Quotients of group completions by spherical subgroups

N. Ressayre

Abstract. Let G be a semi-simple algebraic group and let H be a spherical subgroup. The ground field k is algebraically closed and of characteristic zero. This article is concerned by projective embeddings Y of spherical homogeneous spaces G/H. Our approach in the study of such a variety Y is to realize them as quotients under the action of H of projective embeddings of G. First, we give a more precise sense to this project by defining the quotient of a G-variety by a spherical subgroup H. Then, we give a condition, in terms of G-invariant valuations, under which Y can be obtained by quotient of an embedding of G. Finally, if the index of H in its normalizer is finite, we show that an important class of embeddings of G/H (toroidal and liftable) geometric quotients of embeddings of G.

1 Introduction

Let G be a semi-simple algebraic group and let H be a spherical subgroup. The ground field k is algebraically closed and of characteristic zero. This article is concerned with the G-equivariant projective embeddings Y of the homogeneous space G/H. Indeed, whereas the Luna-Vust Theory classifies these embeddings by combinatorial objects (namely colored fans), important questions about the geometry and the topology of these varieties remains unsolved.

We explain our strategy. Note that apart from flag varieties and toric varieties, the best understood spherical varieties Y are the embeddings of the group G viewed as a $G\times G$-homogeneous space (see [Bri98, CP83, LP90, BCP90]...). Our aim in this article is to realize projective embeddings Y of G/H as quotients of $G\times G$-equivariant embeddings X of G. Indeed, such a construction combined with equivariant cohomology methods should give explicit generators of the cohomology ring of Y. On the other hand, the orbit closures of a Borel subgroup B of G in Y play a key role in the geometry of Y. Preliminary results (see [Res00]) show that the closures of the $B\times H$-orbits in X are simpler than B-orbit closures in Y. This is another motivation of this article and the subject of an forthcoming paper.

In Section 2, we collect notation and results about the Theory of Spherical Embeddings. In Section 3, we give essentially known auxiliary results about the moment polytopes (see Section 3.1 for the definition) of a projective spherical variety. The properties of the embeddings of G used through the paper are collected in Section 4. In Section 5, we fix a projective variety X endowed with an action of G. Then, as in Geometric Invariant Theory (see [MFK94]), we associate to any ample H-linearized line bundle on X a "quotient" $X^{\rm ss}(\mathcal{L})/\!/H$ of an open subset $X^{\rm ss}(\mathcal{L})$ of X by H, even if H is not reductive. In Section 6, we prove our first main result:

Theorem *Assume that the kernel of the action of G on G/H is finite. Let Y be a projective embedding of G/H. Then, the following conditions are equivalent:*

(i) There exist a projective $G \times G$-equivariant embedding X of G and an ample $G \times H$-linearized line bundle $\mathcal{L}$ on X such that $Y = X^{\mathrm{ss}}(\mathcal{L})//H$.

(ii) For any G-orbit $\mathcal{O}$ of codimension one in Y, there exists a $G \times G$-equivariant embedding $X_{\mathcal{O}}$ of G and a $G \times \{1\}$-equivariant and $\{1\} \times H$-invariant surjective rational map:

$$\phi : X_{\mathcal{O}} \dashrightarrow G/H \cup \mathcal{O} \subset Y.$$

Assertion (ii) can be expressed in term of the valuations used in the Luna-Vust Theory (see Theorem 1 below). The spherical homogeneous spaces G/H such that any projective embedding of G/H can be realized as quotients of a group completion are said to be *liftable*. As examples, we show that if H is symmetric or if H is solvable and of finite index in its normalizer then G/H is liftable.

The former theorem is not sufficient for applications. Indeed, the quotient $X^{\mathrm{ss}}(\mathcal{L})//H$ is not in general an orbit space but only a categorical quotient. In Sections 7 to 9, we obtain embeddings of G/H as spaces of $\{1\} \times H$-orbits in some open subset of X. First in Section 7, we prove auxiliary results about the divisors of embeddings of G which are stable by left multiplication by a Borel subgroup of G and by right multiplication by H. In Section 8, we fix a projective embedding X of G and an ample $G \times H$-linearized line bundle $\mathcal{L}$ on X. Then, we study the quotient by H: $\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})//H$, in relation with moment polytopes of X and $X^{\mathrm{ss}}(\mathcal{L})//H$. Our main result is contained in Section 9. To state it, we need a definition: an embedding Y of G/H is said to be *toroidal* if $Y - (G/H)$ is an union of G-stable prime divisors and if any G-orbit closure of Y can be obtained by intersecting properly G-stable prime divisors (see 2.3 for an equivalent definition). These embeddings play a key role since any embedding of G/H is the image of a toroidal embedding by a G-equivariant proper morphism (see Proposition 2.6.6 below). Our main result is the following

Theorem *Let G/H be a liftable spherical homogeneous space such that the index of H in its normalizer is finite. Let Y be a toroidal projective embedding of G/H. Then, there exist a toroidal projective embedding X of G and an ample $G \times H$-linearized line bundle $\mathcal{L}$ on X such that the quotient,*

$$\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})//H,$$

of X by $\{1\} \times H$ associated to $\mathcal{L}$ satisfies:

(i) π is surjective and G-equivariant.

(ii) the fibers of π are the orbits of $\{1\} \times H$ in $X^{\mathrm{ss}}(\mathcal{L})$.

Acknowledgments I would like to thank S. Pin for useful discussions. I am especially grateful to M. Brion who gave me precious advice during the preparation of this article.

For an easier reading, we give in the following table the defining occurence of each notation.

Index of Symbols

2 The embeddings of a spherical homogeneous space

Let G be a semi-simple algebraic group and let H be a closed subgroup of G. We assume that H is *spherical*, that is, a Borel subgroup of G has a dense orbit in G/H. Let X be a normal algebraic variety endowed with an algebraic action of G. Then X is said to be an embedding of G/H if it is endowed with an open and G-equivariant immersion of G/H in X. The image of the point H/H by the immersion is called *the base point* of X. In this section, we collect the notions and results of Theory of Spherical Embeddings (see [LV83, Kno91] or [Bri97]) which will be used throughout this paper. We are particularly interested in the classification and the local geometry of these embeddings.

2.1 — In this paragraph, we introduce some material necessary to classify the embeddings of G/H. Let us fix a Borel subgroup B of G such that BH is dense in G; such a B is said to be *opposite* to H.

We denote by $k(G/H)^{(B)}$ the set of all rational functions on G/H which are eigenvectors for B. If Γ is an algebraic group, we denote by $\mathcal{X}(\Gamma)$ the group $\mathrm{Hom}(\Gamma, k^*)$ of its multiplicative characters. We set $\mathcal{X}(B)^{B\cap H} := \{\gamma \in \mathcal{X}(B) \ : \ \gamma_{|B\cap H} = 1\}$. Associating to a function of $k(G/H)^{(B)}$ its weight in $\mathcal{X}(B)$, we obtain an exact sequence:

$$0 \longrightarrow k^* \longrightarrow k(G/H)^{(B)} \longrightarrow \mathcal{X}(B)^{B\cap H} \longrightarrow 0.$$

The rank of $\mathcal{X}(B)^{B\cap H}$ is called the *rank of* G/H.

Let $\nu \ : \ k(G/H) \longrightarrow \mathbb{Z}$ be a k-valuation of the field $k(G/H)$. Then, for all f in $k(G/H)^{(B)}$, $\nu(f)$ only depends on the weight of f in $\mathcal{X}(B)^{B\cap H}$. Thus, the restriction of ν to $k(G/H)^{(B)}$ induces a group homomorphism $\overline{\nu} \ : \ \mathcal{X}(B)^{B\cap H} \longrightarrow \mathbb{Z}$.

Then, the map $\nu \mapsto \overline{\nu}$ defines an injection (see [Kno91] or [Bri97]) from the set $\mathcal{V}(G/H)$ of the G-invariant discrete k-valuations of $k(G/H)$ into $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q})$.

In this article, a convex subset of a real or rational vector space, stable by multiplication by non negative scalars, is called a *cone*. We denote by $\mathcal{CV}(G/H)$ the cone in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q})$ generated by the image of $\mathcal{V}(G/H)$.

A prime B-stable divisor of G/H is called a *color* of G/H. The set of colors of G/H is denoted by $\mathcal{D}(G/H)$; it is finite. If $D \in \mathcal{D}(G/H)$, we denote by ν_D the valuation of $k(G/H)$ with center D which maps onto $\mathbb{Z}$.

2.2 — In this paragraph, we associate to each color an equation.

First, we endow G with the action of $B \times H$ defined by: $(b, h).g = bgh^{-1}$. We consider the set $k(G)^{(B\times H)}$ of all rational functions on G which are eigenvectors for $B \times H$. Then, associating to each element of $k(G)^{(B\times H)}$ its weight, we obtain the following exact sequence:

$$0 \longrightarrow k^* \longrightarrow k(G)^{(B\times H)} \longrightarrow \mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H) \longrightarrow 0,$$

where $\mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H) = \{(\gamma, \chi) \in \mathcal{X}(B) \times \mathcal{X}(H) \ : \ \gamma_{|B\cap H} = -\chi_{|B\cap H}\}$. Moreover, if (γ, χ) belongs to $\mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H)$ the formula $f(b^{-1}h) = \gamma(b)\chi(h)$ defines an element (denoted by $[\gamma : \chi]$) of $k(G)^{(B\times H)}$ of weight (γ, χ). Then, the map $(\gamma, \chi) \mapsto [\gamma : \chi]$ splits the exact sequence.

Consider the universal covering $\zeta \ : \ \widetilde{G} \longrightarrow G$. Then, $k[\widetilde{G}]$ is a unique factorization domain. Set $\widetilde{B} = \zeta^{-1}(B)$ and $\widetilde{H} = \zeta^{-1}(H)$. Then $\widetilde{G}$ acts transitively on G/H which identifies with $\widetilde{G}/\widetilde{H}$. Moreover, ζ induces an inclusion of $\mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H)$ into $\mathcal{X}(\widetilde{B}) \times_{\mathcal{X}(\widetilde{B}\cap\widetilde{H})} \mathcal{X}(\widetilde{H})$.

Note that $\mathcal{D}(G/H)$ identifies canonically with $\mathcal{D}(\widetilde{G}/\widetilde{H})$. Let $D \in \mathcal{D}(G/H)$. The pullback $\widetilde{D}$ of D in $\widetilde{G}$ by the orbit-map is a $\widetilde{B} \times \widetilde{H}$-stable divisor. Thus, there exists a unique f_D in $k(\widetilde{G})$ such that $\mathrm{div}(f_D) = \widetilde{D}$ and $f_D(1) = 1$. Then, there exists (γ_D, χ_D) in $\mathcal{X}(\widetilde{B}) \times_{\mathcal{X}(\widetilde{B}\cap\widetilde{H})} \mathcal{X}(\widetilde{H})$ such that $f_D = [\gamma_D : \chi_D]$. We call f_D *the equation* of D. Since $k[\widetilde{G}]$ is a UFD, one easily checks the following

Lemma 2.2.1 *The map*

$$\begin{array}{ccc} \mathcal{X}(\widetilde{B}) \times_{\mathcal{X}(\widetilde{B}\cap\widetilde{H})} \mathcal{X}(\widetilde{H}) & \longrightarrow & \bigoplus_{D\in\mathcal{D}(G/H)} \mathbb{Z}\widetilde{D} \\ (\gamma, \chi) & \longmapsto & \mathrm{div}([\gamma : \chi]) \end{array}$$

is an isomorphism of groups.

2.3 — Let again X be an embedding of G/H and let $\mathcal{O}$ be an orbit of G in X. Set $X_{\mathcal{O}} := \{x \in X \,:\, \overline{G.x} \text{ contains } \mathcal{O}\}$. Then $X_{\mathcal{O}}$ is a G-stable open subset of X containing $\mathcal{O}$ as its unique closed orbit. As a consequence, X is covered by embeddings of G/H containing a unique closed orbit (such an embedding is said to be *simple*). Simple embeddings are quasi-projective, see [Kno91].

An element of $\mathcal{D}(G/H)$ which contains $\mathcal{O}$ in its closure is called a *color of the orbit* $\mathcal{O}$. Let $\mathcal{D}(X,\mathcal{O})$ denote the set of colors of $\mathcal{O}$. The orbit $\mathcal{O}$ is said to be *colorless* if $\mathcal{D}(X,\mathcal{O})$ is empty. We say that X is *toroidal* if all orbits $\mathcal{O}$ of G in X are colorless. The term "toroidal" will be explained by Proposition 2.6.7 below.

Consider the cone $\mathcal{CV}(X,\mathcal{O})$ in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H},\mathbb{Q})$ generated by the G-invariant valuations which have a center in $X_{\mathcal{O}}$ and by the valuations $\overline{\nu}_D$ with $D \in \mathcal{D}(X,\mathcal{O})$.

Definitions

(i) Let $\mathcal{C}$ be a strictly convex cone in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H},\mathbb{Q})$ and $\mathcal{D}$ be a subset of $\mathcal{D}(G/H)$. Then, $(\mathcal{C},\mathcal{D})$ is called a *colored cone* if the two following conditions hold:

- The convex cone $\mathcal{C}$ is generated by the $\overline{\nu_D}$ with $D \in \mathcal{D}$, and by a finite number of elements of $\mathcal{CV}(G/H)$.
- The relative interior of $\mathcal{C}$ intersects $\mathcal{CV}(G/H)$.

(ii) A *colored face* of a colored cone $(\mathcal{C},\mathcal{D})$ is a colored cone $(\mathcal{C}',\mathcal{D}')$ such that $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{D}' = \{D \in \mathcal{D} \,:\, \overline{\nu_D} \in \mathcal{C}'\}$.

A link between the colored cones and the simple embeddings of G/H is the following:

Proposition 2.3.2 *Let X be a simple embedding of G/H with closed orbit Z. Then, $(\mathcal{CV}(X,Z),\mathcal{D}(X,Z))$ is a colored cone. Moreover, the map $\mathcal{O} \mapsto (\mathcal{CV}(X,\mathcal{O}),\mathcal{D}(X,\mathcal{O}))$ is a bijection between the set of all G-orbits in X and the set of all colored faces of $(\mathcal{CV}(X,Z),\mathcal{D}(X,Z))$.*

For any embedding X of G/H, we set

$$\mathcal{F}(X) := \Big\{\big(\mathcal{C}(X,\mathcal{O}),\mathcal{D}(X,\mathcal{O})\big) \,:\, \mathcal{O} \text{ is an orbit of } G \text{ in } X\Big\}.$$

To explain the structure of $\mathcal{F}(X)$, we need the following:

Definition A *colored fan* is a set $\mathcal{F}$ of colored cones which satisfy the two following conditions:

- Any colored face of a colored cone of $\mathcal{F}$ belongs to $\mathcal{F}$.
- For any $\nu \in \mathcal{CV}(G/H)$, there exists at most one colored cone $(\mathcal{C},\mathcal{D})$ in $\mathcal{F}$ such that $\mathcal{C}$ contains ν in its relative interior.

Then, the following classification statement holds:

Proposition 2.3.3 *The map $X \mapsto \mathcal{F}(X)$ is a bijection between the set of isomorphism classes of embeddings of G/H and the set of colored fans.*

2.4 — Let X be an embedding of G/H and $\mathcal{O}$ be an orbit of G in X. One can check that $\mathcal{O}$ is spherical; let $\mathcal{O}_B^\circ$ denote the open orbit of B in $\mathcal{O}$. Set:

$$X_{\mathcal{O},B} := \{x \in X \ : \ \overline{B.x} \text{ contains } \mathcal{O}\}.$$

Then, it is proved in [Kno91] or [Bri97] that $X_{\mathcal{O},B}$ is an affine 111 open subset in X containing $\mathcal{O}_B^\circ$ as its unique closed B-orbit. One easily checks the following characterizations of $X_{\mathcal{O},B}$ (see for example Proposition 2.4.1 of [Res00]):

Proposition 2.4.4 *(i) The complement of $X_{\mathcal{O},B}$ in X is the union of the closures of the $D \in \mathcal{D}(G/H)$ which do not contain $\mathcal{O}$.*

(ii) The subset $X_{\mathcal{O},B}$ is the intersection of the open B-stable subsets of X which intersect $\mathcal{O}$.

2.5 — Let $P_{G/H}$ denote the stabilizer in G of the open subset BH/H of G/H. Then, $P_{G/H}$ is a parabolic subgroup of G containing B. Let $P_{G/H}^u$ denote its unipotent radical. The next proposition (see [BP90] or [Bri97]) defines Levi subgroups of $P_{G/H}$ in special position with respect to H:

Proposition 2.5.5 *There exist Levi subgroups L of $P_{G/H}$ satisfying the following two conditions:*

(i) If $[L;L]$ denotes the derived subgroup of L, then:

$$P_{G/H} \cap H = L \cap H \supseteq [L;L].$$

(ii) Let C denote the connected center of L. Then, for any embedding X of G/H with base point x, the set $P_{G/H}^u.\overline{C.x}$ (where $\overline{C.x}$ denotes the closure of $C.x$ in X) contains a non-empty open subset of any orbit of G in X.

Such a Levi subgroup of $P_{G/H}^u$ is said to be adapted to H.

2.6 — In this section, we fix our attention on the toroidal embeddings of G/H. Proposition 2.4.2 of [Bri97] explains the key role of these embeddings:

Proposition 2.6.6 *Let X be an embedding of G/H. Then, there exists a toroidal embedding $\widetilde{X}$ of G/H and a G-equivariant birational projective morphism $\pi : \widetilde{X} \longrightarrow X$.*

The next proposition (see [Bri97] or [Bri89]) describes the local structure of the toroidal embeddings of G/H.

Proposition 2.6.7 *Let X be an embedding of G/H with base point x and $\mathcal{O}$ be a colorless orbit of G. Let L be an adapted Levi subgroup of $P_{G/H}$. Let S denote the closure in $X_{\mathcal{O},B}$ of $L.x$. Then, we have:*

(i) The map:

$$\begin{array}{ccc} P^u_{G/H} \times S & \longrightarrow & X_{\mathcal{O},B} \\ (g,x) & \longmapsto & gx \end{array}$$

is a $P_{G/H}$−equivariant isomorphism.

(ii) The group $[L;L]$ acts trivially on S. The induced action of $L/[L;L]$ endows S with a structure of an affine toric variety.

(iii) Each orbit of G in $X_{\mathcal{O}}$ intersects S transversely in a unique orbit of L.

Proposition 2.6.7 means that the local structure of the orbits of G in X looks like the orbits in a toric variety. A common feature between the toric varieties and the toroidal embeddings of G/H is the following easy lemma (see [Res00]):

Lemma 2.6.8 *Let X be an embedding of G/H and $\mathcal{O}$ be a colorless orbit of G in X. Let* $\mathrm{rk}(\mathcal{O})$ *denote the rank of the spherical homogeneous space $\mathcal{O}$. Let* $\dim(G/H)$ *(resp.* $\dim(\mathcal{O})$*) denote the dimension of G/H (resp. $\mathcal{O}$).*

Then, we have $\mathrm{rk}(G/H) - \mathrm{rk}(\mathcal{O}) = \dim(G/H) - \dim(\mathcal{O})$.

2.7 — Now we introduce an important class of spherical homogeneous spaces. Proposition 4.4.1 of [Bri97] is

Proposition 2.7.9 *For a spherical homogeneous space G/H, the following conditions are equivalent:*

(i) The index of H in its normalizer in G is finite.

(ii) There exists a simple complete embedding of G/H.

Such a spherical homogeneous space is said to be sober.

Let G/H be a sober spherical subgroup of G. Then there exists a unique simple complete toroidal embedding Y of G/H: we call it the *canonical embedding of G/H*. Note that Y is projective.

3 Moment polyhedron

3.1 — Let X be a quasiprojective embedding of G/H and $\mathcal{L}$ be an ample G-linearized line bundle on X. In this section, we recall the notion of moment polyhedron associated to $\mathcal{L}$. After recalling the classical properties of these polyhedra (see [Bri97]), we fix our attention on the case when $X = G/H$.

If Γ is an Abelian group, we denote by $\Gamma_{\mathbb{Q}}$ its tensor product with $\mathbb{Q}$. Let P^+ denote the set of dominant weights for (G,B). For $\gamma \in P^+$, we denote by V_γ the irreducible G-module of highest weight γ for B.

For each positive integer n, the set $\Gamma(X, \mathcal{L}^{\otimes n})$ of sections of $\mathcal{L}^{\otimes n}$ is a rational G-module. Set

$$\mathbf{P}(X, \mathcal{L}) := \{p \in \mathcal{X}(B)_{\mathbb{Q}} \,:\, \exists n > 0,\, np \in P^+,\, V_{np} \hookrightarrow \Gamma(X, \mathcal{L}^{\otimes n})\},$$

where $V_{np} \hookrightarrow \Gamma(X, \mathcal{L}^{\otimes n})$ means that the G-module V_{np} is a sub-module of $\Gamma(X, \mathcal{L}^{\otimes n})$.

The convex hull of a finite number of points in a vector space will be called a *polytope*. A subset of a vector space defined by a finite number of linear inequalities will be called a *polyhedron*. Then, we have (see [Bri97]; 1.2 or [Bri89]):

Proposition 3.1.1 *The set $\mathbf{P}(X, \mathcal{L})$ is a polyhedron in $\mathcal{X}(B)_{\mathbb{Q}}$; the differences of elements of $\mathbf{P}(X, \mathcal{L})$ spans $\mathcal{X}(B)_{\mathbb{Q}}^{B\cap H}$. If moreover X is projective, then $\mathbf{P}(X, \mathcal{L})$ is a polytope.*

We call $\mathbf{P}(X, \mathcal{L})$ the *moment polyhedron* (resp. *moment polytope* if X is projective) of X associated to $\mathcal{L}$.

3.2 — If X' is a locally closed G-stable subset of X, we set $\mathbf{P}(X', \mathcal{L}) := \mathbf{P}(X', \mathcal{L}_{|X'})$. One easily proves (see [Bri97];5.3.2 and [Res00])

Proposition 3.2.2 *With above notation, we have:*

(i) If X' is a G-stable open subset of X, then $\mathbf{P}(X, \mathcal{L})$ is contained in $\mathbf{P}(X', \mathcal{L})$.

(ii) Moreover, $\mathbf{P}(X, \mathcal{L}) = \bigcap_Z \mathbf{P}(X_Z, \mathcal{L})$, intersection over all closed orbits Z of G in X.

To give a more precise description of $\mathbf{P}(X, \mathcal{L})$, we introduce more notation. Let $\mathcal{V}(X)$ denote the set of the k-valuations of $k(G/H)$ associated to the G-stable prime divisors of X. If $\nu \in \mathcal{V}(X)$, we denote by X_ν its center. Let us fix a section σ_0 of $\mathcal{L}$, B-eigenvector of weight $\gamma(\sigma_0)$. Then, we have $\mathrm{div}(\sigma_\circ) = \sum_{\nu\in\mathcal{V}(X)} n_\nu X_\nu + \sum_{D\in\mathcal{D}(G/H)} n_D \overline{D}$, where the n_ν and the n_D are non-negative integers. We recall Proposition 5.3.1 of [Bri97] (see also Proposition 3.3 of [Bri89]):

Proposition 3.2.3 *With the above notation and those of Paragraph 2.1, $\mathbf{P}(X, \mathcal{L})$ is the set of all $\gamma(\sigma_\circ) + p$ where $p \in \mathcal{X}(B)_{\mathbb{Q}}^{B\cap H}$ satisfies:*

(i) $\overline{\nu}(p) + n_\nu \geq 0 \qquad \forall \nu \in \mathcal{V}(X)$.

(ii) $\overline{\nu_D}(p) + n_D \geq 0 \qquad \forall D \in \mathcal{D}(G/H)$.

3.3 — In this paragraph, we assume that X is projective. Consider a face $\mathbf{F}$ of $\mathbf{P}(X, \mathcal{L})$ and a point p in the relative interior of $\mathbf{F}$. Set $-p + \mathbf{P}(X, \mathcal{L}) := \{-p + q \,:\, q \in \mathbf{P}(X, \mathcal{L})\}$. Proposition 3.1.1 shows that $-p + \mathbf{P}(X, \mathcal{L})$ is contained in $\mathcal{X}(B)_{\mathbb{Q}}^{B\cap H}$. Moreover, the set of all linear forms in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q})$, non-negative on $-p+\mathbf{P}(X, \mathcal{L})$, is called the dual cone of $-p+\mathbf{P}(X, \mathcal{L})$ and is denoted by $(-p+\mathbf{P}(X, \mathcal{L}))^\vee$. One checks that $(-p+\mathbf{P}(X, \mathcal{L}))^\vee$ only depends on $\mathbf{F}$. The latter cone is called the *dual cone of* $\mathbf{P}(X, \mathcal{L})$ *from* $\mathbf{F}$ and is denoted by $\mathcal{C}(\mathbf{F})$.

Let n be a positive integer and σ be a section of $\mathcal{L}^{\otimes n}$, B-eigenvector of weight np. We consider $X_\sigma = \{x \in X \,:\, \sigma(x) \neq 0\}$. Then, X_σ only depends on $\mathbf{F}$ and is denoted by $X_{\mathbf{F}}$. Moreover, there exists a unique orbit $\mathcal{O}(\mathbf{F})$ of G which meets $X_{\mathbf{F}}$ and which is minimal for the order defined by the inclusion of closures.

Proposition 3.3.4 *Keep notation as above. If X is projective, we have:*

(i) If $\mathcal{O}$ is an orbit of G in X, then $\mathbf{P}(\overline{\mathcal{O}}, \mathcal{L})$ is a face of $\mathbf{P}(X, \mathcal{L})$. Such a face is said to be orbital.

(ii) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two orbits of G in X, then:

$$\mathbf{P}(\overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2}, \mathcal{L}) = \mathbf{P}(\overline{\mathcal{O}_1}, \mathcal{L}) \cap \mathbf{P}(\overline{\mathcal{O}_2}, \mathcal{L}).$$

(iii) $\mathbf{P}(\overline{\mathcal{O}(\mathbf{F})}, \mathcal{L})$ is the unique minimal orbital face of $\mathbf{P}(X, \mathcal{L})$ which contains $\mathbf{F}$.

(iv) If $\mathbf{F} = \mathbf{P}(\overline{\mathcal{O}}, \mathcal{L})$, for an orbit $\mathcal{O}$ of G in X, then:

$$\mathcal{O}(\mathbf{F}) = \mathcal{O} \quad , \quad X_{\mathbf{F}} = X_{\mathcal{O},B} \quad \text{and} \quad \mathcal{C}(\mathbf{F}) = \mathcal{C}(X, \mathcal{O}).$$

(v) The map $\mathcal{O} \mapsto \mathbf{P}(\overline{\mathcal{O}}, \mathcal{L})$ is a bijection from the set of orbits of G in X onto the set of those faces $\mathbf{F}$ of $\mathbf{P}(X, \mathcal{L})$ such that the relative interior of $\mathcal{C}(\mathbf{F})$ intersects $\mathcal{CV}(G/H)$.

Proof: Assertions $(i), (v)$ and (iv) are Proposition 5.3.2 of [Bri97]. Assertions $(ii), (iii)$ and (iv) are proved in Proposition 2.6.4 of [Res00]. □

3.4 — In this paragraph, we are interested in the moment polyhedra of an orbit G/H. First, we recall the description of all G-linearized line bundles on G/H.

Let χ be a character of H. We endow $G \times k$ with an action of $G \times H$ by the formula: $(g, h).(g', \tau) = (gg'h^{-1}, \chi(h)\tau)$. Then the quotient by $\{1\} \times H$ exists and is a G-linearized line bundle on G/H denoted by $\mathcal{L}_\chi$. It is shown in [KKV84]:

Lemma 3.4.5 *The map $\chi \longmapsto \mathcal{L}_\chi$ is an isomorphism of groups between $\mathcal{X}(H)$ and the group of all G-linearized line bundles on G/H.*

Before describing $\mathbf{P}(G/H, \mathcal{L}_\chi)$, we introduce some notation. If V is a G-module, we set $V^{H,\chi} := \{v \in V \ : \ \forall h \in H \ h.v = \chi(h)v\}$, $V^H := V^{H,0}$ and $\Gamma_\chi := \left\{\gamma \in P^+ \ : \ (V_\gamma^*)^{H,\chi} \neq 0\right\}$. Then we have

Proposition 3.4.6 *With above notation and those of Section 2.2, we have:*

$$\Gamma_\chi = \{\gamma \in \mathcal{X}(B) \ : \ (\gamma, \chi) \in \bigoplus_{D \in \mathcal{D}(G/H)} \mathbb{N}(\gamma_D, \chi_D)\}.$$

Moreover, for all γ in Γ_χ the dimension of $(V_\gamma^*)^{H,\chi}$ equals one.

Proof: Let $\gamma \in \Gamma_\chi$ and ν be a non-zero vector in $(V_\gamma^*)^{H,\chi}$. Let $v \in V_\gamma^{(B)}$. Consider $f \in k[G]$ defined by $f(g) = \nu(gv)$. Since $f \in k(G)^{(B \times H)}$, (γ, χ) belongs to $\mathcal{X}(B) \times_{\mathcal{X}(B \cap H)} \mathcal{X}(H)$. Moreover, since f is regular on G, Lemma 2.2.1 implies that (γ, χ) belongs to $\bigoplus_{D \in \mathcal{D}(G/H)} \mathbb{N}(\gamma_D, \chi_D)$. The first inclusion is proved.

Moreover, since (γ, χ) determines f up to a multiplicative constant, the same is true for ν. So, the dimension of $(V_\gamma^*)^{H,\chi}$ equals one.

Conversely, let $\gamma \in \mathcal{X}(B)$ such that (γ, χ) belongs to $\bigoplus_{D\in\mathcal{D}(G/H)} \mathbb{N}(\gamma_D, \chi_D)$. Then, the function $[\gamma : \chi]$ is regular on G. But, by Frobenius' theorem, the $G \times G$-module $k[G]$ is isomorphic to $\bigoplus_{\lambda\in P^+} V_\lambda \otimes V_\lambda^*$. By this isomorphism, $k[G]^{(B\times H)}$ identifies with the disjoint union of the $V_\lambda^{(B)} \otimes V_\lambda^{*(H)}$. Now, $[\gamma : \chi]$ belongs to $k[G]^{(B\times H)}$ implies that γ belongs to Γ_χ. $\square$

Now we can describe $\mathbf{P}(G/H, \mathcal{L}_\chi)$ in

Proposition 3.4.7 *Keep notation as above. Then, we have:*

(i) $\mathbf{P}(G/H, \mathcal{L}_\chi) = \{\gamma \in \mathcal{X}(B)_\mathbb{Q} \, : \, (\gamma, -\chi) \in \bigoplus_{D\in\mathcal{D}(G/H)} \mathbb{Q}^{\geq 0}(\gamma_D, \chi_D)\}$.

(ii) Let $\mathbf{F}$ *be a face of* $\mathbf{P}(G/H, \mathcal{L}_\chi)$*. Let* I *be the minimal subset of* $\mathcal{D}(G/H)$ *such that* $\bigoplus_{D\in I} \mathbb{Q}^{\geq 0}(\gamma_D, \chi_D)$ *contains* $\mathbf{F}$*. With the notation of Section 3.3, we have:*

$$(G/H)_\mathbf{F} = G/H - \cup_{D\in I} D.$$

Proof: If $f \in k[G]$ satisfies $h.f = \chi(h)^{-1} f$, the map $G \longrightarrow G \times k$, $g \mapsto (g, f(g))$ induces a section of $\mathcal{L}_\chi$. So, we identify $\Gamma(G/H, \mathcal{L}_\chi)$ with the set of those $f \in k[G]$ such that for all h in H we have $h.f = \chi(h)^{-1} f$ (see [KKV84]). Now, Frobenius' Theorem yields an isomorphism of G-modules between $\Gamma(G/H, \mathcal{L}_{n\chi})$ and $\bigoplus_{\gamma\in P^+} V_\gamma \otimes (V_\gamma^*)^{H, -n\chi}$. Then, Assertion (i) follows from Proposition 3.4.6.

Let us consider $\mathbf{F}$ and I as in Assertion (ii) and fix a point p in the relative interior of $\mathbf{F}$. Let n be a positive integer and σ a section of $\mathcal{L}^{\otimes n}$, B-eigenvector of weight np. Then, $(np, n\chi) = \sum_{D\in I} k_D(\gamma_D, \chi_D)$ for some positive integers k_D. With the notation of Section 2.2, we consider $f = \prod_{D\in I} [\gamma_D : \chi_D]^{k_D}$. Then, the map $G \longrightarrow G \times k$, $g \mapsto (g, f(g))$ induces a section of $\mathcal{L}_\chi$. Then, the latter section is a scalar multiple of σ. Assertion (ii) follows. $\square$

3.5 — In this paragraph, we apply the description of the moment polyhedra of G/H given by Proposition 3.4.7 to the description of the moment polytopes of a projective embedding X of G/H.

By Lemma 3.4.5, there exists a character χ of H such that the restriction of $\mathcal{L}$ to G/H is $\mathcal{L}_\chi$. If $\mathbf{P}$ is a polytope in $\mathcal{X}(B)_\mathbb{Q}$, we set $\mathbf{P} \times \chi := \{(p, \chi) \in \mathcal{X}(B)_\mathbb{Q} \times \mathcal{X}(H)_\mathbb{Q} \, : \, p \in \mathbf{P}\}$. If $\mathbf{F}$ is a face of $\mathbf{P}(X, \mathcal{L})$, we set $\mathcal{D}_\mathbf{F} := \{D \in \mathcal{D}(G/H) \, : \, D \text{ intersects } X_\mathbf{F}\}$, where $X_\mathbf{F}$ denotes the open subset of X defined in Paragraph 3.3.

Proposition 3.5.8 *With preceding notation, if* $\mathbf{F}$ *is a face of* $\mathbf{P}(X, \mathcal{L})$*, we have:*

(i) $\mathbf{F} \times \chi = \left(\mathbf{P}(\overline{\mathcal{O}(\mathbf{F})}, \mathcal{L}) \times \chi\right) \cap \oplus_{D\notin\mathcal{D}_\mathbf{F}} \mathbb{Q}^{\geq 0}(\gamma_D, \chi_D)$.

(ii) If in addition $\mathbf{F} = \mathbf{P}(\overline{\mathcal{O}}, \mathcal{L})$*, then* $\mathcal{D}_\mathbf{F} = \mathcal{D}(X, \mathcal{O})$.

(iii) Moreover,

$$\mathbf{P}(X, \mathcal{L}) = \mathbf{P}(G/H, \mathcal{L}_\chi) \cap \bigcap_Z \left(\mathbf{P}(Z, \mathcal{L}) + \mathcal{C}(X, Z)^\vee\right),$$

intersection over all closed orbits Z *of* G *in* X*.*

Proof: Propositions 3.3.4 and 3.4.7 show that $\mathbf{F}\times\chi$ is contained in the intersection of Assertion (i). Let (p,χ) belong to this intersection. Then, there exist a positive integer n and a section σ of $\mathcal{L}^{\otimes n}$, B-eigenvector of weight np. Since σ is non-zero on $\mathcal{O}(\mathbf{F})$, Assertion (ii) of Proposition 3.4.7 shows that $X_{\mathbf{F}}$ is contained in X_σ. Now, Assertion (i) follows from Proposition 3.3.4.

If $\mathbf{F}=\mathbf{P}(\overline{\mathcal{O}},\mathcal{L})$, then $X_{\mathbf{F}}=X_{\mathcal{O},B}$ by Proposition 3.3.4. Now, Assertion (ii) follows from Proposition 3.4.7.

The inclusion of $\mathbf{P}(X,\mathcal{L})$ in the intersection of Assertion (iii) follows from Propositions 3.2.2 and 3.3.4. Let p belong to this intersection. Replacing $\mathcal{L}$ by a positive power if necessary, we can assume that there exist a section σ_0 of $\mathcal{L}$, B-eigenvector of weight $\gamma(\sigma_0)$ and a rational function f on G/H, B-eigenvector of weight $p-\gamma(\sigma_0)$. Then, with the notation of Proposition 3.2.3, p belongs to $\mathbf{P}(G/H,\mathcal{L}_\chi)$ implies that:

$$\forall D\in\mathcal{D}(G/H)\qquad \langle\overline{\nu_D},p-\gamma(\sigma_\circ)\rangle+n_D\geq 0.$$

Let $\nu\in\mathcal{V}(X)$ and let Z be a closed orbit of G in X_ν. Then, since p belongs to $\mathbf{P}(Z,\mathcal{L})+\mathcal{C}(X,Z)^\vee$, we have $\langle\overline{\nu},p-\gamma(\sigma_\circ)\rangle+n_\nu\geq 0$. Now, Proposition 3.2.3 completes the proof. □

4 The embeddings of the group

4.1 — Recall that G denotes a semi-simple algebraic group. We endow G with the action of $G\times G$ by the formula: $(g_1,g_2).g=g_1gg_2^{-1}$. In this article the $G\times G$-equivariant embeddings of G play a key role: in this section, we collect the results about these embeddings which will be used through this paper.

Set $\mathbf{G}:=G\times G$ and $\mathbf{H}=\{(g,g)\,:\,g\in G\}$. Then, G is the homogeneous space $\mathbf{G}/\mathbf{H}$.

Let B and B^- be two opposite Borel subgroup of G and let T denote their intersection. Set $\mathbf{B}:=B\times B^-$. Then, by the Bruhat decomposition $\mathbf{B}.\mathbf{H}/\mathbf{H}$ is dense in $\mathbf{G}/\mathbf{H}$. So, $\mathbf{H}$ is a spherical subgroup of $\mathbf{G}$, and $\mathbf{B}$ is opposite to $\mathbf{H}$.

First, note that $\mathcal{X}(\mathbf{B})^{\mathbf{B}\cap\mathbf{H}}=\{(\gamma,-\gamma)\,:\,\gamma\in\mathcal{X}(T)\}$. From now on, we identify $\mathcal{X}(\mathbf{B})^{\mathbf{B}\cap\mathbf{H}}$ with $\mathcal{X}(B)$ by $(\gamma,-\gamma)\mapsto\gamma$. Then, $\mathrm{Hom}(\mathcal{X}(\mathbf{B})^{\mathbf{B}\cap\mathbf{H}},\mathbb{Q})$ identifies with $\mathrm{Hom}(\mathcal{X}(B),\mathbb{Q})$.

Let Σ denote the set of simple roots of (B,T) and $\alpha\in\Sigma$. Let $W:=N(T)/T$ denote the Weyl group of T. We denote by s_α the simple reflection of W associated to α, by $\alpha^\vee$ the coroot associated to α, and by $\omega_{\alpha^\vee}$ the fundamental weight of the coroot $\alpha^\vee$. So, $(\omega_{\alpha^\vee})_{\alpha\in\Sigma}$ is the dual basis of the basis $(\alpha)_{\alpha\in\Sigma}$ of $\mathcal{X}(T)_{\mathbb{Q}}$. Let D_α denote the closure of $Bs_\alpha B^-$ in G. Then, by the Bruhat decomposition $\mathcal{D}(\mathbf{G}/\mathbf{H})=\{D_\alpha\,:\,\alpha\in\Sigma\}$. Moreover, with the notation of Section 2.2, the equation of D_α is the function $[\omega_\alpha:-\omega_\alpha]$. Then, under the identification of $\mathrm{Hom}(\mathcal{X}(\mathbf{B})^{\mathbf{B}\cap\mathbf{H}},\mathbb{Q})$ with $\mathrm{Hom}(\mathcal{X}(B),\mathbb{Q})$, the image of the valuation ν_{D_α} identifies with the coroot $\alpha^\vee$.

Set $\mathbf{T}=T\times T$. With the notation of Proposition 2.5.5, we have $P_{\mathbf{G}/\mathbf{H}}=\mathbf{B}$, and $\mathbf{T}$ is a Levi subgroup of $P_{\mathbf{G}/\mathbf{H}}$ adapted to $\mathbf{H}$. Finally, the valuation cone $\mathcal{CV}(\mathbf{G}/\mathbf{H})$ is identified with the negative Weyl chamber: $\mathcal{CV}(\mathbf{G}/\mathbf{H})=\bigoplus_{\alpha\in\Sigma}\mathbb{Q}^{\leq 0}\omega_{\alpha^\vee}$ (see [Bri97]; 4.1).

4.2 — Now, we study moment polyhedra of embeddings of $\mathbf{G}/\mathbf{H}$.

Let $P^+_{\mathbb{Q}}=\oplus_{\alpha\in\Sigma}\mathbb{Q}^{\geq 0}\omega_\alpha$ denote the cone generated by P^+ in $\mathcal{X}(B)_{\mathbb{Q}}$; this is the positive Weyl chamber. Note that the only $\mathbf{G}$-linearized line bundle on $\mathbf{G}/\mathbf{H}$ is the trivial one

$\mathcal{L}_0$. Moreover, we have: $\mathbf{P}(\mathbf{G}/\mathbf{H}, \mathcal{L}_0) = \{(p, -p) : p \in P^+_{\mathbb{Q}}\}$. From now on, we embed $\mathbf{P}(\mathbf{G}/\mathbf{H}, \mathcal{L}_0)$ (and more generally any moment polyhedron of an embedding of $\mathbf{G}/\mathbf{H}$) into $\mathcal{X}(B)_{\mathbb{Q}}$, by $(p, -p) \mapsto p$.

Let X be a projective toroidal embedding of $\mathbf{G}/\mathbf{H}$ and $\mathcal{L}$ be an ample $\mathbf{G}$-linearized line bundle on X. Consider the corresponding moment polytope $\mathbf{P}(X, \mathcal{L}) \subset \mathcal{X}(B)_{\mathbb{Q}}$. When $\mathbf{F}$ runs over the faces of $\mathbf{P}(X, \mathcal{L})$, the cones $\mathcal{C}(\mathbf{F})$ defined in Section 3.3 form a fan in $\mathrm{Hom}(\mathcal{X}(B), \mathbb{Q})$ denoted by $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$.

If $\mathcal{O}$ is an orbit of $\mathbf{G}$ in X and I is a subset of Σ, we denote by $\mathcal{C}(I, \mathcal{O})$ the cone of $\mathrm{Hom}(\mathcal{X}(B), \mathbb{Q})$ generated by $\mathcal{C}(X, \mathcal{O})$ and by the $\alpha^\vee$ for $\alpha \in I$. The following proposition describes the fan $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$:

Proposition 4.2.1 *With preceding notation, the cones of $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$ are the cones $\mathcal{C}(I, \mathcal{O})$, where $\mathcal{O}$ is an orbit of $\mathbf{G}$ in X and I is a subset of Σ such that $\mathcal{C}(X, \mathcal{O})$ is contained in $\bigoplus_{\beta \notin I} \mathbb{Q}\omega_{\beta^\vee}$.*

Proof: If $\mathcal{O}$ is an orbit of $\mathbf{G}$ in X, Proposition 3.3.4 shows that $\mathcal{C}(X, \mathcal{O})$ belongs to $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$. Moreover, Proposition 3.5.8 gives:

$$\mathbf{P}(X, \mathcal{L}) = P^+_{\mathbb{Q}} \cap \bigcap_{\text{closed orbit } Z \text{ of } \mathbf{G} \text{ in } X} \mathbf{P}(Z, \mathcal{L}) + \mathcal{C}(X, Z)^\vee.$$

In particular, every extremal ray of $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$ is either $\mathbb{Q}^{\geq 0}\alpha^\vee$ for some $\alpha \in \Sigma$ or an extremal ray of $\mathcal{F}(X)$. Let $\mathcal{C}$ be a cone in $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$. Then, there exists a $I \subset \Sigma$ and an orbit $\mathcal{O}$ of $\mathbf{G}$ in X such that $\mathcal{C} = \mathcal{C}(I, \mathcal{O})$.

If I is empty there is nothing to prove. If I is non-empty, since X is toroidal, Proposition 3.3.4 shows that the relative interior of $\mathcal{C}$ does not intersect $\mathcal{CV}(\mathbf{G}/\mathbf{H}) = \oplus_{\beta \in \Sigma} \mathbb{Q}^{\leq 0} \omega_{\beta^\vee}$. Since $\alpha^\vee$ is orthogonal to $\omega_{\beta^\vee}$ for all simple roots $\beta \neq \alpha$, we deduce that $\mathcal{C}(X, \mathcal{O})$ is contained in $\bigoplus_{\beta \notin I} \mathbb{Q}\omega_{\beta^\vee}$.

Conversely, let I and $\mathcal{O}$ be as in the proposition. Then,

$$\mathcal{C}(I, \mathcal{O}) \cap \oplus_{\beta \in \Sigma} \mathbb{Q}^{\geq 0} \beta^\vee = \oplus_{\beta \in I} \mathbb{Q}^{\geq 0} \beta^\vee, \quad \text{and} \quad \mathcal{C}(I, \mathcal{O}) \cap \mathcal{CV}(\mathbf{G}/\mathbf{H}) = \mathcal{C}(X, \mathcal{O}).$$

It follows easily that $\mathcal{C}(I, \mathcal{O})$ belongs to $\mathcal{F}(\mathbf{P}(X, \mathcal{L}))$. □

4.3 — In this section, we are interested in the isotropy subgroups of the action of $\mathbf{G}$ in X. We begin with some notation.

Let λ be a one parameter subgroup of T. Set:

$$P(\lambda) := \{g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t^{-1}) \text{ exists in } G\}.$$

For example, if α is a simple root then $P(\alpha^\vee)$ is the usual minimal parabolic subgroup P_α associated to α. In general, by [MFK94], $P(\lambda)$ is a parabolic subgroup of G with unipotent radical:

$$P^u(\lambda) := \{g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t^{-1}) = 1\}.$$

Moreover, $P(\lambda)$ and $P(-\lambda)$ are opposite and their intersection $L(\lambda)$ is the centralizer of the image of λ. Set $\Delta L(\lambda) := \{(l, l) \in \mathbf{G} : l \in L(\lambda)\}$. Denote by $C(\lambda)$ the connected center of $L(\lambda)$.

The proof of Theorem A1 in [Bri98] shows

Proposition 4.3.2 *Let X be an embedding of $\mathbf{G}/\mathbf{H}$ and $\mathcal{O}$ be a colorless orbit of $\mathbf{G}$ in X. Then, there exists a one parameter subgroup λ of T such that $\lim_{t\to 0}\lambda(t)$ exists in X and belongs to $\mathcal{O}$. Set $z := \lim_{t\to 0}\lambda(t)$.*

The isotropy subgroup of z in $\mathbf{G}$ is generated by $P^u(\lambda)\times P^u(-\lambda)$ and $\Delta L(\lambda).(C(\lambda)\times\{1\})_z$. In particular, the conjugacy class of $P(\lambda)$ only depends on $\mathcal{O}$; its representative containing B^- is denoted by $P(\mathcal{O})$.

The parabolic subgroups $P(\mathcal{O})$ can be read off the moment polytopes of X by

Lemma 4.3.3 *Assume that X is projective and toroidal. Let $\mathcal{L}$ be an ample $\mathbf{G}$-linearized line bundle on X. Let $\alpha\in\Sigma$. Then, the following are equivalent:*

(i) $\{p\in\mathcal{X}(T)_{\mathbb{Q}}\,:\,\alpha^\vee(p)=0\}\cap\mathbf{P}(\overline{\mathcal{O}},\mathcal{L})\neq\emptyset$.

(ii) $P_\alpha\subseteq P(\mathcal{O})$.

Proof: By Proposition 4.2.1, Assertion (i) is equivalent to the fact that $\mathcal{C}(X,\mathcal{O})$ is contained in $\oplus_{\beta\neq\alpha}\mathbb{Q}\omega_{\beta^\vee}$.

Let λ be as in Proposition 4.3.2. Replacing λ by a conjugated one parameter subgroup, we can assume that $P(\lambda)\supset B^-$, that is, $P(\lambda)=P(\mathcal{O})$. Then, one checks that $-\lambda$ belongs to the relative interior of $\mathcal{C}(X,\mathcal{O})$. So, Assertion (i) is equivalent to: λ belongs to $\oplus_{\beta\neq\alpha}\mathbb{Q}\omega_{\beta^\vee}$. The lemma follows easily. □

4.4 — In this paragraph, X is a simple toroidal embedding of $\mathbf{G}/\mathbf{H}$ such that the closed orbit Z is projective. We recall some results (see [Bri97] or [Bri89]) about the Picard group, $\mathrm{Pic}(X)$ of X.

Consider the universal covering $\zeta\,:\,\tilde{G}\longrightarrow G$ of G. As in Section 2.2, if Γ is a subgroup of G, $\tilde{\Gamma}$ denotes its preimage in $\tilde{G}$.

If Γ is an algebraic group acting on a variety Y, we denote by $\mathrm{Pic}^\Gamma(Y)$ the group of all Γ-linearized line bundles on Y. Then we have canonical isomorphisms: $\mathrm{Pic}^{\tilde{G}\times\tilde{G}}(X)\simeq\mathrm{Pic}(X)=\bigoplus_{\alpha\in\Sigma}\mathbb{Z}[\overline{Bs_\alpha B^-}]$.

By Proposition 4.3.2, the orbit Z is isomorphic to $\tilde{G}/\tilde{B}\times\tilde{G}/\tilde{B}^-$. Then, Lemma 3.4.5 allows us to identify $\mathrm{Pic}^{\tilde{G}\times\tilde{G}}(Z)$ with $\mathcal{X}(\tilde{B})\times\mathcal{X}(\tilde{B}^-)$. Let $\rho_Z\,:\,\mathrm{Pic}^{\tilde{G}\times\tilde{G}}(X)\longrightarrow\mathrm{Pic}^{\tilde{G}\times\tilde{G}}(Z)$ be the restriction homomorphism. Then, by the preceding isomorphisms, ρ_Z induces a morphism $\overline{\rho}_Z\,:\,\mathrm{Pic}(X)\longrightarrow\mathcal{X}(\tilde{B})\times\mathcal{X}(\tilde{B}^-)$. Then, we have (see [Bri97] and [Res00]):

Proposition 4.4.4 *With above notation (X is simple and toroidal), we have:*

(i) The morphism $\overline{\rho}_Z$ induces an isomorphism

$$\mathrm{Pic}(X)\stackrel{\sim}{\longrightarrow}\{(\lambda,-\lambda)\,:\,\lambda\in\mathcal{X}(\tilde{B})\}.$$

If $\lambda\in\mathcal{X}(\tilde{B})$, we denote by $\mathcal{L}_\lambda$ the $\tilde{G}\times\tilde{G}$-linearized line bundle such that $\overline{\rho}_Z(\mathcal{L}_\lambda)=(\lambda,-\lambda)$.

(ii) If $\lambda\in\mathcal{X}(\tilde{B})$, $\mathcal{L}_\lambda$ is generated by its global sections (resp. ample) if and only if λ is dominant (resp. dominant regular).

5 GIT-quotient by a spherical subgroup

5.1 — In this section, X denotes a normal projective variety endowed with an action of a semi-simple group G, and H denotes a spherical subgroup of G. As in Geometric Invariant Theory, to each ample H-linearized line bundle on X, we will associate an open subset of X which admits a categorical quotient by H in the category of affine morphisms.

Let $r_H : \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}^H(X)$ and $r : \mathrm{Pic}^H(X) \longrightarrow \mathrm{Pic}(X)$ denote the morphisms of restriction of the actions. A character of H induces a linearization of the trivial line bundle. This defines an embedding i of $\mathcal{X}(H)$ into $\mathrm{Pic}^H(X)$.

With these notation, it is shown in [KKV84] that the following sequence:

$$0 \longrightarrow \mathcal{X}(H)_{\mathbb{Q}} \stackrel{i}{\longrightarrow} \mathrm{Pic}^H(X)_{\mathbb{Q}} \stackrel{r}{\longrightarrow} \mathrm{Pic}(X)_{\mathbb{Q}} \longrightarrow 0$$

is exact. Applying this to G and H, we easily obtain

Lemma 5.1.1 *The morphism*

$$\begin{array}{rrcl} \phi : & \mathrm{Pic}^G(X)_{\mathbb{Q}} \times \mathcal{X}(H)_{\mathbb{Q}} & \longrightarrow & \mathrm{Pic}^H(X)_{\mathbb{Q}} \\ & (\mathcal{L}, \chi) & \longmapsto & r_H(\mathcal{L}) \otimes i(\chi) \end{array}$$

is surjective.

Now, we can prove the fundamental lemma of this section:

Lemma 5.1.2 *Let $\mathcal{L}$ be an H-linearized line bundle on X. Then, the algebra $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})^H$ of H-invariant sections is finitely generated.*

Proof: By Lemma 5.1.1, there exist a positive integer m, $\mathcal{L}_0$ in $\mathrm{Pic}^G(X)$ and χ in $\mathcal{X}(H)$ such that $\mathcal{L}^{\otimes m} = r_H(\mathcal{L}_0) \otimes i(\chi)$. Then, with the notation of Section 3.4, we have a canonical isomorphism:

$$\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes mn})^H \simeq \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}_0^{\otimes n})^{H,-n\chi}.$$

The grading of $\oplus_{n\geq 0}\Gamma(X, \mathcal{L}_0^{\otimes n})$ and the G-linearization of $\mathcal{L}_0$ define an action of $G \times k^*$ on $\oplus_{n\geq 0}\Gamma(X, \mathcal{L}_0^{\otimes n})$. Consider $H_\chi = \{(h, \chi(h)) : h \in H\}$. Then, we have

$$\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}_0^{\otimes n})^{H,-n\chi} = \left(\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}_0^{\otimes n})\right)^{H_\chi}.$$

Moreover, H_χ is a spherical subgroup of $G \times k^*$ and the algebra $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}_0^{\otimes n})$ is finitely generated. Then, Theorem 9.3. of [Gro97] shows that $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes mn})^H$ is finitely generated.

On the other hand, the ring $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})^H$ is integral on $\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes mn})^H$. We conclude by Theorem 2, Chap. V (§3.2) of [Bou64]. □

By Lemma 5.1.2, if $\mathcal{L}$ is an ample H-linearized line bundle , we set, as in GIT for reductive groups:

$$
\begin{array}{ll}
Y(\mathcal{L}) = \mathrm{Proj}\left(\bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes n})^H\right), & \\
X^{\mathrm{ss}}(\mathcal{L}) = \{x\in X\ :\ \exists n>0, \sigma\in\Gamma(X,\mathcal{L}^{\otimes n})^H\ :\quad \sigma(x)\neq 0\}, \text{ and} & \\
\pi\ :\ X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow Y(\mathcal{L}) & \text{the morphism induced by the inclusion of} \bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes n})^H \\
 & \text{in } \bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes n}).
\end{array}
$$

5.2 — The preceding construction has the following properties:

Proposition 5.2.3 *Keep notation as above; in particular, $\mathcal{L}$ is ample. Then, we have:*

(i) *The map π is affine. Moreover, for every affine open subset U in $Y(\mathcal{L})$, we have $k[\pi^{-1}(U)]^H = \pi^*(k[U])$.*

(ii) *If Y is a variety and $\phi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow Y$ is an affine H-invariant map then there exists a map $\widetilde{\phi} : Y(\mathcal{L}) \longrightarrow Y$ such that the following diagram is commutative:*

$$
\begin{array}{ccc}
X^{\mathrm{ss}}(\mathcal{L}) & \xrightarrow{\phi} & Y \\
{\scriptstyle\pi}\downarrow & \nearrow_{\widetilde{\phi}} & \\
Y(\mathcal{L}) & &
\end{array}
$$

In particular, $Y(\mathcal{L})$ only depends on $X^{\mathrm{ss}}(\mathcal{L})$, and is denoted by $X^{\mathrm{ss}}(\mathcal{L})//H$.

(iii) *The variety $X^{\mathrm{ss}}(\mathcal{L})//H$ is normal.*

(iv) *The map π is surjective in codimension one.*

(v) *Let Z be a G-stable closed subvariety of X. If $\mathcal{L}_{|Z}$ denotes the restriction of the H-linearized line bundle $\mathcal{L}$ to Z, then we have $Z^{\mathrm{ss}}(\mathcal{L}_{|Z}) = Z\cap X^{\mathrm{ss}}(\mathcal{L})$. Moreover the restriction of π to $Z\cap X^{\mathrm{ss}}(\mathcal{L})$ identifies canonically with the quotient of Z by H.*

Proof: The proofs of Assertions (i) and (iii) are the same as for reductive quotients (see [Res00] for details). Assertion (ii) is a direct consequence of the first one.

To prove Assertion (iv) let us fix a prime divisor D in $X^{\mathrm{ss}}(\mathcal{L})//H$. Since $X^{\mathrm{ss}}(\mathcal{L})//H$ is normal, there exists an affine open subset U and a regular function f on U such that $D\cap U$ is non-empty and equal to $\{x\in U\ :\ f(x)=0\}$. Let π_U^* denote the inclusion of $k[U]^H$ in $k[U]$. Consider $\widetilde{D} = \pi^{-1}(D\cap U) = \{x\in\pi^{-1}(U)\ :\ \pi_U^*(f)(x)=0\}$.

If A is a ring and a belongs to A, then we denote by $a.A$ the ideal generated by a. Since $\pi_U^*(k[U]) = k[\pi^{-1}(U)]\cap k(X)^H$ and $f\in k(X)^H$, we have:

$$\left(\pi_U^*(f).k[\pi^{-1}(U)]\right)\cap k(X)^H = \pi_U^*(f.k[U]).$$

This shows that $k[U]/f.k[U]$ embeds into $k[\pi^{-1}(U)]/f.k[\pi^{-1}(U)]$ and so $\pi(\widetilde{D})$ is dense in D. Assertion (iv) follows.

Let us prove Assertion (v). By Lemma 5.1.1, replacing $\mathcal{L}$ by a power if necessary, we can assume that there exist $\mathcal{L}_0 \in \mathrm{Pic}^G(X)$ and $\chi \in \mathcal{X}(H)$ such that $\mathcal{L} = r_H(\mathcal{L}_0) \otimes i(\chi)$. Replacing $\mathcal{L}$ by a power again, we can assume that the restriction morphism $\rho : \Gamma(X, \mathcal{L}_0) \longrightarrow \Gamma(Z, \mathcal{L}_{0|Z})$ is surjective. Since G is reductive and ρ is G-equivariant, there exists a sub-module M of $\Gamma(X, \mathcal{L}_0)$ such that ρ induces an isomorphism of G-modules between M and $\Gamma(Z, \mathcal{L}_{0|Z})$. In particular, ρ induces a surjection from $\Gamma(X, \mathcal{L}_0)^{H,\chi}$ onto $\Gamma(Z, \mathcal{L}_{0|Z})^{H,\chi}$. Assertion (v) follows easily. □

Remark: 1) If H is reductive, then Assertion (ii) holds without assuming that ϕ is affine. But in general, this assumption cannot be omitted. Indeed, one can easily find an example where H is a Borel subgroup of G (see [Res00]).

2) If H is reductive, then the quotient morphism is surjective; but this does not hold in general. Consider, indeed, the additive group $\mathbb{G}_a$ of the field k. Let M_2 denote the vector space of 2×2-matrices and let $\mathbb{P}(M_2)$ be the corresponding projective space. We define an action of $\mathbb{G}_a$ on $\mathbb{P}(M_2)$ by:

$$\tau.[m] = [\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} m] \qquad \forall \tau \in k \text{ and } m \in M_2,\ m \neq 0$$

One checks easily that the quotient of $\mathbb{P}(M_2)$ by $\mathbb{G}_a$ associated to $\mathcal{L} = \mathcal{O}(1)$ is not surjective.

5.3 — Now, we set: $X^{\mathrm{s}}(\mathcal{L}) := \{x \in X^{\mathrm{ss}}(\mathcal{L}) \ : \ \pi^{-1}(\pi(x)) = H.x\}$. Points in $X^{\mathrm{s}}(\mathcal{L})$ are said to be *stable* for $\mathcal{L}$.

Remark: Assume that there exists a point in X with finite isotropy in H. Then, one checks easily that any stable point x has a finite stabilizer in H and a closed H-orbit in $X^{\mathrm{ss}}(\mathcal{L})$. When H is reductive the converse is also true. But this converse is false in general (see [Res00]; 5.2 for an example).

We have the following criterion for existence of stable points:

Proposition 5.3.4 *Let d be the dimension of the general orbits of H in X. Then, the following assertions are equivalent:*

(i) $\dim(X^{\mathrm{ss}}(\mathcal{L})/\!/H) + d{=}\dim(X)$.

(ii) Each general fiber of π contains a unique dense orbit of H.

Proof: The implication $(ii) \Rightarrow (i)$ is trivial. Let us prove the converse. Consider an affine open subset U_H in $X^{\mathrm{ss}}(\mathcal{L})/\!/H$ and set $U = \pi^{-1}(U_H)$. We claim that the quotient field $\mathrm{Frac}(k[U]^H)$ equals the field $k(U)^H$ of invariant rational functions on U.

By Rosenlicht's Theorem (see [PV89]; Theorem 2.3), the transcendence degree of $k(U)^H$ equals $\dim(U) - d$. Since $k[U]^H = k[U_H]$, the transcendence degree of $\mathrm{Frac}(k[U]^H)$ is the dimension of $X^{\mathrm{ss}}(\mathcal{L})/\!/H$. So, $k(U)^H$ is a finite extension of $\mathrm{Frac}(k[U]^H)$.

Let $f \in k(U)^H$. Then, there exist $a_0, \cdots, a_k \in k[U]^H$ such that $a_0 f^k + a_1 f^{k-1} + \cdots + a_k = 0$. Multiplying by a_0^{k-1}, we obtain that $a_0 f$ is integral on $k[U]$. Now, the normality of X (and so of U) implies that $a_0 f \in k[U]$. Then f belongs to $\mathrm{Frac}(k[U]^H)$. This proves the claim.

The other part of Rosenlicht's Theorem then shows that there exists a restriction of π to an open subset of $X^{\mathrm{ss}}(\mathcal{L})$ which is a geometric quotient. Assertion (*ii*) follows. $\square$

5.4 — In this paragraph, we show that some power of $\mathcal{L}$ descends to an ample line bundle on $X^{\mathrm{ss}}(\mathcal{L})//H$.

Since the graded algebra $\bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes n})^H$ is finitely generated, Proposition 3 of Chapter III of [Bou61] shows that there exists a positive integer m such that $\bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes mn})^H$ is generated by $\Gamma(X,\mathcal{L}^{\otimes m})^H$. Then, we consider the map:

$$\begin{array}{rccl} \phi\ : & X^{\mathrm{ss}}(\mathcal{L})//H & \longrightarrow & \mathbb{P}\left((\Gamma(X,\mathcal{L}^{\otimes m})^H)^*\right) \\ & y & \longmapsto & \left\{\sigma\in\Gamma(X,\mathcal{L}^{\otimes m})^H\ :\ \sigma(y)=0\right\}. \end{array}$$

We set $\mathcal{L}^{\otimes m}//H := \phi^*(\mathcal{O}(1))$. Then, $\mathcal{L}^{\otimes m}//H$ is a very ample line bundle on $X^{\mathrm{ss}}(\mathcal{L})//H$. Moreover, $\pi^*(\mathcal{L}^{\otimes m}//H)=\mathcal{L}^{\otimes m}{}_{|X^{\mathrm{ss}}(\mathcal{L})}$ and we have a canonical isomorphism:

$$\bigoplus_{n\geq 0}\Gamma(X^{\mathrm{ss}}(\mathcal{L})//H,(\mathcal{L}^{\otimes m}//H)^{\otimes n})\simeq\bigoplus_{n\geq 0}\Gamma(X,\mathcal{L}^{\otimes mn})^H.$$

6 Projective embeddings of G/H as quotients of completions of G: a criterion

6.1 — We fix again a semi-simple group G, a spherical subgroup H of G and a projective embedding Y of G/H. The following theorem answers the question: can Y be obtained as a quotient by $\{1\}\times H$ of a $G\times G$-equivariant projective embedding of G ?

An action of G is said to be *quasi-faithful* if its kernel is finite.

Theorem 1 *Assume that the action of G on G/H is quasi-faithful. Let Y be a projective embedding of G/H. Then, the two following conditions are equivalent:*

(i) There exist a projective $G\times G$-equivariant embedding X of G and an ample $G\times H$-linearized line bundle $\mathcal{L}$ on X such that $Y=X^{\mathrm{ss}}(\mathcal{L})//H$.

(ii) For any G-stable prime divisor D in Y, the valuation ν_D of the field $k(G/H)$ with center D extends to a $G\times G$-invariant valuation of $k(G)$.

Proof: (*i*) $\Rightarrow$ (*ii*): Consider the quotient-morphism $\pi\ :\ X^{\mathrm{ss}}(\mathcal{L})\longrightarrow Y$. Let D be a G-stable prime divisor of Y.

Since π is surjective in codimension one, there exists a prime divisor E of X such that $\pi(E\cap X^{\mathrm{ss}}(\mathcal{L}))=D$. Let $\pi^*\ :\ k(Y)\longrightarrow k(X)$ denote the map induced by π. Then, $\nu_D=\nu_E\circ\pi^*$.

Since π is G-equivariant, we have $\pi(G)=\pi(G/H)$. Then, E is contained in $X-G$ and so is stable by $G\times G$. Moreover, the map π^* is the canonical embedding of $k(G/H)$ in $k(G)$.

Now, Assertion (*ii*) follows from the relation $\nu_D=\nu_E\circ\pi^*$.

(*ii*) $\Rightarrow$ (*i*): Let $\mathcal{M}$ be a very ample G-linearized line bundle on Y. Set $V=\Gamma(Y,\mathcal{M})^*$. Then, Y is embedded in $\mathbb{P}(V)$. By Exercise 5.1.4 of [Har77], replacing $\mathcal{M}$ by a power if necessary, we can assume that the affine cone $\tilde{Y}$ over Y is normal. Let y be the base point of Y and $\tilde{y}$ be a lift of y in V. The scalar multiplication on the G-module V gives an action of $G \times k^*$ on V. There exists a character χ of H such that the isotropy of $\tilde{y}$ in $G \times k^*$ is equal to $H_\chi = \{(h, \chi(h)) \,:\, h \in H\}$.

We denote by ρ the action map $G \longrightarrow \mathrm{PGL}(V)$ and by G^1 its image. Consider the closure X^1 of G^1 in $\mathbb{P}(\mathrm{End}(V))$, $\widetilde{X^1}$ the corresponding affine cone in $\mathrm{End}(V)$, and the map:

$$\begin{array}{cccc} \tilde{\psi} : & \widetilde{X^1} & \longrightarrow & \tilde{Y} \\ & m & \longmapsto & m.\tilde{y} \end{array}$$

We claim that $\tilde{\psi}$ is surjective in codimension one. Otherwise, there exists a prime divisor $\widetilde{D}$ in the closure of $\tilde{Y} - \mathrm{Im}(\tilde{\psi})$. Then, $\widetilde{D}$ is stable by $G \times k^*$ and is the affine cone over a G-stable divisor of Y.

Let X^1_{ss} be the image in X^1 of the pullback in $\widetilde{X^1}$ of $\tilde{Y} - \{0\}$ by $\tilde{\psi}$. Then, $\tilde{\psi}$ restricts to $\psi \,:\, X^1_{\mathrm{ss}} \longrightarrow Y$.

By assumption, there exists a $G \times G$-invariant valuation ν of $k(G)$ whose restriction to $k(G)^H$ is ν_D. Since X^1 is complete, $\nu \circ \rho^*$ has a center Z in X^1. So $\nu \circ \psi^* = \nu_D$ and $\psi(Z)$ is dense in D. This contradiction proves the claim.

Via ρ, $k(X^1)$ is embedded into $k(G)$. Let us consider the normalization X of X^1 in $k(G)$ and the corresponding morphism, $\phi \,:\, X \longrightarrow X^1$.

Let $\mathcal{L}^1$ denote the restriction to X^1 of $\mathcal{O}(1)$ on $\mathbb{P}(\mathrm{End}(V))$ and $\mathcal{L}$ its pullback by ϕ. Since the action of G on G/H is quasi-faithful, ρ and ϕ are finite. Thus, $\mathcal{L}$ is ample. Replacing $\mathcal{M}$ (and so $\mathcal{L}$) by a power if necessary, we can assume that X is embedded into $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$. Consider the affine cone $\widetilde{X}$ over X in $\Gamma(X, \mathcal{L})^*$ and the k^*-equivariant map $\tilde{\phi} \,:\, \widetilde{X} \longrightarrow \widetilde{X^1}$ over ϕ.

Note that $\widetilde{X}$ is endowed with an action of $G \times G$ such that $\tilde{\phi}$ is $G \times G$-equivariant. Then, $\tilde{\phi} \circ \tilde{\psi} \,:\, \widetilde{X} \longrightarrow \tilde{Y}$ induces a commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\;\tilde{\phi} \circ \tilde{\psi}\;} & \tilde{Y} \\ & {\scriptstyle \pi} \searrow \quad \nearrow {\scriptstyle \eta} & \\ & \mathrm{Spec}(k[\widetilde{X}]^{H_\chi}) & \end{array}$$

Since η is $G \times k^*$-equivariant, the stabilizer of $\pi(1)$ in $G \times k^*$ is contained in H_χ. So it is equal to H_χ, since π is H_χ-invariant. In particular, η is birational.

Moreover, the claim implies that η is surjective in codimension one. Since $\tilde{Y}$ is normal, Richardson's Lemma (see [PV89]) shows that η is an isomorphism. Then, Y equals $X^{\mathrm{ss}}(\mathcal{L} \otimes \chi)/\!/H$. $\square$

Remarks 1- Note that a G-invariant valuation ν of $k(G/H)$ always extends to a G-invariant (for the left multiplication) valuation of $k(G)$ (see [Kno91] or [Bri97]). But, as shown by the example in Appendix A of [Res00], a $G \times G$-invariant extension of ν may not exist. In particular, Condition (ii) of Theorem 1 may not hold.

2- The construction used in the proof of Theorem 1 is essentially due to L. Renner (see [Ren89]). But, in his article L. Renner forgot an essential assumption (that is, Condition (ii) of Theorem 1). Moreover, he assumed that H is semi-simple.

6.2 — Theorem 1 motivates the following

Definition A spherical homogeneous space G/H is said to be *liftable* if any G-invariant valuation of $k(G/H)$ extends to a $G \times G$-invariant valuation of $k(G)$.

Proposition 6.2.1 *Let G be a semi-simple group. Then, we have:*

(i) Let $H_1 \subset H_2$ be two spherical subgroups of G. Then, if G/H_1 is liftable then so is G/H_2.

(ii) Let $H_1 \subset H_2$ be two spherical subgroups of G such that the index of H_1 in H_2 is finite. Then, G/H_1 is liftable if and only if G/H_2 is.

(iii) If H is symmetric (i.e. the set of the fix points of an automorphism of G of order 2) then G/H is liftable.

Proof: Assertion (i): Let ν be a G-invariant valuation of $k(G/H_2)$. Then, by Corollary 3.1.1 of [Bri97] or by [Kno91] there exists a $G \times \{1\}$-invariant valuation $\overline{\nu}$ of $k(G)$ such that ν is the restriction of $\overline{\nu}$ to $k(G/H_2)$. Since, G/H_1 is liftable, the restriction of $\overline{\nu}$ to $k(G/H_1)$ extends to a $G \times G$-invariant valuation of $k(G)$. Assertion (i) follows.
Assertion (ii): Let B be a Borel subgroup of G opposite to H_2. With the assumptions of Assertion (ii), $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H_1}, \mathbb{Q})$ identifies canonically with $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H_2}, \mathbb{Q})$. Moreover, Corollary 3.1.1 of [Bri97] (see also [Kno91]) shows that $\mathcal{CV}(G/H_1)$ maps onto $\mathcal{CV}(G/H_2)$. Since for $i = 1$ or 2, $\mathcal{CV}(G/H_i)$ embeds in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H_i}, \mathbb{Q})$, this implies that $\mathcal{CV}(G/H_1)$ identifies with $\mathcal{CV}(G/H_2)$. Assertion (ii) follows.
Assertion (iii): By Assertion (ii), we can assume that G is adjoint and that G/H has a canonical embedding $\underline{X}$, with the notation of [CP83] (note that $\underline{X}$ is called the *wonderful* compactification of X). Let $\mathcal{L}$ be an ample G-linearized line bundle on $\underline{X}$. Consider the vertices p of the moment polytope $\mathbf{P}(\underline{X}, \mathcal{L})$ corresponding to the unique closed orbit of G in $\underline{X}$. Then, Proposition 8.2 of [CP83] describes the cone generated by $-p + \mathbf{P}(\underline{X}, \mathcal{L})$. Indeed, this cone is the intersection of $\mathcal{X}(B)_{\mathbb{Q}}^{B\cap H}$ and the cone of $\mathcal{X}(B)_{\mathbb{Q}}$ generated by the opposite of simple roots. Thus, the dual cone of $-p + \mathbf{P}(\underline{X}, \mathcal{L})$ in $\mathrm{Hom}(\mathcal{X}(B)_{\mathbb{Q}}^{B\cap H}, \mathbb{Q})$ is the image by the restriction of the negative Weyl chamber of $\mathrm{Hom}(\mathcal{X}(B)_{\mathbb{Q}}, \mathbb{Q})$. By Proposition 3.3.4, this implies that the cone $\mathcal{CV}(G/H)$ is the image of the cone $\mathcal{CV}(G)$ generated by the $G \times G$-invariant valuations of $k(G)$. □

Remark: In [Kan99], S. Kannan showed that the canonical embedding of a symmetric space is a GIT-quotient of the canonical embedding of the group. This also follows from Theorem 1 and Proposition 6.2.1.

The symmetric spaces are a first family of liftable spherical homogeneous spaces. The following proposition gives another one:

Proposition 6.2.2 *Let G/H be a sober spherical homogeneous space. If H is solvable then G/H is liftable.*

Proof: Let Y be the canonical embedding of G/H and y be its base point. Let B^- be a Borel subgroup of G containing H. Then, by [Kno91] or [Bri97], the canonical map $G/H \longrightarrow G/B^-$ extends to a G-equivariant map $\phi : Y \longrightarrow G/B^-$. Consider the B^--variety $\Sigma = \phi^{-1}(B^-/B^-)$.

Let B be a Borel subgroup of G opposite to H and hence to B^-. Set $T = B \cap B^-$. Denote by U the unipotent radical of B. Consider the following commutative diagram:

$$\begin{array}{ccc} U \times \Sigma & \xrightarrow[\text{isomorphism}]{(u,s)\mapsto us} & U.\Sigma \\ {\scriptstyle (u,s)\mapsto u}\downarrow & & \downarrow{\scriptstyle \phi} \\ U & \xrightarrow{u\mapsto uB^-/B^-} & G/B^-. \end{array}$$

The subset $U.\Sigma$ is open in Y. Then, $T.y$ is dense in Σ which is a toric variety. Since Y contains a unique closed orbit of G, Σ contains a unique closed orbit of B^-. This orbit being projective, it is a fixed point denoted by z. Consider the unique affine T-stable and open subset Σ_z of Σ containing z. Then, by the previous diagram $U \times \Sigma_z$ is isomorphic to $Y_{G.z,B}$. We deduce that the cone $\mathcal{CV}(G/H)$ identifies with the cone $\mathcal{C}$ associated to the affine toric variety Σ_z.

We want to determine the rays of the cone $\mathcal{C}^\vee$ generated by the weights of the action of T in $k[\Sigma_z]$. Let x be a point in Σ_z such that $\dim(T.x)$ equals one. Consider the restriction morphism $\rho : k[\Sigma_z]^{(T)} \longrightarrow k[\overline{T.x}^{\Sigma_z}]^{(T)}$.

Since ρ is surjective, the half-line generated by a weight of T in $k[\overline{T.x}^{\Sigma_z}]$ is contained in $\mathcal{C}^\vee$. Moreover, the classical theory of toric varieties (see [Ful93] or [Oda88]) shows that this half-line is a ray of $\mathcal{C}^\vee$ and that conversely all rays of $\mathcal{C}^\vee$ are obtained in this way. Thus, it remains to compute the weights of the action of T in $k[\overline{T.x}^{\Sigma_z}]$.

Consider the closure S of $T.x$ in Σ. Since Y is toroidal, all T-stable divisors in Σ containing z are stable by B^-. Then, S is stable by B^-. On the other hand, as a projective toric variety of dimension one, S is isomorphic to $\mathbb{P}^1$. Moreover, $B^-.x$ is either isomorphic to k or k^*. If $B^-.x$ is isomorphic to k^*, then B^- has two fixed point in S; that is not possible. We deduce that $B^-.x$ is isomorphic to k.

Let B^-_x (resp. T_x) be the stabilizer of x in B^- (resp. in T_x). Since $B^-.x$ is isomorphic to k, B^-_x does not contain the unipotent radical of B^-. Then, there exists a simple root α of (B^-, T) such that the unipotent one parameter subgroup U_α of B^- associated to α does not fix x.

We claim that the restriction of α to T_x is trivial. Indeed, let $\xi : k \longrightarrow U_\alpha$ be the canonical isomorphism. Since $T.x$ is open in S, there exist $\epsilon \in k^*$ and $t_0 \in T$ such that $\xi(\epsilon).x = t_0.x$. Let $t \in T_x$. Then, we have:

$$\begin{aligned} \xi(\alpha^\vee(t)\epsilon).x &= t\xi(\epsilon)t^{-1}.x = t.\xi(\epsilon).x \\ &= tt_0.x = t_0t.x \\ &= \xi(\epsilon). \end{aligned} \tag{1}$$

Moreover, since k has no non-trivial subgroup, $B_x \cap U_\alpha$ is trivial. Then, Equality (1) implies that $\alpha(t) = 1$. This proves the claim.

By the claim, α or $-\alpha$ is a weight of the T-module $k[\Sigma_z \cap S]$. On the other hand, it is shown in [BP90] (see also Section 4.2 of [Bri97]) that $\mathcal{C}^\vee$ is contained in the cone generated by the negative roots. We deduce that α is a weight of the T-module $k[\Sigma_z \cap S]$.

We just proved that the rays of the dual cone of $\mathcal{CV}(G/H)$ contain the simple roots of (B^-, T). The proposition follows easily. □

6.3 — Consider $G = \mathrm{PGL}(3)$ and the symmetric subgroup $H = \mathrm{PSO}(3)$. Then G/H is the set of (non-degenerated) conics in $\mathbb{P}^2$. Associating to each conic its equation defines an embedding Y of G/H into $\mathbb{P}^5$.

Theorem 1 and Proposition 6.2.1 show that Y is the quotient of a projective embedding of G. But it is not a geometric quotient of any projective embedding of G (see [Res00]; 7.5.2 for details). Note that the embedding Y is not toroidal.

In the sequence of this article, our main aim is to obtain projective embeddings of spherical homogeneous spaces as geometric quotients of projective embeddings of the group. So, the preceding example explain why we pay now a particular attention to colorless embeddings (and colorless orbits).

7 $B \times H$-stable divisors in embeddings of G

7.1 — Let X be an embedding of G and $\mathcal{O}$ be a colorless orbit of $G \times G$ in X. Let $D \in \mathcal{D}(G/H)$. We denote by $\overline{D}^X$ the closure in X of the set of all $g \in G$ such that $gH/H \in D$. The aim of this section is to determine the intersection of $\mathcal{O}$ and $\overline{D}^X$.

7.2 — The first step is to show that $\mathcal{O}$ contains an open $B \times H$-orbit whose complement in $\mathcal{O}$ is a divisor.

By Proposition 4.3.2, there exist two opposite parabolic subgroups P and Q of G and a point x in $\mathcal{O}$ such that the isotropy I of x in $G \times G$ is $(P^u \times Q^u).(\Delta L.(\{1\} \times C))$ where $L = P \cap Q$, $\Delta L = \{(l, l) \;:\; l \in L\}$ and C is a subgroup of the connected center of L. Moreover, replacing x by another point in $\mathcal{O}$ if necessary, we can assume that Q contains B.

The inclusion of I in $P \times G$ defines a $G \times \{1\}$-equivariant fibration $p \,:\, \mathcal{O} \longrightarrow G/P$. The fiber F over P/P is the $P \times G$-homogeneous space $(P \times G)/I$. Note that F is homogeneous under the action of $\{1\} \times G$. Moreover, the inclusion of $I \cap (\{1\} \times G)$ in $\{1\} \times B$ induces a $\{1\} \times G$-equivariant fibration $q \,:\, F \longrightarrow G/B$. We obtain a diagram:

$$\begin{array}{ccc} F & \longrightarrow & \mathcal{O} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ G/B & & G/P \end{array} \qquad (7.2.1)$$

Let $E \in \mathcal{D}(G/P)$ be a prime B-stable divisor in G/P. Then, $p^{-1}(E)$ is a prime $B \times H$-divisor of $\mathcal{O}$ denoted by $E_{\mathcal{O}}$. Now, we consider $\mathcal{O} - \bigcup_{E \in \mathcal{D}(G/P)} E_{\mathcal{O}}$. Each orbit of $B \times H$ in $\mathcal{O} - \bigcup_{E \in \mathcal{D}(G/P)} E_{\mathcal{O}}$ intersects F in a unique orbit of $(P \cap B) \times H$; so, for all $y \in \mathcal{O}$, $(B \times H).y \cap F$ is either the empty set or the preimage by q of a unique orbit of H in G/B.

If $D \in \mathcal{D}(G/H)$, we set $D_B = \{gB/B \in G/B \, : \, g^{-1}H/H \in D\}$. We denote by $D_{\mathcal{O}}$ the closure of $(B \times H).q^{-1}(D_B)$ in $\mathcal{O}$. Then, $D_{\mathcal{O}}$ is a $B \times H$-stable divisor in $\mathcal{O}$.

The previous discussion shows

Lemma 7.2.1 *With the previous notation (in particular $B \subset Q$), we have:*

$$\mathcal{O} = (B \times H).x \cup \bigcup_{D \in \mathcal{D}(G/H)} D_{\mathcal{O}} \cup \bigcup_{E \in \mathcal{D}(G/P)} E_{\mathcal{O}}.$$

7.3 — Let $\Sigma(L)$ denote the set of simple roots of $(B \cap L, T)$. Then, the Bruhat decomposition yields $\mathcal{D}(G/P) = \{\overline{Bs_\alpha P/P} \, : \, \alpha \in \Sigma - \Sigma(L)\}$. We set $E^\alpha = \overline{Bs_\alpha P/P}$. Returning to the situation of Section 7.1, we can now formulate a description of $\overline{D}^X \cap \mathcal{O}$:

Proposition 7.3.2 *Let X be an embedding of G and $\mathcal{O}$ a colorless orbit of $G \times G$ in X. Let P, Q and L be as in Section 7.2. Let $D \in \mathcal{D}(G/H)$ and let γ_D be the B-weight of its equation. Write $\gamma_D = \sum_{\alpha\in\Sigma} k_\alpha \omega_\alpha$, with $k_\alpha \in \mathbb{N}$.*

Then, with the notation of Lemma 7.2.1, we have:

$$\overline{D}^X \cap \mathcal{O} = D_{\mathcal{O}} \cup \bigcup_{\alpha \in \Sigma - \Sigma(L) \text{ s.t. } k_\alpha \neq 0} E^\alpha_{\mathcal{O}}.$$

7.4 — From Paragraph 7.4 to Paragraph 7.7, we will prove Proposition 7.3.2. First, we define and calculate"equations" of $D_{\mathcal{O}}$ and $E_{\mathcal{O}}$ as we have defined the equations of elements of $\mathcal{D}(G/H)$ in Section 2.2.

Consider the universal covering $\zeta \, : \, \widetilde{G} \longrightarrow G$ and the map $q_{G\times G} \, : \, \widetilde{G} \times \widetilde{G} \longrightarrow \mathcal{O}$, $(g_1, g_2) \mapsto (\zeta(g_1, g_2)).x$. If $D \in \mathcal{D}(G/H)$, we denote by $f_{D_{\mathcal{O}}}$ the unique equation of $q^{-1}_{G\times G}(D_{\mathcal{O}})$ in $k[\widetilde{G} \times \widetilde{G}]$ such that $f_{D_{\mathcal{O}}}(1) = 1$. We define $f_{E_{\mathcal{O}}}$ similarly.

To compute $f_{D_{\mathcal{O}}}$ and $f_{E_{\mathcal{O}}}$, we fix our attention on $\mathcal{O}$. Considering the action of $\widetilde{G} \times \widetilde{G}$ on $\mathcal{O}$, we can assume that G is simply connected. Moreover, the inclusion of $(P^u \times Q^u).\Delta L$ in I induces a commutative diagram:

$$\begin{array}{ccc} & G \times G/((P^u \times Q^u).\Delta L) & \\ \nearrow & & \searrow^{\phi} \\ G \times G & \xrightarrow[q_{G\times G}]{} & \mathcal{O} \end{array}$$

Then, applying Lemma 7.4.3 below to ϕ, we can assume that C is trivial.

Lemma 7.4.3 *Let Γ be a linear algebraic group, Γ_1 and Γ_2 two closed subgroups of Γ such that $\Gamma_1 \subset \Gamma_2$. Consider the natural map $\phi \, : \, \Gamma/\Gamma_1 \longrightarrow \Gamma/\Gamma_2$. Let D be a prime divisor in Γ/Γ_2. Then, the pullback $\phi^*(D)$ of D by ϕ is the sum of the irreducible components of $\phi^{-1}(D)$, with multiplicity being one.*

Proof: In this proof, if Y is a variety and y is a point in Y, we denote by $\mathcal{O}_{Y,y}$ the local ring of rational functions in Y defined at y. By absurd, we assume that there exists an irreducible component E of $\phi^{-1}(D)$ such that $\phi^*(D) - 2E$ is effective. Since all fibers of ϕ have the same dimension, $\phi(E)$ is dense in D. In particular, there exists $x \in E$ such that $\phi(x)$ is smooth in D. Then, there exists a local equation $f \in \mathcal{O}_{\Gamma/\Gamma_2,\phi(x)}$ of D at $\phi(x)$. There also exists a local equation $g \in \mathcal{O}_{\Gamma/\Gamma_1,x}$ of E at x. Since, $\phi^*(D) - 2E$ is effective, $h := \frac{f\circ\phi}{g^2}$ belongs to $\mathcal{O}_{\Gamma/\Gamma_1,x}$. So, the differential of $f \circ \phi$ at x is zero. But, since ϕ is equivariant, its differential is surjective at any point of Γ/Γ_1. Then, the differential of f at $\phi(x)$ is zero. This is impossible because of smoothness of D at $\phi(x)$. The lemma is proved. □

To compute $f_{D_\mathcal{O}}$ and $f_{E_\mathcal{O}}$, we will also use the following

Lemma 7.4.4 *Assume that G is simply connected and C is trivial. Let $D \in \mathcal{D}(G/H)$. Set $\widetilde{D}^{-1} := \{g \in G \,:\, g^{-1}H/H \in D\}$. Consider*

$$\begin{array}{rccc} q_2 : & G & \longrightarrow & \mathcal{O} \\ & g & \longmapsto & (1,g).x. \end{array}$$

Then, the pullback $q_2^*(D_\mathcal{O})$ of $D_\mathcal{O}$ by q_2 equals $\widetilde{D}^{-1}$.

Proof: Let us use the notation of Diagram (7.2.1). For this proof, we set $U := p^{-1}(BP/P)$. Since q_2 is a fibration, and U is an open subset of $\mathcal{O}$ which intersects $D_\mathcal{O}$ it suffices to determine $q_2^*(U \cap D_\mathcal{O})$.

Consider the action of $B \cap L$ on F by right multiplication. Then, the quotient of $B \times F$ by the diagonal action of $B \cap L$ exists and is denoted by $B \times_{B\cap L} F$. With obvious notation, we set

$$\begin{array}{rccc} \xi : & B \times_{B\cap L} F & \longrightarrow & U \\ & \overline{(b,f)} & \longmapsto & b.f \end{array}$$

One easily shows that ξ is bijective; then, the normality of U implies that ξ is an isomorphism. Then, we have $\xi^*(D_\mathcal{O} \cap U) = 1.\left(B \times_{B\cap L} q_2(\widetilde{D}^{-1})\right)$. Consider now:

$$\begin{array}{rccc} i : & F & \longrightarrow & B \times_{B\cap L} F \\ & f & \longmapsto & \overline{(1,f)}. \end{array}$$

We have $i^*\left(B \times_{B\cap L} q_2(\widetilde{D}^{-1})\right) = 1.q_2(\widetilde{D}^{-1})$. To conclude, we factor q_2 as

$$G \longrightarrow F \stackrel{i}{\longrightarrow} B \times_{B\cap L} F \stackrel{\xi}{\longrightarrow} U \hookrightarrow \mathcal{O}.$$

and use Lemma 7.4.3. □

Consider on $G \times G$ the action of $B \times H \times I$ defined by: $(b,h,i).(g_1,g_2) = (bg_1,hg_2)i$ for all $b \in B$, $h \in H$, $i \in I$ and $(g_1,g_2) \in G \times G$. Then, if $D \in \mathcal{D}(G/H)$ (resp. $E \in \mathcal{D}(G/P)$), the equation $f_{D_\mathcal{O}}$ (resp. $f_{E_\mathcal{O}}$) is an eigenvector for the induced action of $B \times H \times I$ on $k[G \times G]$. The corresponding character in $\mathcal{X}(B) \times \mathcal{X}(H) \times \mathcal{X}(I)$ which determines $f_{D_\mathcal{O}}$ (resp. $f_{E_\mathcal{O}}$) is still denoted by $f_{D_\mathcal{O}}$ (resp. $f_{E_\mathcal{O}}$).

Note that by the restriction homomorphism, $\mathcal{X}(I)$ identifies with $\mathcal{X}(\Delta L)$, that is, with $\mathcal{X}(L)$. Similarly, $\mathcal{X}(P)$ and $\mathcal{X}(Q)$ identify with $\mathcal{X}(L)$. Moreover, $\mathcal{X}(L)$ is canonically embedded into $\mathcal{X}(B)$. From now on, we make these identifications implicitly.

Now, we can describe $f_{D_\mathcal{O}}$ and $f_{E_\mathcal{O}}$ as follows:

Lemma 7.4.5 *With the preceding notation, we have:*

(i) Let $E^\alpha \in \mathcal{D}(G/P)$ with $\alpha \in \Sigma - \Sigma(L)$. Then, the weight of $f_{E^\alpha_\mathcal{O}}$ in $\mathcal{X}(B)\times\mathcal{X}(H)\times\mathcal{X}(L)$ is $(\omega_\alpha, 0, -\omega_\alpha)$.

(ii) Let D belong to $\mathcal{D}(G/H)$ and $[\gamma_D : \chi_D] \in \mathcal{X}(B) \times_{\mathcal{X}(B\cap H)} \mathcal{X}(H)$ be its equation. Write $\gamma_D = \sum_{\alpha\in\Sigma} k_\alpha\omega_\alpha$ with $k_\alpha \in \mathbb{N}$. Then,

$$f_{D_\mathcal{O}} = (\sum_{\alpha\in\Sigma(L)} k_\alpha\omega_\alpha, \chi_D, \sum_{\alpha\in\Sigma-\Sigma(L)} k_\alpha\omega_\alpha).$$

Proof: Since $\alpha \in \Sigma - \Sigma(L)$, $\omega_\alpha \in \mathcal{X}(T)$ extends to P. Moreover, the equation of $\overline{Bs_\alpha P}$ in G is a $B \times P$-eigenvector of weight $(\omega_\alpha, -\omega_\alpha)$. On the other hand, by Lemma 7.4.3, we have $q^*_{G\times G}(E_\mathcal{O}) = \overline{Bs_\alpha P} \times G$. Assertion (i) follows.

Consider the rational function f on $G \times G$ defined on $B \times H.I$ by the formula:

$$f((b,h).i) = \gamma(b)\chi(h) \qquad \forall b \in B\,, h \in H \text{ and } i \in I. \tag{2}$$

Indeed, one easily verifies that for all $b \in B$ and $h \in H$ such that $(b,h) \in I$, we have $\gamma_D(b)\chi_D(h) = 1$; that is, (2) makes sense.

Set D_f =div(f). One easily shows that for all $b \in B$ and $h \in H$, $f(1, bh) = \gamma_D(b^{-1})\chi_D(h)$. So, $D_f\cap(\{1\}\times G) = \widetilde{D}^{-1}$ (with notation as in Lemma 7.4.4). Since D_f is stable by $B \times H\times I$, using Lemmas 7.2.1 and 7.4.4, it follows that:

$$D_f = q^*_{G\times G}(D_\mathcal{O}) + \sum_{E\in\mathcal{D}(G/P)} n_E.q^*_{G\times G}(E_\mathcal{O}), \tag{3}$$

where the n_E are integers. Denote by λ the character of P such that the equation of $\sum_{E\in\mathcal{D}(G/P)} n_E E$ is $[\lambda : -\lambda]$ (with the notation of Section 2.2 for $H = P$). Then, Assertion (i) and Equation (3) imply that $f_{D_\mathcal{O}} = (\gamma_D - \lambda, \chi_D, \lambda)$.

Let $\alpha \in \Sigma - \Sigma(L)$. We claim that $D_\mathcal{O}$ is stable by $P_\alpha \times \{1\}$. Indeed, since $D_\mathcal{O}$ is stable by $B \times \{1\}$, $(P_\alpha \times \{1\}).D_\mathcal{O}$ is closed in $\mathcal{O}$ and thus equals $D_\mathcal{O}$ or $\mathcal{O}$. But, looking at Diagram (7.2.1), we see that $((P_\alpha \times \{1\}).D_\mathcal{O}) \cap F = ((P \cap P_\alpha) \times \{1\}).(D_\mathcal{O} \cap F)$. Since $\alpha \in \Sigma - \Sigma(L)$, $P \cap P_\alpha$ equals B. So, $((P_\alpha \times \{1\}).D_\mathcal{O}) \cap F = D_\mathcal{O} \cap F$. The claim follows.

The claim shows that $\gamma_D - \lambda \in \mathcal{X}(B)$ extends to P_α for all $\alpha \in \Sigma - \Sigma(L)$. That is, $\gamma_D - \lambda$ is a linear combination of the ω_α for $\alpha \in \Sigma(L)$. Moreover, λ is a character of P, that is, a linear combination of the ω_α for $\alpha \in \Sigma - \Sigma(L)$. We deduce that $\lambda = \sum_{\alpha\in\Sigma-\Sigma(L)} k_\alpha\omega_\alpha$. Assertion (ii) follows. □

7.5 — The next step in the proof of Proposition 7.3.2 is to find an equation for $\overline{D}^X$.

For this, we make some reductions. The theory of embeddings of G (see [Res00] or [Bri97] and Section 4) shows that there exists a simple toroidal $G \times G$-equivariant embedding X'

of G which contains $X_{\mathcal{O}}$ and a projective $G \times G$-orbit Z. Replacing X by X' if necessary, to prove Proposition 7.3.2, we can assume (in the sequence of Section 7) that X is simple, toroidal, with projective closed orbit Z. Then, we have the following

Lemma 7.5.6 *Keep notation as above. Consider the $\widetilde{G} \times \widetilde{G}$-linearized line bundle $\mathcal{L}_{\gamma_D}$ defined in Proposition 4.4.4.*

Then, there exists a section σ of $\mathcal{L}_{\gamma_D}$ (unique up to scalar multiplication) such that σ is an eigenvector for $\widetilde{B} \times \widetilde{H}$ of weight (γ_D, χ_D). Moreover, $\overline{D}^X = \mathrm{div}(\sigma)$.

Proof: The uniqueness of σ follows from Proposition 3.4.6 and from the fact that $\Gamma(X, \mathcal{L}_{\gamma_D})$ is a multiplicity-free $\widetilde{G} \times \widetilde{G}$-module.

Since γ_D is dominant, $\mathcal{L}_{\gamma_D}$ is generated by its global sections. As a consequence, the restriction morphism from $\Gamma(X, \mathcal{L}_{\gamma_D})$ to $\Gamma(Z, \mathcal{L}_{\gamma_D|Z})$ is non-zero. But, the $\widetilde{G} \times \widetilde{G}$-module $\Gamma(Z, \mathcal{L}_{\gamma_D|Z})$ is isomorphic to $V_{\gamma_D} \otimes V^*_{\gamma_D}$. Thus, there exists a $\widetilde{G} \times \widetilde{G}$-equivariant embedding i of $V_{\gamma_D} \otimes V^*_{\gamma_D}$ into $\Gamma(X, \mathcal{L}_{\gamma_D})$ (unique up to scalar multiplication).

By Proposition 3.4.6, there exists $\nu \in V^*_{\gamma_D}$, $\widetilde{H}$-eigenvector of weight χ_D. Let $v \in V_{\gamma_D}$ be a B-eigenvector. Then $\sigma = i(v \otimes \nu)$ satisfies the first assertion of the lemma.

Since the restriction of σ to Z is non-zero, no component of $\mathrm{div}(\sigma)$ is stable by $G \times G$. Then, each component of $\mathrm{div}(\sigma)$ intersects G. So, to prove the lemma it suffices to determine $\mathrm{div}(\sigma) \cap G$.

Consider $\zeta : \widetilde{G} \longrightarrow G$. Since $\mathrm{Pic}(\widetilde{G})$ is trivial, $\Gamma(\widetilde{G}, \zeta^*(\mathcal{L}_{\gamma_D|G}))$ is isomorphic as a $\widetilde{G} \times \widetilde{G}$-module to $k[\widetilde{G}]$. Then, $\sigma \circ \zeta$ equals $[\gamma_D : \chi_D]$ up to a scalar multiplication. Thus, $\mathrm{div}(\sigma_{|G} \circ \zeta) = \widetilde{D}$. Moreover, Lemma 7.4.3 shows that $\zeta^*\left(\zeta(\widetilde{D})\right) = \widetilde{D}$. Then, we have $\mathrm{div}(\sigma_{|G}) = \zeta(\widetilde{D})$. The lemma follows. □

7.6 — Now, we want to understand the restriction of the equation of $\overline{D}^X$ (given by Lemma 7.5.6) to an orbit $\mathcal{O}$ of $G \times G$.

Consider the restriction morphism $\mathrm{Pic}^{\widetilde{G}\times\widetilde{G}}(X) \longrightarrow \mathrm{Pic}^{\widetilde{G}\times\widetilde{G}}(\mathcal{O})$. Then, by Proposition 4.4.4, we identify $\mathrm{Pic}^{\widetilde{G}\times\widetilde{G}}(X)$ with $\mathcal{X}(\widetilde{B})$. Set $\widetilde{I} = (\zeta, \zeta)^{-1}(I)$. Then, by Lemma 3.4.5, the group $\mathrm{Pic}^{\widetilde{G}\times\widetilde{G}}(\mathcal{O})$ identifies with $\mathcal{X}(\widetilde{I})$, that is, with $\mathcal{X}(\widetilde{L}) \times \mathcal{X}(\widetilde{C})$. If $(\lambda, \gamma) \in \mathcal{X}(\widetilde{L}) \times \mathcal{X}(\widetilde{C})$, then we denote by $\mathcal{L}_{(\lambda,\gamma)}$ the corresponding $\widetilde{G} \times \widetilde{G}$-linearized line bundle on $\mathcal{O}$. Then, we have the

Lemma 7.6.7 *Let $\lambda \in \mathcal{X}(\widetilde{B})$. Consider the $\widetilde{G} \times \widetilde{G}$-linearized line bundle $\mathcal{L}_\lambda$ on X defined by Proposition 4.4.4. Then, the restriction of $\mathcal{L}_\lambda$ to $\mathcal{O}$ equals $\mathcal{L}_{(0, -\lambda_{|\widetilde{C}})}$ with preceding notation (where $(0, -\lambda_{|\widetilde{C}}) \in \mathcal{X}(\widetilde{L}) \times \mathcal{X}(\widetilde{C})$).*

Proof: Let $X_{Z,B\times B^-}$ be the unique affine open $B \times B^-$-stable and open subset of X that intersects Z. Let S be the closure of T in $X_{Z,B\times B^-}$. Then, by Proposition 2.6.7, S intersects Z in a unique point z and $X_{Z,B\times B^-}$ is isomorphic to $U \times U^- \times S$ as $B \times B^-$-variety . The variety S is an affine $T \times T$-equivariant embedding of T, in particular its Picard group is trivial. Thus, the restriction of $\mathcal{L}_\lambda$ to S is trivial as a line bundle on S (without linearization). Furthermore, the $\widetilde{T} \times \widetilde{T}$-linearization of $\mathcal{L}_{\lambda|S}$ obtained by restricting the $\widetilde{G} \times \widetilde{G}$-linearization of $\mathcal{L}_\lambda$ defines a character of $\widetilde{T} \times \widetilde{T}$: via the action of $\widetilde{T} \times \widetilde{T}$ on the fiber of $\mathcal{L}_\lambda$ at z. Since z is the point of Z fixed by $B^- \times B$ and $\mathcal{L}_{\lambda|Z} = \mathcal{L}_{(\lambda,-\lambda)}$, the group $\widetilde{T} \times \widetilde{T}$ acts on $(\mathcal{L}_\lambda)_z$ by $(\lambda, -\lambda)$.

By Proposition 4.3.2, there exists x in $S \cap \mathcal{O}$ fixed by $\widetilde{I}$. But now, since $\mathcal{L}_{\lambda|S}$ is trivial, the stabilizer of x in $\widetilde{T} \times \widetilde{T}$ acts on the fiber at x by $(\lambda, -\lambda)$. We deduce that $\Delta\widetilde{L}$ acts trivially on this fiber and $\widetilde{C}$ acts by $-\lambda_{|\widetilde{C}}$. The proposition follows. □

7.7 — Now, we can complete the proof of Proposition 7.3.2:

Let $D \in \mathcal{D}(G/H)$ and $\mathcal{O}$ a colorless orbit of $G \times G$ in X. The aim is to determine $\overline{D}^X \cap \mathcal{O}$. As noted just before Lemma 7.5.6, we can assume that X is a simple toroidal embedding of G and contains a projective orbit Z of $G \times G$.

Consider the weight $(\gamma, \chi, \lambda) \in \mathcal{X}(\widetilde{B}) \times \mathcal{X}(\widetilde{H}) \times \mathcal{X}(\widetilde{L})$ of the equation in $k[\widetilde{G} \times \widetilde{G}]$ of the pullback of $\overline{D}^X \cap \mathcal{O}$ in $\widetilde{G} \times \widetilde{G}$ (see Lemma 7.4.5). Let $\mathcal{L}_{\gamma_D}$ and σ be as in Lemma 7.5.6. Since $\sigma_{|\mathcal{O}}$ is an equation of $\overline{D}^X \cap \mathcal{O}$, we have: $\gamma = \gamma_D$ and $\chi = \chi_D$. Moreover, by Lemma 7.6.7, the restriction of $\mathcal{L}_{\gamma_D}$ to $\mathcal{O}$ is $\mathcal{L}_{(0, -\gamma_{D_{|\widetilde{C}}})}$. We deduce that $\lambda = 0$. Then, Proposition 7.3.2 follows easily from Lemma 7.4.5. □

8 Quotients of projective embeddings of the group by a spherical subgroup

8.1 — In Section 6, we started with a projective embedding of G/H and tried to realize it as a quotient of a projective embedding of G. Conversely, in this section, we start with a $G \times G$-equivariant projective embedding X of G.

Let $\mathcal{L}$ be an ample $G \times H$-linearized line bundle on X and let χ be a character of H. We use the notation of Lemma 5.1.1 for the subgroup $G \times H$ of $G \times G$. Since $\mathcal{X}(G)$ is trivial, $r_{G\times H}$ is injective. For simplicity, we denote $r_{G\times H}(\mathcal{L}) \otimes i(\chi)$ by $\mathcal{L} \otimes \chi$. Then, Lemma 5.1.1 shows that any $G \times H$-linearized line bundle has a non-zero tensor power of the form $\mathcal{L} \otimes \chi$ for some $\mathcal{L}$ and χ.

We fix our attention on the quotient of X by $\{1\} \times H$ associated to $\mathcal{L} \otimes \chi$, as in Section 5.1:

$$\pi : X^{\mathrm{ss}}(\mathcal{L} \otimes \chi) \longrightarrow X^{\mathrm{ss}}(\mathcal{L} \otimes \chi)/\!/H.$$

Then, the action of $G \times \{1\}$ on X descends to an action of G on $X^{\mathrm{ss}}(\mathcal{L} \otimes \chi)/\!/H$ which becomes a spherical variety (since $B \times H$ has a dense orbit in G).

Moreover, replacing $\mathcal{L} \otimes \chi$ by a power, we can assume (see Section 5.4) that there exists a "quotient" line bundle $(\mathcal{L} \otimes \chi)/\!/H$. This line bundle has a natural G-linearization induced by the $G \times \{1\}$-one on $\mathcal{L}$. For simplicity, we set:

$$Y := X^{\mathrm{ss}}(\mathcal{L} \otimes \chi)/\!/H \quad \text{and} \quad \mathcal{L}_Y := (\mathcal{L} \otimes \chi)/\!/H.$$

Consider the quotient:

$$\begin{array}{ccc} \mathcal{L}\otimes\chi & \longrightarrow & \mathcal{L}_Y \\ \downarrow & & \downarrow \\ X^{\mathrm{ss}}(\mathcal{L}\otimes\chi) & \xrightarrow{\pi} & Y \end{array}$$

8.2 — Let B be a Borel subgroup of G opposite to H and B^- be another one, opposite to B. As in Section 4.2, we embed $\mathbf{P}(X,\mathcal{L})$ in $\mathcal{X}(B)_{\mathbb{Q}}$. Then, we can describe the moment polytope $\mathbf{P}(Y,\mathcal{L}_Y)$ by:

Theorem 2 *Keep notation as above. Then, we have:*

$$\mathbf{P}(Y,\mathcal{L}_Y)=\mathbf{P}(X,\mathcal{L})\cap\mathbf{P}(G/H,\mathcal{L}_\chi).$$

Proof: We have:

$$\mathbf{P}(X,\mathcal{L})=\{p\in\mathcal{X}(B)_{\mathbb{Q}}\,:\,\exists n>0,\ np\in P^+,\quad V_{np}\otimes V_{np}^*\hookrightarrow\Gamma(X,\mathcal{L}^{\otimes n})\}.$$

Moreover,

$$\mathbf{P}(Y,\mathcal{L}_Y)=\{p\in\mathcal{X}(B)_{\mathbb{Q}}\,:\,\exists n>0,\ np\in P^+,\quad V_{np}\hookrightarrow\Gamma(X,(\mathcal{L}\otimes\chi)^{\otimes n})^H\}.$$

With the notation of Proposition 3.4.6, we deduce that:

$$\mathbf{P}(Y,\mathcal{L}_Y)=\{p\in\mathcal{X}(B)_{\mathbb{Q}}\,:\,\exists n>0,\ np\in\Gamma_{n\chi}\text{ et }V_{np}\otimes V_{np}^*\hookrightarrow\Gamma(X,\mathcal{L}^{\otimes n})\}.$$

But by definition $\mathbf{P}(G/H,\mathcal{L}_\chi)$ is the set of $p\in\mathcal{X}(B)_{\mathbb{Q}}$ such that there exists a positive integer n such that $np\in\Gamma_{n\chi}$. The theorem follows. $\square$

8.3 — In the sequel of the section, we show how to read properties of Y and π on the polytopes $\mathbf{P}(X,\mathcal{L})$ and $\mathbf{P}(G/H,\mathcal{L}_\chi)$. For example, the following corollary gives a criterion for Y to be an embedding of G/H.

Corollary 8.3.1 *The G-variety Y is an embedding of G/H if and only if $\mathbf{P}(X,\mathcal{L})$ intersects the relative interior of $\mathbf{P}(G/H,\mathcal{L}_\chi)$.*

Proof: The necessary condition follows easily from Theorem 2 and Proposition 3.1.1.

Conversely, assume that $\mathbf{P}(X,\mathcal{L})$ intersects the relative interior of $\mathbf{P}(G/H,\mathcal{L}_\chi)$. Let I denote the isotropy in G of $\pi(1)$. Obviously, I contains H; in particular, I is a spherical subgroup and Y is an embedding of G/I. By Proposition 3.1.1, the interior of $\mathbf{P}(X,\mathcal{L})$ in $\mathcal{X}(B)_{\mathbb{Q}}$ is not empty. Then, the diffenrences of elements of $\mathbf{P}(Y,\mathcal{L}_Y)$ span $\mathcal{X}(B)_{\mathbb{Q}}^{B\cap H}$. In particular, the ranks of G/I and G/H equal. Then, Theorem 3.4.3 in [Bri97] (see also [Kno91]) shows that the index of H in I is finite. So, Proposition 5.3.4 implies that each fiber of π over G/I contains a unique open orbit of H. We deduce that $H=I$. $\square$

Remark: Note that if Y satisfies Corollary 8.3.1, we can determine the fan $\mathcal{F}(Y)$ by Theorem 2 and Propositions 3.3.4 and 3.5.8 (see [Res00]; 7.2.3 for examples).

8.4 — The following proposition describes the image by π of an orbit of $G\times G$ in X:

Proposition 8.4.2 *Let $\mathcal{O}$ be an orbit of $G \times G$ in X which intersects $X^{\rm ss}(\mathcal{L} \otimes \chi)$. Then, there exists a dense orbit $(G \times H).x$ of $G \times H$ in $\mathcal{O}$. Moreover,*

(i) $\overline{\pi(\mathcal{O} \cap X^{\rm ss}(\mathcal{L} \otimes \chi))} = \overline{G.\pi(x)}$, *and*

(ii) $\mathbf{P}(\overline{G.\pi(x)}, \mathcal{L}_Y) = \mathbf{P}(\overline{\mathcal{O}}, \mathcal{L}) \cap \mathbf{P}(G/H, \mathcal{L}_\chi)$.

Proof: Proposition 2.6.6 shows that there exists a toroidal embedding $\widetilde{X}$ of G and a $G \times G$-equivariant surjective morphism $\widetilde{X} \longrightarrow X$. Moreover, any orbit of $G \times G$ in $\widetilde{X}$ contains a dense orbit of $G \times H$ by Lemma 7.2.1. Then, there exists a dense orbit of $G \times H$ in $\mathcal{O}$.

Since the variety $\overline{\pi(\mathcal{O} \cap X^{\rm ss}(\mathcal{L} \otimes \chi))}$ is irreducible and stable by G, it is the closure of an orbit of G in Y. But $(G \times H).x$ is dense in $\mathcal{O}$, and $G.\pi(x)$ is dense in $\overline{\pi(\mathcal{O} \cap X^{\rm ss}(\mathcal{L} \otimes \chi))}$. Assertion (i) follows.

By Proposition 5.2.3, the restriction of π to $\mathcal{O} \cap X^{\rm ss}(\mathcal{L} \otimes \chi)$ is the quotient by H of $\overline{\mathcal{O}}^{\rm ss}(\mathcal{L} \otimes \chi)$. Then, the proof of Assertion (ii) is the same as that of Theorem 2. ☐

8.5 — Let us fix an orbit Ω of G in Y. We are now interested in the preimage of Ω by π. Recall that $\mathcal{D}(Y, \Omega)$ denotes the set of colors of Ω in Y. Set:

$$\mathbf{F}(\mathcal{D}(Y, \Omega)) := \{\gamma \in \mathbf{P}(G/H, \mathcal{L}_\chi) \ : \ (\gamma, \chi) \in \sum_{D \notin \mathcal{D}(Y,\Omega)} \mathbb{Q}.(\gamma_D, \chi_D)\}.$$

Then by Proposition 3.5.8, $\mathbf{F}(\mathcal{D}(Y, \Omega)$ is the minimal face of $\mathbf{P}(G/H, \mathcal{L}_\chi)$ which contains $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$.

Proposition 8.5.3 *With the preceding notation, we have:*

(i) There exists a minimal orbit (for the order induced by the inclusion of the closures) among the orbits $\mathcal{O}$ of $G \times G$ in X such that the closure of $\pi(\mathcal{O} \cap X^{\rm ss}(\mathcal{L} \otimes \chi))$ contains Ω. We denote this minimal orbit by $\mathcal{O}(\Omega)$.

(ii) $\mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L})$ *is the minimal orbital face of* $\mathbf{P}(X, \mathcal{L})$ *which contains* $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$.

(iii) $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y) = \mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L}) \cap \mathbf{F}(\mathcal{D}(Y, \Omega))$.

(iv) If Ω is colorless in Y, then $\overline{\pi(\mathcal{O}(\Omega) \cap X^{\rm ss}(\mathcal{L} \otimes \chi))} = \overline{\Omega}$. In particular, the image of π contains Ω.

Proof: Let $\mathcal{O}$ be an orbit of $G \times G$ in X and let x be a point in the open orbit of $G \times H$ in $\mathcal{O}$. Proposition 8.4.2 shows that $\overline{\pi(\mathcal{O}(\Omega) \cap X^{\rm ss}(\mathcal{L} \otimes \chi))}$ contains Ω if and only if $\overline{G.\pi(x)}$ contains Ω. On the other hand, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two orbits of $G \times G$ in X, Proposition 3.3.4 shows that:

$$\mathbf{P}(\overline{\mathcal{O}_1}, \mathcal{L}) \cap \mathbf{P}(\overline{\mathcal{O}_2}, \mathcal{L}) = \mathbf{P}(\overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2}, \mathcal{L}).$$

Then, we deduce that:

$$\begin{aligned} \Omega \subseteq \overline{\pi(\mathcal{O}) \cap X^{\rm ss}(\mathcal{L} \otimes \chi)} &\iff \mathbf{P}(\overline{\Omega}, \mathcal{L}_Y) \subseteq \mathbf{P}(\overline{G.\pi(x)}, \mathcal{L}_Y) \\ &\iff \mathbf{P}(\overline{\Omega}, \mathcal{L}_Y) \subseteq \mathbf{P}(\overline{\mathcal{O}}, \mathcal{L}) \cap \mathbf{P}(G/H, \mathcal{L}_\chi) \\ &\iff \mathbf{P}(\overline{\Omega}, \mathcal{L}_Y) \subseteq \mathbf{P}(\overline{\mathcal{O}}, \mathcal{L}). \qquad (1) \end{aligned}$$

Then, Proposition 3.3.4 shows that there exists an orbit $\Omega(G)$ of $G \times G$ in X satisfying (1) and minimal for this property. This proves Assertions (i) and (ii).

Moreover, $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$ is contained in $\mathbf{F}(\mathcal{D}(Y,\Omega))$. Since, $\mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L}) \cap \mathbf{F}(\mathcal{D}(Y,\Omega))$ is a face of $\mathbf{P}(Y, \mathcal{L}_Y)$, we deduce that $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$ is a face of $\mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L}) \cap \mathbf{F}(\mathcal{D}(Y,\Omega))$. But, $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$ intersects the relative interior of $\mathbf{F}(\mathcal{D}(Y,\Omega))$. So, there exists a face $\mathbf{F}$ of $\mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L})$ such that:

$$\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y) = \mathbf{F} \cap \mathbf{F}(\mathcal{D}(Y,\Omega)).$$

With the notation of Proposition 3.3.4, the minimality of $\mathcal{O}(\Omega)$ implies that $\mathcal{O}(\mathbf{F}) = \mathcal{O}(\Omega)$.

Let $P^+_{\mathbf{F}}$ denote the minimal face of P^+ intersecting $\mathbf{F}$. Since $\mathbf{P}(G/H, \mathcal{L}_\chi)$ is contained in P^+ and $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$ is contained in $\mathbf{F}$, $\mathbf{F}(\mathcal{D}(Y,\Omega))$ is contained in $P^+_{\mathbf{F}}$. But now, Proposition 3.3.4 implies that:

$$\mathbf{F} \cap \mathbf{F}(\mathcal{D}(Y,\Omega)) = \mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L}) \cap \mathbf{F}(\mathcal{D}(Y,\Omega)).$$

Assertion (iii) is proved. If Ω is colorless, we have, by Assertion (iii):

$$\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y) = \mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L}) \cap \mathbf{P}(G/H, \mathcal{L}_\chi).$$

Then, Assertion (iv) follows from Proposition 8.4.2. □

Remark: If H is reductive, then π is surjective. Moreover, if y is a point in Ω and x is a point in the unique closed orbit of $\{1\} \times H$ in $\pi^{-1}(y)$, then $\mathcal{O}(\Omega)$ is the orbit of x by $G \times G$.

Let $\rho : \mathrm{Hom}(\mathcal{X}(B), \mathbb{Q}) \longrightarrow \mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q})$ be the restriction map. Then, a connection between the colored fans of X and Y is the following

Lemma 8.5.4 *With the notation of Proposition 8.5.3, we have:*

$$\rho\left(\mathcal{C}(X, \mathcal{O}(\Omega))\right) \subseteq \mathcal{C}(Y,\Omega).$$

Proof: Let p be a point in the relative interior of $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$. By Proposition 3.5.8, the cone $\mathcal{C}(Y,\Omega)$ is dual to $-p + \mathbf{P}(Y, \mathcal{L}_Y)$ in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q})$. Since $\mathbf{P}(Y, \mathcal{L}_Y)$ is contained in $\mathbf{P}(X, \mathcal{L})$, $\mathcal{C}(Y,\Omega)$ contains the image by ρ of the dual in $\mathrm{Hom}(\mathcal{X}(B), \mathbb{Q})$ of $-p + \mathbf{P}(X, \mathcal{L})$. Since p belongs to $\mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L})$, applying Proposition 3.5.8 to X and $\mathcal{O}(\Omega)$ completes the proof of the lemma. □

8.6 — Denote again by Ω an orbit of G in Y. The next proposition gives a description of the preimage by π of the minimal affine B-stable open subset of Y intersecting Ω, namely $Y_{\Omega,B}$ (see Proposition 2.4.4).

Proposition 8.6.5 *With preceding notation, we have:*

$$\pi^{-1}(Y_{\Omega,B}) = X_{\mathcal{O}(\Omega)} - \bigcup_{D \notin \mathcal{D}(Y,\Omega)} \overline{D}^X.$$

Proof: Let p be a point in the relative interior of $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$. Let n be a positive integer and σ be a section of $\mathcal{L}_Y^{\otimes n}$ which is a B-eigenvector of weight np. Then, by Proposition 3.3.4, we have $Y_{\Omega,B} = Y - \{y \in Y \ : \ \sigma(y) = 0\}$.

But σ belongs to $\Gamma(X, (\mathcal{L}\otimes\chi)^{\otimes n}))$, and $\pi^{-1}(Y_{\Omega,B}) = X - \{x \in X \,:\, \sigma(x) = 0\}$. Moreover, by definition of $\mathcal{O}(\Omega)$ (see Proposition 8.5.3), the set $\pi^{-1}(Y_\Omega)$ is contained in $X_{\mathcal{O}(\Omega)}$. So, we have:

$$\pi^{-1}(Y_{\Omega,B}) = X_{\mathcal{O}(\Omega)} - \{x \in X \,:\, \sigma(x) = 0\}. \tag{4}$$

We consider the set of those $x \in X$ such that $\sigma(x) = 0$. Let M be an irreducible component of this set which does not intersect G. Since the codimension of M equals one, M is stable by $G \times G$. But, $\pi^{-1}(Y_{\Omega,B})$ intersects $\mathcal{O}(\Omega)$. Then, Equality (4) shows that M is contained in $X - X_{\mathcal{O}(\Omega)}$. We deduce that:

$$\pi^{-1}(Y_{\Omega,B}) = X_{\mathcal{O}(\Omega)} - \overline{\{x \in G \,:\, \sigma(x) = 0\}}.$$

Moreover, since p belongs to the relative interior of $\mathbf{P}(\overline{\Omega}, \mathcal{L}_Y)$, we have:

$$\{y \in G/H \,:\, \sigma(y) = 0\} = \bigcup_{D \notin \mathcal{D}(Y,\Omega)} D.$$

The proposition follows. □

8.7 — The main result of this section is a criterion (expressed orbit by orbit) in terms of polytopes to decide if $X^{\rm s}(\mathcal{L}\otimes\chi)$ equals $X^{\rm ss}(\mathcal{L}\otimes\chi)$, with notation of Section 5.3. We start by the following

Lemma 8.7.6 *Let Ω be an orbit of G in Y. With the notation of Proposition 8.5.3, we assume that Ω and $\mathcal{O}(\Omega)$ are colorless. Then, $\pi^{-1}(\Omega) \cap \mathcal{O}(\Omega)$ is the open orbit of $G \times H$ in $\mathcal{O}(\Omega)$.*

Proof: Assertion (*iv*) of Proposition 8.5.3 shows that $\pi^{-1}(\Omega) \cap \mathcal{O}(\Omega)$ contains the open orbit of $G \times H$ in $\mathcal{O}(\Omega)$. Moreover, by Proposition 8.6.5, we have: $\pi^{-1}(Y_{\Omega,B}) = X_{\mathcal{O}(\Omega)} - \bigcup_{D\notin\mathcal{D}(Y,\Omega)} \overline{D}^X$. But, since G is connected, an orbit of $G \times \{1\}$ is contained in $\bigcup_{D\notin\mathcal{D}(Y,\Omega)} \overline{D}^X$ if and only if it is contained in some $\overline{D}^X$. We deduce that $\pi^{-1}(Y_\Omega) = X_{\mathcal{O}(\Omega)} - \bigcup_{D\notin\mathcal{D}(Y,\Omega)} \Omega_{\overline{D}^X}$, where $\Omega_{\overline{D}^X}$ denotes the union of all orbits of $G \times \{1\}$ contained in $\overline{D}^X$.

Moreover, with the notation of Diagram (7.2.1), each orbit of $G \times \{1\}$ in $\mathcal{O}$ intersects F in a unique orbit of $P \times \{1\}$; that is, in a fiber of the natural map

$$\phi \,:\, F \xrightarrow{q} G/B \longrightarrow G/Q.$$

This identifies the set of all orbits of $G \times \{1\}$ contained in $D_\mathcal{O}$ with the set of all orbits of Q contained in D. In particular, any non open orbit of $G \times H$ in $\mathcal{O}$ is contained in $D_\mathcal{O}$ for some D in $\mathcal{D}(G/H)$.

But, by Proposition 7.3.2, for any $D \in \mathcal{D}(G/H)$, $\overline{D}^X \cap \mathcal{O}$ contains $D_\mathcal{O}$. Then, no non-open orbit of $G \times H$ in $\mathcal{O}$ is contained in $\pi^{-1}(Y_\Omega)$. This completes the proof of the proposition.
□

Let $\mathcal{E}$ be a finite-dimensional vector space and let $\mathbf{P}$ be a polyhedron in $\mathcal{E}$. The dimension of the affine subspace generated by $\mathbf{P}$ in $\mathcal{E}$ is called the *dimension of* $\mathbf{P}$ and is denoted by $\dim(\mathbf{P})$. If $\mathbf{Q}$ is another polyhedron in $\mathcal{E}$, we say that the intersection of $\mathbf{P}$ and $\mathbf{Q}$ is *transversal* if $\dim(\mathbf{P}\cap\mathbf{Q}) = \dim(\mathcal{E}) - \dim(\mathbf{P}) - \dim(\mathbf{Q})$.

Proposition 8.7.7 *Let Ω be an orbit of G in Y. With the notation of Proposition 8.5.3, we assume that Ω and $\mathcal{O}(\Omega)$ are colorless. Then, the three following conditions are equivalent:*

(i) $\pi^{-1}(Y_\Omega)$ *is contained in* $X^{\rm s}(\mathcal{L})$.

(ii) $\pi^{-1}(\Omega)$ *is contained in* $X^{\rm s}(\mathcal{L})$.

(iii) The intersection of $\mathbf{P}(\overline{\mathcal{O}(\Omega)},\mathcal{L})$ *and* $\mathbf{P}(G/H,\mathcal{L}_\chi)$ *is transversal.*

Proof: Note that Proposition 3.1.1 shows that the rank of Ω (resp. $\mathcal{O}(\Omega)$) equals the dimension of $\mathbf{P}(\overline{\Omega},\mathcal{L}_Y)$ (resp. $\mathbf{P}(\overline{\mathcal{O}(\Omega)},\mathcal{L})$). Then, Condition (*iii*) is equivalent to $\mathrm{rk}(G/H)-\mathrm{rk}(\Omega)=\mathrm{rk}(G)-\mathrm{rk}(\mathcal{O}(\Omega))$. By Lemma 2.6.8, this is also equivalent to:

$$\dim(G/H)-\dim(\Omega)=\dim(G)-\dim(\mathcal{O}(\Omega)). \tag{5}$$

On the other hand, Proposition 8.5.3 shows that: $\overline{\pi(\mathcal{O}(\Omega)\cap X^{\rm ss}(\mathcal{L}\otimes\chi))}=\overline{\Omega}$. We deduce that the dimension of the general fibers of π over $\overline{\Omega}$ equals $\dim(\mathcal{O}(\Omega))-\dim(\Omega)$. In particular, the fiber over any point y in Ω has this dimension. But now, Condition (*iii*) (that is, Equality (5)) is equivalent to:

$$\forall y\in\Omega\qquad \dim(\pi^{-1}(y))=\dim(H).$$

Now, using the Remark of Section 5.3, we conclude that Condition (*ii*) implies (*iii*). Since (*i*) implies trivially (*ii*), it remains to prove: "(*ii*) implies (*i*)".

Assume that $\dim(\pi^{-1}(y))=\dim(H)$. Then , Proposition 8.4.2 implies easily that $\pi^{-1}(y)$ is contained in $\mathcal{O}(\Omega)$.

Consider the stabilizer $P_{G/H}$ in G of BH/H. Let us fix a Levi subgroup $L_{G/H}$ of $P_{G/H}$ adapted to H (see Proposition 2.5.5) and a maximal torus $T_{G/H}$ of $L_{G/H}$ contained in B.

By Proposition 4.3.2, there exists a point x in $\mathcal{O}$ such that the isotropy of x in $G\times G$ is:

$$I=(P^u\times Q^u)\ltimes(\Delta L\times(\{1\}\times C)),$$

where Q is a parabolic subgroup of G containing B, P is the parabolic subgroup of G containing $T_{G/H}$ and opposite to Q, L is the intersection of P and Q and C is a subgroup of the connected center of L.

We claim that $P_{G/H}$ is contained in Q.

Let α be a simple root of $(B,T_{G/H})$ such that P_α is contained in $P_{G/H}$. Then, since the complement of $P_{G/H}H/H$ in G/H is the union of the colors of G/H, P_α stabilizes each color of G/H. Thus, the B-weight of the equation f_D of any color D of G/H is orthogonal to the coroot $\alpha^\vee$. Then, Lemma 4.3.3 shows that $\mathbf{P}(G/H,\mathcal{L}_\chi)$ is contained in the orthogonal space to $\alpha^\vee$. In particular, $\mathbf{P}(\mathcal{O}(\Omega),\mathcal{L})$ which contains $\mathbf{P}(\overline{\Omega},\mathcal{L}_Y)$ intersects the orthogonal of $\alpha^\vee$. Thus, Lemma 4.3.3 shows that Q contains P_α. The claim follows.

Now, we claim that the isotropy of $\{1\}\times H$ at a general point of $\mathcal{O}(\Omega)$ is finite.

Indeed, since $(G\times H).I/I$ is open in $\mathcal{O}(\Omega)$, we have to prove that the intersection of I and $\{1\}\times H$ is finite. By the first claim, Q^u is contained in $P^u_{G/H}$. Thus, since C is contained in $L_{G/H}$, Proposition 2.6.7 implies that $H\cap Q^uC=H\cap C$. On the other hand, by Proposition 3.1.1, the differences of elements of $\mathbf{P}(G/H,\mathcal{L}_\chi)$ span $\mathcal{X}(B)^{B\cap H}_{\mathbb{Q}}$, and that of elements of $\mathbf{P}(\overline{\mathcal{O}(\Omega)}, \mathcal{L})$ span $\mathcal{X}(B)_{\mathbb{Q}}^{B\cap C}$. The assumption implies that the intersection of $\mathcal{X}(T_{G/H})^{T_{G/H}\cap H}$ and $\mathcal{X}(T_{G/H})^C$ is finite; hence $H \cap C$ is finite. This proves the second claim.

Let us fix an affine open subset U of Ω. The claims show that the general fibers of π over U and the general orbits of H in $\mathcal{O}(\Omega)$ have the same dimension. But, by Lemma 8.7.6, $\pi^{-1}(U)$ is contained in the open orbit of $G \times H$ in $\mathcal{O}(\Omega)$. In particular, $\pi^{-1}(U)$ is smooth. Then, the proof of Proposition 5.3.4 shows that every general fiber of π over U contains a unique open dense orbit of H. But now, the fact $\pi^{-1}(U)$ is contained in the open orbit of $G \times H$ in $\mathcal{O}(\Omega)$ implies that the fibers of π over U are orbits of H. This implies Assertion (ii).

But, Condition (iii) holds for all orbits of G in Y_Ω. Thus, the same is true for (ii). Then, Condition (i) holds. □

9 Toroidal embeddings as geometric quotients

9.1 — In Section 9, G/H is supposed to be sober and liftable. Fix a projective embedding Y of G/H. As in Section 6, we want to obtain Y as a quotient of a $G \times G$-equivariant projective embedding X of G for an ample $G \times H$-linearized line bundle $\mathcal{L}$. But now, we want to have: $X^{\rm ss}(\mathcal{L}) = X^{\rm s}(\mathcal{L})$; that is, a geometric quotient. We start with the case when Y is the canonical embedding of G/H:

Theorem 3 *Assume that G/H is sober and liftable. Consider the canonical embedding Y (resp. X) of G/H (resp. G). Then, there exists an ample $G \times H$-linearized line bundle $\mathcal{L}$ on X such that the quotient $\pi : X^{\rm ss}(\mathcal{L}) \longrightarrow X^{\rm ss}(\mathcal{L})//H$ of X by $\{1\} \times H$ associated to $\mathcal{L}$ satisfies the following conditions:*

(i) $X^{\rm ss}(\mathcal{L})//H = Y$,

(ii) $X^{\rm ss}(\mathcal{L}) = X^{\rm s}(\mathcal{L})$,

(iii) π *is surjective.*

Before proving Theorem 3, we illustrate the ideas of the proof by examples.

9.2 — In this paragraph, G is PGL(3) and H is the subgroup of G consisting of matrices of the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

It is easy to see that G/H is spherical and identifies with the pairs $(p, d) \in \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ (a point and a line in $\mathbb{P}^2$) such that p does not belong to d. Set $Y = \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ viewed as an embedding of G/H. Then Y is the canonical embedding of G/H.

We fix an ample G-linearized line bundle $\mathcal{M}$ on Y. The proof of Theorem 1 is constructive: it gives the canonical embedding X of G and an ample $G \times H$-linearized line bundle $\mathcal{L} \otimes \chi$ on X depending on $\mathcal{M}$. The polytopes $\mathbf{P}(X, \mathcal{L})$ and $\mathbf{P}(Y, \mathcal{M})$ look like the following picture:

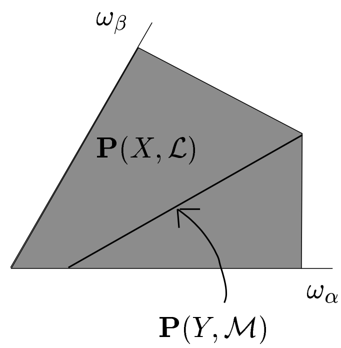

Figure 1: The polytopes $\mathbf{P}(Y,\mathcal{M})$ and $\mathbf{P}(X,\mathcal{L})$

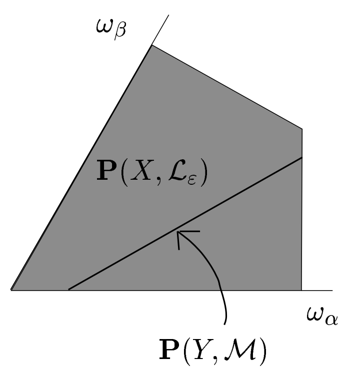

Figure 2: The polytopes $\mathbf{P}(Y,\mathcal{M})$ and $\mathbf{P}(X,\mathcal{L}_\varepsilon)$

By Proposition 8.7.7, we do not have $X^{\mathrm{ss}}(\mathcal{L}\otimes\chi)=X^{\mathrm{s}}(\mathcal{L}\otimes\chi)$. Yet, if we move a little bit $\mathcal{L}$ to $\mathcal{L}_\varepsilon$ we obtain the situation of Figure 9.2. Then Y is the geometric quotient of $X^{\mathrm{ss}}(\mathcal{L}_\varepsilon\otimes\chi)$.

9.3 — The proof of Theorem 3 will be a generalization of the preceding example. More precisely, we start with an ample G-linearized line bundle on the canonical embedding Y of G/H. Then, the construction used in the proof of Theorem 1 gives a $G\times H$-linearized line bundle $\mathcal{L}$ on the canonical embedding X of G. Then, using Proposition 8.7.7, we are going to prove that we can move $\mathcal{L}$ a little bit and obtain Y as a geometric quotient of X.

Two difficulties can appear. First, as shown by the example when $G=\mathrm{SL}(4)$ and $H=\mathrm{Sp}(4)$ (see [Res00]) the construction of the proof of Theorem 1 may not give the canonical embedding of G; or equivalently, the line bundle $\mathcal{L}$ on X may not be ample.

On the other hand, if the rank of G/H is less than that of G, replacing $\mathcal{L}$ by a nearby $\mathcal{L}_\varepsilon$, we may change the "shape" of the moment polytope of the quotient. That is, we can change the quotient variety $X^{\mathrm{ss}}(\mathcal{L}\otimes\chi)/\!/H$ (see [Res00], for an example). We will show that these problems can be avoided by moving $\mathcal{L}$ carefully.

9.4 — In this paragraph, we obtain some technical results about the cone of valuations of G/H.

Let us fix a W-invariant scalar product $\langle .,.\rangle$ on $\mathcal{X}(T)\otimes\mathbb{R}$. Denote by $\Sigma(P_{G/H})$ the set of simple roots α such that $P_{G/H}$ contains P_α. Consider the basis $(\alpha^*)_{\alpha\in\Sigma}$ of $\mathrm{Hom}(\mathcal{X}(B),\mathbb{Q})$ dual to the basis $(\alpha)_{\alpha\in\Sigma}$ of $\mathcal{X}(B)_{\mathbb{Q}}$. Then, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\sum_{\alpha\in\Sigma}\mathbb{Q}^{\geq 0}\alpha^* & \hookrightarrow & \mathrm{Hom}(\mathcal{X}(B),\mathbb{Q})\\
\uparrow & & \downarrow \rho_P^B\\
\sum_{\alpha\notin\Sigma(P_{G/H})}\mathbb{Q}^{\geq 0}\alpha^* & \longrightarrow & \mathrm{Hom}(\mathcal{X}(P_{G/H}),\mathbb{Q}),
\end{array}$$

where ρ_P^B is the restriction homomorphism induced by the inclusion of $\mathcal{X}(P_{G/H})$ into $\mathcal{X}(B)$. With this notation, we have:

Lemma 9.4.1 *(i) The set* $\left(\rho_P^B(\alpha^*)\right)_{\alpha\notin\Sigma(P_{G/H})}$ *is a basis of the vector space* $\mathrm{Hom}(\mathcal{X}(P_{G/H}),\mathbb{Q})$.

(ii) Moreover,

$$\rho_P^B\left(\sum_{\alpha\in\Sigma}\mathbb{Q}^{\geq 0}\alpha^*\right)=\sum_{\alpha\notin\Sigma(P_{G/H})}\mathbb{Q}^{\geq 0}\rho_P^B(\alpha^*).$$

Proof: Let us consider the dual statements. The dual space of $\sum_{\alpha\notin\Sigma(P_{G/H})}\mathbb{Q}\alpha^*$ identifies with $\sum_{\alpha\in\Sigma(P_{G/H})}\mathbb{Q}\alpha$, that is, with the orthogonal of $\mathcal{X}(P_{G/H})$ for the W-invariant scalar product. Assertion (i) follows easily.

Now, to prove Assertion (ii), it is sufficient to show that the dual cones in $\mathcal{X}(P_{G/H})_{\mathbb{Q}}$ of $\sum_{\alpha\notin\Sigma(P_{G/H})}\mathbb{Q}^{\geq 0}\rho_P^B(\alpha^*)$ and of $\rho_P^B\left(\sum_{\alpha\in\Sigma}\mathbb{Q}^{\geq 0}\alpha^*\right)$ are equal; that is, to show that:

$$\mathcal{X}(P_{G/H})_{\mathbb{Q}}\cap\left(\sum_{\alpha\notin\Sigma(P_{G/H})}\mathbb{Q}^{\geq 0}\alpha+\sum_{\alpha\in\Sigma(P_{G/H})}\mathbb{Q}\alpha\right)=\mathcal{X}(P_{G/H})_{\mathbb{Q}}\cap\left(\sum_{\alpha\in\Sigma}\mathbb{Q}^{\geq 0}\alpha\right).$$

The inclusion of the right side in the left one is obvious. Conversely, let us fix $\gamma = \sum_{\alpha\in\Sigma} x_\alpha \alpha$ with $x_\alpha \in \mathbb{Q}^{\geq 0}$ if $\alpha \notin \Sigma(P_{G/H})$, and $x_\alpha \in \mathbb{Q}$ if $\alpha \in \Sigma(P_{G/H})$.

Since $\mathcal{X}(P_{G/H})_{\mathbb{Q}}$ is the orthogonal space of $\Sigma(P_{G/H})$, we have $\langle \beta^\vee, \gamma\rangle = 0$ for all β in $\Sigma(P_{G/H})$. Thus, for all β in $\Sigma(P_{G/H})$, we have:

$$\langle \beta^\vee, \sum_{\alpha\in\Sigma(P_{G/H})} x_\alpha \alpha\rangle = - \sum_{\alpha\notin\Sigma(P_{G/H})} \langle \beta^\vee, \alpha\rangle x_\alpha.$$

On the other hand, for all distinct α and β in Σ, $\langle \beta^\vee, \alpha\rangle$ is non-positive. Moreover, for all $\alpha \notin \Sigma(P_{G/H})$, x_α is non-negative. As a consequence, we have:

$$\forall \beta \in \Sigma(P_{G/H}) \qquad \langle \beta^\vee, \sum_{\alpha\in\Sigma(P_{G/H})} x_\alpha \alpha\rangle \geq 0.$$

Then, we can apply Lemma 6 of Chapter 5, no 3.5 of [Bou68] to the basis $\Sigma(P_{G/H})$. We obtain that x_α is non-negative for all α. Assertion (ii) of the lemma follows. □

Since $P_{G/H}$ is parabolic, it is connected. Then, each D in $\mathcal{D}(G/H)$ is stable by $P_{G/H}$; and the character γ_D for the action of B of the equation of D extends to $P_{G/H}$. Then, by Lemma 2.2.1, we have the following inclusions: $\mathcal{X}(B)^{B\cap H} \subseteq \mathcal{X}(P_{G/H}) \subseteq \mathcal{X}(B)$. Taking the dual, we obtain:

$$\mathrm{Hom}(\mathcal{X}(B), \mathbb{Q}) \xrightarrow{\rho_P^B} \mathrm{Hom}(\mathcal{X}(P_{G/H}), \mathbb{Q}) \longrightarrow \mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q}).$$

We denote by $\mathcal{CV}(G)$ the valuation cone associated to the $G\times G$-homogeneous space G. Consider also the restriction map $\rho : \mathcal{CV}(G) \longrightarrow \mathcal{CV}(G/H)$ induced by the inclusion of $k(G/H)$ in $k(G)$. Then, we have

Lemma 9.4.2 *Keep notation as above . We assume in addition that G/H is sober and liftable (i.e. $\mathcal{CV}(G/H)$ is strictly convex and ρ is surjective).*

Then, there exists a subset $\Sigma_{G/H}$ of $\Sigma - \Sigma(P_{G/H})$ such that, in the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{CV}(G) & \hookrightarrow & \mathrm{Hom}(\mathcal{X}(B), \mathbb{Q}) & \hookleftarrow & \sum_{\alpha\in\Sigma} \mathbb{Q}^{\leq 0}\alpha^* \\
\Big\downarrow \rho & & \Big\downarrow \rho_P^B & & \Big\uparrow \\
 & & \mathrm{Hom}(\mathcal{X}(P_{G/H}), \mathbb{Q}) & \longleftarrow & \sum_{\alpha\notin\Sigma(P_{G/H})} \mathbb{Q}^{\leq 0}\alpha^* \\
\Big\downarrow & & \Big\downarrow & & \Big\uparrow \\
\mathcal{CV}(G/H) & \hookrightarrow & \mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q}) & \hookleftarrow & \sum_{\alpha\in\Sigma_{G/H}} \mathbb{Q}^{\leq 0}\alpha^*
\end{array}$$

the images in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q})$ of the cones $\mathcal{CV}(G/H)$ and $\sum_{\alpha\in\Sigma_{G/H}} \mathbb{Q}^{\leq 0}\alpha^*$ are equal. Moreover, the hooked arrows $\hookrightarrow$ are injective.

Proof: By Lemma 9.4.1, the assumption that ρ is surjective implies that the images in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H},\mathbb{Q})$ of $\sum_{\alpha\notin\Sigma(P_{G/H})}\mathbb{Q}^{\leq 0}\alpha^*$ and of $\mathcal{CV}(G/H)$ are equal. Indeed, $\mathcal{CV}(G)$ equals $\sum_{\alpha\in\Sigma}\mathbb{Q}^{\leq 0}\alpha^*$ (see Section 4).

Moreover, by [Bri90], since G/H is sober, the cone $\mathcal{CV}(G/H)$ is simplicial. For all rays $\mathbb{Q}^{\geq 0}\gamma$ of $\mathcal{CV}(G/H)$, there exists a root α in $\Sigma-\Sigma(P_{G/H})$ such that $\mathbb{Q}^{\leq 0}\alpha^*$ maps to $\mathbb{Q}^{\geq 0}\gamma$ in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H},\mathbb{Q})$ by the diagram of the lemma. Choosing such an α for all rays of $\mathcal{CV}(G/H)$, we obtain a set $\Sigma_{G/H}$ contained in $\Sigma-\Sigma(P_{G/H})$ which satisfies the condition of the lemma. □

9.5 — We can now give the:

Proof of Theorem 3: Let $\mathcal{M}$ be an ample G-linearized line bundle on Y. Let $P^-_{G/H}$ be the parabolic subgroup of G containing T and opposite to $P_{G/H}$. Then, by Proposition 2.6.7, the closed G-orbit Z in Y is isomorphic to $G/P^-_{G/H}$. Denote by γ_0 the character of $P^-_{G/H}$ such that, with the notation of Lemma 3.4.5, the restriction of $\mathcal{M}$ to Z is $\mathcal{L}_{\gamma_0}$. We start by proving:

Claim 1: the set $\gamma_0+\sum_{\alpha\notin\Sigma_{G/H}}\mathbb{Q}^{>0}\alpha$ contains a rational regular dominant weight.

Since, $\Sigma_{P_{G/H}}$ is contained in $\Sigma-\Sigma_{G/H}$, it is sufficient to prove Claim 1 for the cone $\gamma_0+\sum_{\alpha\in\Sigma(P_{G/H})}\mathbb{Q}^{>0}\alpha$.

Note that, since $\mathcal{M}$ is ample, $\mathcal{L}_{\gamma_0}$ is ample and γ_0 belongs to the relative interior of the cone generated by $P^+\cap\mathcal{X}(P_{G/H})$. Recall that $P^+_{\mathbb{Q}}$ denotes the cone of $\mathcal{X}(B)_{\mathbb{Q}}$ generated by P^+. Then, $\mathbb{Q}^{\leq 0}\alpha^\vee$ is the dual cone of $P^+_{\mathbb{Q}}$ from the face $\sum_{\beta\neq\alpha}\mathbb{Q}^{\geq 0}\omega_\beta$. We deduce that the dual cone of $-\gamma_0+P^+_{\mathbb{Q}}$ equals $\sum_{\alpha\in\Sigma(P_{G/H})}\mathbb{Q}^{\leq 0}\alpha^\vee$.

If, by absurd, $\gamma_0+\sum_{\alpha\notin\Sigma_{G/H}}\mathbb{Q}^{>0}\alpha$ does not intersect the interior of $P^+_{\mathbb{Q}}$, then the interior of $-\gamma_0+P^+_{\mathbb{Q}}$ does not intersect $\sum_{\alpha\notin\Sigma_{G/H}}\mathbb{Q}^{>0}\alpha$. Thus, there exists $\sigma\in\sum_{\alpha\in\Sigma(P_{G/H})}\mathbb{Q}^{\leq 0}\alpha^\vee$ which is negative on $\sum_{\beta\notin\Sigma_{G/H}}\mathbb{Q}^{>0}\beta$. This contradicts the fact that $\langle\alpha^\vee,\beta\rangle$ is non-positive and proves Claim 1.

Replacing $\mathcal{M}$ by a power, Claim 1 shows that there exists $\eta\in\mathcal{X}(B)$ which belongs to $\gamma_0+\sum_{\alpha\notin\Sigma_{G/H}}\mathbb{Q}^{>0}\alpha$ and to the relative interior of $P^+_{\mathbb{Q}}$. Consider the $\tilde{G}\times\tilde{G}$-linearized line bundle $\mathcal{L}_\eta$ on X with the notation of Proposition 4.4.4. Replacing $\mathcal{M}$ and hence $\mathcal{L}_\eta$ by a power if necessary, we can assume that the $\tilde{G}\times\tilde{G}$-linearization of $\mathcal{L}_\eta$ induces a $G\times G$-linearization. Denote by χ the character of H such that the restriction of $\mathcal{M}$ to G/H is $\mathcal{L}_\chi$ (see Lemma 3.4.5). We are going to prove that $\mathcal{L}=\mathcal{L}_\eta\otimes\chi$ has the properties announced in the theorem.

First, since η is regular dominant, Proposition 4.4.4 shows that $\mathcal{L}$ is ample. Note that Proposition 3.5.8 shows here that:

$$\mathbf{P}(X,\mathcal{L}_\eta)=P^+_{\mathbb{Q}}\cap\left(\eta+\sum_{\alpha\in\Sigma}\mathbb{Q}^{\leq 0}\alpha\right).$$

In particular, γ_0 belongs to $\mathbf{P}(X,\mathcal{L}_\eta)\cap\mathbf{P}(G/H,\mathcal{L}_\chi)$.

Let us denote by $\mathcal{O}_0$ the unique orbit of $G\times G$ in X such that $\mathcal{C}(X,\mathcal{O}_0)=\sum_{\alpha\in\Sigma_{G/H}}\mathbb{Q}^{\leq 0}\alpha^*$. Then, we have:

Claim 2: Any orbit $\mathcal{O}$ of $G\times G$ in X such that $\mathbf{P}(\overline{\mathcal{O}},\mathcal{L}_\eta)$ contains γ_0 is contained in $X_{\mathcal{O}}$.

Let $\mathcal{O}$ be such an orbit. If α is a simple root, we denote by X_α the center in X of the valuation of $\mathcal{CV}(G)$ which maps to $-\omega_{\alpha^\vee}$ in $\mathrm{Hom}(\mathcal{X}(B),\mathbb{Q})$. Then, there exists a subset I of Σ such that:

$$\overline{\mathcal{O}} = \bigcap_{\alpha\in I} X_\alpha$$

(see for example [CP83]). Thus, we have:

$$\mathbf{P}(\overline{\mathcal{O}},\mathcal{L}_\eta) = \left(\eta + \sum_{\alpha\notin I}\mathbb{Q}\alpha\right) \cap \mathbf{P}(X,\mathcal{L}_\eta),$$

and by Claim 1:

$$\gamma_0 - \eta \in \left(\sum_{\alpha\notin I}\mathbb{Q}\alpha\right) \cap \left(\sum_{\alpha\notin\Sigma_{G/H}}\mathbb{Q}^{<0}\alpha\right).$$

We conclude that $\Sigma - \Sigma_{G/H}$ is contained in $\Sigma - I$; that is I is contained in $\Sigma_{G/H}$. Claim 2 follows.

The next step is to prove
Claim 3: $\mathbf{P}(\overline{\mathcal{O}_0},\mathcal{L}_\eta)\cap\mathbf{P}(G/H,\mathcal{L}_\chi) = \{\gamma_0\}$.

The diffenrences of elements of $\mathbf{P}(\overline{\mathcal{O}_0},\mathcal{L}_\eta)$ span $\sum_{\alpha\notin\Sigma_{G/H}}\mathbb{Q}\alpha$. The one spanned by the diffenrences of elements of $\mathbf{P}(G/H,\mathcal{L}_\chi)$ is $\mathcal{X}(B)^{B\cap H}_{\mathbb{Q}}$. Moreover, the intersection of these two vector subspaces is $\{0\}$, since by Lemma 9.4.2 $\sum_{\alpha\in\Sigma_{G/H}}\mathbb{Q}^{\leq 0}\alpha^*$ embeds in $\mathrm{Hom}(\mathcal{X}(B)^{B\cap H},\mathbb{Q})$. Then, $\mathbf{P}(\overline{\mathcal{O}_0},\mathcal{L}_\eta)\cap\mathbf{P}(G/H,\mathcal{L}_\chi)$ is either the empty set or a single point. On the other hand, the proof of Claim 2 shows that γ_0 belongs to this intersection. Claim 3 is proved.

Now, we can prove Assertion (i).

Consider the quotient morphism $\pi : X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi) \longrightarrow X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi)/\!/H$. Replacing $\mathcal{M}$ by a power, we can assume that $\mathcal{L}_\eta\otimes\chi$ admits a quotient by $\{1\}\times H$ denoted by $\mathcal{L}_Y$ (see Section 5.4). Let us fix a point x in the open orbit of $B\times H$ in $\mathcal{O}_0$. By Proposition 8.4.2, we have: $\mathbf{P}(\overline{G.\pi(x)},\mathcal{L}_Y) = \mathbf{P}(\overline{\mathcal{O}_0},\mathcal{L}_\eta)\cap\mathbf{P}(G/H,\mathcal{L}_\chi)$. Thus, Claim 3 and Proposition 3.1.1 show that the rank of $G.\pi(x)$ equals 0. Thus, $G.\pi(x)$ is projective, hence closed in $X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi)/\!/H$.

Moreover, Proposition 3.5.8 applied to the closed orbit of G in Y implies that (γ_D,χ_D) belongs to the relative interior of the cone $\sum_{D\in\mathcal{D}(G/H)}\mathbb{Q}^{\geq 0}(\gamma_D,\chi_D)$. Now, Proposition 3.5.8 shows that $G.\pi(x)$ is colorless in $X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi)/\!/H$.

Moreover, if $\rho^B_{G/H} : \mathrm{Hom}(\mathcal{X}(B),\mathbb{Q}) \longrightarrow \mathrm{Hom}(\mathcal{X}(B)^{B\cap H},\mathbb{Q})$ denotes the restriction morphism, we have:

$$\begin{aligned}\left(-\gamma_0 + \mathbf{P}(X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi)/\!/H,\mathcal{L}_Y)\right)^\vee &\supseteq \rho^B_{G/H}\left((-\gamma_0+\mathbf{P}(\overline{\mathcal{O}_0},\mathcal{L}_\eta))^\vee\right)\\ &\supseteq \rho^B_{G/H}\left(\textstyle\sum_{\alpha\in\Sigma_{G/H}}\mathbb{Q}^{\leq 0}\alpha^*\right).\end{aligned}$$

Then, Proposition 3.3.4 shows that $G.\pi(x)$ is the unique closed orbit of G in $X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi)/\!/H$. This easily implies that $X^{\mathrm{ss}}(\mathcal{L}_\eta\otimes\chi)/\!/H = Y$. The fact that $\mathcal{L}_\eta\otimes\chi$ satisfies Assertion (i) is proved.

Since Y is toroidal, Proposition 8.5.3 implies Assertion (iii). It remains to prove

Claim 5: $X^{ss}(\mathcal{L}_\eta \otimes \chi) = X^s(\mathcal{L}_\eta \otimes \chi)$.

We noted in the proof of Claim 3 that the subspaces spaned by the differences of elements of $\mathbf{P}(G/H, \mathcal{L}_\chi)$ and of $\mathbf{P}(\overline{\mathcal{O}_0}, \mathcal{L}_\eta)$ intersect in $\{0\}$. We conclude by Proposition 8.7.7. □

9.6 — We come to our main theorem. It asserts that if G/H is sober and liftable then any toroidal embedding of G/H can be obtained as a geometric quotient of a toroidal embedding of G. A toroidal embedding of a spherical homogeneous space is said to be *simplicial* if its fan is simplicial.

Theorem 4 *Let G/H be a sober, liftable spherical homogeneous space. Let Y be a projective, toroidal embedding of G/H.*

Then, there exist a projective, toroidal $G \times G$-equivariant embedding X of G and an ample $G \times H$-linearized line bundle $\mathcal{L}$ on X such that the quotient,

$$\pi : X^{ss}(\mathcal{L}) \longrightarrow X^{ss}(\mathcal{L})//H$$

of X by $\{1\} \times H$ associated to $\mathcal{L}$ satisfies:

(i) $X^{ss}(\mathcal{L})//H = Y$,

(ii) $X^{ss}(\mathcal{L}) = X^s(\mathcal{L})$,

(iii) π *is surjective.*

If in addition Y is simplicial, then there exists a simplicial embedding X of G satisfying the preceding conditions.

Proof: The first step of the proof is to construct the colored fan of X (see Proposition 2.3.3).

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{X}(B), \mathbb{Q}) & \xrightarrow{\widetilde{\rho}} & \mathrm{Hom}(\mathcal{X}(B)^{B\cap H}, \mathbb{Q}) \\ \uparrow & & \uparrow \\ \mathcal{CV}(G) & \xrightarrow{\rho} & \mathcal{CV}(G/H), \end{array}$$

where $\widetilde{\rho}$ denotes the restriction map.

Let $\underline{X}$ denote the canonical embedding of G. Let $\mathcal{L}_1$ be an ample $G \times H$-linearized line bundle on $\underline{X}$ which satisfies Theorem 3. Let Ω_0 be the closed orbit of G in the canonical embedding $\underline{Y}$ of G/H. Consider the orbit $\mathcal{O}(\Omega_0)$ of $G \times G$ in $\underline{X}$ defined in Proposition 8.5.3.

Then $\mathcal{CV}(\underline{X}, \mathcal{O}(\Omega_0))$ is a face of $\mathcal{CV}(G)$; the latter is mapped by ρ isomorphically onto $\mathcal{CV}(G/H)$. Consider the fan $\mathcal{F}_1$ of $\mathcal{CV}(\underline{X}, \mathcal{O}(\Omega_0))$ obtained from $\mathcal{F}(Y)$ by ρ. Let $\mathbf{F}$ denote the face of $\mathcal{CV}(G)$ generated by those extremal rays of $\mathcal{CV}(G)$ which do not belong to $\mathcal{CV}(\underline{X}, \mathcal{O}(\Omega_0))$. Consider the fan $\mathcal{F}$ with maximal cones generated by $\mathbf{F}$ and the maximal cones of $\mathcal{F}_1$. Then, $\mathcal{F}$ is the fan of a complete toroidal $G \times G$-equivariant embedding X of G.

We claim that X is projective. This will follow from a projectivity criterion of an embedding of a spherical homogeneous space in term of its fan (see Corollary 5.2.2 of [Bri97] or see [Bri89]). When applied to Y, this criterion shows existence of a function $l : \mathcal{CV}(G/H) \longrightarrow \mathbb{Q}$, which is linear on each cone of $\mathcal{F}(Y)$ and strictly convex (as defined in [Ful93] p.68). Then, there exists a unique function $\tilde{l} : \mathcal{CV}(G) \longrightarrow \mathbb{Q}$, which equals zero on $\mathbf{F}$, which equals $l \circ \rho$ on $\mathcal{CV}(\underline{X}, \mathcal{O}(\Omega_0))$ and whose restrictions to the cones of $\mathcal{F}$ are linear. One checks that $\tilde{l}$ is strictly convex. Since X is complete, Corollary 5.2.2 of [Bri97] shows that X is projective.

The next step is to chose an ample line bundle on X.

Replacing $\mathcal{L}_1$ by a power if necessary, we can write $\mathcal{L}_1 = \mathcal{L}_2 \otimes \chi$, with $\mathcal{L}_2 \in \mathrm{Pic}^{G\times G}(X)$ and $\chi \in \mathcal{X}(H)$. By [Bri97], Theorem 2 (see also [Kno91]) there exists a $G \times G$-equivariant birational morphism $\psi : X \longrightarrow \underline{X}$. We fix our attention on $\psi^*(\mathcal{L}_2)$.

By Theorem 3, $\underline{X}^{\mathrm{ss}}(\mathcal{L}_2 \otimes \chi)$ is contained in $\underline{X}_{\mathcal{O}(\Omega_0)}$ and intersects $\mathcal{O}(\Omega_0)$. We deduce that for any orbit $\mathcal{O}$ of $G \times G$ in X, the following equivalence holds:

$$\mathbf{P}(\overline{\mathcal{O}}, \psi^*(\mathcal{L}_2)) \cap \mathbf{P}(G/H, \mathcal{L}_\chi) \neq \emptyset \iff \mathcal{O} \subseteq \psi^{-1}(\underline{X}_{\mathcal{O}(\Omega_0)}). \tag{6}$$

Moreover, the intersection in (6) is transversal. We deduce that there exists a neighborhood U of $\psi^*(\mathcal{L}_2)$ in $\mathrm{Pic}^{G\times G}(X)_{\mathbb{Q}}$ such that for all $\mathcal{M}$ in U, we have:

$$\mathbf{P}(\overline{\mathcal{O}}, \mathcal{M}) \cap \mathbf{P}(G/H, \mathcal{L}_\chi) \neq \emptyset \iff \mathcal{O} \subseteq \psi^{-1}(\underline{X}_{\mathcal{O}(\Omega_0)}). \tag{7}$$

On the other hand, the line bundle $\psi^*(\mathcal{L}_2)$ is generated by its global sections. Then, by Proposition 4.4.4, U contains an ample line bundle $\mathcal{L}_2^\varepsilon$.

We claim that $(X, \mathcal{L} = \mathcal{L}_2^\varepsilon \otimes \chi)$ satisfies the three conditions of the theorem.

Consider the quotient $\pi : X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/H$. Let $\mathcal{O}$ be an orbit of $G \times G$ which is closed in $\psi^{-1}(\underline{X}_{\mathcal{O}(\Omega_0)})$. Denote by Ω the open G-orbit in $\pi(\mathcal{O} \cap X^{\mathrm{ss}}(\mathcal{L}))$. Since $X^{\mathrm{ss}}(\mathcal{L})$ is contained in $\psi^{-1}(\underline{X}_{\mathcal{O}(\Omega_0)})$, we have $\mathcal{O}(\Omega) = \mathcal{O}$. Thus, Lemma 8.5.4 shows that the cone $\mathcal{C}(X^{\mathrm{ss}}(\mathcal{L})/\!/H, \Omega)$ contains $\rho(\mathcal{C}(X, \mathcal{O}))$. Moreover, the restriction of ρ to $\mathcal{C}(\underline{X}, \mathcal{O}(\Omega_0))$ is injective. Then, the interior of the cone $\mathcal{C}(X^{\mathrm{ss}}(\mathcal{L})/\!/H, \Omega)$ in $\mathcal{CV}(G/H)$ is not empty. It follows that Ω is projective and that $\pi(\mathcal{O} \cap X^{\mathrm{ss}}(\mathcal{L})) = \Omega$. Then, by Lemma 8.5.4 and by construction of X, $X^{\mathrm{ss}}(\mathcal{L})/\!/H$ is isomorphic to Y.

Since the cones $\rho(\mathcal{C}(X, \mathcal{O}))$, where $\mathcal{O}$ is an orbit of $G \times G$ in X as above, cover $\mathcal{CV}(G/H)$, we have established a correspondence between closed orbits of $G \times G$ in $\psi^{-1}(\underline{X}_{\mathcal{O}(\Omega_0)})$ and complete orbits of G in $X^{\mathrm{ss}}(\mathcal{L})/\!/H$. It is now easy to prove that $X^{\mathrm{ss}}(\mathcal{L}) = X^{\mathrm{s}}(\mathcal{L})$, by using Proposition 8.7.7.

Moreover, Proposition 8.5.3 show that π is surjective.

If in addition Y is simplicial, then by construction X is simplicial too. $\square$

Now, we can apply Theorem 4 and describe the isotropy subgroups of the action of G in Y (with the notation of Theorem 4). So, the following corollary extends results that C. DeConcini and C. Procesi (see [CP83]) obtained when H is symmetric.

Corollary 9.6.3 *Let G/H and Y be as in Theorem 4. Let y be a point in Y. Then, there exist two opposite parabolic subgroups P and Q of G such that*

$$G_y = P^u.C_y.\Big(L \cap Q^u(Q \cap H)\Big),$$

where $L = P \cap Q$, C is the connected center of L and, P^u and Q^u denote respectively the unipotent radicals of P and Q.

Proof: Let X be a $G \times G$-equivariant embedding of G and $\mathcal{L}$ be a ample $G \times H$-linearized line bundle satisfying Theorem 4. Let x in X such that $\pi(x) = y$. Then, by Proposition 4.3.2, there exists two opposite parabolic subgroups P and Q of G and a subgroup C of the connected center of $L = P \cap Q$ such that the isotropy of x in $G \times G$ is

$$I := (P^u \times Q^u) \ltimes (\Delta L \times (C \times \{1\})).$$

Since $\pi^{-1}(y) = (\{1\} \times H).x$, we have:

$$G_y = \{g \in G \ : \ (g,1)I \cap (\{1\} \times H) \neq \emptyset\}.$$

The corollary follows. □

With preceding notation, Corollary 9.6.3 implies that $L \cap Q^u(Q \cap H)$ is a spherical subgroup of L. Moreover, G_y is obtained by parabolic induction (see [Bri98], [Lun96] or [Was96] for a precise definition) from the latter spherical subgroup of L.

Note that Corollary 9.6.3 does not necessary hold if G/H is not liftable (see Example 10.7.3 of [Res00]).

References

[BCP90] E. Bifet, C. De Concini, and C. Procesi. Cohomology of regular embeddings. *Adv. Math.*, 82(1):1–34, 1990.

[Bou61] N. Bourbaki. *Éléments de mathématique. Fascicule XXVIII. Algèbre commutative. Chapitre 3: Graduations, filtrations et topologies. Chapitre 4: Idéaux premiers associés et décomposition primaire.* Hermann, Paris, 1961. Actualités Scientifiques et Industrielles, No. 1293.

[Bou64] N. Bourbaki. *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations.* Hermann, Paris, 1964. Actualités Scientifiques et Industrielles, No. 1308.

[Bou68] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre VI: systèmes de racines.* Hermann, Paris, 1968.

[BP90] M. Brion and C. Procesi. Action d'un tore dans une variété projective. In A. Joseph A.Connes, M.Duflo and R.Rentschler, editors, *Operator algebras, unitary representations, enveloping algebras, and invariant theory*, pages 509–539. Birkhäuser, 1990.

[Bri89] M. Brion. Groupe de Picard et nombres caractéristiques des variétés sphériques. *Duke Math. J.*, 58(2):397–424, 1989.

[Bri90] M. Brion. Vers une généralisation des espaces symétriques. *J. Algebra*, 134(1):115–143, 1990.

[Bri97] M. Brion. Variétés Sphériques. *notes de la session de la S. M. F. "Opérations hamiltoniennes et opérations de groupes algébriques"*, pages 1–60, 1997. http://www-fourier.ujf-grenoble.fr/ mbrion/notes.html.

[Bri98] M. Brion. The behaviour at infinity of the Bruhat decomposition. *Comment. Math. Helv.*, 73(1):137–174, 1998.

[CP83] C. De Concini and C. Procesi. Complete symmetric varieties. In *Invariant theory (Montecatini, 1982)*, pages 1–44. Springer, Berlin, 1983.

[Ful93] W. Fulton. *Introduction to toric varieties.* Princeton University Press, NJ, 1993.

[Gro97] J.D. Grosshans. *Algebraic Homogeneous Spaces and Invariant Theory.* Lecture Notes 1673, Springer, New York, 1997.

[Har77] R. Hartshorne. *Algebraic Geometry.* Springer Verlag, New York, 1977.

[Kan99] S. Kannan. Remarks on the wonderful compactification of semisimple algebraic groups. *Proc. Indian Acad. Sci. Math. Sci.*, 109(3):241–256, 1999.

[KKV84] F. Knop, H. Kraft, and T. Vust. The Picard group of a G-variety. In *Algebraic transformation group and invariant theory*, pages 77–87. Birkhäuser, 1984.

[Kno91] F. Knop. The Luna-Vust theory of spherical embeddings. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, pages 225–249, Madras, 1991. Manoj Prakashan.

[LP90] P. Littelmann and C. Procesi. Equivariant cohomology of wonderful compactifications. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, pages 219–262. Birkhäuser Boston, Boston, MA, 1990.

[Lun96] D. Luna. Toute variété magnifique est sphérique. *Transform. Groups*, 1(3):249–258, 1996.

[LV83] D. Luna and T. Vust. Plongements d'espaces homogènes. *Comment. Math. Helv.*, 58(2):186–245, 1983.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory.* Springer Verlag, New York, 3d edition, 1994.

[Oda88] T. Oda. *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, Translated from the Japanese.* Springer-Verlag, Berlin, 1988.

[PV89] V.L. Popov and È. B. Vinberg. Invariant theory. In *Algebraic geometry, 4 (Russian)*, pages 137–314, 315. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989.

[Ren89] Lex E. Renner. Reductive embeddings. In *Group actions and invariant theory (Montreal, PQ, 1988)*, pages 175–192. Amer. Math. Soc., Providence, RI, 1989.

[Res00] N. Ressayre. Variations de Quotients Géométriques et Applications. *Thèse, Université Grenoble I*, pages 1–214, 2000. http://www-fourier.ujf-grenoble.fr/THESE/these_daterev.html.

[Was96] B. Wasserman. Wonderful varieties of rank two. *Transform. Groups*, 1(4):375–403, 1996.

- ◇ -

Nicolas Ressayre
Université MONTPELLIER II
Dépatement des Sciences Mathématiques
Case courrer 051. Place Eugène Bataillon
34095 MONTPELLIER CEDEX 5
FRANCE
e-mail: `ressayre@math.univ-montp2.fr`