A quadratic bound for the determinant and permanent problem

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Abstract

The *determinantal complexity* of a polynomial f is defined here as the minimal size of a matrix M with affine entries such that $f = \det M$. This function gives a minoration of the more traditional *size* of an arithmetical formula.

Consider the polynomial "permanent" Perm_d of a $d \times d$ matrix with entries $X_{i,j}$. A conjecture in complexity theory says that the determinantal complexity (dc) of Perm_d should not be polynomial in d.

In this article we prove that $dc(\mathrm{Perm}_d) \geq d^2/2$, improving the previously known minoration, $\sqrt{2}d$. We also begin a systematic study of the function dc, and compute it for the homogeneous polynomials of degree 2.

1 Introduction

The *size* of an arithmetical formula is the number of symbols $(+, \times)$ which it contains. The *complexity* of a polynomial defined over a field $\mathbf{k}$ is the minimum size of formulas defining it (see [11]). Using this notion of complexity, Valiant gave algebraic analogs to algorithmic complexity problems such as $P \neq NP$ ([11], [12], [13]). In this context, we would like to find lower bounds to the complexity of certain sequences of polynomials. Such a sequence is given by the *permanent* Perm_n of a matrix $M = (m_{i,j})$ of size $n \times n$:

$$\mathrm{Perm}_n(M) = \sum_{\sigma \in \Sigma_n} \prod_{i=1}^{n} m_{i,\sigma(i)},$$

where Σ_n is the permutation group of the set $\{1, \cdots, n\}$. A central conjecture is:

Conjecture 1.1 *The complexity of* Perm_n *is not bounded by a polynomial function in* n.

To approach this kind of problem, Valiant makes use of determinants: A polynomial $P \in \mathbf{k}[X_1, \ldots, X_m]$ is called *affine projection* of a determinant of size n if there exists an affine function $F : \mathbf{k}^m \longrightarrow M_n(\mathbf{k})$ such that: $P = \det \circ F$. In [11], Valiant prove that, if P is a polynomial of $\mathbf{k}[X_1, \ldots, X_m]$ of complexity c, then P is affine projection of a determinant of size $2c$.

Thanks to this result, we can give the

Definition 1.1 *The* determinantal complexity *of a polynomial P over $\mathbf{k}$ is the smallest integer n such that P is affine projection of a determinant of size n. It is denoted by $dc(P)$.*

The Valiant's result tells us that the determinantal complexity of a polynomial is less than or equal to the double of its complexity. A conjecture is:

Conjecture 1.2 *The function $dc(\mathrm{Perm}_n)$ is not polynomial in n.*

Conjecture 1.1 follows from Conjecture 1.2, but these two assertions are not equivalent. The smallest known arithmetical formula to write the determinant is of size $n^{O\ln(n)}$, which only gives: complexity$(P) \leq dc(P)^{O(\ln(dc(P)))}$. Moreover, the complexity of the determinant is conjectured not to be polynomial, even if algorithms (Strassen's algorithm for example) are able to compute the determinant of a given matrix in $O(n^{2,81})$ steps (no such algorithm exists for the permanent).

In this article, we prove some results on the determinantal complexity of homogeneous polynomials. For an homogeneous polynomial of three variables (or less) the determinantal complexity is known (see Section 2). We determine the determinantal complexity of the homogeneous polynomials of degree 2:

Theorem 1 *We assume that $\mathbf{k}$ is algebraicaly closed. Let q be a polynomial of degree 2 defining a quadratic form of rank r, then:*

$$dc(q) = \begin{cases} 2 & \text{if } r \leq 4 \\ \lceil \frac{r+1}{2} \rceil & \text{else,} \end{cases}$$

where $\lceil x \rceil$ denotes the unique integer such that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$, for all real number x.

In 1913, Polya (see [9]) already asked if the permanent can be written as a determinant. After works of Szegö (see [10]) on the Polya's question, Marcus and Minc shown in 1961 (see [7]) that $dc(\mathrm{Perm}_n) > n$. In spite of recent work of Mulmuley and Sohoni (see [8]), the best known lower bound for $dc(\mathrm{Perm}_n)$ was linear. Actually, improving a previous result of Zur Gathen [5], Jin-Yi Cai proved in [3] that: $dc(\mathrm{Perm}_n) \geq \sqrt{2}n$. The main result of this article is a quadratic lower bound to the function $dc(\mathrm{Perm}_n)$:

Theorem 2 *If the characteristic of $\mathbf{k}$ is zero, $dc(\mathrm{Perm}_n) \geq \frac{n^2}{2}$.*

Consider the restriction SPerm_n of Perm_n to the symmetric matrices of size $n \times n$; so that, SPerm_n is a polynomial in $\frac{n(n+1)}{2}$ variables. In [6], it is shown that for any $n \geq 3$ we have $dc(\mathrm{SPerm}_n) > n$. Here, we obtain the following improvement:

Theorem 3 *If the characteristic of* $\mathbf{k}$ *is zero,* $dc(\mathrm{SPerm}_n) \geq \frac{n(n+1)}{4}$.

The proof of Theorem 2 comes from the following observation: Let S be the hypersurface of $M_n(\mathbf{k})$ defined by the polynomial "determinant". For every X in S, there exist affine subspaces of big dimension (precisely, of codimension less than $n+1$) contained in S and passing through X. From this, the rank of the second fundamental form (which takes account of infinitesimal variations of the tangent space) is small (precisely, less than $2n+1$).

On the other hand, if S' is the "permanent" hypersurface of $M_m(\mathbf{k})$ and X' a general point of S' we will prove that the rank of the second fundamental form in X' equals m^2.

If an affine function $F : \mathbf{k}^{m\times m} \longrightarrow \mathbf{k}^{n\times n}$ exists, such that $\mathrm{Perm} = \det \circ F$, then the second fundamental forms ω and ω' satisfy the inequality:

$$\mathrm{rk}_X \omega' \leq \mathrm{rk}_{F(X)} \omega,$$

and Theorem 2 follows. Theorem 3 is shown by the same method.

2 Determinantal complexity: low degree or dimension

We assume in this section that $\mathbf{k}$ is an algebraically closed field.

Let V be a $\mathbf{k}$-vector space. The space $\mathbf{k}[V]$ of polynomials functions on V is graded by the degree. If f belongs to $\mathbf{k}[V]$, we denote by $[f]_k$ its component of degree k.

In this paper, we are only concerned with determinantal complexity of *homogeneous* polynomials. This restriction allows effective use of the graduation.

Let f be an homogeneous polynomial. For degree reasons, one have:

$$dc(f) \geq \deg(f).$$

Assume that $dc(f) = \deg(f) = d$.

Let $F : V \longrightarrow M_d(\mathbf{k})$ be an affine function such that $f = \det \circ F$, and let $\tilde{F} : V \longrightarrow M_d(\mathbf{k})$ be the linear part of F: $\tilde{F}$ is obtained from F by omitting the constant part.

Since f is homogeneous of degree equal to the size of the matrix, it comes $f = \det \circ \tilde{F} = \det \circ F$. In that case, we will say that f is *linear determinantal.* The zero locus of f is a projective hypersurface of $\mathbb{P}(V)$ which will also be said *linear determinantal.* Such hypersurfaces have been studied since a long time in algebraic geometry.

If $\dim V$ equals to 1, 2 or 3, every homogeneous polynomial f is linear determinantal: The case of dimension 1 is obvious. If $\dim V = 2$ we have (on any field $\mathbf{k}$):

$$\begin{vmatrix} a_0 y & -x & & \\ \vdots & y & \ddots & \\ \vdots & & \ddots & -x \\ a_{d-1}y + a_d x & & & y \end{vmatrix} = a_0 y^d + a_1 y^{d-1} x + \cdots + a_d x^d.$$

If $\dim V = 3$, we refer to [4], where it is proved that any plane curve is linear determinantal.

Partial results are known in bigger dimension: In [2], Brundun and Logar show that every cubic surface of $\mathbb{P}^3$ is linear determinantal except for the cubic containing a single line (unique, up to the action of the projective group). This last surface exactly have one singular point. (see also [1] for a different proof for *smooth* cubic surfaces).

Remark: This last result shows that, contrary to what we may expect from a "complexity" function and from the behaviour of dc in the quadratic case (see Theorem 1), the dc function is *not semicontinuous.*

Summarizing what has been said above, we write down

Proposition 2.1 *If* $\dim V \leq 3$ *or* $\deg f = 1$*, then* f *is linear determinantal and* $dc(f) = \deg(f)$.

Let us consider now the case of quadratic polynomials.

Theorem 1 *Let* q *be a non zero quadratic form on* V*. Let* r *denote the rank of* q*. Then,*

$$dc(q) = \begin{cases} 2 & \text{if } r \leq 4 \\ \lceil \frac{r}{2} + 1 \rceil & \text{else} \end{cases}$$

Before proving the theorem, we show

Lemma 2.1 *Let* $n \geq 2$ *be an integer and* $F : V \longrightarrow \mathrm{M}_n(\mathbf{k})$ *be an affine map. We assume that* $\det \circ F$ *is a quadratic form on* V *and denote by* r *its rank.*

If $n = 2$ *then* $r \leq 4$ *and if* $n \leq 3$ *then* $r \leq 2(n-1)$.

PROOF. Set $q = \det \circ F$ and $M = F(0)$. Let s denote the rank of M. Since $q(0) = 0$, $s \leq n - 1$. There exists two invertible matrices A and B such that

$$AMB^{-1} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & I_s \end{array}\right) =: M_0,$$

where I_s is the identity matrix of size $s \times s$. Moreover, there exist such matrices A and B such that $\det(A) = \det(B)$. Set $\tilde{F} : V \to \mathrm{M}_n(k)$, $v \mapsto AF(v)B^{-1}$.

Then, $q = \det \circ \tilde{F}$ and $\tilde{F}(0) = M_0$. So, we may assume that $M = M_0$.

We can write $F = (F_{ij})_{1 \le i,j \le n}$, where the F_{ij} are n^2 affine forms on V. Set $k = n - s$. Since $M = M_0$, all the F_{ij} are linear forms excepted $F_{k+1\,k+1}, \cdots, F_{nn}$.

In particular, for any $\sigma \in \Sigma_n$, $F_{1\sigma(1)} \cdots F_{k\sigma(k)}$ is homogeneous of degree k. But, $q = \sum_{\sigma \in \Sigma_n} F_{1\sigma(1)} \cdots F_{n\sigma(n)}$; and so $k \le 2$.

Let us assume that $k = 2$. We have:

$$
\begin{aligned}
q = \det \circ F = [\det \circ F]_2 &= \textstyle\sum_{\sigma \in \Sigma_n} [F_{1\sigma(1)} \cdots F_{n\sigma(n)}]_2 \\
&= \textstyle\sum_{\sigma \in \Sigma_n} F_{1\sigma(1)} F_{2\sigma(2)} [F_{3\sigma(3)} \cdots F_{n\sigma(n)}]_0.
\end{aligned}
$$

But, for all $i \ge 3$, if $\sigma(i) \ne i$, $[F_{3\sigma(3)} \cdots F_{n\sigma(n)}]_0 = 0$ since $F_{i\sigma(i)}$ is a linear form. Moreover, $[F_{33} \cdots F_{nn}]_0 = 1$. One easily deduces that $q = F_{11}F_{22} - F_{21}F_{12}$; and so that $r \le 4$.

Let us now assume that $k = 1$. In the same way as above, one easily checks that $[\det \circ F]_1 = F_{11}$; and so that $F_{11} = 0$. If $\sigma \in \Sigma_n$, we set $P_\sigma = [F_{2\sigma(2)} \cdots F_{n\sigma(n)}]_1$. We have

$$
q = \det \circ F = \sum_{\sigma \in \Sigma_n} F_{1\sigma(1)} P_\sigma. \tag{i}
$$

If there exists two $i \ge 2$ such that $\sigma(i) \ne i$ then $P_\sigma = 0$. On the other hand, if $\sigma(1) = 1$ then $F_{1\sigma(1)} = 0$. Finally, in Sum (i), we can only keep the transpositions $(1, i)$ for $i = 2, \cdots, n$. We obtain $q = \sum_{i=2}^{n} F_{1i}F_{i1}$, and so $r \le 2(n-1)$. $\square$

We can now prove Theorem 1.

PROOF. Since the degree of q is two, $dc(q) \ge 2$. Let us assume that $r \le 4$. Then, by Gauss' Theorem, there exist four linear forms $\varphi_1, \cdots, \varphi_4$ on V such that $q = \varphi_1\varphi_2 + \varphi_3\varphi_4$. In particular, $q = \begin{vmatrix} \varphi_1 & -\varphi_3 \\ \varphi_4 & \varphi_2 \end{vmatrix}$. So, $dc(q) \le 2$ and the theorem follows in this case.

We may now assume that $r \ge 5$. Then, Lemma 2.1 shows that $dc(q) \ge \frac{r}{2} + 1$.

If $r = 2k$ is even, by Gauss' Theorem there exist $2k$ linear forms φ_i and ψ_i such that $q = \varphi_1\psi_1 + \cdots + \varphi_k\psi_k$. In this case, we have:

$$
q = \begin{vmatrix} 0 & \varphi_1 & \cdots & \varphi_k \\ \psi_1 & 1 & & \\ \vdots & & \ddots & \\ \psi_k & & & 1 \end{vmatrix}.
$$

So, $dc(q) \le \frac{r}{2} + 1$ and the theorem follows in this case.

Now, if $r = 2k+1$ is odd, there exist $2k+1$ linear forms φ_i, ψ_i and ρ such that $q = \varphi_1\psi_1 + \cdots + \varphi_k\psi_k + \rho^2$. Then, we have:

$$q = \begin{vmatrix} 0 & \varphi_1 & \cdots & \varphi_k & \rho \\ \psi_1 & 1 & & & \\ \vdots & & \ddots & & \\ \psi_k & & & \ddots & \\ \rho & & & & 1 \end{vmatrix}.$$

So, $dc(q) \leq k+2 = \left\lceil \frac{r}{2} + 1 \right\rceil$ and the theorem follows in this case. □

3 Determinantal complexity: the Permanent

We assume in this section that the characteristic of $\mathbf{k}$ is zero.

3.1 Second fundamental form and restriction

Let $\mathcal{E}$ be a $\mathbf{k}$-affine space of dimension N. Let E denote the vector space associated to $\mathcal{E}$ and E^* its dual. Let $f : \mathcal{E} \longrightarrow \mathbf{k}$ be a polynomial function.

We consider the tangent map Tf to f:

$$\begin{array}{rcl} Tf : \mathcal{E} & \longrightarrow & E^* \\ x & \longmapsto & T_x f. \end{array}$$

With good coordinates (i.e. by considering a base in E and its dual base in E^*), we have: $Tf = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_N})$. We now consider the tangent map $T^2 f$ to Tf:

$$\begin{array}{rcl} T^2 f : \mathcal{E} & \longrightarrow & \mathrm{Hom}(E, E^*) \\ x & \longmapsto & T_x^2 f. \end{array}$$

If v belongs to E, $T_x^2 f(v) \in E^*$ denotes the evaluation of $T_x^2 f$ at v; if w is another vector of E, $T_x^2 f(v, w)$ denotes the evaluation of $T_x^2 f(v)$ at w. With a good choice of coordinates, we have: $T^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \leq i,j \leq N}$.
At a smooth point x of the zero locus of f, $T_x f$ is the Gauss map, and $T_x^2 f$ is the second fundamental form of this hypersurface.

Let $\mathcal{F}$ be an affine subspace of $\mathcal{E}$ and g denote the restriction of f to $\mathcal{F}$. We denote by F the vector space associated to $\mathcal{F}$ and F^* its dual. We consider Tg and $T^2 g$ as before.

Lemma 3.1 *For all $x \in \mathcal{F}$, the rank of $T_x^2 g$ is less than those of $T_x^2 f$.*

PROOF. Let $\rho : E^* \longrightarrow F^*$ be the restriction map. Since Tg is the restriction of $\rho \circ Tf$ to $\mathcal{F}$ and ρ is linear, for all $x \in \mathcal{F}$, $T_x^2 g$ is the restriction to F of $\rho \circ T_x^2 f$. The lemma follows. □

3.2 Second fundamental form of the permanent

3.2.1 A general formula

Let G be the universal matrix

$$G = \begin{pmatrix} X_{1,1} & \cdots & X_{1,d} \\ \vdots & & \vdots \\ X_{d,1} & \cdots & X_{d,d} \end{pmatrix} \in M_d(\mathbf{k}[X_{i,j}, 1 \leq i,j \leq d])$$

and $P = \mathrm{Perm}_d G$.

Let i, i', j, j', be four integers between 1 and d, such that $i \neq i'$ and $j \neq j'$. We denote by $G_{i,j}$ the submatrix of G obtained by omitting the i-th line and j-th column. We denote by $G_{\{i,i'\},\{j,j'\}}$ the submatrix of G obtained by omitting the two lines i, i' and the two columns j, j'. We also define polynomials $P_{i,j}$ and $P_{\{i,i'\},\{j,j'\}}$ as follows:

$$\begin{array}{rcl} P_{i,j} & = & \mathrm{Perm}_{d-1}(G_{i,j}) \\ P_{\{i,i'\},\{j,j'\}} & = & \left\{ \begin{array}{cc} 0 & \text{if } i = i' \text{ or } j = j' \\ \mathrm{Perm}_{d-2}(G_{\{i,i'\},\{j,j'\}}) & \text{else.} \end{array} \right. \end{array}$$

Let e_{ij} denote the $d \times d$-matrix with coefficient 1 at the entry (i,j) and 0 anywhere else. We call $(e_{11}, \cdots, e_{1n}, e_{21}, \cdots, e_{2n}, \cdots, e_{nn})$ the canonical base of $\mathrm{M}_n(\mathbf{k})$, and its dual the canonical base of $\mathrm{M}_n(\mathbf{k})^*$.

Lemma 3.2 *Let J be the matrix of $T^2\mathrm{Perm}_d$ in the canonical bases of $\mathrm{M}_n(\mathbf{k})$ and $\mathrm{M}_n(\mathbf{k})^*$. Then, the matrix J is symmetric and:*

$$J = \begin{pmatrix} 0 & J_{1,2} & \cdots & J_{1,d} \\ J_{1,2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & J_{d-1,d} \\ J_{1,d} & \cdots & J_{d-1,d} & 0 \end{pmatrix}$$

where $J_{i,i'}$ is the following symmetric matrix of size $d \times d$:

$$J_{i,i'} = \begin{pmatrix} 0 & P_{\{i,i'\},\{1,2\}} & \cdots & P_{\{i,i'\},\{1,d\}} \\ P_{\{i,i'\},\{2,1\}} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & P_{\{i,i'\},\{d-1,d\}} \\ P_{\{i,i'\},\{d,1\}} & \cdots & P_{\{i,i'\},\{d,d-1\}} & 0 \end{pmatrix}.$$

PROOF. We have to prove that, for all i, i', j, j' between 1 and d:

$$\frac{\partial^2 P}{\partial X_{i,j} \partial X_{i',j'}} = P_{\{i,i'\},\{j,j'\}}.$$

By expanding along the i^{th} line, one easily checks that $\frac{\partial P}{\partial X_{ij}}$ equals P_{ij}. In particular, if $i = i'$ or $j = j'$, $\frac{\partial P}{\partial X_{ij}}$ is independent of $X_{i'j'}$ and $\frac{\partial^2 P}{\partial X_{i,j} \partial X_{i',j'}} = 0 = P_{\{i,i'\},\{j,j'\}}$. If $i \neq i'$ and $j \neq j'$, the same computation as above shows that $\frac{\partial^2 P}{\partial X_{i,j} \partial X_{i',j'}} = P_{\{i,i'\},\{j,j'\}}$. $\square$

3.2.2 Evaluation at a special point

We assume here that $d \geq 3$. Consider the following $d \times d$-matrix:

$$A = \begin{pmatrix} 1-d & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

The goal of this subsection is

Proposition 3.1 *The permanent of A equals zero and the rank of $T_A^2\mathrm{Perm}_d$ equals d^2.*

Let us start with some computation. Let N_k be the $k \times k$-matrix with all coefficients equal to 1. Set $n_k = \mathrm{Perm}_k(N_k)$.

Lemma 3.3 *We have:*

$$\begin{aligned} n_k &= k! && \text{(ii)} \\ \mathrm{Perm}_d(A) &= 0 && \text{(iii)} \end{aligned}$$

PROOF. By expanding along the first line, one obtains $n_k = kn_{k-1}$. Equality (ii) follows by an immediate induction.

By expanding along the first line, one obtains $\mathrm{Perm}_d(A) = (1-d)n_{d-1} + (d-1)n_{d-1}$. The second equality follows. □

Lemma 3.4 *Let i, i', j, j' be four integers between 1 and d such that $i \neq i'$ and $j \neq j'$. We have:*

$$P_{i,j}(A) = \begin{cases} (d-1)! & \text{if } 1 \in \{i,j\} \\ -(d-2)! & \text{else} \end{cases} \qquad \text{(iv)}$$

$$P_{\{i,i'\},\{j,j'\}}(A) = \begin{cases} (d-2)! & \text{if } 1 \in \{i,j,i',j'\} \\ -2(d-3)! & \text{else} \end{cases} \qquad \text{(v)}$$

PROOF. *Computation of $P_{i,j}(A)$:* If $1 \in \{i,j\}$, $P_{ij}(A) = \mathrm{Perm}_{d-1}(N_{d-1}) = (d-1)!$. Else, by expanding along the first line one obtains $P_{ij}(A) = (1-d)n_{d-2} + (d-2)n_{d-2} = -(d-2)!$.

Computation of $P_{\{i,i'\},\{j,j'\}}(A)$: If $1 \in \{i,j,i',j'\}$, $P_{\{i,i'\},\{j,j'\}}(A)$ is the permanent of N_{d-2} and the lemma follows in this case. Else, by expanding along the first line one obtains $P_{\{i,i'\},\{j,j'\}}(A) = (1-d)n_{d-3} + (d-3)n_{d-3}$. □

Lemmas 3.2, 3.3 and 3.4 allow us to compute easily the matrix $J(A)$ of $T_A^2\mathrm{Perm}_d$:

Lemma 3.5 *With above notation, we have:*

$$J(A) = (d-3)! \begin{pmatrix} 0 & B & B & \dots & B \\ B & 0 & C & \dots & C \\ B & C & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & C \\ B & C & \cdots & C & 0 \end{pmatrix}$$

where B and C are the following matrices of size $d \times d$:

$$B = (d-2) \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} C = \begin{pmatrix} 0 & d-2 & d-2 & \dots & d-2 \\ d-2 & 0 & -2 & \dots & -2 \\ d-2 & -2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -2 \\ d-2 & -2 & \cdots & -2 & 0 \end{pmatrix}.$$

To show Proposition 3.1, it remains to prove that $J(A)$ is invertible. For this, we will use the following

Lemma 3.6 *Let Q, R be two invertible matrices of size $a \times a$ ($a \in \mathbb{N}$) and $b \in \mathbb{N}$. Then, the matrix*

$$M = \begin{pmatrix} 0 & Q & Q & \dots & Q \\ Q & 0 & R & \dots & R \\ Q & R & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & R \\ Q & R & \cdots & R & 0 \end{pmatrix},$$

of size $ab \times ab$ is invertible.

Proof. By multiplying on the left and on the right by the matrix diagonal by blocks with diagonal $(Q^{-1}, I, \cdots, I)$ (I denotes the identity matrix of size $a \times a$), we may assume that $Q = I$. Let $\begin{pmatrix} U_1 \\ \vdots \\ U_b \end{pmatrix}$ a vector of the kernel of M (each U_i is a column vector of size a). We have:

$$\begin{cases} & U_2 + \cdots + U_b & = & 0 \\ U_1 \quad + & RU_3 + \cdots + RU_b & = & 0 \\ & \cdots & & \\ U_1 \quad + & RU_2 + \cdots + RU_{b-1} & = & 0, \end{cases}$$

and so

$$\begin{cases} U_2 + \cdots + U_b & = & 0 \\ U_1 - RU_2 & = & 0 \\ & \cdots & \\ U_1 - RU_b & = & 0 \end{cases}$$

By multiplying the first line by R one obtains that $(b-1)U_1 = 0$, and since the characteristic is zero, one obtains $U_1 = 0$. Now, the following lines imply that $U_2 = \cdots = U_b = 0$. $\square$

Proof.[of Proposition 3.1] We apply Lemma 3.6 three times: firstly to obtain that the matrices B and C are invertible, and to the matrix $J(A)$. $\square$

3.3 Second fundamental form of determinant

Proposition 3.2 *For any non invertible matrix $A \in \mathrm{M}_n(\mathbf{k})$, the rank of $T^2_A \det_n$ is less than or equal to $2n$.*

Proof. Let A be as in the proposition. Let P and Q be two invertible $n \times n$-matrices. Since the map $\mathrm{M}_n(\mathbf{k}) \longrightarrow \mathrm{M}_n(\mathbf{k})$, $B \longrightarrow PBQ^{-1}$ multiply the determinant by a non zero constant (namely, $\det_n(P)\det_n(Q)^{-1}$), the rank of $T^2_A \det_n$ equals those of $T^2_{PAQ^{-1}} \det_n$. So, we may assume that A is a diagonal matrix with diagonal of the form $(0, \cdots, 0, 1, \cdots, 1)$. Now, we achieve the proof by computations analogue with that made in Section 3.2.1. $\square$

3.4 The Permanent and Determinant problem

Here comes our main result:

Theorem 2 *We have $dc(\mathrm{Perm}_d) \geq \frac{d^2}{2}$.*

Let $F : \mathrm{M}_d(\mathbf{k}) \longrightarrow \mathrm{M}_n(\mathbf{k})$ be an affine map such that $\mathrm{Perm}_d = \det_n \circ F$. We have to prove that $n \geq \frac{d^2}{2}$.

Firstly, by using the second fundamental form, we obtain a new proof of a lemma of Jin-Yi Cai (see [3, p125]):

Lemma 3.7 *With above notation, F is injective.*

Proof. By absurd, we assume that there exists a non zero vector $v \in \mathrm{M}_d(\mathbf{k})$ in the kernel of the linear part of F. For all $x \in \mathrm{M}_d(\mathbf{k})$ and $t \in \mathbf{k}$, we have:

$$\mathrm{Perm}_d(x + tv) = \det{}_n \circ F(x + tv) = \det{}_n \circ F(x) = \mathrm{Perm}_d(x).$$

So, $T_x\mathrm{Perm}_d$ evaluated to v equals zero; in other words, $T_x\mathrm{Perm}_d$ belongs to $H := \{\varphi \in \mathrm{M}_d(\mathbf{k})^* \ : \ \varphi(v) = 0\}$. Since v is non zero, H is an hyperplane in $\mathrm{M}_d(\mathbf{k})^*$. So, for all $y \in \mathrm{M}_d(\mathbf{k})$ the image of $T^2_y\mathrm{Perm}_d \in \mathrm{Hom}(\mathrm{M}_d(\mathbf{k}), \mathrm{M}_d(\mathbf{k})^*)$ is contained in H; in particular the rank of $T^2_y\mathrm{Perm}_d$ is less than $d^2 - 1$. This contradicts Proposition 3.1. $\square$

Proof.[of Theorem 2] Set $\mathcal{F}$ denote the image of F. By Lemma 3.7, the restriction g of $\det_n$ at $\mathcal{F}$ is affinely isomorphic to Perm_d. So, by Proposition 3.1, there exists x in $\mathcal{F}$ such that the rank of $T^2_x g$ equals d^2. But, by Lemmas 3.1, this rank is less than or equal to those of $T^2_x \det_n$ which is by Proposition 3.2 less than or equal to $2n$. Finally, we have $d^2 \leq 2n$. $\square$

3.5 The symmetric permanent

Consider the polynomial SPerm_d in $\frac{d(d+1)}{2}$ variables obtained from Perm_d by restriction to the symmetric matrices. We have:

Theorem 3 *We have,* $dc(\mathrm{SPerm}_d) \geq \frac{d(d+1)}{4}$.

We prove Theorem 3 exactly as Theorem 2 with the following proposition in place of Proposition 3.1.

Proposition 3.3 *The permanent of A equals zero and the rank of $T_A^2\mathrm{SPerm}_d$ equals $\frac{d(d+1)}{2}$.*

PROOF. Let Sym_d (resp. ASym_d) denote the set of symmetric (resp. antisymmetric) matrices of size $d \times d$. We claim that ASym_d is the orthogonal $\mathrm{Sym}_d^\perp$ of Sym_d for the bilinear form $T_A^2\mathrm{Perm}_d$ on $\mathrm{M}_d(\mathbf{k})$.

Firstly, we notice that

$$\begin{aligned} & T_A^2\mathrm{Perm}_d(e_{ij}+e_{ji}, e_{kl}-e_{lk}) \\ = \; & P_{\{ik\}\{jl\}}(A) + P_{\{jk\}\{il\}}(A) - P_{\{jl\}\{ik\}}(A) - P_{\{il\}\{jk\}}(A) \\ = \; & 0. \end{aligned}$$

Indeed, since A is symmetric, we have $P_{\{ik\}\{jl\}}(A) = P_{\{jl\}\{ik\}}(A)$ and $P_{\{jk\}\{il\}}(A) = P_{\{il\}\{jk\}}(A)$. This prove that ASym_d is contained in $\mathrm{Sym}_d^\perp$. But, by Proposition 3.1, the dimensions of these vector subspaces are equal. The claim follows.

The claim implies in particular that $\mathrm{Sym}_d \cap \mathrm{Sym}_d^\perp = \{0\}$. The proposition is proved since the bilinear form $T_A^2\mathrm{SPerm}$ is the restriction of $T_A^2\mathrm{Perm}$ to Sym_d. $\square$

Bibliography

[1] A. BEAUVILLE – "Determinantal hypersurfaces", *Michigan Math. J.* **48** (2000), p. 39–64, Dedicated to William Fulton on the occasion of his 60th birthday.

[2] M. BRUNDU & A. LOGAR – "Parametrization of the orbits of cubic surfaces", *Transform. Groups* **3** (1998), no. 3, p. 209–239.

[3] J.-Y. CAI – "A note on the determinant and permanent problem", *Inform. and Comput.* **84** (1990), no. 1, p. 119–127.

[4] D. EISENBUD, F.-O. SCHREYER & J. WEYMAN – "Resultants and Chow forms via exterior syzygies", *J. Amer. Math. Soc.* **16** (2003), no. 3, p. 537–579 (electronic).

[5] J. von zur Gathen – "Permanent and determinant", *Linear Algebra Appl.* **96** (1987), p. 87–100.

[6] M. H. Lim – "A note on the relation between the determinant and the permanent", *Linear and Multilinear Algebra* **7** (1979), no. 2, p. 145–148.

[7] M. Marcus & H. Minc – "On the relation between the determinant and the permanent", *Illinois J. Math.* **5** (1961), p. 376–381.

[8] K. D. Mulmuley & M. Sohoni – "Geometric complexity theory. I. An approach to the P vs. NP and related problems", *SIAM J. Comput.* **31** (2001), no. 2, p. 496–526 (electronic).

[9] G. Polya – "Aufgabe 424", *Archiv der Mathematik und Physik (3)* **20** (1913), p. 271.

[10] G. Szegö – "Losung zu 424", *Archiv der Mathematik und Physik (3)* **21** (1913), p. 291–292.

[11] L. G. Valiant – "Completeness classes in algebra", in *Conference Record of the Eleventh Annual ACM Symposium on Theory of Computing (Atlanta, Ga., 1979)*, ACM, New York, 1979, p. 249–261.

[12] — , "The complexity of computing the permanent", *Theoret. Comput. Sci.* **8** (1979), no. 2, p. 189–201.

[13] — , "Reducibility by algebraic projections", in *Logic and algorithmic (Zurich, 1980)*, Monograph. Enseign. Math., vol. 30, Univ. Genève, Geneva, 1982, p. 365–380.

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