GIT-cones and quivers

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Abstract

In this work, we improve results of [Res07, Res08a] about GIT-cones associated to the action of a reductive group G on a projective variety X. These results are applied to give a short proof of the Derksen-Weyman theorem which parametrizes bijectively the faces of a rational cone associated to any quiver without oriented cycle. An important example of such a cone is the Horn cone.

Contents

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1 Introduction

We work over the field $\mathbb{C}$ of complex numbers. Let $Q = (Q_0, Q_1)$ be a quiver without oriented cycle. Here, Q_0 is the set of vertexes and Q_1 the set of arrows. Let $\beta = (\beta(s))_{s\in Q_0}$ be a vector dimension of Q and $\mathrm{Rep}(Q,\beta)$ be the vector space of the representations of dimension vector β. The group $\mathrm{GL}(\beta) = \prod_{s\in Q_0} \mathrm{GL}(\beta(s))$ acts naturally on $\mathrm{Rep}(Q,\beta)$. We consider the group Γ of the characters of $\mathrm{GL}(\beta)$; it is isomorphic to $\mathbb{Z}^{Q_0}$. We consider the cone $\Sigma(Q,\beta)$ in $\Gamma\otimes\mathbb{Q}$ generated by the elements $\sigma\in\Gamma$ such that there exists a non zero regular function $f\in\mathbb{C}[\mathrm{Rep}(Q,\beta)]$ such that $g.f = \sigma(g)f$ for any $g\in\mathrm{GL}(\beta)$. It is a convex polyhedral cone. In [DW00, DW06], Derksen-Weyman showed that the Horn cones can be obtained in such a way from well chosen quiver and vector dimension. This is an important motivation to the study $\Sigma(Q,\beta)$. Here, we use general methods of Geometric Invariant Theory to give a proof of the Derksen-Weyman theorem (see [DW06]) which parametrizes bijectively the faces of $\Sigma(Q,\beta)$.

In Section 2, we improve results of [Res07] about GIT-cones in general. In particular, Theorem 1 is an improvement of [Res07, Theorem 4], and Theorem 2 of [Res07, Theorem 7]. After each statement we have include a remark to make clear the improvement. It was very interesting for myself to see that the example $\Sigma(Q,\beta)$ enlightens phenomenons which did not occur in the example studied in [Res07, Res08a].

The Horn cones can also be obtained as GIT-cones for the action of the linear group on a product of complete flag varieties. This point of view was used in [BS00, BK06, Res07, Res08a]. Whereas, in the literature this GIT approach of the Horn problem was distinct from the quiver one, this work intends to show that the GIT-cones are a very useful generalization of these two approaches.

It may be not so easy to see the differences between the proof presented here and the Derksen-Weyman one. The most obvious one is the context in which the technical part of the work is made. The author guess that the general context of GIT makes more clear the key argues. A most fundamental difference is that here and in [Res07] the points outside the cone play a crucial role. Moreover, we do not use here the fact that a base of the space of semiinvariant functions on $\mathrm{Rep}(Q,\beta)$ is known.

The Fulton conjecture about the Littlewood-Richardson coefficients is as follows

$$c_{\lambda\,\mu}^{\nu} = 1 \Rightarrow c_{N\lambda\,N\mu}^{N\nu} = 1 \ \ \forall N \geq 1.$$

The generalization to quiver setting is $\alpha \circ \beta = 1 \Rightarrow \alpha \circ (N\beta) = 1$. This generalization is proved in [DW06] and I cite, "is crucial for that paper". Here, this result is not used. In contrast, Theorem 3 and the method used in [Res08c] allow to reprove this result.

2 Well covering pairs and GIT-cones

2.1 Well covering pairs

Let G be a connected reductive group acting on a smooth projective variety X. Let λ be a one parameter subgroup of G. Let G^λ denote the centralizer of λ in G. We consider the usual parabolic subgroup $P(\lambda)$ associated to λ with Levi subgroup G^λ:

$$P(\lambda) = \left\{ g \in G \,:\, \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Let C be an irreducible component of the fixed point set X^λ of λ in X. We also consider the Białynicki-Birula cell C^+ associated to C:

$$C^+ = \{x \in X \mid \lim_{t \to 0} \lambda(t)x \in C\}.$$

Then, C is stable by the action of G^λ and C^+ by the action of $P(\lambda)$.

Consider over $G \times C^+$ the action of $G \times P(\lambda)$ given by the formula (with obvious notation): $(g,p).(g',y) = (gg'p^{-1}, py)$. Consider the quotient $G \times_{P(\lambda)} C^+$ of $G \times C^+$ by the action of $\{e\} \times P(\lambda)$. The class of a pair $(g,y) \in G \times C^+$ in $G \times_{P(\lambda)} C^+$ is denoted by $[g:y]$.

The action of $G \times \{e\}$ induces an action of G on $G \times_{P(\lambda)} C^+$. Moreover, the first projection $G \times C^+ \longrightarrow G$ induces a G-equivariant map $\pi \,:\, G \times_{P(\lambda)} C^+ \longrightarrow G/P(\lambda)$ which is a locally trivial fibration with fiber C^+. Consider also the G-equivariant map

$$\eta \,:\, G \times_{P(\lambda)} C^+ \longrightarrow X,\ [g:y] \mapsto gy.$$

Definition. The pair (C, λ) is said to be *dominant* if η is. The pair (C, λ) is said to be *well covering* if η induces an isomorphism over an open subset of X intersecting C.

Let $\mathcal{L} \in \mathrm{Pic}^G(X)$. Let x be any point in C. Since λ fixes x, it induces a linear action of the group $\mathbb{K}^*$ on the fiber $\mathcal{L}_x$. There exists an integer $\mu^{\mathcal{L}}(x, \lambda)$ such that:

$$\forall \tilde{x} \in \mathcal{L}_x \quad \forall t \in \mathbb{C}^* \quad \lambda(t)\tilde{x} = t^{-\mu^{\mathcal{L}}(x,\lambda)}\tilde{x}.$$

One easily checks that $\mu^{\mathcal{L}}(x, \lambda)$ does not depends on $x \in C$; it will be denoted by $\mu^{\mathcal{L}}(C, \lambda)$.

2.2 Total cones and well covering pair

Notation. If Γ is an abelian group, we set $\Gamma_{\mathbb{Q}} = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. If Γ is any algebraic group, we denote by $Y(\Gamma)$ the set of one parameter subgroups of Γ. If $\mathcal{F}$ is a part of a vector space E we denote by $\mathrm{Span}(\mathcal{F})$ the subspace of E spanned by $\mathcal{F}$.

2.2.1 — Consider the convex cones $\mathcal{TC}^G(X)$ generated in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ by the $\mathcal{L}$'s in $\mathrm{Pic}^G(X)$ which have non zero G-invariant sections. We will denote by $X^{\mathrm{ss}}(\mathcal{L})$ the open subset of the x's in X such that for some positive integer n, there exists a G-invariant section of $\mathcal{L}^{\otimes n}$ such that $\sigma(x) \neq 0$. Note that this definition is standard, only if $\mathcal{L}$ is ample. Since $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{ss}}(\mathcal{L}^{\otimes n})$ (for any positive integer n), one can define $X^{\mathrm{ss}}(\mathcal{L})$ if $\mathcal{L} \in \mathrm{Pic}^G(X)_{\mathbb{Q}}$.

Let (C, λ) be a dominant pair. Since $\mathcal{L} \mapsto \mu^{\mathcal{L}}(C, \lambda)$ is a group morphism, it induces a linear map from $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ to $\mathbb{Q}$, also denoted by $\mu^{\mathcal{L}}(C, \lambda)$. By [Res07, Lemma 4], $\mathcal{TC}^G(X)$ is contained in the half-space $\mu^{\mathcal{L}}(C, \lambda) \leq 0$. In particular, intersecting $\mathcal{TC}^G(X)$ with the hyperplane $\mu^{\mathcal{L}}(C, \lambda) = 0$, one obtains a face $\mathcal{F}(C)$ of $\mathcal{TC}^G(X)$. Indeed, the following lemma shows that the face only depends on C (see [Res07, Lemma 4]):

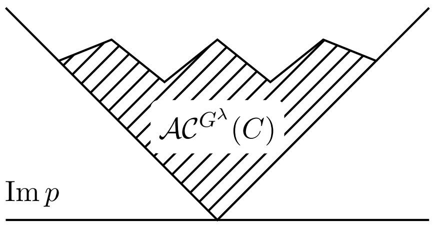
Lemma 1 *Let (C, λ) be a dominant pair. Then, $\mathcal{F}(C)$ is the set of $\mathcal{L} \in \mathrm{Pic}^G(X)_{\mathbb{Q}}$ such that $X^{\mathrm{ss}}(\mathcal{L})$ intersects C.*

The following theorem is an improvement of [Res07, Theorem 4].

Theorem 1 *We assume that the rank of $\mathrm{Pic}^G(X)$ is finite and consider $\mathcal{TC}^G(X)$. Let (C, λ) be a well covering pair.*

Then the rank of $\mathrm{Pic}^{G^\lambda}(C)$ is finite and the codimension of $\mathcal{F}(C)$ in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ equals the codimension of $\mathcal{TC}^{G^\lambda}(C)$ in $\mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}$. More precisely, the restriction morphism induces an isomorphism from $\mathrm{Pic}^G(X)_{\mathbb{Q}}/\mathrm{Span}(\mathcal{F}(C))$ onto $\mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}/\mathrm{Span}(\mathcal{TC}^{G^\lambda}(C))$.

Remark. Let $p : \mathrm{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}$ be the restriction morphism and let $\mathrm{Im}\, p$ denote its image. At first glance, it is not so obvious to see why Theorem 1 improves [Res07, Theorem 4]. In particular, if p is surjective (as in the example considered in [Res07]) the two statements are equivalent. The main difference is that the first one allows the following picture in $\mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}$ and second one no:



Proof. Let Ω be a G-stable open subset of X such that the natural map $G \times_{P(\lambda)} (C^+ \cap \Omega) \longrightarrow \Omega$ is an isomorphism. Since (C, λ) is well covering one can find such an Ω intersecting C. By [Res07, Lemma 5], $\mathrm{Pic}^G(\Omega)$ is isomorphic to $\mathrm{Pic}^{G^\lambda}(C \cap \Omega)$.

Let $E_1, \cdots, E_s$ (resp. $D_1, \cdots, D_t$) be the irreducible components of codimension one of $X - \Omega$ (resp. $C - \Omega$). Since G and G^λ are connected the E_i's and the D_i's are respectively G and G^λ-stable. We consider the associated G and G^λ-linearized line bundles $\mathcal{L}_{E_i}$ and $\mathcal{L}_{D_i}$.

Consider the following diagram:

$$\begin{tikzcd}
\oplus_i\mathbb{Q}\mathcal{L}_{E_i} \arrow[r] & \mathrm{Pic}^G(X)_{\mathbb{Q}} \arrow[dd] \arrow[dr, "\pi_X"] & & \\
& & \mathrm{Pic}^G(\Omega)_{\mathbb{Q}} \simeq \mathrm{Pic}^{G^\lambda}(C \cap \Omega)_{\mathbb{Q}} \arrow[r] & 0. \\
\oplus_i\mathbb{Q}\mathcal{L}_{D_i} \arrow[r] & \mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}} \arrow[ur, "\pi_C"'] & &
\end{tikzcd}$$

Since X and so C are smooth, the restriction maps π_X and π_C are surjective. By construction, $\mathcal{L}_{D_i}$ belongs to $\mathcal{TC}^{G^\lambda}(C)$. Moreover, each $\mathcal{L}_{E_i}$ has a G-invariant section with E_i has zero locus. Since E_i does not contains C, this proves that $\mathcal{L}_{E_i} \in \mathcal{F}(C)$. So, it is sufficient to prove that $\pi_X(\mathcal{F}(C)) = \pi_C(\mathcal{TC}^{G^\lambda}(C))$.

Let $\mathcal{L} \in \mathcal{F}(C)$. Since $X^{\rm ss}(\mathcal{L})$ intersects C, $\mathcal{L}_{|C}$ belongs to $\mathcal{TC}^{G^\lambda}(C)$. So, $\pi_X(\mathcal{F}(C)) \subset \pi_C(\mathcal{TC}^{G^\lambda}(C))$.

Conversely, let $\mathcal{L}$ be a G^λ-linearized line bundle on C which belongs to $\mathcal{TC}^{G^\lambda}(C)$. Up to changing $\mathcal{L}$ by a positive power, there exists a non zero G^λ-invariant section σ of $\mathcal{L}$. Let $\tilde{\mathcal{L}}$ be the G-linearized line bundle on Ω associated to $\pi_C(\mathcal{L})$, and $\tilde{\sigma}$ the G-invariant section of $\tilde{\mathcal{L}}$ associated to σ. Now, let $\mathcal{M} \in \mathrm{Pic}^G(X)$ such that $\pi_X(\mathcal{M}) = \tilde{\mathcal{L}}$. The section $\tilde{\sigma}$ induces a non zero G-invariant rational section of $\mathcal{M}$, and so a non zero regular G-invariant section σ' of $\mathcal{M}' = \mathcal{M} + \sum_i \mathcal{L}_{E_i}^{\otimes n_i}$ for some non negative integers n_i. Since no E_i contains C, σ' is not identically zero on C; in particular, $\mathcal{M}' \in \mathcal{F}(C)$. Since, $\pi_X(\mathcal{M}') = \pi_X(\mathcal{M}) = \pi_C(\mathcal{L})$, it follows that $\pi_X(\mathcal{F}(C)) \supset \pi_C(\mathcal{TC}^{G^\lambda}(C))$. Note that details about the above argue can be found in the proof of [Res07, Theorem 4]. $\square$

2.2.2 — We are now interested in the span of $\mathcal{AC}^G(X)$. Let H be a connected normal subgroup of G acting trivially on X. Then, H has to act trivially on any line bundle in $\mathcal{TC}^G(X)$. We denote by $\mathrm{Pic}^{G/H}(X)$ the subgroup consisting of the $\mathcal{L} \in \mathrm{Pic}^G(X)$ on which H acts trivially. We have: $\mathcal{AC}^G(X) \subset \mathrm{Pic}^{G/H}(X)_{\mathbb{Q}}$. Consider now:

Definition. Let K be the neutral component of the kernel of the G-action on X. The cone $\mathcal{AC}^G(X)$ is said to be non degenerated if it spans $\mathrm{Pic}^G(X)_{\mathbb{Q}}$.

2.2.3 — Whereas $Y(\Gamma)$ is not a group, one can construct (see [MFK94]) a map $\|\cdot\| : Y(G) \longrightarrow \mathbb{R}$ which is invariant by conjugacy and measures the length of λ. Moreover precisely, for any subtorus S of G the restriction of $\|\cdot\|$ to $Y(T)_{\mathbb{Q}}$ is the norm associated to a scalar product defined on $\mathbb{Q}$. This norm is used to normalize the number $\mu^{\mathcal{L}}(x,\lambda)$. Using $\frac{\mu^{\mathcal{L}}(x,\lambda)}{\|\lambda\|}$, Kempf defined (see [Kem78]) an optimal destabilizing one parameter subgroup.

2.2.4 — The set of ample G-linearized line bundles generate an open convex $\mathrm{Pic}^G(X)^+_{\mathbb{Q}}$ in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$. We set:

$$\mathcal{AC}^G(X) = \mathrm{Pic}^G(X)^+_{\mathbb{Q}} \cap \mathcal{TC}^G(X).$$

If $\mathcal{F}$ is a face of $\mathcal{TC}^G(X)$, $\mathcal{F}^0$ denotes its intersection with $\mathrm{Pic}^G(X)^+_{\mathbb{Q}}$.

To any ample $\mathcal{L} \in \mathrm{Pic}^G(X)_{\mathbb{Q}}$ which does not belong to $\mathcal{AC}^G(X)$, using mainly the Kempf theorem, we associated in [Res07] a well covering pair (C, λ) defined up to conjugacy (and depending on $\| \cdot \|$). The face $\mathcal{F}(C)$ is also denoted by $\mathcal{F}(\mathcal{L})$. [Res07, Theorem 7] asserts that any face of $\mathcal{AC}^G(X)$ equals $\mathcal{F}^\circ(\mathcal{L})$ for some ample $\mathcal{L} \notin \mathcal{AC}^G(X)$. Here, we need an improvement of this result.

Theorem 2 *Let $\mathcal{F}$ be a face of $\mathcal{AC}^G(X)$. Consider the set $\Delta(\mathcal{F})$ of ample $\mathcal{L} \notin \mathcal{AC}^G(X)$ such that $\mathcal{F} = \mathcal{F}^\circ(\mathcal{L})$.*

Then,

(i) There exists $\mathcal{L} \in \Delta(\mathcal{F})$ such that the associated pair (C, λ) has a non degenerated cone $\mathcal{AC}^{G^\lambda}(C)$.

(ii) $\Delta(\mathcal{F})$ has non empty interior in $\mathrm{Pic}^G(X)_{\mathbb{Q}}$.

Remark. [Res07, Theorem 7] asserts that any $\Delta(\mathcal{F})$ is not empty. It is clear that Assertion (ii) of Theorem 2 improves this statement. The first assertion is useful to elimenate unuseful inequalities. Actually, if $X = G/B \times \hat{G}/\hat{B}$ (here $\hat{G}$ is a connected reductive group containing G) as in [Res07], for every well covering pair (C, λ) $\mathcal{AC}^{G^\lambda}(C)$ has non empty interior; and so, this improvement is not useful. We will see that this property of $\mathcal{AC}^{G^\lambda}(C)$ explains the role of the Schur roots relatively to the cones $\Sigma(Q, \beta)$.

Let G be a simple group and P, Q and R three parabolic subgroups of G. Theorem 2 asks for an algorithm to decide if $\mathcal{AC}^G(G/P \times G/Q \times G/R)$ has non empty interior. In type A, Proposition 4 gives a interpretation of the question in terms of the canonical decomposition of a vector dimension of a quiver. In [DW02], Derksen-Weyman gives an algorithm which answers the question. In general, it seems to be unknown.

Proof. Let $\mathcal{L} \in \Delta(\mathcal{F})$ and (C, λ) be an associated well covering pair. Let K^λ denote the neutral component of the kernel of action of G^λ on C.

We start by constructing a kind of projection from $\mathrm{Pic}^{G^\lambda}(C)_{\mathbb{Q}}$ onto $\mathrm{Pic}^{G^\lambda/K^\lambda}(C)_{\mathbb{Q}}$. Let S be a maximal torus of K^λ and $T \supset S$ be a maximal torus of G^λ. Let S' be the subtorus of T such that $Y(S)_{\mathbb{Q}}$ is orthogonal with $Y(S')_{\mathbb{Q}}$ for $\|\cdot\|$. Note that the product induces an isogeny $S \times S' \longrightarrow T$. Let H^λ be the subgroup of G^λ containing S' such that the product induces an isogeny $K^\lambda \times H^\lambda \longrightarrow G^\lambda$. Now, we can identify $\mathrm{Pic}^{G^\lambda/K^\lambda}(C)_{\mathbb{Q}}$ with $\mathrm{Pic}^{H^\lambda}(C)_{\mathbb{Q}}$. In particular, we obtain a restriction map:

$$p : \mathrm{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^{H^\lambda}(C)_{\mathbb{Q}} \simeq \mathrm{Pic}^{G^\lambda/K^\lambda}(C)_{\mathbb{Q}}.$$

By [Res07, Lemma 10], $p(\mathcal{L})$ belongs to $\mathcal{AC}^{H^\lambda}(C)$. Let us first assume that $p(\mathcal{L})$ belongs to the interior of $\mathcal{AC}^{H^\lambda}(C)$ in $\mathrm{Pic}^{H^\lambda}(C)_{\mathbb{Q}}$. The first assertion is automatic. The hyperplane $\mu^\bullet(C,\lambda)=0$ cuts $\mathrm{Pic}^G(X)_{\mathbb{Q}}$ in two half spaces, the one $\mathcal{H}^+$ contains $\mathcal{L}$ and the closure of the other one contains $\mathcal{AC}^G(X)$. [Res07, Lemma 11] asserts that $\Delta(\mathcal{F})$ contains the set of ample $\mathcal{L}'\in\mathcal{H}^+$ such that $p(\mathcal{L}')\in\mathcal{AC}^{H^\lambda}(C)$. This implies the second assertion.

Let Im p denote the image of p. Note that the above proof of Assertion (ii) also works if $p(\mathcal{L})$ belongs to the interior of $\mathcal{AC}^{H^\lambda}(C)\cap \mathrm{Im}\ p$ in Im p. But, in this case, Theorem 1 shows that $\mathcal{AC}^{H^\lambda}(C)$ has non empty interior in $\mathrm{Pic}^{H^\lambda}(C)_{\mathbb{Q}}$; and, the first assertion follows.

We now assume that $\mathcal{AC}^{H^\lambda}(C)\cap \mathrm{Im}\ p$ has non empty interior in Im p. Since $p(\mathcal{L})$ belongs to $\mathcal{AC}^{H^\lambda}(C)\cap\mathrm{Im}\ p$, one can find a neighbor $\mathcal{L}_\epsilon$ of $\mathcal{L}$ such that $p(\mathcal{L}_\epsilon)$ belongs to the interior of $\mathcal{AC}^{H^\lambda}(C)\cap \mathrm{Im}\ p$. Then, $\mathcal{F}(\mathcal{L}_\varepsilon)=\mathcal{F}$ and we are in the preceding case.

We now assume that $\mathcal{AC}^{H^\lambda}(C)\cap\mathrm{Im}\ p$ has empty interior in Im p. One can find a neighbor $\mathcal{L}_\epsilon$ of $\mathcal{L}$ such that $p(\mathcal{L}_\epsilon)$ does not belong to $\mathcal{AC}^{H^\lambda}(C)\cap\mathrm{Im}\ p$; and, such that $\mathcal{F}(p(\mathcal{L}_\epsilon))=\mathcal{AC}^{H^\lambda}(C)$. By [Res07, Lemma 12], $\mathcal{F}(\mathcal{L}_\varepsilon)=\mathcal{F}$. Let $(C_\varepsilon,\lambda_\varepsilon)$ be the pair associated to $\mathcal{L}_\varepsilon$. Up to conjugacy, C_ε is strictly contained in C. Now, one has to restart the proof with $\mathcal{L}_\varepsilon$. The procedure will finish since C_ε is strictly contained in C. □

3 Application to quiver representations

3.1 Definitions

In this section, we fix some classical notation about quiver representations.

Let Q be a quiver (that is, a finite oriented graph) with vertexes Q_0 and arrows Q_1. An arrow $a\in Q_1$ has initial vertex ia and terminal one ta. A representation R of Q is a family $(V(s))_{s\in Q_0}$ of finite dimensional vector spaces and a family of linear maps $u(a)\in\mathrm{Hom}(V(ia),V(ta))$ indexed by $a\in Q_1$. The dimension vector of R is the family $(\dim(V(s)))_{s\in Q_0}\in\mathbb{N}^{Q_0}$.

Let us fix $\alpha\in\mathbb{N}^{Q_0}$ and a vector space $V(s)$ of dimension $\alpha(s)$ for each $s\in Q_0$. Set
$$\mathrm{Rep}(Q,\alpha)=\bigoplus_{a\in Q_1}\mathrm{Hom}(V(ia),V(ta)).$$

Consider also the groups:

$$\mathrm{GL}(\alpha) = \prod_{s\in Q_0} \mathrm{GL}(V(s)) \text{ and } \mathrm{SL}(\alpha) = \prod_{s\in Q_0} \mathrm{SL}(V(s).$$

They acts naturally on $\mathrm{Rep}(Q,\alpha)$.

The character group of $\mathrm{GL}(\alpha)$ identifies with $\Gamma = \mathbb{Z}^{Q_0}$; to $\sigma \in \mathbb{Z}^{Q_0}$, we associate the character χ_σ defined by $\chi_\sigma\left((g(s))_{s\in Q_0}\right) = \prod_{s\in Q_0} \det(g(s))^{\sigma(s)}$.

3.2 Three cones

3.2.1 — Consider the algebra $\mathbb{C}[\mathrm{Rep}(Q,\alpha)]$ of the regular functions on $\mathrm{Rep}(Q,\alpha)$ endowed with the $\mathrm{GL}(\alpha)$-action. For $\sigma \in \mathbb{Z}^{Q_0}$, we denote $\mathbb{C}[\mathrm{Rep}(Q,\alpha)]_\sigma$ the set of $f \in \mathbb{C}[\mathrm{Rep}(Q,\alpha)]$ such that for all $g \in \mathrm{GL}(\alpha)$, $g.f = \chi_\sigma(g)f$. We embed $\Gamma = \mathbb{Z}^{Q_0}$ in $\Gamma_{\mathbb{Q}} := \mathbb{Q}^{Q_0}$. Let $\Sigma(Q,\alpha)$ denote the convex cone of $\Gamma_{\mathbb{Q}}$ generated by the points $\sigma \in \Gamma$ such that $\mathbb{C}[\mathrm{Rep}(Q,\alpha)]_{-\sigma}$ is non reduced to $\{0\}$.

3.2.2 — Consider the projective space $X = \mathbb{P}(\mathrm{Rep}(Q,\alpha) \oplus \mathbb{C})$. The formula

$$g.(R,t) = (gR,t) \quad \forall g \in \mathrm{GL}(\alpha)\,, R \in \mathrm{Rep}(Q,\alpha) \text{ and } t \in \mathbb{C},$$

defines an action of $\mathrm{GL}(\alpha)$ on X and a $\mathrm{GL}(\alpha)$-linearization $\mathcal{L}_0 \in \mathrm{Pic}^{\mathrm{GL}(\alpha)}(X)$ of the line bundle $\mathcal{O}(1)$ on X. We are now interested in the GIT-cone $\mathcal{AC}^{\mathrm{GL}(\alpha)}(X)$. Since any line bundle on X admitting non zero sections is ample, $\mathcal{AC}^{\mathrm{GL}(\alpha)}(X) = \mathcal{TC}^{\mathrm{GL}(\alpha)}(X)$.

For $n \in \mathbb{Z}$ and $\sigma \in \Gamma$, set $\mathcal{L}(n,\sigma) = \mathcal{L}_0^{\otimes n} \otimes \sigma \in \mathrm{Pic}^{\mathrm{GL}(\alpha)}(X)$. Note that $\mathcal{L}(n,\sigma) = \mathcal{O}(n)$ as a line bundle. We have the following obvious

Lemma 2 *The map $\mathbb{Z}\times\Gamma \longrightarrow \mathrm{Pic}^{\mathrm{GL}(\alpha)}(X)$, $(n,\sigma) \mapsto \mathcal{L}(n,\sigma)$ is an isomorphism of groups. Moreover, $\mathcal{L}(n,\sigma)$ is ample if and only if n is positive.*

Lemma 2 allows to embed $\mathcal{AC}^{\mathrm{GL}(\alpha)}(X)$ in $\mathbb{Q} \times \Gamma_{\mathbb{Q}}$. Set $\mathcal{P}(Q,\alpha) = \mathcal{AC}^{\mathrm{GL}(\alpha)}(X) \cap \{1\} \times \Gamma_{\mathbb{Q}}$. General properties of $\mathcal{AC}^{\mathrm{GL}(\alpha)}(X)$ imply that $\mathcal{P}(Q,\alpha)$ is closed convex rational and polyhedral. We claim that it is compact. Consider the center Z of $\mathrm{GL}(\alpha)$ and the set Wt of these weights for its action on $\mathrm{Rep}(Q,\alpha) \oplus \mathbb{C}$. One can easily prove that $\mathcal{P}(Q,\alpha)$ is contained in the convex hull of the Wt. Finally, $\mathcal{P}(Q,\alpha)$ is a rational polytope in $\Gamma_{\mathbb{Q}}$. Moreover, $\mathcal{AC}^{\mathrm{GL}(\alpha)}(X)$ is the pointed convex cone generated by $\mathcal{P}(Q,\alpha)$ in such a way the faces of $\mathcal{AC}^{\mathrm{GL}(\alpha)}(X)$ and of $\mathcal{P}(Q,\alpha)$ correspond bijectively.

3.2.3 — We consider $\mathrm{Rep}(Q,\alpha)$ as an open subset of X by $R \mapsto (R,1)$; and we identify the complement with $\mathbb{P}(\mathrm{Rep}(Q,\alpha))$.

Proposition 1 *(i) We have: $X^{\mathrm{ss}}(\mathcal{L}_0) = \mathrm{Rep}(Q,\alpha)$.*

(ii) The point $0 \in \Gamma_{\mathbb{Q}}$ is a vertex of $\mathcal{P}(Q,\alpha)$.

(iii) The cone of $\Gamma_{\mathbb{Q}}$ generated by $\mathcal{P}(Q,\alpha)$ is $\Sigma(Q,\alpha)$.

Proof. Since Q has no oriented cycle, one can chose a numeration of the vertexes such that the index of ta is greater than the index of ia for all $a \in Q_1$. Consider the one parameter subgroup λ_0 of $\mathrm{GL}(\alpha)$ acting on the vector space corresponding to the vertex indexed by i as an homothety of coefficient t^i.

The point $0 \in \mathrm{Rep}(Q,\alpha) \subset X$ is an isolated fixed point of λ_0. Set $C_0 = \{0\}$. One easily checks that $C_0^+ = \mathrm{Rep}(Q,\alpha)$ and that λ_0 is central in $\mathrm{GL}(\alpha)$: it follows that (C_0,λ_0) is a well covering pair. Let $\mathcal{F}(C_0)$ (resp. $\mathcal{P}(C_0)$) denote the face of $\mathcal{TC}^{\mathrm{GL}(\alpha)}(X)$ (resp. $\mathcal{P}(Q,\alpha)$) associated to (C_0,λ_0).

Let $\mathcal{L} \in \mathcal{F}(C_0)$. Then, 0 is semistable for $\mathcal{L}$. But 0 is fixed by $\mathrm{GL}(\alpha)$; in particular its center has to act trivially on the fiber in $\mathcal{L}$ over 0. This implies $\mathcal{F}(C_0)$ is contained in $\mathbb{Q}^+\mathcal{L}_0$.

Since $(R,t) \mapsto t$ is a $\mathrm{GL}(\alpha)$-invariant section of $\mathcal{L}_0$, $\mathrm{Rep}(Q,\alpha) \subset X^{\mathrm{ss}}(\mathcal{L}_0)$. Then, $\mathcal{F}(C_0) = \mathbb{Q}^+\mathcal{L}_0$.

Since 0 is the only closed $\mathrm{GL}(\alpha)$-orbit in $\mathrm{Rep}(Q,\alpha)$, $X^{\mathrm{ss}}(\mathcal{L}_0)/\!/\mathrm{GL}(\alpha)$ is a point. So, $X^{\mathrm{ss}}(\mathcal{L}_0)$ contains only one closed orbit $\mathcal{O}$ which is contained in the closure $\mathrm{GL}(\alpha).0$. We deduce that $\mathcal{O} = \{0\}$ and that $X^{\mathrm{ss}}(\mathcal{L}_0) = \mathrm{Rep}(Q,\alpha)$.

The last assertion of the proposition is a direct application of [Res08a, Theorem 4]. $\square$

3.2.4 — Consider now the projective space $D = \mathbb{P}(\mathrm{Rep}(Q,\alpha))$ endowed with the $\mathrm{GL}(\alpha)$-action. We are now interested in the GIT-cone $\mathcal{AC}^{\mathrm{GL}(\alpha)}(D)$.

We have the following obvious

Lemma 3 *The restriction map $\rho_D : \mathrm{Pic}^{\mathrm{GL}(\alpha)}(X) \longrightarrow \mathrm{Pic}^{\mathrm{GL}(\alpha)}(D)$ is an isomorphism of groups. Moreover, $\rho_D(\mathcal{L})$ is ample if and only if $\mathcal{L}$ is.*

Lemma 3 allows to embed $\mathcal{AC}^{\mathrm{GL}(\alpha)}(D)$ in $\mathbb{Q} \times \Gamma_{\mathbb{Q}}$. Set $\mathcal{P}(D,Q,\alpha) = \mathcal{AC}^{\mathrm{GL}(\alpha)}(D) \cap \{1\} \times \Gamma_{\mathbb{Q}}$. Obviously, $\mathcal{P}(D,Q,\alpha)$ is a rational polytope in $\Gamma_{\mathbb{Q}}$. Via the identification of Lemma 3, the relation between $\mathcal{P}(D,Q,\alpha)$ and $\mathcal{P}(Q,\alpha)$ is as follows:

Proposition 2 *The polytope $\mathcal{P}(Q,\alpha)$ is the convex hull of* 0 *and* $\mathcal{P}(D,Q,\alpha)$.

If in addition Q is a tree then $\mathcal{P}(D,Q,\alpha)$ is a face of $\mathcal{P}(Q,\alpha)$. In particular, $\mathcal{P}(D,Q,\alpha)$ is an affine section of $\Sigma(Q,\alpha)$.

Proof. Let $\sigma \in \Gamma_{\mathbb{Q}}$. It is clear that $\sigma \in \mathcal{P}(D,Q,\alpha)$ if and only if $X^{\rm ss}(\sigma)$ intersects D, if and only if $X^{\rm ss}(\sigma)$ is not contained in $X^{\rm ss}(0)$. By [Res00], this is equivalent to the fact that the closure of the GIT-class of σ does not contain 0. In particular, all the vertexes of $\mathcal{P}(Q,\alpha)$ excepted 0 belong to $\mathcal{P}(D,Q,\alpha)$; the first assertion follows.

With the additional assumption, one can easily construct a central one parameter subgroup $\lambda(t)$ of $G(\alpha)$ which acts on each $\mathrm{Hom}(V(ia),V(ta))$ (for $a \in Q_1$) by multiplication by t. Then, (D,λ) is a well covering pair; this implies that $\mathcal{P}(D,Q,\alpha)$ is a face of $\mathcal{P}(Q,\alpha)$. $\square$

The four last statements can be summarized by the following pictures. On the third picture, Q is assumed to be a tree.

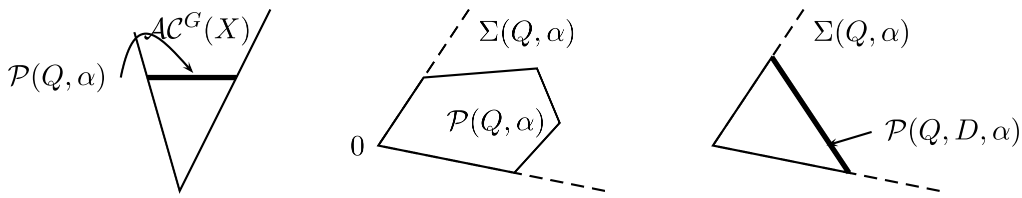


Figure 1: Positions of $\mathcal{AC}^G(X)$, $\Sigma(Q,\alpha)$, $\mathcal{P}(Q,\alpha)$ and $\mathcal{P}(Q,D,\alpha)$

3.2.5 — Propositions 1 and 2 prove that when Q is a tree, the descriptions of the three cones are equivalent. From now on, we are mainly interested in $\Sigma(Q,\alpha)$ viewed as the cone generated by $\mathcal{P}(Q,\alpha)$; that is to the faces of $\mathcal{P}(Q,\alpha)$ containing the vertex 0:

Lemma 4 *Let (C,λ) be a dominant pair. Then, $\mathcal{F}(C)$ contains* 0 *if and only if C contains* 0.

Proof. The point is that $\{0\}$ is the only closed orbit in $X^{\rm ss}(0)$. Actually, if $\mathcal{F}(C)$ contains 0, C has to contains 0 by [Res07, Proposition 9]. Conversely, if C contains 0, $\mu^0(C,\lambda)=0$; and so, 0 belongs to $\mathcal{F}(C)$. $\square$

3.3 Dominant pairs

3.3.1 — Let $\sigma \in \Gamma$ and α be a vector dimension. We set:

$$\sigma(\alpha) := \sum_{s \in Q_0} \sigma(s)\alpha(s).$$

We consider the one parameter subgroup λ_α of $\mathrm{GL}(\alpha)$ acting on $V(s)$ by $t.\mathrm{Id}$ for any $s \in Q_0$: $\sigma(\alpha)$ is simply the composition $\sigma \circ \lambda_\alpha$. Note that λ_α acts trivially on $\mathrm{Rep}(Q,\alpha)$. This implies that $\mathcal{P}(Q,\alpha)$ is contained in the hyperplane $\mathcal{H}(\alpha)$ consisting of the σ's such that $\sigma(\alpha) = 0$.

Definition. The dimension vector α is called a *rational Schur root* if $\mathcal{P}(Q,\alpha)$ or equivalently $\Sigma(Q,\alpha)$ has non empty interior in $\mathcal{H}(\alpha)$.

If there exists $R \in \mathrm{Rep}(Q,\alpha)$ whose the stabilizer in $\mathrm{GL}(\alpha)$ has dimension one, α is said to be a *Schur root*.

The second notion is very classical (see [Kac82]) and the first one very natural in our context, in particular reading Theorem 2. We will explain the relation between these two notions in Paragraph 3.3.5.

3.3.2 — **Decompositions of dimension vectors.** Let α be a vector dimension of Q.

Definition. A $\mathbb{Z}$*-decomposition* of α is a family of dimension vectors α_i indexed by $\mathbb{Z}$ such that $\alpha_i = 0$ with finitely many exceptions and $\alpha = \sum_{i\in\mathbb{Z}} \alpha_i$. An *ordered decomposition* of α, is a sequence $(\beta_1, \cdots, \beta_s)$ of non-zero vector dimensions such that $\alpha = \beta_1 + \cdots + \beta_s$. We denote the ordered decomposition by $\alpha = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$.

3.3.3 — Let λ be a one parameter subgroup of $\mathrm{GL}(\alpha)$. For any $i \in \mathbb{Z}$ and $s \in Q_0$, we set $V_i(s) = \{v \in V(s) \,|\, \lambda(t)v = t^i v\}$ and $\alpha_i(s) = \dim V_i(s)$. Obviously, $\alpha = \sum_{i\in\mathbb{Z}} \alpha_i$ form a $\mathbb{Z}$-decomposition of α which determines λ up to conjugacy.

The parabolic subgroup $P(\lambda)$ of $\mathrm{GL}(\alpha)$ associated to λ is the set of $(g(s))_{s\in Q_0}$ such that for all $i \in \mathbb{Z}$ and $s \in Q_0$ we have $g(s)(V_i(s)) \subset \oplus_{j\leq i} V_j(s)$.

Now, $\mathrm{Rep}(Q,\alpha)^\lambda$ is the set of the $(u(a))_{a\in Q_1}$'s such that for any $a \in Q_1$ and for any $i \in \mathbb{Z}$, $u(a)(V_i(ia)) \subset V_i(ta)$. It is isomorphic to $\bigoplus_i \mathrm{Rep}(Q,\alpha_i)$. In particular, the irreducible component C of X^λ containing 0 is isomorphic to $\mathbb{P}(\bigoplus_i \mathrm{Rep}(Q,\alpha_i) \oplus \mathbb{C})$.

Moreover, $C^+ \cap \mathrm{Rep}(Q,\alpha)$ is the set of the $(u(a))_{a\in Q_1}$'s such that for any $a \in Q_1$ and for any $i \in \mathbb{Z}$, $u(a)(V_i(ia)) \subset \oplus_{j\leq i} V_j(ta)$.

Consider the morphism $\eta_\lambda : G \times_{P(\lambda)} C^+ \longrightarrow \mathrm{Rep}(Q,\alpha)$. Note that, $P(\lambda)$, C and C^+ only depend (up to conjugacy) on the ordered decomposition of α induced by the $\mathbb{Z}$-decomposition $\sum_i \alpha_i$ in an obvious way. From now on, we will consider the *map* $\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ associated to the ordered decomposition of α; it is defined up to conjugacy. We will say that the ordered decomposition is dominant respectively birational if $\eta_{\beta_1\tilde{+}\cdots\tilde{+}\beta_s}$ is. We will say that the decomposition is *well covering* if (C,λ) is.

Let us assume that our decomposition is dominant. Using Lemma 1, this decomposition gives a face of $\mathcal{AC}^{G(\alpha)}(X)$ and so one of $\mathcal{P}(Q,\alpha)$. This last face is denoted by $\mathcal{F}_{\mathcal{P}}(\beta_1\tilde{+}\cdots\tilde{+}\beta_s)$. Lemma 4 implies that it contains 0. Now, Proposition 1 shows that this face generate a face $\mathcal{F}_\Sigma(\beta_1\tilde{+}\cdots\tilde{+}\beta_s)$ of $\Sigma(Q,\alpha)$.

Lemma 5 *Let $\beta=\beta_1\tilde{+}\cdots\tilde{+}\beta_s$ be a dominant ordered decomposition. Then,*

$$\mathcal{F}_\Sigma(\beta_1\tilde{+}\cdots\tilde{+}\beta_s)=\mathcal{H}(\beta_1)\cap\cdots\cap\mathcal{H}(\beta_s)\cap\Sigma(Q,\beta).$$

Proof. Let (C,λ) be a dominant pair associated to $\beta=\beta_1\tilde{+}\cdots\tilde{+}\beta_s$. Let us fix $\underline{V}=(V(s))_{s\in Q_0}$ of dimension β. Let $\underline{V}=\underline{V}_1\oplus\cdots\oplus\underline{V}_s$ be a decomposition such that $\underline{V}_i$ has dimension β_i. The torus $(\mathbb{C}^*)^s$ acts on $\underline{V}$ as follows; the $i^{\rm th}$ component acts by homothety on $\underline{V}_i$. The induced action of $(\mathbb{C}^*)^s$ on C is trivial. So, $(\mathbb{C}^*)^s$ has to acts trivially on any point in $\mathcal{F}(C)$; it follows that, $\mathcal{F}_\Sigma(C)$ is contained in $\mathcal{H}(\beta_1)\cap\cdots\cap\mathcal{H}(\beta_s)$.

Conversely, let $\sigma\in\mathcal{H}(\beta_1)\cap\cdots\cap\mathcal{H}(\beta_s)\cap\Sigma(Q,\beta)$. Since (C,λ) is covering, $X^{\rm ss}(\mathcal{L}(1,\sigma))$ intersects C^+. But, since $\sigma\in\mathcal{H}(\beta_1)\cap\cdots\cap\mathcal{H}(\beta_s)$, λ acts trivially on $\mathcal{L}(1,\sigma)_{|C}$. By [Res07, Lemma 4], this implies that $X^{\rm ss}(\mathcal{L}(1,\sigma))$ intersects C. $\square$

3.3.4 — Let $\alpha,\beta\in\mathbb{N}^{Q_0}$. Following Derksen-Schofield-Weyman (see [DSW07]), we define $\alpha\circ\beta$ to be the number of α-dimensional subrepresentations of a general representation of dimension $\alpha+\beta$ if it is finite, and 0 otherwise.

We now deduce from [Res08b] a description of the well covering ordered decomposition:

Proposition 3 *The ordered decomposition $\beta=\beta_1\tilde{+}\cdots\tilde{+}\beta_s$ is well covering if and only if*

$$\forall i<j\quad \beta_i\circ\beta_j=1.$$

Proof. For simplicity we assume that $s=3$; there is no more difficulty for bigger s. By [Res08b, Lemma 10], we have $\langle\beta_1,\beta_2\rangle=\langle\beta_1,\beta_3\rangle=\langle\beta_2,\beta_3\rangle=0$. So, we can apply [Res08b, Theorem 3] and obtain that $1=(\beta_1\circ(\beta_2+\beta_3)).(\beta_2\circ\beta_3)$. We obtain $\beta_2\circ\beta_3=1$. Now, [Res08b, Corollary 2] implies that $1=(\beta_1\circ(\beta_2+\beta_3))=(\beta_1\circ\beta_2).(\beta_1\circ\beta_3)$. The conclusion follows.

Conversely, [Res08b, Theorem 3 and Corollary 2] imply that the degree of $\eta_{\beta_1\tilde{+}\beta_2\tilde{+}\beta_3}$ is one. So, [Res08b, Lemma 10] implies that $\eta_{\beta_1\tilde{+}\beta_2\tilde{+}\beta_3}$ is well covering. $\square$

3.3.5 — We can now explain the name "rational Schur root". Let us first reprove two well known lemmas:

Lemma 6 *If $\alpha \circ \beta \neq 0$ and $\alpha \circ \gamma \neq 0$ then $\alpha \circ (\beta + \gamma) \neq 0$.*

Proof. In [DSW07], Derksen-Schofield-Weyman proved that $\alpha \circ \beta$ is the dimension of $\mathbb{C}[\mathrm{Rep}(Q, \alpha)]_\sigma$ for well chosen weight σ. With this characterization, the lemma just follows from the fact that $\mathbb{C}[\mathrm{Rep}(Q, \alpha)]^{\mathrm{SL}(\alpha)}$ has no zero divisors. In this work, $\alpha \circ \beta$ is always understood as the degree of a map η; in particular, we include a proof using this point of view.

Consider a pair (C, λ) (resp. (C', λ')) associated to the ordered decomposition $\alpha\tilde{+}\beta$ (resp. $\alpha\tilde{+}\gamma$) in $\mathrm{Rep}(Q, \alpha + \beta)$ and $\mathrm{Rep}(Q, \alpha + \gamma)$. Since $\alpha \circ \beta \neq 0$, $\eta_{\alpha\tilde{+}\beta}$ is generically finite. Moreover, by [Res08b, Lemma 9], λ acts trivially on the restriction to C of the determinant bundle of η. It follows that for general $x \in C = \mathrm{Rep}(Q, \alpha) \oplus \mathrm{Rep}(Q, \beta)$, the differential of $\eta_{\alpha\tilde{+}\beta}$ at x is an isomorphism. In the same way, the differential of $\eta_{\alpha\tilde{+}\gamma}$ is an isomorphism for x' general in $\mathrm{Rep}(Q, \alpha) \oplus \mathrm{Rep}(Q, \beta)$. A direct computation implies now that $\eta_{\alpha\tilde{+}(\beta+\gamma)}$ is an isomorphism for y general in $\mathrm{Rep}(Q, \alpha) \oplus \mathrm{Rep}(Q, \beta) \oplus \mathrm{Rep}(Q, \gamma) \subset \mathrm{Rep}(\alpha + \beta + \gamma)$. In particular, $\alpha \circ (\beta + \gamma) \neq 0$. $\square$

Let us recall the following well known

Lemma 7 *We have:*

$$\Sigma(Q, \beta) = \{\sigma \in \Gamma \,:\, \sigma(\beta) = 0 \text{ and } \sigma(\alpha) \leq 0 \ \forall \alpha \text{ s.t. } \alpha \circ (\beta - \alpha) \neq 0\}.$$

Proof. Let $\sigma \in \Sigma(Q, \beta)$. We already saw that $\sigma(\beta) = 0$. Let α be such that $\alpha \circ (\beta - \alpha) \neq 0$. Since $\eta_{\alpha\tilde{+}(\beta-\alpha)}$ is dominant, $\sigma(\alpha) \leq 0$.

The converse inclusion is a direct consequence of [Kin94] (see [DW00, Remark 5]). $\square$

Here, comes a variant of the Derksen-Weyman saturation theorem. Note that this variant is much more easy:

Lemma 8 *We have:*

$$\Sigma(Q, k\beta) = \Sigma(Q, \beta).$$

Proof. The inclusion $\Sigma(Q, k\beta) \subset \Sigma(Q, \beta)$ is a direct consequence of Lemmas 6 and 7.

The converse inclusion follows from the Derksen-Weyman Reciprocity Property (see [DW00, Corollary 1]). We include here a simpler proof. Let $(V(s))_{s\in Q_0}$ be vector spaces of dimension vector β. Consider the family $\mathrm{Hom}(\mathbb{C}^k, V(s)$ of vector spaces indexed by $s \in Q_0$ of dimension vector $k\beta$. Then, for the natural inclusion $\mathrm{Rep}(Q, \beta) \subset \mathrm{Rep}(Q, k\beta)$, $\mathrm{Rep}(Q, \beta)$ is the fix point set of $H = (\mathrm{GL}_k)^{Q_0} \subset \mathrm{GL}(k\beta)$. Moreover, the centralizer of $(\mathrm{GL}_k)^{Q_0}$ in $\mathrm{GL}(k\beta)$ is isomorphic to $\mathrm{GL}(\beta)$. By a Luna theorem (see [Lun75]), for any linearized ample line bundle a point $x \in \mathrm{Rep}(Q, \beta)$ is semistable for $\mathcal{L}$ and the action of $\mathrm{GL}(\beta)$ if and only if it is for the action of $\mathrm{GL}(k\beta)$. It follows that $\mathcal{P}(Q, \beta) \subset \mathcal{P}(Q, k\beta)$. The lemma is proved. $\square$

Proposition 4 *A vector dimension α is a rational Schur root if and only if it is positively proportional to a Schur root.*

Proof. The Ringle form is denoted by $\langle\cdot,\cdot\rangle$. Let β be a Schur root. By [Sch92, Theorem 6.1], X contains stable points for the action of $\mathrm{GL}(\beta)/\mathrm{Im}(\lambda_0)$ and the line bundle $\mathcal{L}(1, \langle\beta,\cdot\rangle - \langle\cdot,\beta\rangle)$. It follows that $\Sigma(Q, \beta)$ has non empty interior in $\mathcal{H}(\beta)$. By Lemma 8, $k\beta$ is a rational Schur root for any positive integer k.

Conversely, let β be a rational Schur root. Let d denote the gcd of the $\beta(s)$ for $s \in Q_0$. By Lemma 8, $\overline{\beta} = \beta/d$ is a rational Schur root. Consider the canonical decomposition $\overline{\beta} = \beta_1 + \cdots + \beta_s$ of β (see [Kac82]). Then, $\Sigma(Q, \overline{\beta})$ is contained in $\mathcal{H}(\beta_1) \cap \cdots \cap \mathcal{H}(\beta_s)$. Since $\Sigma(Q, \overline{\beta})$ spans the hyperplane $\mathcal{H}(\beta)$, it follows that $\mathcal{H}(\beta) = \mathcal{H}(\beta_1) = \cdots = \mathcal{H}(\beta_s)$. So, the β_i's are proportional; since, $\overline{\beta}$ is indivisible, it follows that $s = 1$ and that $\overline{\beta}$ is a Schur root. $\square$

3.4 The Derksen-Weyman theorem

3.4.1 — The ordered decomposition $\beta = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$ is called an *ordered decomposition by rational Schur roots* if $\beta_1, \cdots, \beta_s$ are rational Schur roots. To any such decomposition we associate the (unordered) set $\{\beta_1, \cdots, \beta_s\} \subset \mathbb{N}^{Q_0}$. Let $\mathcal{W}_s(\beta)$ denote the set of subsets obtained in such a way from well covering ordered decomposition by s rational Schur roots.

We can now state and reprove the Derksen-Weyman theorem:

Theorem 3 *Let β be a vector dimension. We denote by d the dimension of $\Sigma(Q, \beta)$ and by n the cardinality of Q_0. For any $s = n - d, \cdots, 0$, the map*

$$\begin{array}{rccc} \Theta : & \mathcal{W}_s(\beta) & \longrightarrow & \{\text{faces of } \Sigma(Q,\beta) \text{ of codimension } s\} \\ & \{\beta_1, \cdots, \beta_s\} & \longmapsto & \mathcal{H}(\beta_1) \cap \cdots \cap \mathcal{H}(\beta_s) \cap \Sigma(Q, \beta), \end{array}$$

is a bijection. Moreover, the family $(\beta_1, \cdots, \beta_s)$ *is linearly independent.*

Proof. Let $\beta = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$ be a well covering ordered decomposition by rational Schur roots. Then, by Lemma 5, $\mathcal{H}(\beta_1) \cap \cdots \cap \mathcal{H}(\beta_s) \cap \Sigma(Q, \beta)$ equals $\mathcal{F}_\Sigma(\beta_1 \tilde{+} \cdots \tilde{+} \beta_s)$. Since the β_i's are rational Schur roots, Theorem 1 shows that $\mathcal{F}_\Sigma(\beta_1 \tilde{+} \cdots \tilde{+} \beta_s)$ has codimension s. Let us recall that $\mathcal{AC}^{\mathrm{GL}(\beta)}(X) = \mathcal{TC}^{\mathrm{GL}(\beta)}(X)$. This proves that Θ is well defined.

From now on, we prefer to consider the faces of $\mathcal{P}(Q, \beta)$ containing 0 rather than faces of $\Sigma(Q, \beta)$. By Proposition 1, this is equivalent.

We are going to prove that Θ is surjective. Let us fix a face $\mathcal{F}$ of $\mathcal{P}(Q, \alpha)$ of codimension d and containing 0. By Theorem 2, there exists an open subset U in $\mathrm{Pic}^G(X)^+_{\mathbb{Q}} - \mathcal{AC}^G(X)$ such that $\mathcal{F} = \mathcal{F}(\mathcal{L})$ for all $\mathcal{L} \in U$. Let (C, λ) be a well covering pair associated to a line bundle $\mathcal{L} \in U$. By Lemma 4, C contains 0. Let $\beta = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$ be the ordered decomposition associated to λ. By Paragraph 3.3.3, $\eta_{(C,\lambda)} = \eta_{\beta_1 \tilde{+} \cdots \tilde{+} \beta_s}$. The Kernel of the G^λ-action on (C, λ) contains the central subtorus S of dimension s; and, $\mathcal{AC}^{G^\lambda}(C)$ is contained in $\mathrm{Pic}^{G^\lambda/S}(C)_{\mathbb{Q}}$.

We claim that the β_i's are rational Schur roots. Let us fix $i \in \{1, \cdots, s\}$. Let λ_{β_i} be the central one parameter subgroup of $\mathrm{GL}(\beta_i)$ defined in Paragraph 3.3.1; and, S_i be the codimension one subtorus of the center of $\mathrm{GL}(\beta_i)$ such that $Y(S_i)$ is orthogonal to λ_{β_i}. Consider the subgroup H_i of $\mathrm{GL}(\beta_i)$ generated by the S_i and $\mathrm{SL}(\beta_i)$. We embed $\mathbb{P}(\mathrm{Rep}(Q, \beta_i) \oplus \mathbb{C})$ in X in an obvious way and consider the restriction morphism:

$$p_i \,:\, \mathrm{Pic}^G(X)_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^{H_i}(\mathbb{P}(\mathrm{Rep}(Q, \beta_i) \oplus \mathbb{C}))_{\mathbb{Q}}.$$

By construction, the restriction of p_i to $\mathcal{H}(\beta_i)$ is surjective. Moreover, by [Res07, Lemma 11], $p_i(U)$ is contained in $\mathcal{P}(Q, \beta_i, H_i)$. Since p_i is an open map, this implies that $\mathcal{P}(Q, \beta_i)$ has codimension one in $X(\mathrm{GL}(\beta_i))_{\mathbb{Q}}$. So, the β_i's are rational Schur roots and Θ is surjective.

Let $\beta = \beta_1 \tilde{+} \cdots \tilde{+} \beta_s$ be any well covering ordered decomposition by rational Schur roots. By Theorem 1, the intersection $\mathcal{H}(\beta_1) \cap \cdots \cap \mathcal{H}(\beta_s)$ has codimension s. This means that the β_i are linearly independent.

Let us fix $\sigma \in \Gamma_{\mathbb{Q}}$ such that $\mathcal{L}(1, \sigma)$ belongs to the relative interior of $\mathcal{F} := \mathcal{F}_{\mathcal{P}}(\beta_1 \tilde{+} \cdots \tilde{+} \beta_s)$. Since, β_i are rational Schur roots, Theorem 1 shows that the codimension of $\mathcal{F}$ equals s. Note that, $\Delta(\mathcal{F})$ contains $\mathcal{L}(1, \sigma)$ in its closure. We claim that $p_i(\sigma)$ belongs to the relative interior of $\Sigma(Q, \beta_i)$. Assuming it does not, one can find σ_ϵ in $\Delta(\mathcal{F})$ such that $p_i(\sigma_\epsilon)$ does not belongs to $\Sigma(Q,\beta_i)$. Then, the ordered decomposition associated to σ_ϵ contains strictly more than s vector dimensions. By Theorem 1 this implies that the codimension of $\mathcal{F}$ is strictly greater than s; which is a contradiction.

We now want to prove the injectivity of Θ. Let $\beta = \beta_1\tilde{+}\cdots\tilde{+}\beta_s$ be a well covering ordered decomposition by rational Schur roots and $\mathcal{F}$ be the associated face. We want to obtain the decomposition of β from $\mathcal{F}$. By Proposition 4 and [Sch92, Theorem 3.2], the canonical decomposition of $\beta_i = a_i\overline{\beta}_i$ for some positive integer a_i and some Schur root $\overline{\beta_i}$. Set $C = \mathbb{P}(\oplus_i \mathrm{Rep}(Q,\beta_i)\oplus\mathbb{C})$ and $C_0 = \mathbb{P}(\oplus_i \mathrm{Rep}(Q,\overline{\beta}_i)^{\oplus a_i}\oplus\mathbb{C})$; and fix embeddings $C_0\subset C\subset X$.

Let $\mathcal{L} := \mathcal{L}(1,\sigma)$ be a point in the relative interior of $\mathcal{F}$. Let x be a general point in C_0. Since $p_i(\sigma)$ belongs to the relative interior of $\Sigma(Q,\overline{\beta}_i)$, [Sch92, Theorem 6.1] implies that the orbit of x by the group $\prod_i \mathrm{GL}(\overline{\beta}_i)^{a_i}$ is closed in $X^{\mathrm{ss}}(\mathcal{L})$. By [Lun75], this implies that $\mathrm{GL}(\beta).x$ is closed in $X^{\mathrm{ss}}(\mathcal{L})$. Conversely, by [Res07, Proposition 9], any general closed orbit in $X^{\mathrm{ss}}(\mathcal{L})$ intersects C and so C_0. This proves that a general closed orbit in $X^{\mathrm{ss}}(\mathcal{L})$ contains a general point of C_0. In particular, any point in a general closed orbit of $X^{\mathrm{ss}}(\mathcal{L})$ decompose as a sum of a_1 indecomposable representations of dimension $\overline{\beta}_1\ldots$ and a_s indecomposable representations of dimension $\overline{\beta}_s$. Moreover, such a decomposition is unique and the $\overline{\beta}_i$'s are pairwise distinct (the family is free). The injectivity follows. □

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