 \\ 0 & 0 & x_3 & x_4 & x_9 & x_{10} & 0 & -x_8 \\ 0 & 0 & 0 & x_5 & x_{11} & 0 & -x_{10} & -x_7 \\ 0 & 0 & 0 & 0 & 0 & -x_{11} & -x_9 & -x_6 \\ \hline 0 & 0 & 0 & 0 & 0 & -x_5 & -x_4 & -x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

in Lie(U) and set $u = \exp(\xi)$. The matrix $M(v, w, w, e, u, e)$ is

	α_4	$\alpha_2+\alpha_4$	$\alpha_1+\alpha_2+\alpha_4$	$\alpha_2+\alpha_3+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$
α_4	1	$-x_3$	$\frac{1}{2}x_0x_3 - x_1$	$\frac{1}{2}x_3x_5 + x_4$	$-\frac{1}{3}x_0x_3x_5 - \frac{1}{2}x_0x_4 + \frac{1}{2}x_1x_5 + x_2$
α_2	0	$-x_{11}$	x_0x_{11}	x_5x_{11}	$-x_0x_5x_{11}$
$\alpha_2+\alpha_4$	0	1	$-x_0$	$-x_5$	x_0x_5
$\alpha_1+\alpha_2+\alpha_4$	0	0	1	0	$-x_5$
$\alpha_2+\alpha_3+\alpha_4$	0	0	0	1	$-x_0$
$\alpha_1+2\alpha_2+\alpha_3+\alpha_4$	0	0	0	0	0

Its kernel is nontrivial since two rows are proportional. The authors did not find such a general raison to prove that the kernel of any M is nontrivial. Instead, we use the fact the result is known in the case of Poincaré duality and we reduce to it.

6.2 The case when a submatrix is strictly triangular

Define the height $h : \Phi \longrightarrow \mathbb{Z}$, by $h(\sum_{\alpha\in\Delta} n_\alpha \alpha) = \sum_{\alpha\in\Delta} n_\alpha$. For $h \in \mathbb{Z}$, we set $\Phi(w)_h = \{\varphi \in \Phi(w) : h(\varphi) = h\}$ and $\Phi(w)_{\leq h} = \{\varphi \in \Phi(w) : h(\varphi) \leq h\}$. Note that if $\varphi \leq \psi$, then $h(\varphi) \leq h(\psi)$.

The following lemma is well-known:

Lemma 22. *Let $\gamma \in \Phi$. For any $g \in U$ and $\xi \in \mathfrak{g}_{-\gamma}$, we have*

$$g\xi \in \xi + \sum_{\psi<\gamma} \mathfrak{g}_{-\psi}.$$

As an immediate consequence of Lemma 22, we get that

$$\begin{array}{lcl} M_{\beta\gamma} = 1 & \text{if} & \beta = \gamma, \\ M_{\beta\gamma} \neq 0 & \text{implies} & \beta \leq \gamma. \end{array}$$

We improve this fact as follows.

Lemma 23. *If at least one of the following assertions holds:*

1. *$\exists h \in \mathbb{N}^*$ such that $\sharp\Phi(v)_{\leq h} > \sharp\Phi(w)_{\leq h}$;*

2. *$\exists h \in \mathbb{N}^*$ such that $\sharp\Phi(v)_{\leq h} = \sharp\Phi(w)_{\leq h}$ and $\Phi(v)_{h+1} \not\subset \Phi(w)$;*

then $\mathrm{Ker}M \neq \{0\}$.

Proof. First, number the elements of $\Phi(w)$ (and independently $\Phi(v)$) in such a way that the map $\beta \mapsto h(\beta)$ is nondecreasing. Let N be the submatrix of M with rows and columns in $\Phi(w)_{\leq h}$ and $\Phi(v)_{\leq h}$ respectively. Lemma 22 implies that M has the following form

$$\begin{pmatrix} N & \star \\ 0 & \star \end{pmatrix} \tag{15}$$

With the first assumption of the lemma, N has more columns than rows; hence its kernel is not reduced to zero. By (15), that of M too.

Assume now that we are in the second case. Then, N is a square matrix and we can fix $\gamma \in \Phi(v)_{h+1}$ and $\gamma \notin \Phi(w)$. Up to renumbering, assume that γ is the first root in $\Phi(v)_{h+1}$.

Let $\tilde{N}$ be the submatrix of M with rows and columns in $\Phi(w)_{\leq h}$ and $\Phi(v)_{\leq h} \cup \{\gamma\}$ respectively. Lemma 22 implies that M is block triangular as in (15) with $\tilde{N}$ in place of N. Hence the kernel M is not reduced to zero. □

6.3 The case when no submatrix is strictly triangular

We now assume that Lemma 23 does not apply; that is that:

(H1) $\forall h \in \mathbb{N}^* \qquad \sharp\Phi(v)_{\leq h} \leq \sharp\Phi(w)_{\leq h}$; and

(H2) $\forall h \in \mathbb{N}^*$ such that $\sharp\Phi(v)_{\leq h} = \sharp\Phi(w)_{\leq h}$ we have $\Phi(v)_{h+1} \subset \Phi(w)$.

Observe that (H2) can be re-written as

$$\text{(H2')}\ \forall h \in \mathbb{N}^* \qquad \sharp\Phi(v)_{<h} = \sharp\Phi(w)_{<h} \implies \Phi(v)_h \subset \Phi(w).$$

Set

$$\begin{array}{c} \Phi(w) - \Phi(v) = \{\beta_0, \beta_1, \dots, \beta_s\} \\ \Phi(v) - \Phi(w) = \{\gamma_0, \gamma_1, \dots, \gamma_t\} \end{array}$$

by labeling the elements by nondecreasing height.

A key result to understand the matrix M is the following

Proposition 24. *With above notation and assuming (H1) and (H2), we have*

1. $w = vs_{\beta_0}$;
2. $s = t + 1$;
3. $h(\beta_0) < h(\gamma_0) < h(\beta_1) < h(\gamma_1) < \cdots < h(\beta_s)$;
4. *for any* $i = 0, \dots, t$, *there exists* $k_i \in \mathbb{N}^*$ *such that* $\beta_{i+1} = \gamma_i + k_i\beta_0$.

Proof. Since $v \leq^1 w$, there exists $\beta \in \Phi^+$ such that $w = vs_\beta$. By an immediate induction, it is sufficient to prove the following three assertions:

(P0) $\Phi(w)_{\leq h(\beta)} = \Phi(v)_{\leq h(\beta)} \sqcup \{\beta\}$, and $\beta_0 = \beta$.

(P1) If

- (a) $h(\beta_0) < h(\gamma_0) < \cdots < h(\beta_i)$,
- (b) $\Phi(w)_{\leq h(\beta_i)} - \Phi(v) = \{\beta_0, \beta_1, \dots, \beta_i\}$,
- (c) $\Phi(v)_{\leq h(\beta_i)} - \Phi(w) = \{\gamma_0, \gamma_1, \dots, \gamma_{i-1}\}$,

(d) $\forall 0 \leq j < i \qquad \exists k \in \mathbb{N} \qquad \beta_{j+1} - k\beta_0 = \gamma_j$, and

(e) $\Phi(w)_{>h(\beta_i)} \neq \Phi(v)_{>h(\beta_i)}$

then

(f) $h(\gamma_i) > h(\beta_i)$,

(g) $\Phi(w)_{\leq h(\gamma_i)} - \Phi(v) = \{\beta_0, \beta_1, \ldots, \beta_i\}$, and

(h) $\Phi(v)_{\leq h(\gamma_i)} - \Phi(w) = \{\gamma_0, \gamma_1, \ldots, \gamma_i\}$.

(P2) If

(a) $h(\beta_0) < h(\gamma_0) < \cdots < h(\gamma_i)$,

(b) $\Phi(w)_{\leq h(\gamma_i)} - \Phi(v) = \{\beta_0, \beta_1, \ldots, \beta_i\}$,

(c) $\Phi(v)_{\leq h(\gamma_i)} - \Phi(w) = \{\gamma_0, \gamma_1, \ldots, \gamma_i\}$, and

(d) $\forall 0 \leq j < i \qquad \exists k \in \mathbb{N} \qquad \beta_{j+1} - k\beta_0 = \gamma_j$

then

(e) $h(\beta_{i+1}) > h(\gamma_i)$,

(f) $\Phi(w)_{\leq h(\beta_{i+1})} - \Phi(v) = \{\beta_0, \beta_1, \ldots, \beta_{i+1}\}$,

(g) $\Phi(v)_{\leq h(\beta_{i+1})} - \Phi(w) = \{\gamma_0, \gamma_1, \ldots, \gamma_i\}$, and

(h) $\exists k \in \mathbb{N} \qquad \beta_{i+1} - k\beta_0 = \gamma_i$.

Proof of (P0). Fix a positive root $\theta \neq \beta$. It is well known that there exist integers $p \leq q$ such that

$$(\theta + \mathbb{Z}\beta) \cap \Phi^+ = \{\theta + k\beta \,:\, k \in [p; q] \cap \mathbb{Z}\}.$$

Since $\beta \in \Phi(w)$, the convexity of $\Phi(w)$ implies that

$$(\theta + \mathbb{Z}\beta) \cap \Phi(w) = \{\theta + k\beta \,:\, k \in [r; q] \cap \mathbb{Z}\},$$

for some integer $p \leq r \leq q + 1$. Then

$$(\theta + \mathbb{Z}\beta) \cap \Phi(v) = \{\theta + k\beta \,:\, k \in [p; s] \cap \mathbb{Z}\}, \quad \text{where } s = p + q - r. \tag{16}$$

In other words, $(\theta + \mathbb{Z}\beta) \cap \Phi(w)$ consists in the $q - r + 1$ last elements of $(\theta + \mathbb{Z}\beta) \cap \Phi^+$, whereas $(\theta+\mathbb{Z}\beta)\cap\Phi(v)$ consists in the $q-r+1$ first elements. In [Res18], there is a geometric proof of equality (16). For completeness, we include a combinatorial one. By coconvexity of $\Phi(v)$ it is sufficient to prove the following lemma.

Lemma 25. *Let $w \in W$, $\beta \in \Phi^+$ and $v = ws_\beta \leq^1 w$. For any $\theta \in \Phi^+ \setminus \{\beta\}$ we have:*

$$\sharp((\theta + \mathbb{Z}\beta) \cap \Phi(w)) = \sharp((\theta + \mathbb{Z}\beta) \cap \Phi(v)).$$

Proof. Enumerate the simple roots, that is $\Delta = \{\alpha_1, \dots \alpha_r\}$. Let $w = s_{i_m} \dots s_{i_1}$, for some integers $1 \le i_j \le r$, be a reduced expression of w. Then there exists a unique $1 \le k \le m$ such that

$$\beta = \begin{cases} s_{i_1} \dots s_{i_{k-1}} \alpha_{i_k} & \text{if} \quad 1 < k \\ \alpha_{i_1} & \text{otherwise.} \end{cases}$$

Moreover $v = s_{i_m} \dots s_{i_{k+1}} s_{i_{k-1}} \dots s_{i_1}$. If $k = m$, the statement is obvious since $v \le_L^1 w$, hence $\Phi(w) = \Phi(v) \cup \{\beta\}$. Otherwise, let $w' = s_{i_m} w$ and $v' = s_{i_m} v$. We have that $v' \le^1 w'$ and

$$\Phi(w) = \Phi(w') \sqcup \{(w')^{-1}\alpha_{i_m}\} \quad \text{and} \quad \Phi(v) = \Phi(v') \sqcup \{(v')^{-1}\alpha_{i_m}\}.$$

But, paying attention if $k = 1$,

$$\begin{aligned} (w')^{-1}\alpha_{i_m} =& s_{i_1} \dots s_{i_{m-1}} \alpha_{i_m} \\ =& s_{1_1} \dots s_{i_{k-1}} s_{i_k} s_{i_{k-1}} \dots s_{i_1} (v')^{-1} \alpha_{i_m} \\ =& s_\beta (v')^{-1} \alpha_{i_m} \in (v')^{-1} \alpha_{i_m} + \mathbb{Z}\beta. \end{aligned}$$

The statement follows easily by induction on $m - k$. □

Going back to the proof, we have that $\Phi(w)_{<h(\beta)} \subset \Phi(v)$ and $\Phi(w)_{h(\beta)} - \{\beta\} \subset \Phi(v)$. Using $(H1)$, we get $\Phi(w)_{<h(\beta)} = \Phi(v)_{<h(\beta)}$. Now, (H2) implies that $\Phi(v)_{h(\beta)} \subset \Phi(w)$. Hence $\Phi(w)_{\le h(\beta)} = \Phi(v)_{\le h(\beta)} \sqcup \{\beta\}$.

Then, β is the unique element of minimal height in $(\Phi(v) \cup \Phi(w)) - (\Phi(v) \cap \Phi(w))$ and it belongs to $\Phi(w)$. Hence $\beta_0 = \beta$.

Proof of (P1). Let $\theta \in (\Phi(v) \cup \Phi(w)) - (\Phi(v) \cap \Phi(w))$ such that $h(\theta) > h(\beta_i)$ and of minimal height with these properties. Such a θ exists by Hypothesis (P1e). Roughly speaking θ is the next difference.

Consider $\theta + \mathbb{Z}\beta$. By Hypothesis (P1d), for any $0 \le j < i$, $\gamma_j \in \theta + \mathbb{Z}\beta$ if and only if $\beta_{j+1} \in \theta + \mathbb{Z}\beta$. Hence, by (16), θ, that is the next difference in $\theta + \mathbb{Z}\beta$, belongs to $\Phi(v) - \Phi(w)$.

The assumptions imply that $\sharp\Phi(v)_{<h(\beta_i)} = \sharp\Phi(w)_{<h(\beta_i)}$. Hence (H2) gives $\Phi(v)_{h(\beta_i)} \subset \Phi(w)$. Thus $h(\theta) \ne h(\beta_i)$ and $h(\theta) > h(\beta_i)$.

The set $\{\gamma_0, \dots, \gamma_{i-1}, \theta\} \sqcup \Phi(w)_{\le h(\theta)}$ is contained in $\{\beta_0, \dots, \beta_i\} \sqcup \Phi(v)_{\le h(\theta)}$, and even equal by (H1). Then $\gamma_i = \theta$. This ends the proof of (P1).

Proof of (P2). Let $\theta \in (\Phi(v) \cup \Phi(w)) - (\Phi(v) \cap \Phi(w))$ such that $h(\theta) > h(\gamma_i)$ and of minimal height with these properties. Such a θ exists since $\sharp\Phi(w) = \sharp\Phi(v) + 1$.

The assumptions imply that $\sharp\Phi(v)_{<h(\theta)} = \sharp\Phi(w)_{<h(\theta)}$. Then (H2) gives $\Phi(v)_{h(\theta)} \subset \Phi(w)$ and $\theta \in \Phi(w)$.

Consider $\theta + \mathbb{Z}\beta$ and recall (16). For any $j < i$, $\gamma_j \in \theta + \mathbb{Z}\beta$ if and only if $\beta_{j+1} \in \theta + \mathbb{Z}\beta$. We deduce that there exists $k \in \mathbb{N}$ such that $\theta - k\beta_0$ belongs to $\Phi(v) - \{\gamma_0, \dots, \gamma_{i-1}\}$ and not in $\Phi(w)$. But γ_i is the only such element of height less than $h(\theta)$. Hence $\theta - k\beta_0 = \gamma_i$.

What we have just proved also implies that $\theta + \mathbb{Z}\beta_0 = \gamma_i + \mathbb{Z}\beta_0$ and that θ is the only element in $\Phi(w) - \Phi(v)$ of its height. It follows that $\theta = \beta_{i+1}$ and $\Phi(v)_{h(\theta)} \sqcup \{\theta\} = \Phi(w)_{h(\theta)}$. This ends the proof of (P2). □

To emphasise the structure of M in blocks, let us set, for any $i = 0, \dots, s$

$$\begin{aligned} \Phi_i^- &= \{\theta \in \Phi(v) - \{\gamma_0, \dots, \gamma_{s-1}\} \mid h(\gamma_{i-1}) \le h(\theta) \le h(\beta_i)\} \\ \Phi_i^+ &= \{\theta \in \Phi(v) - \{\gamma_0, \dots, \gamma_{s-1}\} \mid h(\beta_i) < h(\theta) < h(\gamma_i)\}, \end{aligned}$$

where, by convention $h(\gamma_{-1}) = 0$ and $h(\gamma_s) = +\infty$.

For $i = 0, \dots, s$, denote by M_i^- the submatrix of M whose rows and columns indices of its entries belong to Φ_i^-. For $i = 0, \dots, s-1$, denote by M_i^+ the submatrix of M corresponding to the rows $\Phi_i^+ \sqcup \{\beta_i\}$ and columns $\Phi_i^+ \sqcup \{\gamma_i\}$. Finaly, denote by M_s^+ the submatrix of M corresponding to the rows $\Phi_s^+ \sqcup \{\beta_s\}$ and columns Φ_s^+.

Observe that all the $M_i^\pm$ are square matrices except M_s^+. Recall that the elements of $\Phi(v)$ and $\Phi(w)$ are numbered by nondecreasing height. Then, Lemma 22 implies easily that M is upper triangular by blocs with $M_0^-, M_0^+, M_1^-, \dots, M_s^+$ as diagonal blocks. The same lemma also implies that each M_i^- is upper triangular with 1's on the diagonal and that

$$M_s^+ = \begin{pmatrix} * & \cdots & * & \cdots & * \\ 1 & & * & & \\ & & \ddots & & \\ & & 0 & & 1 \end{pmatrix}.$$

On the example in Section 6.1, $s = 1$, M_0^- is the identity matrix of size 1, M_0^+ is the (4×4)-submatrix with rows in $\{2, 3, 4, 5\}$ and columns in $\{2, 3, 4, 5\}$. The matrix M_0^+ is empty in this case since Φ_1^+ is.

Then, elementary linear algebra gives

Lemma 26. *With above notation, we have* $\mathrm{Ker} M \neq \{0\}$ *if and only if there exists* $i \in \{0, \dots, s-1\}$ *such that* M_i^+ *is not invertible.*

6.4 The trick using Poincaré duality

Recall that the matrix M depends on the choice of a pair (g_x, g_y) of elements in the unipotent group U, nevertheless the sets $\Phi_i^\pm$, and the existence of a corresponding block subdivision of M, only depend on v and w.

Proof of Proposition 21. First consider the case of Poincaré duality: $M(v, w, w, e, g, g')$ for g, g' in U. Since $\Phi(e) = \emptyset$, the matrix is independent of g'.

By Proposition 18, the divisor associated to $(v, w^\vee, w_0)$ is contacted by η. Then the Kernel of $M(v, w, w, e, g, g')$ is nonzero for any $g \in U$. By Lemma 26 this implies that $\prod_i \det M_i^+(v, w, w, e, g, g') = 0$ for any $g \in U$. Since U is irreducible as a variety, this implies that there exists i_0 such that $\det M_{i_0}^+(v, w, w, e, g, g') = 0$ for any $g, g' \in U$.

Consider now the matrix $M(v, w, x, y, g_x, g_y)$.

By Proposition 56 of Section 9, the only coefficients occurring in $\det M^+_{i_0}(v,w,x,y,g_x,g_y)$ are indexed by roots in the interval $[\beta_{i_0};\gamma_{i_0}]$. By Proposition 24 there exists a nonnegative integer k such that $\gamma_{i_0}+k\beta_0 \notin \Phi(w)$ and $\gamma_{i_0}+(k+1)\beta_0 \in \Phi(w)$. Moreover $[\beta_{i_0},\gamma_{i_0}] \subseteq [\beta_0;\gamma_{i_0}+k\beta_0]$ by Statement 4 of Proposition 24.

Then we can apply Theorem 3 with $\beta=\beta_0$ and $\gamma=\gamma_{i_0}+k\beta_0$, and we deduce that $\det M^+_{i_0}(v,w,x,y,g_x,g_y)$ is equal to either $\det M^+_{i_0}(v,w,w,e,g_x,g_x)$ or $\det M^+_{i_0}(v,w,e,w,g_y,g_y)$. By the first part of the proof, $\det M^+_{i_0}(v,w,w,e,g,g')=0$ for any $g,g'\in U$.

But $\det M^+_{i_0}(v,w,e,w,g',g)=\det M^+_{i_0}(v,w,w,e,g,g')$. Thus $\det M^+_{i_0}(v,w,x,y,g_x,g_y)=0$. Since $M^+_{i_0}(v,w,x,y,g_x,g_y)$ is one of the diagonal blocks of $M(v,w,x,y,g_x,g_y)$, this completes the proof. □

7 The proofs of the corollaries

Proof of Corollary 4. Theorem 1 implies that $[X_{w_1}]\cdot[X_{w_2}]\cdot[X_{w_3}]=[pt]$. On the other hand ${w_1}^{-1}X_{w_1}$, ${w_2}^{-1}X_{w_2}$ and ${w_3}^{-1}X_{w_3}$ intersect transversally at the point B/B. It follows that

$$ {w_1}^{-1}X_{w_1}\cap {w_2}^{-1}X_{w_2}\cap {w_3}^{-1}X_{w_3}=\{B/B\}. \qquad (17) $$

Given $x\in W$, we have $x\in {w_1}^{-1}X_{w_1}$ if and only if $w_1x\leq w_1$. Then the statement translates the fact that eB/B is the only point in the intersection (17). □

Proof of Corollary 5. Let $E_1,\dots,E_s$ be the irreducible components of $X-\Omega$ of codimension one in X. The same argument of Step 3 of the proof of Proposition 18 implies that $s=2\mathrm{rk}(G)$. By Propositions 10, 11 and the fact that η is proper, it follows that any of the E_i is dominated by exactly one irreducible components of $Y-Y^\circ$ of codimension one in Y. Hence, $2\mathrm{rk}(G)$ is the number of divisors contained in $Y-Y^\circ$ and not contracted by η. Because of Lemma 17 and Proposition 19 there are exactly $\sum_i d(w_i^\vee)$ such divisors.

□

Proof of Corollary 7. The fact that $\mathcal{F}_{(w_1,w_2,w_3)}$ is a regular face of the cone $\mathcal{LR}(G)$ is a direct application of [Res10, Theorem D]. Idem for the fact that any regular face of dimension $2\mathrm{rk}(G)$ is obtained in such a way. The fact that any integral point in $\mathcal{F}_{(w_1,w_2,w_3)}$ belongs to the semigroup $\mathrm{LR}(G)$ is a consequence of the PRV conjecture (see [Kum88, Mat89] or [MPR11]).

Let $E_1,\dots,E_{2\mathrm{rk}(G)}$ be the irreducible components of $X-\Omega$ of codimension one in X. Each E_i gives a line bundle $\mathcal{O}(E_i)$ on X and a section $\sigma_i\in \mathrm{H}^0(X,\mathcal{O}(E_i))$ such that $\mathrm{div}(\sigma_i)=E_i$. Since G is semisimple and simply connected, $\mathcal{O}(E_i)$ admits a unique G-linearisation. Thus, there exists $(\lambda_1^i,\lambda_2^i,\lambda_3^i)\in X(T)^3$ such that $\mathcal{O}(E_i)=\mathcal{L}(\lambda_1^i,\lambda_2^i,\lambda_3^i)$. Since E_i is G-stable and G has no character, σ_i is G-invariant. Then, $(\lambda_1^i,\lambda_2^i,\lambda_3^i)$ belongs to $\mathrm{LR}(G)$.

By construction $\sigma_i([e:x_0])\neq 0$, where $x_0=({w_1}^{-1}B/B,{w_2}^{-1}B/B,{w_3}^{-1}B/B)$. Since $[e:x_0]$ is fixed by the maximal torus T and σ_i is G-invariant, T has to act trivially on the fiber in $\mathcal{O}(E_i)$ over $[e:x_0]$. Thus, ${w_1}^{-1}\lambda_1^i+{w_2}^{-1}\lambda_2^i+{w_2}^{-1}\lambda_3^i=0$ and $(\lambda_1^i,\lambda_2^i,\lambda_3^i)$ belongs to $\mathcal{F}_{(w_1,w_2,w_3)}$.

Conversely, let $(\lambda_1, \lambda_2, \lambda_3)$ in $\mathcal{F}_{(w_1,w_2,w_3)}$. Set $\mathcal{L} = \mathcal{L}(\lambda_1, \lambda_2, \lambda_3)$. By *e.g.* [Res21, Theorem 1.2], the rectriction map induces an isomorphism from $\mathrm{H}(X, \mathcal{L})^G$ onto $\mathrm{H}^0(\{x_0\}, \mathcal{L})^T \simeq \mathbb{C}$. Fix a nonzero element σ in $\mathrm{H}(X, \mathcal{L})^G$. From $\sigma(x_0) \neq 0$, one easily deduces that σ does not vanish on $\Omega = \eta(G \times_B C^+)$ using G-invariance and continuity. Then there exist nonnegative integers n_i such that $\mathrm{div}(\sigma) = \sum_{i=1}^{2\mathrm{rk}(G)} n_i E_i$. In particular $\mathcal{L} = \sum_i n_i \mathcal{O}(E_i)$ and $(\lambda_1, \lambda_2, \lambda_3)$ belongs to the semigroup generated by the triples $(\lambda_1^i, \lambda_2^i, \lambda_3^i)$. □

8 Proof of Theorem 3

8.1 Some reductions

Let us start with this simple observation.

Lemma 27. *If Theorem 3 holds for any irreducible root system, then it holds for any root system.*

Proof. Suppose that $\Phi = \Phi^1 \sqcup \Phi^2$ is a reducible root system, let Δ^i be a set of simple roots for Φ^i, $i = 1, 2$ and $\Delta = \Delta^1 \sqcup \Delta^2$ be the set of simple roots of Φ. The key observation is that if, for $\beta, \gamma \in \Phi^+$ we have that $\beta < \gamma$ and $\beta \in \Phi^1$, then $\gamma \in \Phi^1$. Then, noticing that $\Phi^i_j := \Phi^i \cap \Phi_j$ (for $i = 1, 2$ and $j = 1, 2, 3$) are biconvex in Φ^i and satisfy $\Phi^i_3 = \Phi^i_1 \sqcup \Phi^i_2$ the lemma follows by a straightforward argument. □

Strat with a root φ such that $\beta < \varphi < \gamma$. It remains to prove that $\varphi \notin \Phi_2$. We intensively use the following reduction lemma:

Lemma 28. *Assume that there exists a linear subspace F of $\mathbb{R}\Phi$ such that $\beta \in F$ and $\varphi - \beta$, $\gamma - \varphi$ can be expressed as a positive linear combinaison of roots in $F \cap \Phi^+$.*

If Theorem 3 holds in the root system $F \cap \Phi$, then $\varphi \notin \Phi_2$.

Proof. Observe that, for $i = 1, 2, 3$, $\Phi_i \cap F$ is biconvex in $F \cap \Phi$. Hence, the assumption $\Phi_3 = \Phi_1 \cup \Phi_2$ can be transfered in $F \cap \Phi$. The assumptions of the lemma imply that $\beta < \varphi < \gamma$ holds in $F \cap \Phi$. So, Theorem 3 in $F \cap \Phi$ implies that $\varphi \notin F \cap \Phi_2$; and $\varphi \notin \Phi_2$. □

For later use, notice that if Φ is irreducible and simply laced, then any irreducible component of $F \cap \Phi$ is simply laced.

A triple $\beta < \varphi < \gamma$ in Φ is said to be *irreducible* if there exists no strict F as in Lemma 28. In particular, for any irreducible triple, the support of γ is Δ.

Lemma 29. *In the setting of Theorem 3, $\gamma + \beta \in \Phi_1$.*

Proof. If $\gamma + \beta \notin \Phi_1$ then $\gamma + \beta \in \Phi_2$. But $\beta \notin \Phi_2$ and $\gamma \notin \Phi_2$. Contradiction with the coconvexity of Φ_2. □

8.2 Type ADE

In this section, we prove Theorem 3 for irreducible root systems of type ADE. Notice that if Φ is of type ADE, then any irreducible component of $F \cap \Phi$ is also of type ADE.

Start with an observation excluding type A:

Lemma 30. *In type A, for any pair of positive roots $\beta \leq \gamma$, $\beta + \gamma$ is not a root.*

Proof. It suffices to observe that the coefficients of the positive roots in the basis of simple roots are 0 or 1. □

8.2.1 Reduction through quiver theory

For quiver theory we follow the notation and conventions of [DW11]. The contents and definitions not given here can be found in [DW11][Sections 2.1 to 2.4]. Let $Q = (Q_0, Q_1, h, t)$ be a quiver whose underlying unoriented graph is a Dynkin diagram of type ADE. Let $\mathbb{N}^{Q_0}$ be the space of *dimension vectors*. For $i \in Q_0$, e_i denotes the i-th element of the canonical basis of $\Gamma = \mathbb{Z}^{Q_0}$.
Let $\Delta = \{\alpha_i : i \in Q_0\}$ be a basis of the root system Φ corresponding to the graph underlying Q. By the well known Gabriel's Theorem, the $\mathbb{Z}$-linear map $\Gamma \longrightarrow \mathrm{R} = \mathbb{Z}\Phi$ that sends e_i to α_i is an isomorphism which restricts to a bijection between the set of *Schur roots* of Q and Φ^+. From now on we identify Γ with R as described above.
Let $\langle -, - \rangle : \Gamma \times \Gamma \longrightarrow \mathbb{Z}$ denote the *Euler form of the quiver*. In particular, if α, β are dimension vectors, then

$$\langle \alpha, \beta \rangle = \hom(\alpha, \beta) - \mathrm{ext}(\alpha, \beta).$$

Let $(-, -) : \mathrm{R} \times \mathrm{R} \longrightarrow \mathbb{Z}$ be the *Killing form*, recall that for any $\alpha, \beta \in \Gamma = \mathrm{R}$, $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

Our reduction strategy exploits the *Kac canonical decomposition* [Kac80, DW11]. A dimension vector α can be written uniquely (up to reordering) as $\alpha = \alpha_1 + \cdots + \alpha_s$ where the α_i are Schur roots such that a generic representation of Q of dimension α decomposes into a direct sum of indecomposable representations of dimension α_i. In this case, we write $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_s$ and we call this expression the *canonical decomposition* of α. It's known that $\alpha = \alpha_1 + \cdots + \alpha_s$ is the canonical decomposition of α if and only if $\alpha_1, \ldots, \alpha_s$ are Schur roots such that $\mathrm{ext}(\alpha_i, \alpha_j) = 0$ for any $i \neq j$. Since the identification between Γ and R gives a bijection between the space of dimension vectors and $\mathbb{N}\Phi^+$, we will freely refer to the canonical decomposition of any $\gamma \in \mathrm{R}$ such that $\gamma > 0$. Note also that the canonical decomposition of such a γ gives an explicit way of writing it as a sum of elements of Φ^+.

Lemma 31. *Let $\gamma, \beta \in \Phi^+$ such that $\beta \leq \gamma$. If $\gamma - \beta = \alpha_1 \oplus \cdots \oplus \alpha_s$ is the canonical decomposition of $\gamma - \beta$, then $s \leq 2 - (\gamma, \beta)$.*

Proof. Since Q is an orientation of a Dynkin diagram, the quadratic form $\langle -, - \rangle$ is positive definite on Γ. Moreover, a dimension vector α is a Schur root if and only if $\langle \alpha, \alpha \rangle = 1$. In particular $\mathrm{ext}(\alpha, \alpha) = 0$ for any Schur root (see *e.g.* [Bri12]). Hence

$$\langle \gamma - \beta, \gamma - \beta \rangle = \langle \gamma, \gamma \rangle + \langle \beta, \beta \rangle - (\gamma, \beta) = 2 - (\gamma, \beta)$$

Then, by using the properties of the canonical decomposition we have that

$$\langle \gamma - \beta, \gamma - \beta \rangle = \sum_{i=1}^{s} \langle \alpha_i, \alpha_i \rangle + \sum_{i \neq j} \langle \alpha_i, \alpha_j \rangle = s + \sum_{i \neq j} \mathrm{hom}(\alpha_i, \alpha_j)$$

In particular

$$s = 2 - (\gamma, \beta) - \sum_{i \neq j} \mathrm{hom}(\alpha_i, \alpha_j) \leq 2 - (\gamma, \beta).$$

□

We prove the following easy lemma by lack of a precise reference.

Lemma 32. *Let $\alpha, \beta \in \Phi$ with $\alpha \neq \beta$, then*

1. $(\beta, \alpha) = 1 \iff \beta - \alpha \in \Phi$.
2. $(\beta, \alpha) = 0 \iff \beta - \alpha$ *and* $\beta + \alpha$ *are not roots.*
3. $(\beta, \alpha) = -1 \iff \beta + \alpha \in \Phi$.

Proof. [Hum72, Section 9.4] Recall that we are in type ADE, hence $(\beta, \alpha) \in \{-1, 0, 1\}$. Let $r, q \in \mathbb{N}$ be the greatest integers such that $\beta - r\alpha \in \Phi$ and $\beta + q\alpha \in \Phi$. It's a classical fact that $r - q = (\beta, \alpha)$. Moreover we know that if $(\beta, \alpha) = 1$, then $\beta - \alpha \in \Phi$ and that if $(\beta, \alpha) = -1$ then $\beta + \alpha \in \Phi$. Since $(\beta + q\alpha, \alpha) \leq 1$ we deduce that $2q \leq 1 - (\beta, \alpha)$, while using that $-1 \leq (\beta - r\alpha, \alpha)$ we get that $2r \leq 1 + (\beta, \alpha)$. In particular, we deduce that $r + q \leq 1$, hence the pair (r, q) belongs to $\{(0,0), (0,1), (1,0)\}$. If $\beta - \alpha \in \Phi$, then $(r, q) = (1, 0)$, hence $1 = r - q = (\beta, \alpha)$. Similarly if $\beta + \alpha \in \Phi$ we deduce that $(\beta, \alpha) = -1$. We have proved 1 and 3, then 2 follows. □

Corollary 33. *In type ADE, for any triple of positive roots $\beta < \varphi < \gamma$ such that $\beta + \gamma \in \Phi$, there exist a linear space F as in Lemma 28 satisfying one of the following statment:*

1. $\dim(F) = 6$ *or* 7 *and* $\varphi + \beta \in \Phi$ *and* $\gamma + \varphi \in \Phi$*; or*
2. $\dim(F) = 6$ *and* $\varphi + \beta \in \Phi$ *and* $\gamma \pm \varphi$ *are not roots; or*
3. $\dim(F) = 6$ *and* $\gamma + \varphi \in \Phi$ *and* $\varphi \pm \beta$ *are not roots; or*
4. $\dim F = 4$ *or* 5.

Proof. If $\gamma - \varphi = \alpha_1 \oplus \cdots \oplus \alpha_s$ and $\varphi - \beta = \beta_1 \oplus \cdots \oplus \beta_r$ are the canonical decompositions, it's clear that the space F generated by $\{\beta, \beta_1, \ldots, \beta_r, \alpha_1, \ldots \alpha_s\}$ works. In particular using Lemma 31 we get

$$\dim F \leq 1 + r + s \leq 5 - (\varphi, \beta) - (\gamma, \varphi).$$

Since the Killing form evaluated on two different roots takes values in $\{-1, 0, 1\}$, it follows that $\dim F \leq 7$.

If $\dim F = 7$, $(\varphi, \beta) = (\gamma, \varphi) = -1$. By Lemma 32, we are in the first case.

If $\dim F = 6$, at least one of (φ, β) and (γ, φ) is -1 and the other one is -1 or 0. By Lemma 32, we are in one of the three first cases.

By Lemma 30, $F \cap \Phi$ cannot be of type A. In particular, its rank is at least 4. □

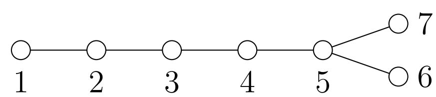
We now improve Corollary 33 excluding the rank 7.

Lemma 34. *There is no irreducible triple $\beta < \varphi < \gamma$ in the root system Φ such that:*

1. *Φ is an irreducible root sytem of type ADE of rank 7;*
2. *$\beta < \varphi < \gamma$ in Φ^+ and $\beta + \gamma \in \Phi$;*
3. *$\varphi + \beta$ and $\varphi + \gamma$ belong to Φ.*

Proof. Assume, for contradiction that such a situation exists. By Lemma 30, the type of Φ is either D_7 or E_7.

Case D_7. Let us number the simple roots of D_7 like in [Bou68]:

D_7 

A root $\varphi = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 + g\alpha_7$ is denoted by $\begin{matrix} & & & & & g \\ a & b & c & d & e & \\ & & & & & f \end{matrix}$.

We also write $a(\varphi) = a \ldots$

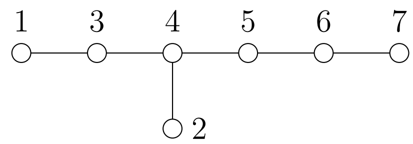
The longest root of D_7 is

$$\alpha_0 = \begin{matrix} & & & & & 1 \\ 1 & 2 & 2 & 2 & 2 & \\ & & & & & 1 \end{matrix}$$

Since $\beta < \gamma$, $\varphi < \gamma$ and $\gamma + \beta$ and $\gamma + \varphi$ are roots, β and φ are supported on the root system generated by $\alpha_2, \ldots, \alpha_5$; hence they lie in a root subsystem of type A_4. In particular $\varphi + \beta$ cannot be a root.

Case E_7. Let us number the simple roots of E_7 like in [Bou68]:

E_7



A root $\varphi = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 + g\alpha_7$ is pictured by $\begin{smallmatrix} a & c & d & e & f \\ & & b & & \end{smallmatrix}$.

Since $\beta < \varphi$, $\varphi + \beta \in \Phi$ has at least one 2 and $b(\varphi) \geq 1$. Similarly, using $\varphi < \gamma$ and $\varphi + \gamma \in \Phi$, we get $b(\varphi) = b(\gamma) = 1$.

In Φ the coefficient g is 0 or 1. Thus $g(\varphi) = 0$ and $g(\gamma) = 1$. Similarly, $b(\gamma) = 1$ implies $a(\gamma) = 0$ or 1. But, the condition on the support implies $a(\gamma) = 1$. Similarly $a(\varphi) \leq 1$.

<u>Case $a(\varphi) = 1$</u>. We have $a(\gamma) = 1$ and $\gamma + \varphi$ is the only root with $a = 2$: $\gamma + \varphi = \begin{smallmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ & & 2 & & & \end{smallmatrix}$ the highest root.

Since the support of φ is connected, $c(\varphi) \neq 0$. Using $c(\varphi) \leq c(\gamma)$, we get $c(\varphi) = 1$ and $c(\gamma) = 2$. If $e(\varphi) = 0$, then the support of φ is of type A_4, but in this case there is no $\beta < \varphi$ such that $\beta + \varphi \in \Phi$. Thus, $e(\varphi) \geq 1$ from which we deduce $e(\varphi) = 1$ and $e(\gamma) = 2$. Hence

$$\varphi = \begin{matrix} 1 & 1 & 1/2 & 1 & 0/1 & 0 \\ & & 1 & & & \end{matrix} \quad \text{and} \quad \gamma = \begin{matrix} 1 & 2 & 3/2 & 2 & 2/1 & 1 \\ & & 1 & & & \end{matrix}$$

Suppose that, $(d(\varphi), f(\varphi)) = (2, 0)$. In D_5, we see that $\beta = \alpha_3$ since $\beta + \varphi \in \Phi$, but $\gamma + \alpha_3 \notin \Phi$ (see [Bou68]). Contradiction.

Suppose that, $(d(\varphi), f(\varphi)) = (1, 0)$. In D_5, we see that $\beta = \alpha_4$ or $\alpha_3 + \alpha_4$. Hence in $\gamma + \beta$, $b = 1$ and $d = 4$. Contradiction (see [Bou68]).

We just proved that $f(\varphi) = 1$. If $d(\varphi) = 1$, then by [Bou68]

$$\varphi = \begin{matrix} 1 & 1 & 1 & 1 & 1 & 0 \\ & & 1 & & & \end{matrix} \quad \text{and} \quad \gamma = \begin{matrix} 1 & 2 & 3 & 2 & 1 & 1 \\ & & 1 & & & \end{matrix}$$

Since $\beta + \varphi \in \Phi$, then $d(\beta) = 1$, hence $d(\beta + \gamma) = 4$, which implies $b(\beta) = 1$. Since $d(\varphi) = 1$, $\beta + \varphi$ is not a root. This implies that $d(\varphi) = 2$, and hence $d(\gamma) = 2$. Expecting the table of E_6 in [Bou68], $\varphi + \beta \in \Phi$ implies that $\beta = \alpha_3, \alpha_5, \alpha_3 + \alpha_4 + \alpha_5$ or $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. In each case $\gamma + \beta$ is not a root of E_7.

<u>Case $a(\varphi) = 0$</u>. Since the support of φ is not of type A, $c(\varphi) \geq 1$. But in E_7, $c \leq 3$, moreover $\varphi < \gamma$ and $\varphi + \gamma \in \Phi$, hence $c(\varphi) = 1$. Similarly $e(\varphi) = 1$. Using the connectness of the support, we get $d(\varphi) = 1$ or 2. Now

$$\varphi = \begin{matrix} 0 & 1 & 1/2 & 1 & 0/1 & 0 \\ & & 1 & & & \end{matrix} \quad \text{and} \quad \gamma = \begin{matrix} 1 & \cdot & \cdot & \cdot & \cdot & 1 \\ & & 1 & & & \end{matrix}$$

Working in D_5, one can see that $(d(\varphi), f(\varphi), \beta)$ belongs to

$$\{(2, 1, \alpha_5), (1, 1, \alpha_4), (1, 1, \alpha_4 + \alpha_5), (1, 0, \alpha_4)\}.$$

If $\beta = \alpha_5$, $d(\varphi) = d(\gamma) = 2$. Hence, for $\varphi + \gamma$, we have $d\ e\ f\ g = 4\ 3\ 2\ 1$. Hence, for γ, we have $d\ e\ f\ g = 2\ 2\ 1\ 1$. In $\gamma + \beta$, $d\ e\ f\ g = 2\ 3\ 1\ 1$, which is a contradiction.

By [Bou68], the possibilities for $(c(\gamma), d(\gamma), e(\gamma), f(\gamma), \beta)$ with $\beta \in \{\alpha_4, \alpha_4 + \alpha_5\}$ with the contraints that $\gamma + \beta \in \Phi$ are

$$\{(2, 2, 1, 1, \alpha_4 + \alpha_5), (2, 2, 2, 1, \alpha_4), (2, 2, 2, 2, \alpha_4), (2, 2, 2, 2, \alpha_4 + \alpha_5)\}.$$

Indeed, notice that since $b(\gamma+\beta) = 1$, $d(\gamma) \leq 2$, hence $d(\gamma) = 2$, and with similar arguments we can obtain the above list. Then, in each case, $d(\varphi + \gamma) \leq 3$. But the difference between two consecutive entries is at most one. One easily checks that this is impossible. □

From now on, to complete the proof of Theorem 3 in type ADE, using Corollary 33, Lemma 34, Lemma 27 and an immediate induction on the rank we assume that the following assumptions hold.

Assumptions: the triple $\beta < \varphi < \gamma$ is irreduible. In particular γ is full supported. And, one of the following property holds:

1. Φ has rank 6 and $\varphi + \beta$ and $\gamma + \varphi$ are roots.
2. Φ has rank 6 and $\varphi + \beta \in \Phi$ but $\gamma \pm \varphi$ are not roots.
3. Φ has rank 6 and $\gamma + \varphi \in \Phi$ but $\varphi \pm \beta$ are not roots.
4. Φ has rank 4 or 5.

8.2.2 Notation

In the sequel, we often consider an irreducible triple (β, φ, γ) in some root system and we want to prove that φ belongs to Φ_1. Then, we assume, for the sake of contradiction, that $\varphi \in \Phi_2$ and we use extensively the biconvexity of the sets Φ_1, Φ_2 and Φ_3 to get a contradiction.

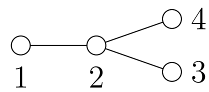
It will be convenient, given θ_1, θ_2 and θ_3 in Φ, to use notations as

$$\begin{array}{lcl} \theta_1 = \theta_2 + \theta_3 & \longmapsto & \theta_1 \in \Phi_2 \\ \theta_1 = \theta_2 + \theta_3 & \longmapsto & \theta_2 \in \Phi_1 \\ \theta_1 = \theta_2 + \theta_3 & \longmapsto & \theta_2 \notin \Phi_3 \end{array}$$

This means that the belonging (or not belonging) of two of the three roots θ_i to the concerned Φ_j is known and that we can deduce the right hand side by convexity or coconvexity. Namely, in the first case if $\theta_2, \theta_3 \in \Phi_2$ we can deduce by convexity of Φ_2 that $\theta_1 \in \Phi_2$. In the second case, if $\theta_1 \in \Phi_1$ and $\theta_3 \notin \Phi_1$, the coconvexity of Φ_1 implies that $\theta_2 \in \Phi_1$. In the last case, if we know that $\theta_1 \notin \Phi_3$ and $\theta_3 \in \Phi_3$ then $\theta_2 \notin \Phi_3$.

We sometimes add a comment like "by Case 3" or "by lower rank" to explain how to recover the information used on the θ_i. The sentence "by lower rank" often implicitly means that Theorem 3 has been applied in a root system as in Lemma 28, to deduce information on a reducible triple.

8.2.3 A proof in type D_4

D_4 

The value of each coefficient of $\gamma+\beta$ along a simple simple root in the support of β is at least 2. Hence $\beta = \alpha_2$ and $\gamma = \sum_{i=1}^4 \alpha_i$.

The group $\mathrm{Aut}(D_4)$ has two orbits in $]\beta;\gamma[$: those of $\alpha_1+\alpha_2$ and $\alpha_1+\alpha_2+\alpha_3$.

Case 1. $\varphi = \alpha_1+\alpha_2$.

Assume, for contradiction, that $\varphi \in \Phi_2$. We have

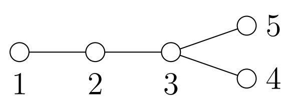
$$\begin{array}{lll} \varphi = \alpha_1 + \beta & \longmapsto & \alpha_1 \in \Phi_2, \\ \gamma = \alpha_1 + (\alpha_2+\alpha_3+\alpha_4) & \longmapsto & \alpha_2+\alpha_3+\alpha_4 \notin \Phi_3, \\ \gamma+\beta = \varphi + (\alpha_2+\alpha_3+\alpha_4) & \longmapsto & \varphi \in \Phi_1. \end{array}$$

Contradiction.

Case 2. $\varphi = \alpha_1+\alpha_2+\alpha_3 \in \Phi_2$.

By Case 1, Φ_2, and hence $\Phi_2 \cap (\mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3)$ contains no root of height 2. By direct verification in type A_3, there is no biconvex subset of the positive roots containing the longest root and no root of height 2. Contradiction.

8.2.4 A proof in type D_5

D_5 

Lemma 35. *Up to $\mathrm{Aut}(D_5)$, there are three irreducible triples $\beta < \varphi < \gamma$ in D_5 such that $\gamma+\varphi \in \Phi$. They are $\beta = \alpha_3$, $\gamma = \sum_{i=1}^5 \alpha_i$ and φ in*

$$\{\alpha_2+\alpha_3, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4+\alpha_5\}.$$

Proof. The longest root being $\begin{smallmatrix} & & & 1 \\ 1 & 2 & 2 & \\ & & & 1 \end{smallmatrix}$, it is easy to check that the only pairs (β,γ) such that $\beta<\gamma$, $\beta+\gamma \in \Phi$ and γ has full support are

$$\begin{array}{lll} a. & \beta = \alpha_2+\alpha_3 & \gamma = \sum_{i=1}^5 \alpha_i \\ b. & \beta = \alpha_2 & \gamma = \alpha_3 + \sum_{i=1}^5 \alpha_i \\ c. & \beta = \alpha_3 & \gamma = \sum_{i=1}^5 \alpha_i \end{array}$$

In Case a, for any $\varphi \in]\beta;\gamma[$, the relations $\beta < \varphi < \gamma$ still hold in the span of $\beta, \alpha_1, \alpha_4, \alpha_5$. Hence there is no irreducible triple.

In Case b, let $\varphi \in]\beta;\gamma[$. If $n_3(\varphi) = 2$ or 0, the relations $\beta < \varphi < \gamma$ still hold in the span F_1 of $\alpha_1, \beta, \alpha_3, \alpha_3 + \alpha_4 + \alpha_5$. If $n_3(\varphi) = 1$, the relations $\beta < \varphi < \gamma$ still hold either in F_1 or in the span of $\alpha_1, \beta, \alpha_3 + \alpha_4, \alpha_3 + \alpha_5$. Hence there is no irreducible triple.

Consider Case c. Let $\varphi \in]\beta;\gamma[$. If $n_4(\varphi) = n_5(\varphi) = 1$, φ is either $\alpha_3 + \alpha_4 + \alpha_5$ or $\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. In the first case, the relations $\beta < \varphi < \gamma$ still hold in the span of $\alpha_1 + \alpha_2, \beta, \alpha_4, \alpha_5$. The second case is in the statement of the lemma.

If $n_4(\varphi) = n_5(\varphi) = 0$, φ is either $\alpha_2 + \alpha_3$ or $\alpha_1 + \alpha_2 + \alpha_3$. The first case appears in the lemma. The second one is not irreducible since $\beta < \varphi < \gamma$ holds in the span of $\alpha_1 + \alpha_2, \beta, \alpha_4, \alpha_5$.

Otherwise, up to symmetry, $n_4(\varphi) = 1$ and $n_5(\varphi) = 0$. Then, $\varphi = \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4$ or $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Only, the second one is irreducible. □

Lemma 36. *Theorem 3 holds for the three triples (β, φ, γ) in Lemma 35.*

Proof. Fix one of the three triples and assume by contradiction that $\varphi \in \Phi_2$.

Case 1: $\varphi = \alpha_2 + \alpha_3$. Set $\eta = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ and $\eta' = \eta + \alpha_3$. We have

$$\begin{array}{lcl} \varphi = \alpha_2 + \beta & \longmapsto & \alpha_2 \in \Phi_2 \\ \alpha_0 = (\gamma + \beta) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3 \\ \alpha_0 = \gamma + \varphi & \longmapsto & \alpha_0 \notin \Phi_1 \end{array}$$

Hence

$$\alpha_0 \in \Phi_2. \tag{18}$$

Moreover, by lower rank, $\alpha_1 + \alpha_2 + \alpha_3 \notin \Phi_3$. Now

$$\begin{array}{lcl} \alpha_0 = (\alpha_1 + \alpha_2 + \alpha_3) + \eta & \longmapsto & \eta \in \Phi_2 \\ \gamma = \alpha_1 + \eta & \longmapsto & \alpha_1 \notin \Phi_3 \\ \gamma + \beta = \alpha_1 + \eta' & \longmapsto & \eta' \in \Phi_1 \\ \eta' = \varphi + (\alpha_3 + \alpha_4 + \alpha_5) & \longmapsto & \alpha_3 + \alpha_4 + \alpha_5 \in \Phi_1 \\ \alpha_0 = (\alpha_1 + \alpha_2) + \eta' & \longmapsto & \alpha_1 + \alpha_2 \in \Phi_2. \end{array}$$

Now

$$\gamma = (\alpha_1 + \alpha_2) + (\alpha_3 + \alpha_4 + \alpha_5)$$

contradicts the convexity of Φ_3.

Case 2: $\varphi = \alpha_2 + \alpha_3 + \alpha_5$.

In $\langle \alpha_2, \alpha_3, \alpha_5 \rangle \cap \Phi$, which is of type A_3, the condition $\alpha_2 + \alpha_3 + \alpha_5 \in \Phi_2$ implies $\sharp(\langle \alpha_2, \alpha_3, \alpha_5 \rangle \cap \Phi_2) \geq 3$. We deduce that $\langle \alpha_2, \alpha_3, \alpha_5 \rangle \cap \Phi_2 = \{\varphi, \alpha_2, \alpha_5\}$. Indeed: $\beta \notin \Phi_2$, $\alpha_2 + \alpha_3 \notin \Phi_2$ by Case 1 and $\alpha_3 + \alpha_5 \notin \Phi_2$ because the triple $\beta < \alpha_3 + \alpha_5 < \gamma$ is reducible.

Now

$$\begin{array}{lcll} \gamma = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \alpha_5 & \longmapsto & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \notin \Phi_3 & \\ \alpha_0 = \varphi + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) & \longmapsto & \alpha_0 \notin \Phi_1 & \\ \alpha_0 = (\gamma + \beta) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3 & \\ \alpha_0 = \gamma + (\alpha_2 + \alpha_3) & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by Case 1.} \end{array}$$

Contradiction.

Case 3. $\varphi = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$. Now

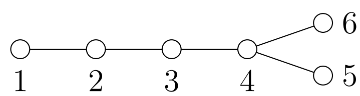
$$\begin{array}{lcll}
\varphi = \alpha_5 + (\alpha_2 + \alpha_3 + \alpha_4) & \longmapsto & \alpha_5 \in \Phi_2 & \text{by Case 2} \\
\varphi = \alpha_4 + (\alpha_2 + \alpha_3 + \alpha_5) & \longmapsto & \alpha_4 \in \Phi_2 & \text{by Case 2} \\
\alpha_0 = \gamma + (\alpha_2 + \alpha_3) & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by Case 1} \\
\varphi = \alpha_2 + (\alpha_3 + \alpha_4 + \alpha_5) & \longmapsto & \alpha_2 \in \Phi_2 & \text{by lower rank} \\
\alpha_0 = (\beta + \gamma) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3 &
\end{array}$$

Hence $\alpha_0 \in \Phi_1$. And

$$\begin{array}{lcl}
\alpha_0 = \varphi + (\alpha_1 + \alpha_2 + \alpha_3) & \longmapsto & \alpha_1 + \alpha_2 + \alpha_3 \in \Phi_1 \\
\gamma = (\alpha_1 + \alpha_2 + \alpha_3) + (\alpha_4 + \alpha_5) & \longmapsto & \alpha_4 + \alpha_5 \notin \Phi_3
\end{array}$$

Contradiction since $\alpha_4, \alpha_5 \in \Phi_3$. □

8.2.5 A proof in type D_6

D_6 

Lemma 37. *There is no irreducible triple $\beta < \varphi < \gamma$ in D_6 such that $\gamma + \varphi \in \Phi$.*

Proof. Since $\gamma + \beta \in \Phi$, the support of β is contained in $\{\alpha_2, \alpha_3, \alpha_4\}$. If $\varphi + \gamma \in \Phi$, the same property holds for φ. Hence $\varphi + \beta \notin \Phi$ (we are in type A_3). Finaly, $\varphi + \gamma \notin \Phi$ or $\varphi + \beta \notin \Phi$.

Combining with Corollary 33, one gets two cases to consider:

Case 1: $\varphi + \gamma \in \Phi$ and $\varphi \pm \beta \notin \Phi$.

Here, φ and β are supported by α_2, α_3 and α_4. Now the assumption $\varphi - \beta \notin \Phi$ implies $\beta = \alpha_3$ and $\varphi = \alpha_2 + \alpha_3 + \alpha_4$. But, $\gamma + \beta \in \Phi$ implies $n_4(\gamma + \beta) = n_4(\gamma) = 2$. This contradicts $\gamma + \varphi \in \Phi$.

Case 2: $\varphi + \beta \in \Phi$ and $\gamma \pm \varphi \notin \Phi$.

Since $\varphi + \beta \in \Phi$ and $\gamma + \beta \in \Phi$, we have

$$\varphi, \gamma \in \begin{array}{ccccc} & & & & 1 \\ 0 \setminus 1 & \cdot & \cdot & \cdot & \\ & & & & 1 \end{array} \qquad \beta \in \begin{array}{ccccc} & & & & 0 \\ 0 & \cdot & \cdot & \cdot & \\ & & & & 0 \end{array}$$

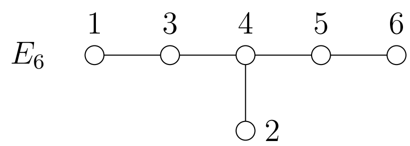
Now $\varphi < \gamma$ and $\gamma - \varphi \notin \Phi$, thus $n_4(\gamma) = 2$. Then $n_4(\beta) = 0$ and the condition $\beta < \varphi < \gamma$ holds in the span of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_4 + \alpha_5 + \alpha_6$.

This proves that there is no irreducible triple in D_6. □

The only remaining case in type ADE is E_6.

8.3 The case E_6

Let us number the simple roots of E_6 like in [Bou68]:



A root $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6$ is denoted by $\begin{smallmatrix} a & c & d & e & f \\ & & b & & \end{smallmatrix}$. The highest root is

$$\alpha_0 = \begin{matrix} 1 & 2 & 3 & 2 & 1 \\ & & 2 & & \end{matrix}.$$

Lemma 38. *There are only two irreducible triples $\beta < \varphi < \gamma$ in E_6 such that $\gamma + \beta \in \Phi$. They are*

$$\beta = \alpha_4 \quad \varphi = \begin{matrix} 0 & 1 & 1 & 1 & 0 \\ & & 0 & & \end{matrix} \qquad \gamma = \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{matrix}$$

$$\beta = \alpha_4 \quad \varphi = \begin{matrix} 0 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{matrix} \qquad \gamma = \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{matrix}$$

Proof. Let $\beta < \varphi < \gamma$ in E_6 be an irreducible triple.

Then the support of γ is Δ. Moreover, $\gamma + \beta \in \Phi$ implies that γ is not the highest root. Hence $a(\gamma) = f(\gamma) = b(\gamma) = 1$ and $a(\beta) = f(\beta) = 0$, $b(\beta) \leq 1$.

<u>Case A:</u> $b(\beta) = 1$.

Then $b(\varphi) = 1$. Moreover $\gamma + \beta$ is the only root with $b = 2$: $\gamma + \beta = \alpha_0$. Since $\gamma + \beta \neq \gamma + \varphi$, then $\gamma + \varphi \notin \Phi$. Similarly, $\beta + \varphi \notin \Phi$. Now, Corollary 33 contradicts the irreducibility of $\beta < \varphi < \gamma$.

<u>Case B:</u> $b(\beta) = 0$.

Let us distinguish two cases on $\gamma + \varphi$:

<u>Case B-I:</u> $\gamma + \varphi \in \Phi$.

Corollary 33 and the irreducibility of the triple of rootsimplies that $\varphi \pm \beta \notin \Phi$. Moreover, $a(\varphi) = f(\varphi) = 0$ and the entries of φ are 0 or 1, otherwise $\gamma + \beta$ should have a coefficient equal to 4. Since $\varphi - \beta \notin \Phi$, we deduce that $\sharp\mathrm{Supp}(\varphi) \geq 3$. Up to $\mathrm{Aut}(E_6)$, we have 3 possibilities for φ:

$$\begin{matrix} 0 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{matrix} \qquad \begin{matrix} 0 & 1 & 1 & 0 & 0 \\ & & 1 & & \end{matrix} \qquad \begin{matrix} 0 & 1 & 1 & 1 & 0 \\ & & 0 & & \end{matrix}$$

In the first case we have $\gamma + \varphi = \alpha_0$ (the only root with $b = 2$) and $\gamma = \begin{smallmatrix} 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{smallmatrix}$. One easily checks that there is no $\beta < \varphi$ such that $\gamma + \beta \in \Phi$ and $\varphi - \beta \notin \Phi$.

In the second case, we still have $\gamma + \varphi = \alpha_0$. Moreover, the only root $\beta < \varphi$ such that $\varphi - \beta \notin \Phi$ is $\beta = \alpha_4$. And $\gamma + \beta \notin \Phi$. Contradiction.

Assume now, φ is the third. Since $\varphi - \beta \notin \Phi$, $\beta = \alpha_4$. Since $\gamma + \varphi \in \Phi$,

$$\gamma = \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{array} \quad \text{or} \quad \gamma = \begin{array}{ccccc} 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{array}$$

In the second case $\gamma + \beta \notin \Phi$, while in the first case we recover the first of the two irreducible triples of the statement.

Case B-II: $\gamma + \varphi \notin \Phi$.

Corollary 33 and the irreducibility implies that $\varphi + \beta \in \Phi$ and $\gamma - \varphi \notin \Phi$.

Up to $\mathrm{Aut}(E_6)$, using that $a(\beta) = b(\beta) = f(\beta) = 0$, we have three possibilities for $\beta < \varphi$:

$$\begin{array}{ccccc} 0 & 1 & 1 & 0 & 0 \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 1 & 0 & 0 \\ & & 0 & & \end{array}$$

that we consider successively below.

Case B-II-1: $\beta = \alpha_3 + \alpha_4$.

Let $\psi \in \Phi$ such that $\beta < \psi$, $b(\psi) = 1$ and $\psi + \beta \in \Phi$. This implies $a(\psi) = c(\psi) = 1$, $d(\psi) = e(\psi) \geq 1$. And, ψ is in the following list

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 2 & 2 & 1 \\ & & 1 & & \end{array}$$

Both φ and γ belong to this list and $\varphi < \gamma$. But for any such pair, $\gamma - \varphi \in \Phi$. Contradiction.

Case B-II-2: $\beta = \alpha_3$.

Like in Case B-II-1, the roots ψ such that $\beta < \psi$, $b(\psi) = 1$ and $\psi + \beta \in \Phi$ are:

$$\begin{array}{ccccc} 1 & 1 & 2 & 1 & 0 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 2 & 1 & 1 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 2 & 2 & 1 \\ & & 1 & & \end{array}$$

Thus, $\gamma - \varphi \notin \Phi$ is not possible.

Case B-II-3: $\beta = \alpha_4$.

There are still four possibilities for ψ:

$$\begin{array}{ccccc} 0 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 1 & 0 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 1 & 2 & 2 & 2 & 1 \\ & & 1 & & \end{array}$$

Since $\gamma - \varphi \notin \Phi$, φ is the first one and γ is the third one. Thus we get the second irreducible triple of the statement. □

We now study the two irreducible triples for E_6.

Lemma 39. *Theorem 3 holds for the two triples (β, φ, γ) in Lemma 38.*

Proof. Fix one of the two triples (β, φ, γ) and assume by contradiction that $\varphi \notin \Phi_2$.

Case 1: $\varphi = \alpha_3 + \alpha_4 + \alpha_5 \in \Phi_2$.

Set $\eta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. We have

$$\begin{array}{lcll}
\eta = (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) & \longmapsto & \eta \notin \Phi_2 & \text{by lower rank} \\
\varphi = (\alpha_3 + \alpha_4) + \alpha_5 & \longmapsto & \alpha_5 \in \Phi_2 & \text{by lower rank} \\
 & & \alpha_3 \in \Phi_2 & \text{by symmetry} \\
\eta = (\gamma + \beta) + \alpha_3 + \alpha_5 & \longmapsto & \eta \in \Phi_3 & \\
\eta = \gamma + \varphi & \longmapsto & \eta \notin \Phi_1. &
\end{array}$$

Contradiction.

Case 2: $\varphi = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \in \Phi_2$.

Here,

$$\begin{array}{lcll}
\varphi = (\varphi - \alpha_2) + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by Case 1} \\
\varphi = (\varphi - \alpha_3) + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by lower rank} \\
 & & \alpha_5 \in \Phi_2 & \text{by symmetry}
\end{array}$$

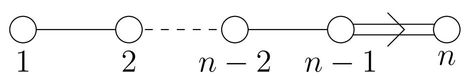
Since $\alpha_2, \ldots, \alpha_5 \in \Phi_3$ and $\gamma + \beta \in \Phi_3$, by convexity $\eta \in \Phi_3$. As in Case 1, $\eta \notin \Phi_2$. Now,

$$\begin{array}{lcl}
\gamma = (\gamma - \alpha_2) + \alpha_2 & \longmapsto & \gamma - \alpha_2 \notin \Phi_3 \\
\eta = \varphi + (\gamma - \alpha_2) & \longmapsto & \eta \notin \Phi_1.
\end{array}$$

We proved that $\eta \in \Phi_3$, $\eta \notin \Phi_1$ and $\eta \notin \Phi_2$. Contradiction. □

8.4 Type B

B_n



We use the same notation of [Bou68] for the associated root system. In Φ^+ we distinguish 3 types of positive roots according to the following list. On the rightmost part we picture the writing of the root as a combinaison of simple roots.

$$\begin{array}{lll}
\text{Type 1} & \varepsilon_i - \varepsilon_j & : 1 \leq i < j \leq n
\end{array}
\qquad
\begin{array}{cccccccc}
0 & \ldots & 0 & 1 & \ldots & 1 & 0 & \ldots \; 0 \\
 & & & i & & j-1 & &
\end{array}$$

$$\begin{array}{lll}
\text{Type 2} & \varepsilon_i & : 1 \leq i \leq n
\end{array}
\qquad
\begin{array}{cccccc}
0 & \ldots & 0 & 1 & \ldots & 1 \\
 & & & i & & n
\end{array}$$

$$\begin{array}{lll}
\text{Type 3} & \varepsilon_i + \varepsilon_j & : 1 \leq i < j \leq n
\end{array}
\qquad
\begin{array}{ccccccccc}
0 & \ldots & 0 & 1 & \ldots & 1 & 2 & \ldots & 2 \\
 & & & i & & j-1 & j & & n
\end{array}$$

8.4.1 Reduction

Lemma 40. *There is no irreducible triple $\beta < \varphi < \gamma$ such that $\beta + \gamma \in \Phi$, in B_ℓ, for any $\ell \geq 6$.*

Proof. Since $\mathrm{Supp}(\gamma) = \Delta$, $\gamma = \varepsilon_1$ or $\gamma = \varepsilon_1 + \varepsilon_j$ for $1 < j \leq n$. We treat the two cases separately. In each case, we find a subspace F of dimension at most 5 that reduces the triple.

Case 1: $\gamma = \varepsilon_1$.
Since $\gamma + \beta \in \Phi$, $\beta = \varepsilon_j$ for a certain $1 < j \leq n$. But $\beta < \varphi$ and $\varphi < \gamma$, thus $\varphi = \varepsilon_i$ for some $1 < i < j$. Then $\gamma - \varphi \in \Phi^+$ and $\varphi - \beta \in \Phi^+$. Hence $F = \langle \beta, \varphi - \beta, \gamma - \varphi \rangle$ works.

Case 2: $\gamma = \varepsilon_1 + \varepsilon_j$ for a certain $1 < j \leq n$.
Since $\gamma + \beta \in \Phi$, $\beta = \varepsilon_i - \varepsilon_j$ with $1 \leq i < j$. We have 3 possibilities for φ according to its type.

Type 1: $\varphi = \varepsilon_k - \varepsilon_l$. Since $\beta \leq \varphi$, we have $k \leq i < j \leq l$. Then

$$\gamma - \varphi = (\varepsilon_1 - \varepsilon_k) + (\varepsilon_j + \varepsilon_l).$$

Moreover $\varepsilon_1 - \varepsilon_k \in \Phi^+ \cup \{0\}$ and $\varepsilon_j + \varepsilon_l \in \Phi^+ \cup 2\Phi^+$. While

$$\varphi - \beta = (\varepsilon_k - \varepsilon_i) + (\varepsilon_j - \varepsilon_l).$$

Here $\varepsilon_k - \varepsilon_i \in \Phi^+ \cup \{0\}$ and $(\varepsilon_j - \varepsilon_l) \in \Phi^+ \cup \{0\}$.
In particular, $F = \langle \beta, \varepsilon_1 - \varepsilon_k, \varepsilon_j + \varepsilon_l, \varepsilon_k - \varepsilon_i, \varepsilon_j - \varepsilon_l \rangle$ works.

Type 2: $\varphi = \varepsilon_k$. Since $\beta < \varphi$, we have $1 \leq k \leq i < j$. Then

$$\gamma - \varphi = (\varepsilon_1 - \varepsilon_k) + \varepsilon_j \quad \text{and} \quad \varphi - \beta = (\varepsilon_k - \varepsilon_i) + \varepsilon_j.$$

Hence $F = \langle \beta, \varepsilon_1 - \varepsilon_k, \varepsilon_j, \varepsilon_k - \varepsilon_i \rangle$ works.

Type 3: $\varphi = \varepsilon_k + \varepsilon_l$ with $k < l$. From $\beta \leq \varphi \leq \gamma$, it follows that $k \leq i < j \leq l$. Then

$$\gamma - \varphi = (\varepsilon_1 - \varepsilon_k) + (\varepsilon_j - \varepsilon_l) \quad \text{and} \quad \varphi - \beta = (\varepsilon_k - \varepsilon_i) + (\varepsilon_j + \varepsilon_l).$$

Then $F = \langle \beta, \varepsilon_1 - \varepsilon_k, \varepsilon_j - \varepsilon_l, \varepsilon_k - \varepsilon_i, \varepsilon_j + \varepsilon_l \rangle$ works. □

8.4.2 Type B_2

The only pair $\beta < \gamma$ with $\beta + \gamma \in \Phi$ is $\beta = \alpha_2$ and $\gamma = \alpha_1 + \alpha_2$. Since $]\beta; \gamma[$ is empty, there is nothing to prove.

8.4.3 Type B_3

Lemma 41. *There are five irreducible triples $\beta < \varphi < \gamma$ in B_3 such that $\gamma + \beta \in \Phi$. They are $\beta = \alpha_3$, $\varphi = \alpha_2 + \alpha_3$ and $\gamma = \alpha_1 + \alpha_2 + \alpha_3$, and $\beta = \alpha_2$, $\gamma = 1\ 1\ 2$ and $\varphi \in \{1\ 1\ 0, 1\ 1\ 1, 0\ 1\ 1, 0\ 1\ 2\}$.*

Proof. Since γ has full support and is not α_0, $\gamma = 1\ 1\ 1$ or $1\ 1\ 2$. In the last case $\beta = \alpha_2$ and $]\beta;\gamma[$ is the set of 4 roots in the statement. One easily checks that they give 4 irreducible triples.

Assume now $\gamma = 1\ 1\ 1$. Since $\gamma + \beta$ is a root, $\beta = \alpha_3$ or $\alpha_2 + \alpha_3$. In the last case, $]\beta;\gamma[$ is empty. So, set $\beta = \alpha_3$. Then $\varphi = 0\ 1\ 1$ is the only root in the open interval and gives the last irreducible triple. □

Lemma 42. *Theorem 3 holds for the five triples (β, φ, γ) in Lemma 41.*

Proof. Fix one of the five triples and assume by contradiction that $\varphi \in \Phi_2$.
Case A. $\beta = 0\ 0\ 1$, $\varphi = 0\ 1\ 1$ and $\gamma = 1\ 1\ 1$.

We have

$$\begin{array}{lll} \varphi = \beta + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2, \\ \gamma = 1\ 1\ 0 + \beta & \longmapsto & 1\ 1\ 0 \notin \Phi_3, \\ \alpha_0 = \alpha_2 + (\gamma + \beta) & \longmapsto & \alpha_0 \in \Phi_3, \\ \alpha_0 = \gamma + \varphi & \longmapsto & \alpha_0 \notin \Phi_1, \end{array}$$

hence $\alpha_0 \in \Phi_2$. Now

$$\begin{array}{lll} \alpha_0 = 0\ 1\ 2 + 1\ 1\ 0 & \longmapsto & 0\ 1\ 2 \in \Phi_2, \\ \gamma + \beta = 0\ 1\ 2 + \alpha_1 & \longmapsto & \gamma + \beta \notin \Phi_1. \end{array}$$

Contradiction.
Case B. $\beta = 0\ 1\ 0$ and $\gamma = 1\ 1\ 2$.

Case B-1. $\varphi = 1\ 1\ 0 \in \Phi_2$. We have:

$$\begin{array}{lll} \varphi = \beta + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2, \\ \beta + \gamma = 0\ 1\ 2 + \varphi & \longmapsto & 0\ 1\ 2 \in \Phi_1. \end{array}$$

Now $\gamma = \alpha_1 + 0\ 1\ 2$ contradicts the convexity of Φ_3.

Case B-2. $\varphi = 1\ 1\ 1 \in \Phi_2$. We have:

$$\begin{array}{llll} \gamma = \varphi + \alpha_3 & \longmapsto & \alpha_3 \notin \Phi_2, & \\ \varphi = \alpha_3 + 1\ 1\ 0 & \longmapsto & \varphi \notin \Phi_2 & \text{by Case B-1.} \end{array}$$

Contradiction.

Case B-3. $\varphi = 0\ 1\ 1 \in \Phi_2$. We have:

$$\begin{array}{lll} \gamma + \beta = \varphi + 1\ 1\ 1 & \longmapsto & 1\ 1\ 1 \in \Phi_1, \\ \varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2, \\ \gamma = \alpha_3 + 1\ 1\ 1 & \longmapsto & \gamma \in \Phi_3. \end{array}$$

Contradiction.

Case B-4. $\varphi = 0\ 1\ 2 \in \Phi_2$. We have:

$$\begin{array}{llll} \varphi = 0\ 1\ 1 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by Case B-3.} \\ \gamma + \beta = \varphi + 1\ 1\ 0 & \longmapsto & 1\ 1\ 0 \in \Phi_1, & \\ \gamma = 2\alpha_3 + 1\ 1\ 0 & \longmapsto & \gamma \in \Phi_3. & \end{array}$$

Contradiction. □

8.4.4 Type B_4

Lemma 43. *There are seven irreducible triples $\beta < \varphi < \gamma$ in B_4 such that $\gamma + \beta \in \Phi$. They are*

1. $\beta = \alpha_3$, $\gamma = 1\ 1\ 1\ 2$ *and* φ *in*
$$\{0\ 1\ 1\ 0, 0\ 1\ 1\ 1, 0\ 1\ 1\ 2\},$$
2. $\beta = \alpha_2$, $\gamma = 1\ 1\ 2\ 2$ *and* φ *in*
$$\{0\ 1\ 1\ 0, 0\ 1\ 1\ 2, 1\ 1\ 1\ 0, 1\ 1\ 1\ 2\}.$$

Proof. Assume first that $\gamma = \sum_{i=1}^{4} \alpha_i$. Then $\gamma + \beta \in \Phi$ implies that there exists $j > 1$ such that $\beta = \sum_{i \geq j}^{4} \alpha_i$. Any $\varphi \in]\beta; \gamma[$ is equal to $\sum_{i \geq j'}^{5} \alpha_i$ for some $1 < j' < j$. In particular whether $]\beta; \gamma[$ is empty, or $\varphi - \beta$ and $\gamma - \varphi$ are roots and the triple $\beta < \varphi < \gamma$ is not irreducible.

There are three pairs $\beta < \gamma$ such that $\beta + \gamma \in \Phi$, $\gamma \neq \sum_i \alpha_i$ and $]\beta; \gamma[$ nonempty. Namely
$$\{(\alpha_2 + \alpha_3, 1\ 1\ 1\ 2), (\alpha_3, 1\ 1\ 1\ 2), (\alpha_2, 1\ 1\ 2\ 2)\}.$$
The first pair gives 4 triples $\beta < \varphi < \gamma$. Considering the linear space $F = \langle \beta, \alpha_1, \alpha_4 \rangle$, one proves that these four triples are reducible.

For $\beta = \alpha_3$ and $\gamma = 1\ 1\ 1\ 2$, the interval $]\beta; \gamma[$ contains 6 roots; 3 of them give reducible triples and 3 of them are in the statement. For example, $F = \langle \beta, \alpha_1 + \alpha_2, \alpha_4 \rangle$ shows that $\varphi = 1\ 1\ 1\ 1$ gives a reducible triple.

For $\beta = \alpha_2$ and $\gamma = 1\ 1\ 2\ 2$, the interval $]\beta; \gamma[$ contains 8 roots; 4 of them give reducible triples. For example, $F = \langle \beta, \alpha_3 + \alpha_4, \alpha_1 \rangle$ shows that $\varphi = 0\ 1\ 1\ 1$ gives a reducible triple. □

Lemma 44. *Theorem 3 holds for the seven triples (β, φ, γ) in Lemma 43.*

Proof. Fix one of the seven triples and assume by contradiction that $\varphi \in \Phi_2$.
<u>Case A:</u> $\beta = \alpha_3$ and $\gamma = 1\ 1\ 1\ 2$.

By reduction to lower rank we have that
$$\Phi_2 \cap \{0\ 0\ 1\ 1, 1\ 1\ 1\ 0, 1\ 1\ 1\ 1\} = \emptyset. \tag{19}$$

<u>Case A-1:</u> $\varphi = 0\ 1\ 1\ 0$. We have:
$$\begin{array}{lll} \varphi = \beta + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2, \\ \alpha_0 = (\gamma + \beta) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3, \\ \alpha_0 = \varphi + \gamma & \longmapsto & \alpha_0 \notin \Phi_1. \end{array}$$
Hence $\alpha_0 \in \Phi_2$. Now
$$\begin{array}{llll} \alpha_0 = 0\ 1\ 1\ 2 + 1\ 1\ 1\ 0 & \longmapsto & 0\ 1\ 1\ 2 \in \Phi_2 & \text{by lower rank}, \\ \gamma = 0\ 1\ 1\ 2 + \alpha_1 & \longmapsto & \alpha_1 \notin \Phi_3, & \\ \gamma + \beta = 0\ 1\ 2\ 2 + \alpha_1 & \longmapsto & 0\ 1\ 2\ 2 \in \Phi_1, & \\ 0\ 1\ 2\ 2 = \varphi + 0\ 0\ 1\ 2 & \longmapsto & 0\ 0\ 1\ 2 \in \Phi_1, & \\ \alpha_0 = 1\ 1\ 0\ 0 + 0\ 1\ 2\ 2 & \longmapsto & 1\ 1\ 0\ 0 \in \Phi_2, & \end{array}$$

Now $\gamma = 1\ 1\ 0\ 0 + 0\ 0\ 1\ 2$ contradicts the convexity of Φ_3.

Case A-2: $\varphi = 0\ 1\ 1\ 1$. We have:

$$\begin{array}{lll}
\alpha_0 = 0\ 1\ 1\ 0 + \gamma & \longmapsto & \alpha_0 \notin \Phi_2 & \text{Case A-1,} \\
\varphi = \alpha_2 + 0\ 0\ 1\ 1 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by lower rank,} \\
\varphi = 0\ 1\ 1\ 0 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{Case A-1,} \\
\gamma = \alpha_4 + 1\ 1\ 1\ 1 & \longmapsto & 1\ 1\ 1\ 1 \notin \Phi_3 & \\
\alpha_0 = 1\ 1\ 1\ 1 + \varphi & \longmapsto & \alpha_0 \notin \Phi_1, & \\
\alpha_0 = (\gamma + \beta) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3. &
\end{array}$$

Contradiction.

Case A-3: $\varphi = 0\ 1\ 1\ 2$. We have:

$$\begin{array}{llll}
\alpha_0 = 1\ 1\ 1\ 1 + 0\ 1\ 1\ 1 & \longmapsto & \alpha_0 \notin \Phi_2 & \text{Case A-2 and lower rank,} \\
\varphi = \alpha_4 + 0\ 1\ 1\ 1 & \longmapsto & \alpha_4 \in \Phi_2, & \\
\gamma = 2\alpha_4 + 1\ 1\ 1\ 0 & \longmapsto & 1\ 1\ 1\ 0 \notin \Phi_3 & \\
\alpha_0 = 1\ 1\ 1\ 0 + \varphi & \longmapsto & \alpha_0 \notin \Phi_1, & \\
\gamma = 1\ 1\ 1\ 0 + 2\alpha_4 & \longmapsto & 1\ 1\ 1\ 0 \notin \Phi_3, & \\
\gamma + \beta = 1\ 1\ 1\ 0 + 0\ 0\ 1\ 2 & \longmapsto & 0\ 0\ 1\ 2 \in \Phi_1, & \\
\varphi = 0\ 0\ 1\ 2 + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2, & \\
\alpha_0 = (\gamma + \beta) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3. &
\end{array}$$

Contradiction.

Case B. $\beta = \alpha_2$ and $\gamma = 1\ 1\ 2\ 2$.

by reduction to lower rank we have

$$\Phi_2 \cap \{0\ 1\ 2\ 2, 1\ 1\ 0\ 0, 1\ 1\ 1\ 1, 0\ 1\ 1\ 1\} = \emptyset. \tag{20}$$

Case B-1: $\varphi = 0\ 1\ 1\ 0$. We have:

$$\begin{array}{lll}
\varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2, \\
\gamma + \beta = 1\ 1\ 1\ 2 + \varphi & \longmapsto & 1\ 1\ 1\ 2 \in \Phi_1.
\end{array}$$

Now, $\gamma = \alpha_3 + 1\ 1\ 1\ 2$ contradicts the convexity of Φ_3.

Case B-2: $\varphi = 0\ 1\ 1\ 2$. We have:

$$\begin{array}{lll}
\varphi = \beta + 0\ 0\ 1\ 2 & \longmapsto & 0\ 0\ 1\ 2 \in \Phi_2, \\
\gamma + \beta = 1\ 1\ 1\ 0 + \varphi & \longmapsto & 1\ 1\ 1\ 0 \in \Phi_1.
\end{array}$$

Now, $\gamma = 0\ 0\ 1\ 2 + 1\ 1\ 1\ 0$ contradicts the convexity of Φ_3.

Case B-3: $\varphi = 1\ 1\ 1\ 0$. We have:

$$\begin{array}{llll}
\varphi = \alpha_1 + 0\ 1\ 1\ 0 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case B-1,} \\
\varphi = \alpha_3 + 1\ 1\ 0\ 0 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by lower rank,} \\
\gamma + \beta = 0\ 1\ 1\ 2 + \varphi & \longmapsto & 0\ 1\ 1\ 2 \in \Phi_1. &
\end{array}$$

Now, $\gamma = 0\ 1\ 1\ 2 + \alpha_1 + \alpha_3$ contradicts the convexity of Φ_3.

Case B-4: $\varphi = 1\ 1\ 1\ 2$. We have:

$$\begin{array}{lcll} \gamma+\beta = 0\ 1\ 1\ 0 + \varphi & \longmapsto & 0\ 1\ 1\ 0 \in \Phi_1, & \\ \varphi = 1\ 1\ 0\ 0 + 0\ 0\ 1\ 2 & \longmapsto & 0\ 0\ 1\ 2 \in \Phi_2 & \text{by lower rank,} \\ 0\ 1\ 2\ 2 = 0\ 0\ 1\ 2 + 0\ 1\ 1\ 0 & \longmapsto & 0\ 1\ 2\ 2 \in \Phi_3 & \\ \varphi = \alpha_1 + 0\ 1\ 1\ 2 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case B-2,} \end{array}$$

Now, $\gamma = 0\ 1\ 2\ 2 + \alpha_1$ contradicts the convexity of Φ_3. □

8.4.5 Type B_5

Lemma 45. *There are two irreducible triples $\beta < \varphi < \gamma$ in B_5 such that $\gamma + \beta \in \Phi$. They are $\beta = \alpha_3$, $\gamma = 1\ 1\ 1\ 2\ 2$ and $\varphi = 0\ 1\ 1\ 1\ 0$ or $0\ 1\ 1\ 1\ 2$.*

Proof. Like for B_4, we easily prove that $\gamma \neq \sum_{i=1}^{5} \alpha_i$.

Assume now that $\gamma = 1\ 1\ 1\ 1\ 2$. Then $\beta = \sum_{j \le i \le 4} \alpha_i$, for $j = 4, 3$ or 2. Moreover, $\gamma = a\alpha_5 + \sum_{j' \le i \le 4} \alpha_i$ for $a \in \{0, 1, 2\}$ and $j' \le j$. Set $\eta = \sum_{j' \le i < j} \alpha_i$ (or 0 if $j = j'$) and $\eta' = \sum_{1 \le i < j'} \alpha_i$ (or 0 if $j' = 1$). Then $F = \langle \eta, \eta', \beta, \alpha_5 \rangle$ shows that the triple $\beta < \varphi < \gamma$ is not irreducible.

Assume now that $\gamma = 1\ 1\ 2\ 2\ 2$. Then $\beta = \alpha_2$. Set $\eta_1 = \sum_{i=3}^{5} n_i(\varphi)\alpha_i$ and $\eta_2 = \sum_{i \ge 3,\ n_i(\varphi)=1} \alpha_i$. Then $\varphi - \beta \in \langle \alpha_1, \eta_1 \rangle$ and $\gamma - \varphi = \langle \alpha_1, \eta_2 \rangle$. Since η_2 and η_1 (or $\frac{1}{2}\eta_1$) are roots, the triple reduces to lower rank.

Assume now that $\gamma = 1\ 1\ 1\ 2\ 2$ and $\beta = \alpha_2 + \alpha_3$. As in the previous case, one can easily find two roots η_1 and η_2 such that $\gamma - \varphi$ and $\varphi - \beta$ belong to $\langle \alpha_1, \eta_1, \eta_2 \rangle$. The triple reduces to the rank 4.

The last case to consider is $\gamma = 1\ 1\ 1\ 2\ 2$ and $\beta = \alpha_3$. If $\varphi = 0\ 0\ 1\ \cdot\ \cdot$ or $1\ 1\ 1\ \cdot\ \cdot$, the same argument as before proves that the triple is reducible. Hence $\varphi = 0\ 1\ 1\ \cdot\ \cdot$. One can check that the triple is reducible if $n_4(\varphi) = n_5(\varphi)$. Hence $\varphi = 0\ 0\ 1\ 1\ 0$ or $0\ 0\ 1\ 1\ 2$, which are the two cases of the statement. □

Lemma 46. *Theorem 3 holds for the two triples (β, φ, γ) in Lemma 45.*

Proof. Fix one of the two triples and assume by contradiction that $\varphi \in \Phi_2$.

Case 1: $\varphi = 0\ 1\ 1\ 1\ 2$. We have

$$\begin{array}{lcll} \alpha_0 = \gamma + 0\ 1\ 1\ 0\ 0 & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by lower rank,} \\ \varphi = \alpha_2 + 0\ 0\ 1\ 1\ 2 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by lower rank,} \\ \alpha_0 = \alpha_2 + (\gamma + \beta) & \longmapsto & \alpha_0 \in \Phi_3. & \end{array}$$

Hence $\alpha_0 \in \Phi_1$. Now

$$\begin{array}{lcll} \varphi = 0\ 0\ 0\ 1\ 2 + 0\ 1\ 1\ 0\ 0 & \longmapsto & 0\ 0\ 0\ 1\ 2 \in \Phi_2 & \text{by lower rank,} \\ \gamma = 0\ 0\ 0\ 1\ 2 + 1\ 1\ 1\ 1\ 0 & \longmapsto & 1\ 1\ 1\ 1\ 0 \notin \Phi_3, & \\ \alpha_0 = 1\ 1\ 1\ 1\ 0 + \varphi & \longmapsto & \alpha_0 \notin \Phi_1. & \end{array}$$

Contradiction.

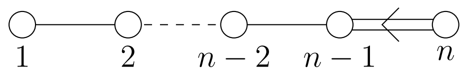
<u>Case 2:</u> $\varphi = 0\ 1\ 1\ 1\ 0$. We have

$$\begin{array}{llll}
\varphi = \alpha_4 + 0\ 1\ 1\ 0\ 0 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by lower rank,} \\
\gamma = \alpha_4 + 1\ 1\ 1\ 1\ 2 & \longmapsto & 1\ 1\ 1\ 1\ 2 \notin \Phi_3, & \\
\alpha_0 = 1\ 1\ 1\ 1\ 2 + \varphi & \longmapsto & \alpha_0 \notin \Phi_1, & \\
\varphi = \alpha_2 + 0\ 0\ 1\ 1\ 0 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by lower rank,} \\
\alpha_0 = (\gamma + \beta) + \alpha_2 & \longmapsto & \alpha_0 \in \Phi_3, & \\
\alpha_0 = \gamma + 0\ 1\ 1\ 0\ 0\ 0 & \longmapsto & \alpha_0 \notin \Phi_2 & \text{by lower rank.}
\end{array}$$

Contradiction. □

8.5 Type C

C_n



Again using the notation of [Bou68] we can distinguish 3 types of positive roots:

Type 1 $\varepsilon_i - \varepsilon_j$ $: 1 \le i < j \le n$

$$\begin{array}{cccccccccc} 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ & & & i & & j-1 & & & \end{array}$$

Type 2 $2\varepsilon_i$ $: 1 \le i \le n$

$$\begin{array}{ccccccc} 0 & \dots & 0 & 2 & \dots & 2 & 1 \\ & & & i & & n-1 & n \end{array}$$

Type 3 $\varepsilon_i + \varepsilon_j$ $: 1 \le i < j \le n$

$$\begin{array}{cccccccccc} 0 & \dots & 0 & 1 & \dots & 1 & 2 & \dots & 2 & 1 \\ & & & i & & j-1 & j & & n-1 & n \end{array}$$

8.5.1 Reduction

Lemma 47. *There is no irreducible triple $\beta < \varphi < \gamma$ such that $\beta + \gamma \in \Phi$, in C_ℓ, for any $\ell \geq 6$.*

Proof. Since $\mathrm{Supp}(\gamma) = \Delta$ and γ is not the longest root, $\gamma = \varepsilon_1 + \varepsilon_j$ for a certain $1 < j \le n$. Since $\beta + \gamma \in \Phi^+$, $\beta = \varepsilon_i - \varepsilon_j$ with $1 \le i < j \le n$. We have 3 possibilities for φ according to the type.

<u>Type 1</u>: $\varphi = \varepsilon_k - \varepsilon_l$. Since $\beta \le \varphi$, we have $k \le i < j \le l$. Then

$$\gamma - \varphi = (\varepsilon_1 - \varepsilon_k) + (\varepsilon_j + \varepsilon_l) \quad \text{and} \quad \varphi - \beta = (\varepsilon_k - \varepsilon_i) + (\varepsilon_j - \varepsilon_l).$$

Hence $F = \langle \beta, \varepsilon_1 - \varepsilon_k, \varepsilon_j + \varepsilon_l, \varepsilon_k - \varepsilon_i, \varepsilon_j - \varepsilon_l \rangle$ works.

Type 2: $\varphi = 2\varepsilon_k$. If $\varphi \leq \gamma$, then $j \leq k$, while if $\beta \leq \varphi$, then $k \leq i$. But $i < j$, so there is no triple that satisfies the hypothesis in this case.

Type 3: $\varphi = \varepsilon_k + \varepsilon_l$ with $k < l$. Since $\beta \leq \varphi$ and $\varphi \leq \gamma$, we deduce that $k \leq i < j \leq l$. Then

$$\gamma - \varphi = (\varepsilon_1 - \varepsilon_k) + (\varepsilon_j - \varepsilon_l) \quad \text{and} \quad \varphi - \beta = (\varepsilon_k - \varepsilon_i) + (\varepsilon_l + \varepsilon_j)$$

Hence $F = \langle \beta, \varepsilon_1 - \varepsilon_k, \varepsilon_j - \varepsilon_l, \varepsilon_k - \varepsilon_i, \varepsilon_l + \varepsilon_j \rangle$ works. □

8.5.2 Type C_2

The only pair $\beta < \gamma$ with $\beta + \gamma \in \Phi$ is $\beta = \alpha_1$ and $\gamma = \alpha_1 + \alpha_2$. Since $]\beta; \gamma[$ is empty, there is nothing to prove.

8.5.3 Type C_3

Lemma 48. *There are four irreducible triples $\beta < \varphi < \gamma$ in C_3 such that $\gamma + \beta \in \Phi$. They are:*

1. $\beta = 1\ 0\ 0$, $\gamma = 1\ 2\ 1$ *and* $\varphi \in \{1\ 1\ 0, 1\ 1\ 1\}$*;*
2. $\beta = 0\ 1\ 0$, $\gamma = 1\ 1\ 1$ *and* $\varphi \in \{1\ 1\ 0, 0\ 1\ 1\}$.

Proof. There are three pairs $\beta < \gamma$ such that the support of γ equals Δ and $\beta + \gamma \in \Phi$. They are

$$1\ 0\ 0 < 1\ 2\ 1 \quad 0\ 1\ 0 < 1\ 1\ 1 \quad 1\ 1\ 0 < 1\ 1\ 1.$$

In the first two cases, the interval $]\beta; \gamma[$ contains the two corresponding roots in the statement. All the triples constructed in this way are easily verified to be irreducible. In the last case $]\beta; \gamma[$ is empty. □

Lemma 49. *Theorem 3 holds for the four triples (β, φ, γ) in Lemma 48.*

Proof. Fix one of the four triples (β, φ, γ). Assume by contradiction that $\varphi \in \Phi_2$.

Case A-1: $\beta = 0\ 1\ 0$, $\varphi = 1\ 1\ 0$ and $\gamma = 1\ 1\ 1$.

We have

$$\begin{array}{lcl} \varphi = \beta + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2, \\ \gamma = 0\ 1\ 1 + \alpha_1 & \longmapsto & 0\ 1\ 1 \notin \Phi_3, \\ \beta + \gamma = 0\ 1\ 1 + \varphi & \longmapsto & \beta + \gamma \notin \Phi_1, \end{array}$$

contradiction.

Case A-2: $\beta = 0\ 1\ 0$, $\varphi = 0\ 1\ 1$ and $\gamma = 1\ 1\ 1$.

We have

$$\begin{array}{lcl} \varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2, \\ \gamma = 1\ 1\ 0 + \alpha_3 & \longmapsto & 1\ 1\ 0 \notin \Phi_3, \\ \beta + \gamma = \varphi + 1\ 1\ 0 & \longmapsto & \beta + \gamma \notin \Phi_1, \end{array}$$

contradiction.

Case B-1: $\beta = 1\ 0\ 0$, $\varphi = 1\ 1\ 0$ and $\gamma = 1\ 2\ 1$.

We have

$$\begin{array}{lll} \varphi = \beta + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2, \\ \gamma = 0\ 1\ 1 + \varphi & \longmapsto & 0\ 1\ 1 \notin \Phi_3, \\ 0\ 1\ 1 = \beta + \alpha_3 & \longmapsto & \alpha_3 \notin \Phi_3, \\ \beta + \gamma = 1\ 1\ 1 + \varphi & \longmapsto & 1\ 1\ 1 \in \Phi_1, \\ 1\ 1\ 1 = \varphi + \alpha_3 & \longmapsto & \varphi \in \Phi_1 \end{array}$$

contradiction.

Case B-2: $\beta = 1\ 0\ 0$, $\varphi = 1\ 1\ 1$ and $\gamma = 1\ 2\ 1$.

We have

$$\begin{array}{lll} \varphi = \beta + 0\ 1\ 1 & \longmapsto & 0\ 1\ 1 \in \Phi_2, \\ \gamma + \beta = 1\ 1\ 0 + \varphi & \longmapsto & 1\ 1\ 0 \in \Phi_1, \\ \gamma = 0\ 1\ 1 + 1\ 1\ 0 & \longmapsto & \gamma \in \Phi_3 \end{array}$$

contradiction. □

8.5.4 Type C_4

Lemma 50. *There are six irreducible triples $\beta < \varphi < \gamma$ in C_4 such that $\gamma + \beta \in \Phi$. They are*

1. $\beta = \alpha_3$, $\gamma = 1\ 1\ 1\ 1$ *and* φ *in*

$$\{0\ 1\ 1\ 0, 0\ 1\ 1\ 1\},$$

2. $\beta = \alpha_2$, $\gamma = 1\ 1\ 2\ 1$ *and* φ *in*

$$\{0\ 1\ 1\ 0, 1\ 1\ 1\ 0, 0\ 1\ 1\ 1, 1\ 1\ 1\ 1\}.$$

Proof. Since the support of γ is Δ, and $\gamma + \beta \in \Phi$,

$$\gamma \in \{1\ 1\ 1\ 1, 1\ 1\ 2\ 1, 1\ 2\ 2\ 1\}.$$

Case A: $\gamma = 1\ 1\ 1\ 1$.

Since $\beta + \gamma \in \Phi$, then $\beta \in \{0\ 0\ 1\ 0, 0\ 1\ 1\ 0, 1\ 1\ 1\ 0\}$.

Case A-1: $\beta = 0\ 0\ 1\ 0$. Then $]\beta; \gamma[= \{0\ 1\ 1\ 0, 0\ 0\ 1\ 1, 1\ 1\ 1\ 0, 0\ 1\ 1\ 1\}$. We easily check that the first and the fourth φ of this list give an irreducible triple. For the second and the third φ, the condition $\beta < \varphi < \gamma$ still holds in the span of $\beta, \alpha_1 + \alpha_2, \alpha_4$.

Case A-2: $\beta = 0\ 1\ 1\ 0$.

Then for any $\varphi \in]\beta; \gamma[$, $\beta < \varphi < \gamma$ still holds in the span of $\beta, \alpha_1, \alpha_4$.

Case A-3: $\beta = 1\ 1\ 1\ 0$. Then $]\beta; \gamma[$ is empty.

Case B: $\gamma = 1\ 1\ 2\ 1$.

Then $\beta \in \{0\ 1\ 0\ 0, 1\ 1\ 0\ 0\}$.

Case B-1: $\beta = 0\ 1\ 0\ 0$.

We have 6 roots in $]\beta; \gamma[$. Four of them correspond to the irreducible triples of the statement. If φ is one of the remaining two roots in the interval, then $\beta < \varphi < \gamma$ holds in the span of $\beta, \alpha_1, 0\ 0\ 2\ 1$.

Case B-2: $\beta = 1\ 1\ 0\ 0$.

Then for any $\varphi \in]\beta; \gamma[$, the condition $\beta < \varphi < \gamma$ holds in the span of $\beta, \alpha_3, \alpha_4$.

Case C: $\gamma = 1\ 2\ 2\ 1$.

In this case $\beta = \alpha_2$ and for any $\varphi \in]\beta; \gamma[$, $\gamma - \varphi \in \Phi^+$ and $\varphi - \beta \in \Phi^+$. Hence any triple is reducible.

□

Lemma 51. *Theorem 3 holds for the six triples (β, φ, γ) in Lemma 50.*

Proof. Fix one of the six triples (β, φ, γ) and assume that $\varphi \in \Phi_2$.

Case A: $\beta = \alpha_3$, $\gamma = 1\ 1\ 1\ 1$.

Case A-1: $\varphi = 0\ 1\ 1\ 1$.

We have

$$\begin{array}{lll} \varphi = 0\ 0\ 1\ 1 + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 \qquad \text{by lower rank,} \\ \gamma = \varphi + \alpha_1 & \longmapsto & \alpha_1 \notin \Phi_3, \\ 0\ 1\ 1\ 0 = \beta + \alpha_2 & \longmapsto & 0\ 1\ 1\ 0 \in \Phi_3. \end{array}$$

Now let $\beta' = 0\ 1\ 1\ 0$, $\varphi' = \varphi$ and $\gamma' = \gamma$. This is a reducible triple that satisfies the hypothesis of Theorem 3 (up to switching Φ_1 and Φ_2). Since $\varphi' \in \Phi_2$ we have that

$$\beta' \in \Phi_2.$$

But $\beta < \beta' < \gamma$ is a reducible triple, hence $\beta' \notin \Phi_2$. Contradiction.

Case A-2: $\varphi = 0\ 1\ 1\ 0$.

We have

$$\begin{array}{lll} \varphi = \beta + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2, \\ 1\ 2\ 2\ 1 = \varphi + \gamma & \longmapsto & 1\ 2\ 2\ 1 \notin \Phi_1, \\ 1\ 2\ 2\ 1 = (\gamma + \beta) + \alpha_2 & \longmapsto & 1\ 2\ 2\ 1 \in \Phi_3, \end{array}$$

Hence $1\ 2\ 2\ 1 \in \Phi_2$. Then

$$1\ 2\ 2\ 1 = 1\ 1\ 1\ 0 + 0\ 1\ 1\ 1 \longmapsto 0\ 1\ 1\ 1 \in \Phi_2 \quad \text{by lower rank}$$

Contradiction by Case A-1.

Case B: $\beta = \alpha_2$, $\gamma = 1\ 1\ 2\ 1$.

Case B-1: $\varphi = 0\ 1\ 1\ 0$.

We have

$$\begin{array}{lll} \varphi = \beta + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 \\ \gamma = 1\ 1\ 1\ 1 + \alpha_3 & \longmapsto & 1\ 1\ 1\ 1 \notin \Phi_3 \\ \gamma + \beta = 1\ 1\ 1\ 1 + \varphi & \longmapsto & 1\ 1\ 1\ 1 \in \Phi_1. \end{array}$$

Contradiction.

Case B-2: $\varphi = 0\ 1\ 1\ 1$.

Here

$$\begin{array}{lll} \varphi = \beta + 0\ 0\ 1\ 1 & \longmapsto & 0\ 0\ 1\ 1 \in \Phi_2 \\ \gamma = 1\ 1\ 1\ 0 + 0\ 0\ 1\ 1 & \longmapsto & 1\ 1\ 1\ 0 \notin \Phi_3 \\ \gamma + \beta = 1\ 1\ 1\ 0 + \varphi & \longmapsto & \gamma + \beta \notin \Phi_1. \end{array}$$

Contradiction. □

Case B-3: $\varphi = 1\ 1\ 1\ 0$.

Here

$$\begin{array}{llll} \varphi = 1\ 1\ 0\ 0 + \alpha_3 & \longmapsto & \alpha_3 \in \Phi_2 & \text{by lower rank} \\ 0\ 1\ 1\ 0 = \beta + \alpha_3 & \longmapsto & 0\ 1\ 1\ 0 \in \Phi_3 & \\ \gamma = 1\ 1\ 1\ 1 + \alpha_3 & \longmapsto & 1\ 1\ 1\ 1 \notin \Phi_3. & \end{array}$$

Then let $\beta' = 0\ 1\ 1\ 0$, $\varphi' = \varphi$ and $\gamma' = 1\ 1\ 1\ 1$. Since $\gamma' + \beta' = \gamma + \beta \in \Phi_3$, the previous triple satisfies the hypothesis of Theorem 3 and is reducible. Since $\varphi' \in \Phi_2$, $\beta' \in \Phi_2$. But by Lemma 29, $\gamma' + \beta' \in \Phi_2$. Contradiction.

Case B-4: $\varphi = 1\ 1\ 1\ 1$.

Here

$$\begin{array}{llll} \gamma + \beta = \varphi + 0\ 1\ 1\ 0 & \longmapsto & 0\ 1\ 1\ 0 \in \Phi_1 & \\ \varphi = 1\ 1\ 0\ 0 + 0\ 0\ 1\ 1 & \longmapsto & 0\ 0\ 1\ 1 \in \Phi_2 & \text{by lower rank} \\ \varphi = 0\ 1\ 1\ 1 + \alpha_1 & \longmapsto & \alpha_1 \in \Phi_2 & \text{by Case B-2} \\ \gamma = 0\ 1\ 1\ 0 + \alpha_1 + 0\ 0\ 1\ 1 & \longmapsto & \gamma \in \Phi_3. & \end{array}$$

Contradiction.

8.5.5 Type C_5

Lemma 52. *There are two irreducible triples $\beta < \varphi < \gamma$ in C_5 such that $\gamma + \beta \in \Phi$. They are:*

$$\beta = \alpha_3 \quad \gamma = 1\ 1\ 1\ 2\ 1 \quad \textit{and} \quad \varphi \in \{0\ 1\ 1\ 1\ 0, 0\ 1\ 1\ 1\ 1\}.$$

Proof. If $\psi \in \Phi$ and $n \in \{1, \ldots, 5\}$ we denote $\psi_{\leq n} = \sum_{i=1}^{n} n_i(\psi)\alpha_i$ and $\psi_{\geq n} = \sum_{i=n}^{5} n_i(\psi)\alpha_i$. Note that $\psi_{\geq n}$ is always a root or zero, while $\psi_{\leq n}$ may not be a root. Since the support of γ is Δ, and $\gamma + \beta \in \Phi$, then

$$\gamma \in \{1\ 1\ 1\ 1\ 1, 1\ 1\ 1\ 2\ 1, 1\ 1\ 2\ 2\ 1, 1\ 2\ 2\ 2\ 1\}.$$

Case A: $\gamma = 1\ 1\ 1\ 1\ 1$.

Then $\beta = \sum_{i=j}^{4} \alpha_i$ for a certain $j \in \{1, \ldots, 4\}$. If $j = 1$, then $]\beta; \gamma[$ is empty. Otherwise, for any $\varphi \in]\beta; \gamma[$, $(\gamma - \varphi)_{\leq 4}$ and $(\varphi - \beta)_{\leq 4}$ are roots and $\beta < \gamma < \varphi$ holds in the span of $\beta, \alpha_5, (\varphi - \beta)_{\leq 4}, (\gamma - \varphi)_{\leq 4}$.

Case B: $\gamma = 1\ 1\ 2\ 2\ 1$.

Here $\beta \in \{\alpha_2, \alpha_1+\alpha_2\}$. For both possible choices of β, for any $\varphi \in]\beta;\gamma[$, $\beta < \varphi < \gamma$ holds in the span of $\beta, \alpha_1, (\varphi-\beta)_{\geq 3}(\gamma-\varphi)_{\geq 3}$.

Case C: $\gamma = 1\ 2\ 2\ 2\ 1$.

Then $\beta = \alpha_1$ and for any $\varphi \in]\beta;\gamma[$, $\beta < \varphi < \gamma$ holds in the span of $\beta, (\varphi-\beta)_{\geq 2}(\gamma-\varphi)_{\geq 2}$.

Case D: $\gamma = 1\ 1\ 1\ 2\ 1$.

Then $\beta \in \{\alpha_3, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$. For the last two possible choices of β, we easily see, as in Case A, that there is no $\varphi \in]\beta;\gamma[$ whose corresponding root is irreducible. If $\beta = \alpha_3$, then there are 10 roots $\varphi \in]\beta;\gamma[$. Two of them correspond to the irreducible triples of the statement. Five of them satisfy $n_2(\varphi) = 0$ or $n_1(\varphi) = 1 = n_2(\varphi)$. In these cases $\beta < \varphi < \gamma$ holds in the span of $\beta, \alpha_1+\alpha_2, \alpha_4, \alpha_5$. The last three φ in the interval satisfy $n_4(\varphi) \in \{0,2\}$. For the corresponding triples, the condition $\beta < \varphi < \gamma$ holds in the span of $\alpha_1, \alpha_2, \beta, 0\ 0\ 0\ 2\ 1$. □

Lemma 53. *Theorem 3 holds for the two triples (β,φ,γ) in Lemma 52.*

Proof. Case A: $\varphi = 0\ 1\ 1\ 1\ 0$.

We have

$$\begin{array}{lcll} \varphi = 0\ 1\ 1\ 0\ 0 + \alpha_4 & \longmapsto & \alpha_4 \in \Phi_2 & \text{by lower rank,} \\ \gamma = 1\ 1\ 1\ 1\ 1 + \alpha_4 & \longmapsto & 1\ 1\ 1\ 1\ 1 \notin \Phi_3, & \\ 0\ 0\ 1\ 1\ 0 = \beta + \alpha_4 & \longmapsto & 0\ 0\ 1\ 1\ 0 \in \Phi_3. & \end{array}$$

We can apply Theorem 3 to the reducible triple $\beta' = 0\ 0\ 1\ 1\ 0$, $\varphi' = \varphi$, $\gamma' = 1\ 1\ 1\ 1\ 1$. Since $\varphi' \in \Phi_2$ we deduce that $\beta' \in \Phi_2$. Then by Lemma 29 we have that $\beta' + \gamma' = \beta + \gamma \in \Phi_2$. Contradiction.

Case B: $\varphi = 0\ 1\ 1\ 1\ 1$.

We have

$$\begin{array}{lcll} \varphi = 0\ 1\ 1\ 1\ 0 + \alpha_5 & \longmapsto & \alpha_5 \in \Phi_2 & \text{by Case A,} \\ \varphi = 0\ 0\ 1\ 1\ 1 + \alpha_2 & \longmapsto & \alpha_2 \in \Phi_2 & \text{by lower rank} \\ \varphi = 0\ 1\ 1\ 0\ 0 + 0\ 0\ 0\ 1\ 1 & \longmapsto & 0\ 0\ 0\ 1\ 1 \in \Phi_2 & \text{by lower rank} \\ 0\ 0\ 1\ 1\ 1 = \beta + 0\ 0\ 0\ 1\ 1 & \longmapsto & 0\ 0\ 1\ 1\ 1 \in \Phi_3. & \end{array}$$

Hence $0\ 0\ 1\ 1\ 1 \in \Phi_1$ by lower rank. Then

$$\begin{array}{lcl} 0\ 0\ 1\ 1\ 1 = 0\ 0\ 1\ 1\ 0 + \alpha_5 & \longmapsto & 0\ 0\ 1\ 1\ 0 \in \Phi_1, \\ 0\ 0\ 1\ 2\ 1 = 0\ 0\ 1\ 1\ 0 + 0\ 0\ 0\ 1\ 1 & \longmapsto & 0\ 0\ 1\ 2\ 1 \in \Phi_3, \\ 0\ 1\ 1\ 2\ 1 = \alpha_2 + 0\ 0\ 1\ 2\ 1 & \longmapsto & 0\ 1\ 1\ 2\ 1 \in \Phi_3. \end{array}$$

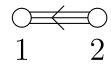
Hence $0\ 1\ 1\ 2\ 1 \in \Phi_1$ by lower rank. Then

$$\begin{array}{lcl} 0\ 1\ 1\ 2\ 1 = \alpha_4 + \varphi & \longmapsto & \alpha_4 \in \Phi_1, \\ \gamma = 1\ 1\ 1\ 1\ 1 + \alpha_4 & \longmapsto & 1\ 1\ 1\ 1\ 1 \notin \Phi_3. \end{array}$$

Then applying Theorem 3 to the reducible triple $\beta' = 0\ 0\ 1\ 1\ 0$, $\varphi' = \varphi$ and $\gamma' = 1\ 1\ 1\ 1\ 1$ we deduce that $\beta' \in \Phi_2$. Contradiction.

□

8.6 Type G_2

G_2 

The highest root is $\alpha_0 = 3\ 2$.

Lemma 54. *The irreducible triples $\beta < \varphi < \gamma$ in G_2 are:*

$$\begin{array}{lll} \beta = \alpha_1 & \varphi = \alpha_1 + \alpha_2 & \gamma = 2\alpha_1 + \alpha_2 \\ \beta = \alpha_2 & \varphi = \alpha_1 + \alpha_2 & \gamma = 3\alpha_1 + \alpha_2 \\ \beta = \alpha_2 & \varphi = 2\alpha_1 + \alpha_2 & \gamma = 3\alpha_1 + \alpha_2 \end{array}$$

Proof. Easy. Left to the reader. □

Lemma 55. *Theorem 3 holds for the three triples (β, φ, γ) in Lemma 54.*

Proof. In the first case,

$$\varphi = \beta + \alpha_2 \ \longmapsto \ \alpha_2 \in \Phi_2.$$

Hence $\alpha_1, \alpha_2 \in \Phi_3$. By convexity $\Phi^+ = \Phi_3$. Contradiction.

The second case is similar: $\varphi = \beta + \alpha_1$ implies $\alpha_1 \in \Phi_2$, and $\Phi_3 = \Phi^+$. Contradiction.

In the last case,

$$\gamma = \varphi + \alpha_1 \ \longmapsto \ \alpha_1 \notin \Phi_3.$$

Hence α_1 and $\alpha_2 = \beta$ do not belong to Φ_2. But any biconvex nonempty subset of Φ^+ contains a simple root. Contradiction. □

8.7 Type F_4

In this case, there are 85 irreducible triples. We checked the theorem with a computer in this case and also wrote a proof (more than 26 pages). See [Res23] for details.

9 A determinant

Fix a poset $\{\varphi_0, \dots, \varphi_k\}$ numbered in such a way that $\varphi_i \leq \varphi_j$ only if $i \leq j$. Let M be a $(k \times k)$-matrix whose rows are labeled by $(\varphi_0, \dots, \varphi_{k-1})$ and columns by $(\varphi_1, \dots, \varphi_k)$. Denote by m_{ij} the entry at row φ_i and column φ_j. We assume that

1. for any $i = 1, \dots, k-1$, $m_{ii} = 1$; and
2. $m_{ij} \neq 0$ implies $\varphi_i \leq \varphi_j$.

Proposition 56. *With above notation, the determinant of M is*

$$\det M = (-1)^{k+1} \sum_{\substack{0 \le s \le k-1 \\ 0 < j_0 < \cdots < j_s < k \\ \varphi_0 < \varphi_{j_0} < \cdots < \varphi_{j_s} < \varphi_k}} (-1)^s m_{0\,j_0} m_{j_0\,j_1} \cdots m_{j_s\,k}.$$

Proof. Start with the expression

$$\det(M) = \sum_{\sigma} \varepsilon(\sigma) m_{\sigma(1)\,1} \ldots m_{\sigma(k)\,k}, \tag{21}$$

where the sum runs over all the bijections $\sigma : [1;k] \longrightarrow [0;k-1]$. Here, $\varepsilon(\sigma)$ is the signature of the bijection $\tilde{\sigma}$ of $[1;k]$ on itself that maps j to $\sigma(j)+1$.

Since M is "almost upper triangular", in the sum (21), we can keep only the bijections σ satisfying: $\sigma(j) \le j$ for any $j \in [1;k]$. Define j_0 by $\sigma(j_0) = 0$. Then, for any $1 \le j < j_0$, we have $\sigma(j) = j$. In other words, the bijection $\tilde{\sigma}$ stabilizes $[1;j_0]$ and acts on it as the cycle $(1, 2, \ldots, j_0)$. Moreover,

$$m_{\sigma(1)\,1} \ldots m_{\sigma(j_0)\,j_0} = m_{0\,j_0},$$

since all the other factors are of the form $m_{jj} = 1$.

Now, an immediate induction shows that the expression of $\tilde{\sigma}$ as a product of disjoint cycles can be obtained by bracketing the word $1\ 2\ \ldots k$. Write

$$\tilde{\sigma} = (1,\ 2, \ldots, j_0)(j_0+1, \ldots, j_1) \cdots (j_s+1, \ldots, k)$$

allowing cycles of length 1. Then the product, associated to σ in (21), is

$$m_{0\,j_0} m_{j_0\,j_1} \ldots m_{j_{s-1} j_s} m_{j_s\,k},$$

and, the signature $\varepsilon(\sigma)$ is $(-1)^{(j_0-1)+(j_1-j_0-1)+\cdots+(k-j_s-1)} = (-1)^{k+s+1}$. The proposition follows. □

Remark. *A determinantal expression of the Möbius function for finite posets.* Let $(P, \le)$ be a finite poset and $[\varphi_0; \varphi_k] = \{\varphi_0, \ldots, \varphi_k\}$ be an interval of P. Let M be the $(k \times k)$-matrix whose rows are labeled by $[0, k-1]$ and columns by $[1, k]$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if} \quad \varphi_i \le \varphi_j, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Proposition 56 shows that

$$\mu([\varphi_0; \varphi_k]) = (-1)^k \det M.$$

The authors do not know if this formula was known before.

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