ADJACENCY OF YOUNG TABLEAUX AND THE SPRINGER FIBERS

N.G.J. PAGNON AND N. RESSAYRE

Abstract. In this article we connect different works about the irreducible components of the Springer fibers of type A. Firstly, we show a relation between the Spaltenstein's partition of the fibers and a total order $\prec$ on the set of standard Young tableau. Next, using a result of Steinberg, we connect a work of the first author to the Robinson-Schensted map. We also perform the Spaltenstein's study of the relative position of the Springer fibers and the $\mathbb{P}^1$-fibrations of the flag manifold. This leads us to consider the adjacency relation on the set of standard Young tableaux and to define oriented and labeled graphs with the standard Young tableaux as vertices. Using this adjacency relation, we describe some smooth irreducible components of the Springer fibers. Finally we show that this graph identify with some *full* subgraphs of the Bruhat graph.

1. Introduction

Let V be an n-dimensional vector space over a field $\mathbf{k}$, $\mathcal{F} = \mathcal{F}(V)$ be the (full) flag manifold of V and x be a nilpotent endomorphism of V. In this article, we perform the study of the Springer fiber

$$\mathcal{F}_x := \{(\{0\} = V_0, V_1, \cdots, V_n) \in \mathcal{F} \;:\; \forall i = 1, \cdots, n \;\; x(V_i) \subseteq V_{i-1}\}.$$

N. Spaltenstein has defined in each irreducible component of $\mathcal{F}_x$ a non singular open subvariety. Moreover, these subvarieties form a partition of $\mathcal{F}_x$ and are parametrized. To explain more precisely the Spaltenstein's results, let us introduce some notation. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ the sizes of the Jordan blocks of x ordered by decreasing. So, $\lambda_1 + \cdots + \lambda_r = n$; in other words, we have a partition of n denoted by $\lambda \vdash n$. This partition is called the *type* of x. To the partition λ is associated its *Young diagram* $Y(\lambda) = Y(x)$ whose the r lines are composed respectively of $\lambda_1, \lambda_2, \ldots, \lambda_r$ squares. A filling of the Young diagram $Y(\lambda)$ with the integers $1, 2, \ldots, n$ such that the entries along any line or column are strictly increasing is called a *standard Young tableau* of *shape* λ. Let St_λ denote the set of standard Young tableaux of shape λ.

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For any flag ξ in $\mathcal{F}_x$, by considering the relative position of the subspaces of ξ and the images of the powers of x, N. Spaltenstein associates to ξ a standard Young tableau σ_ξ. He showed that for any $\sigma \in \mathrm{St}_\lambda$, the set $\mathcal{F}_{x,\sigma}$ of the ξ such that $\sigma_\xi = \sigma$ is a non singular open irreducible subvariety of $\mathcal{F}_x$. Moreover, each irreducible component of $\mathcal{F}_x$ contains exactly one $\mathcal{F}_{x,\sigma}$; that is, the irreducible components of $\mathcal{F}_x$ are the closures $\mathcal{F}_\sigma$ of the $\mathcal{F}_{x,\sigma}$.

Let us recall from [6] the definition of a total order $\prec$ on St_λ. For each $\sigma \in \mathrm{St}_\lambda$ and for $i \in \{1, 2, \dots, n\}$, let $c_\sigma(i)$ (resp. $l_\sigma(i)$) denote the number of the column (resp. line) in which i lies for σ. For $\sigma_1, \sigma_2 \in St_\lambda$, we denote $\sigma_1 \prec \sigma_2$ if for some $1 \leq i_0 \leq n$ we have $l_{\sigma_1}(i_0) < l_{\sigma_2}(i_0)$, and for $i_0 < j \leq n$ we have $l_{\sigma_1}(j) = l_{\sigma_2}(j)$. The notation $\sigma_1 \preceq \sigma_2$ means $\sigma_1 \prec \sigma_2$ or $\sigma_1 = \sigma_2$. We can now state our first result:

Theorem 1. *For any $\sigma \in \mathrm{St}_\lambda$, we have:*

$$\mathcal{F}_\sigma \subseteq \bigcup_{\gamma \preceq \sigma} \mathcal{F}_{x,\gamma}.$$

Let $\sigma_{\max}$ (resp. $\sigma_{\min}$) denote the maximal (resp. minimal) element of St_λ for the order $\prec$. Theorem 1 shows that $\mathcal{F}_{x,\sigma_{\min}} = \mathcal{F}_{\sigma_{\min}}$; thus, by a Spaltenstein's result, this irreducible component of $\mathcal{F}_x$ is smooth.

In Section 4 we connect the result in [6] with the Robinson-Schensted map. Let us fix a base $\mathcal{B}$ such that the matrix of x in $\mathcal{B}$ has Jordan's form with decreasing block sizes. Let B denote the subgroup of $\mathrm{Gl}(V)$ consisting of the endomorphisms with upper triangular matrices in $\mathcal{B}$. The orbits of B in $\mathcal{F}$ are the Schubert cells; they are parametrized by the group $\mathfrak{S}_n$ of the permutations of the set $\{1, \cdots, n\}$. For every $\sigma \in \mathrm{St}_\lambda$, in [7] for the hook case and in [6] for the general one, we can find a description of the element $w_\sigma \in \mathfrak{S}_n$ corresponding to the Schubert cell $\mathcal{C}_{w_\sigma}$ which intersects $\mathcal{F}_\sigma$ in an open dense subset of $\mathcal{F}_\sigma$. In [3] and [10], one can find a combinatorial definition and a geometric interpretation of the Robinson-Schensted bijection $\mathrm{RS} : \bigcup_{\mu \vdash n} \mathrm{St}_\mu \times \mathrm{St}_\mu \longrightarrow \mathfrak{S}_n$, $(\sigma, \gamma) \longmapsto \mathrm{RS}(\sigma, \gamma)$. Our main result in Section 4 is

Theorem 2. *For all $\sigma \in \mathrm{St}_\lambda$, we have*

$$w_\sigma = \mathrm{RS}(\sigma_{\max}, \sigma^\vee).$$

In Section 5 we are interested in the open subset $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_x$ of $\mathcal{F}_x$. Since the map RS is injective, $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_x$ is not dense in $\mathcal{F}_\gamma$ for all $\gamma \neq \sigma$. The following theorem precises this observation:

Theorem 3. *For every $\sigma \in \mathrm{St}_\lambda$, we have $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_x \subseteq \mathcal{F}_{x,\sigma}$. In particular, $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_x$ is smooth.*

Let k be an integer between 1 and $n-1$. We denote by $\mathcal{F}_k$ the variety consisting of the partial flags $(V_1 \subseteq \cdots \subseteq V_{k-1} \subseteq V_{k+1} \subseteq \cdots \subseteq V_n = V)$ such that $\dim(V_i) = i$ for all i. We denote by $\phi_k : \mathcal{F} \longrightarrow \mathcal{F}_k$ the $\mathbb{P}^1$-fibration which omits the subspace V_k. A subset X in $\mathcal{F}$ is said to be ϕ_k-*stable* if $X = \phi_k^{-1}(\phi_k(X))$. We prove again the Spaltenstein's result (see [9]) about the ϕ_k-stability of $\mathcal{F}_\sigma$ and give a more precise description about this (see Proposition 6.1).

In the classical combinatorial theory of standard Young tableaux, the Schützenberger's involution, $\sigma \mapsto \sigma^\vee$ of the set St_λ is known. Combined with the geometric interpretation of this involution due to Van Leeuwen (see [5]), the Spaltenstein's result implies that: $c_\sigma(k-1) \geq c_\sigma(k)$ if and only if $c_{\sigma^\vee}(n-k) \geq c_{\sigma^\vee}(n+1-k)$, for any $2 \leq k \leq n$.

For any $3 \leq k \leq n$, two standard Young tableaux of shape λ are said to be *k-adjacent* if they are obtained one from the other by switching the places of k and $k-1$. From our description of the ϕ_k-stability, we deduce that if σ and γ are k-adjacent then $\mathcal{F}_\sigma$ intersects $\mathcal{F}_\gamma$ in codimension one.

Let Γ_λ be the oriented graph with vertices set St_λ and the edges labeled by the integers $3, \cdots, n$, where σ is joined to γ by an edge labeled by k if σ and γ are k-adjacent and $\sigma \prec \gamma$. The graph Γ_λ is called the *adjacency graph* of λ. We show that $\sigma_{\min}$ (resp. $\sigma_{\max}$) is the unique minimal (resp. maximal) vertex of Γ_λ.

An irreducible component $\mathcal{F}_\sigma$ of $\mathcal{F}_x$ which is homogeneous under the action of a parabolic subgroup of $\mathrm{Gl}(V)$ is called a Richardson component. In Section 7.1, we recall the standard Young tableaux corresponding to the Richardson components. We determine others smooth components:

Theorem 4. *Let σ_0 be a standard Young tableau such that $\mathcal{F}_{\sigma_0}$ is a Richardson component. Let σ be a standard Young tableau k-adjacent to σ_0 such that $\sigma \prec \sigma_0$. Then $\mathcal{F}_\sigma$ is smooth.*

For $i = 1, \cdots, n-1$, let s_i denote the transposition of $\mathfrak{S}_n$ which permutes i and $i+1$. The s_i's generate $\mathfrak{S}_n$; the length $l(w)$ of the element w in $\mathfrak{S}_n$ is the minimal number of s_i necessary to write w. A writing of w is said to be *reduced* if it makes use $l(w)$ transpositions s_i. In Section 8, we show that the expression of w_σ given in [6] is reduced. Generalizing this result, we give a (new ?) reduced canonical writing of the elements of $\mathfrak{S}_n$.

Let Γ_n be the oriented graph with vertices set $\mathfrak{S}_n$ and the edges labeled by the integers $2, \cdots, n$, where w is joined to w' by an edge labeled by k if $w' = ws_{n+1-k}$ and $l(w') = l(w) + 1$. The graph Γ_n is called the *Bruhat graph*. Our main result connects the Bruhat and the adjacency graphs:

Theorem 5 (Main Theorem). *By the map $\mathrm{St}_\lambda \longrightarrow \mathfrak{S}_n$, $\sigma \longmapsto w_\sigma$, the adjacency graph is a full subgraph of the Bruhat graph.*

In particular, for all σ and γ in St_λ, w_σ and w_γ are not joined by an edge labeled by 2 in the Bruhat graph.

Let us recall that in [6], the first author gives an expression of w_σ as a product of simple reflexions in term of the standard Young tableau σ. The study of this writing is central in this article. Our last result gives an alternative reduced expression of w_σ (see Theorem 8.9).

2. Spaltenstein map

In this section, we recall some usual facts about Spaltenstein map. For later use, we also include some proofs. Other proofs can be found in [9, 1, 11].

A base $\mathcal{B}$ is said to be *adapted* to x if the matrix of x in $\mathcal{B}$ is a Jordan matrix with decreasing block sizes. For any base $\mathcal{B}$, the flag whose the i^{th} subspace is spanned by the first i vectors of $\mathcal{B}$ is called *the canonical flag associated to* $\mathcal{B}$. A flag is said to be *adapted to* x if it is the canonical flag associated to an adapted base $\mathcal{B}$.

Let us fix a line V_1 in $\operatorname{Ker} x$. Consider $V' = V/V_1$, the canonical projection $q : V \longrightarrow V'$ and $x' : V' \longrightarrow V', v + V_1 \longmapsto x(v) + V_1$.

Let $(F_1, \cdots, F_n)$ be a flag adapted to x. Set

$$F_i' := \begin{cases} (F_i + V_1)/V_1, & \text{if } 1 \leq i \leq K, \\ F_{i+1}/V_1, & \text{if } K+1 \leq i \leq n-1, \end{cases} \tag{2.1}$$

where K denote the minimum of the i such that $V_1 \subseteq F_i$. Since $V_1 \subseteq \operatorname{Ker} x$, then $K = \lambda_1 + \cdots + \lambda_{i_0} + 1$ for some $i_0 \leq r - 1$ (obviously, if $i_0 = 0$, $\lambda_1 + \cdots + \lambda_{i_0} = 0$).

Lemma 2.1. *With above notation, the flag $(F_1', \cdots, F_{n-1}')$ is adapted to x'.*

Moreover, the partition λ' associated to x' is obtained from λ by replacing λ_{i_1} by $\lambda_{i_1} - 1$ (obviously if λ_{i_1} equals one we have to omit λ_{i_1} to obtain λ'), where $i_1 := \max\{j \mid \lambda_j = \lambda_{i_0}\}$.

Proof. Let $\mathcal{B} = (e_1, \cdots, e_n)$ be a basis of V adapted to x such that $(F_1, \cdots, F_n)$ is the canonical flag associated to $\mathcal{B}$. Consider the following base $\mathcal{B}'$ of V':

$$\mathcal{B}' = (q(e_1), \cdots, q(e_K), q(e_{K+2}), \cdots, q(e_n)).$$

The matrix of x' in $\mathcal{B}'$ has the following form:

$$\left(\begin{array}{c|c|c|c} J_{\lambda_1} & & \begin{matrix}\star \\ 0 \\ \vdots \\ 0\end{matrix} & \\ \hline & \ddots & \begin{matrix}\vdots \\ \vdots\end{matrix} & \\ \hline & & J_{\lambda_{i_0}} & \\ \hline & & & \ddots \end{array}\right)$$

By an easy change of base, one can check the lemma. $\square$

Let $\xi = (V_1, \cdots, V_n) \in \mathcal{F}_x$. For $i = 1, \cdots, n$, we set

$$\alpha_i := \max\{k \, : \, V_i \subseteq V_{i-1} + \operatorname{Im}(x^{k-1})\}. \tag{2.2}$$

Notice that by convention $V_0 = \{0\}$.

Proposition 2.2. *There exists a unique standard Young tableau σ such that $c_\sigma(n+1-i) = \alpha_i$, for all $i = 1, \cdots, n$.*

Proof. The unicity is obvious. Let us prove the existence by induction on n. We use notation of Lemma 2.1. For $i = 1, \cdots, n-1$, set $\beta_i = \max\{k : q(V_{i+1}) \subseteq q(V_i) + \mathrm{Im}(x'^{k-1})\}$. By induction, there exists a standard Young tableau σ' of shape λ' such that $c_{\sigma'}(n-i) = \beta_i$ for all $i = 1, \cdots, n-1$. One easily checks that the standard Young tableau σ obtained from σ' by adding a case indexed by n at the line i_1 satisfies the lemma. $\square$

Corollary 2.3. *Let $\xi = (V_1, \cdots, V_n) \in \mathcal{F}_x$ and $\sigma \in \mathrm{St}_\lambda$. The following are equivalent:*

(1) $V_i \subseteq V_{i-1} + \mathrm{Im}(x^{c_\sigma(n+1-i)-1})$ *for all* $i = 1, \cdots, n-1$.

(2) $\left\{ \begin{array}{l} V_i \subseteq V_{i-1} + \mathrm{Im}(x^{c_\sigma(n+1-i)-1}) \\ V_i \nsubseteq V_{i-1} + \mathrm{Im}(x^{c_\sigma(n+1-i)}) \end{array} \right.$ *for all* $i = 1, \cdots, n-1$.

Proof. We assume that ξ satisfies Condition (1). Consider the integers α_i defined by Formula (2.2). By Proposition 2.2, there exists a standard Young tableau γ such that ξ satisfies Conditions (2) for γ in place of σ. Since ξ satisfies Condition (1), we have for all $i = 1, \cdots, n$, $c_\gamma(n+1-i) = \alpha_i \leq c_\sigma(n+1-i)$. One easily deduces that $\gamma = \sigma$. $\square$

We can now explain the Spaltenstein partition of $\mathcal{F}_x$. For $\sigma \in \mathrm{St}_\lambda$, we denote by $\mathcal{F}_{x,\sigma}$ the set of flags ξ which satisfy the conditions of Corollary 2.3. Proposition 2.2 shows the following result of Spaltenstein: $(\mathcal{F}_{x,\sigma})_{\sigma \in \mathrm{St}_\lambda}$ is a partition of $\mathcal{F}_x$. Moreover, Spaltenstein showed that the $\mathcal{F}_{x,\sigma}$ are smooth open and irreducible subvarieties of $\mathcal{F}_x$. For any $\sigma \in \mathrm{St}_\lambda$, the closure $\mathcal{F}_\sigma$ of $\mathcal{F}_{x,\sigma}$ is an irreducible component of $\mathcal{F}_x$. Let $\mathcal{IF}_x$ denote the set of irreducible components of $\mathcal{F}_x$. Spaltenstein showed in [8] that the map

$$\mathrm{Spal} : \mathrm{St}_\lambda \longrightarrow \mathcal{IF}_x,\ \sigma \longmapsto \mathcal{F}_\sigma$$

is a bijection.

Corollary 2.4. *Let $(V_1, \cdots, V_k)$ be a partial flag stable by x and satisfying Conditions (2) of Corollary 2.3 for $i = 1, \cdots, k$.*

Then there exists $\xi \in \mathcal{F}_{x,\sigma}$ starting with $(V_1, \cdots, V_k)$.

Proof. Firstly we notice that the map

$$\mathcal{F}_{x,\sigma} \to \mathbf{P}(\mathrm{Ker} x \cap \mathrm{Im}(x^{c_\sigma(n)-1})) - \mathbf{P}(\mathrm{Ker}\, x \cap \mathrm{Im}(x^{c_\sigma(n)})),\ (V_i) \longmapsto V_1.$$

is a surjective fibration with a typical fiber isomorphic to $\mathcal{F}_{x',\sigma'}$. If $(V_1, \cdots, V_k)$ is a partial flag stable by x and satisfies Conditions (2) of Corollary 2.3, then the partial flag $(V_1' := V_2/V_1, \cdots, V_{k-1}' := V_k/V_1)$ is stable by x' and satisfies Conditions (2) of Corollary 2.3 for x' and the standard Young tableau σ' obtained by deleting the square labeled by n. By induction on $\dim(V)$, there exists $\xi' = \xi/V_1 \in \mathcal{F}_{x',\sigma'}$ starting with $(V_1', \dots, V_{k-1}')$ and the result follows with ξ. $\square$

3. Closure of Spaltenstein's cells

3.1. Let us recall from [6] the definition of a total order $\prec$ on St_λ. For $\sigma_1, \sigma_2 \in St_\lambda$, we denote $\sigma_1 \prec \sigma_2$ if for some $1 \leq i_0 \leq n$ we have $l_{\sigma_1}(i_0) < l_{\sigma_2}(i_0)$, and for $i_0 < j \leq n$ we have $l_{\sigma_1}(j) = l_{\sigma_2}(j)$.

Let $\sigma_{\max}$ (resp. $\sigma_{\min}$) denote the maximal (resp. minimal) element of St_λ for the order $\prec$. Notice that $\sigma_{\max}$ is obtained by filling first line of the Young diagram $Y(\lambda)$ with the integers $\{1, \dots, \lambda_1\}$, the second one with the integers $\{\lambda_1 + 1, \cdots, \lambda_1 + \lambda_2\}$, and so on... The tableau $\sigma_{\min}$ is obtained by filling the first column of the Young diagram $Y(\lambda)$ with the integers $\{1, \cdots, r\}$, the second one with the integers $r+1, r+2, \dots$, and so on...

Lemma 3.1. *Let σ be a standard Young tableau (different from $\sigma_{\min}$) and γ its predecessor for the order $\prec$. Let i_0 denote the integer such that $l_\gamma(i_0) < l_\sigma(i_0)$, and for $i_0 < j \leq n$ we have $l_\gamma(j) = l_\sigma(j)$.*

Let σ' (resp. γ') denote the standard Young tableau obtained from σ (resp. γ) by deleting the squares labeled by the integers $\{i_0, \dots, n\}$.

Then, we have:

1. *the cases occupied by $i = i_0 + 1, \cdots, n$ are the same in γ and σ.*
2. *Consider the Young diagram Y obtained from σ' (or γ') by adding the square labeled by i_0 in σ (or γ). In Y, there is no corner between the cases labeled by i_0 in σ and γ.*
3. *σ' (resp. γ') is the maximal (resp. minimal) element for the corresponding shape.*

Proof. Assertion 1 is obvious. By omitting the cases occupied by $i_0 + 1, \cdots, n$, we may assume that $i_0 = n$. By absurd, let us assume that there exits a corner between the two squares labeled by n in γ and σ. Let δ be a standard Young tableau for which this corner is occupied by n. We have $\gamma \prec \delta \prec \sigma$; this is absurd and Assertion 2 follows.

Any tableau δ of shape $Y(\lambda)$ with n in the same case as in γ satisfies $\delta \prec \sigma$. Therefore, γ' is maximal. Any tableau δ of shape $Y(\lambda)$ with n in the same case as in σ satisfies $\gamma \prec \delta$. Therefore, σ' is minimal. $\square$

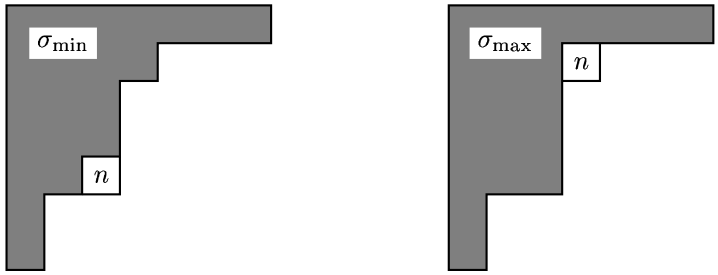
If $i_0 = n$, Lemma 3.1 can be summarized by Figure 1.

3.2. Let $\sigma \in \mathrm{St}_\lambda$ be a standard Young tableau. Define the *boundary* of $\mathcal{F}_\sigma$ as $\partial\mathcal{F}_\sigma := \mathcal{F}_\sigma - \mathcal{F}_{x,\sigma}$. The goal of this section is the following

Theorem 3.2.

$$\partial\mathcal{F}_\sigma \subseteq \bigcup_{\gamma \prec \sigma} \mathcal{F}_\gamma.$$

Firstly, we prove the following



σ Predecessor of σ

FIGURE 1. Predecessor for $\prec$

Lemma 3.3. *There exist integers $d_1, \cdots, d_{n-1}$ such that for all $\xi = (V_1, \cdots, V_n) \in \mathcal{F}_{x,\sigma}$ and for all $i = 1, \cdots, n-1$ we have*

$$\dim \left(V_{i-1} + \operatorname{Im}\left(x^{c_\sigma(n+1-i)-1}\right)\right) = d_i.$$

Proof. We proceed by induction on i. For $i = 1$, there is nothing to prove. Consider V', x', σ' ξ' as before. For $2 \leq i \leq n-1$ we have $c_\sigma(n+1-i) = c_{\sigma'}(n+1-i) = c_{\sigma'}(n-(i-1))$, by induction for all $\xi' = (V'_1, \cdots, V'_{n-1}) \in \mathcal{F}_{x',\sigma'}$ and for every $i = 1, \cdots, n-2$, we have $d'_{i-1} = \dim(V'_{i-2} + \operatorname{Im}(x'^{c_{\sigma'}(n-(i-1))-1}) = \dim(V_{i-1}/V_1 + (\operatorname{Im}(x^{c_\sigma(n+1-i)-1}) + V_1)/V_1) = \dim((V_{i-1} + \operatorname{Im}(x^{c_\sigma(n+1-i)-1}))/V_1) = \dim(V_{i-1} + \operatorname{Im}(x^{c_\sigma(n+1-i)-1})) + 1$. These equalities show that $d_i = d'_{i-1} - 1$ and the proof is complete. $\square$

We also need the following well known

Lemma 3.4. *Let F be a vector subspace of V. Let*

$$\operatorname{Gr}_k(V)_i := \{W \in \operatorname{Gr}_k(V) \ : \ \dim(F + W) = i\}.$$

Then, $\operatorname{Gr}_k(V)_i$ is a locally closed in $\operatorname{Gr}_k(V)$. Moreover, the map $\phi : \operatorname{Gr}_k(V)_i \longrightarrow \operatorname{Gr}_i(V)$, $W \longmapsto F + W$ is a morphism.

We can now prove Theorem 3.2:

Proof. Let $\xi = (V_1, \cdots, V_n) \in \mathcal{F}_\sigma$ and α_i be the integers defined by Formula (2.2). Set $i_0 := \min\{i \ : \ \alpha_i \neq c_\sigma(n+1-i) - 1\}$. Consider the integers d_i defined in Lemma 3.3. Denote by Ω the set of the $(W_1, \cdots, W_n) \in \mathcal{F}$ such that

$$\forall i \leq i_0 \quad \dim\left(W_{i-1} + \operatorname{Im}\left(x^{c_\sigma(n+1-i)-1}\right)\right) = d_i.$$

By Lemma 3.4, Ω is a locally closed subvariety of $\mathcal{F}$. Moreover, for all $i = 1, \cdots, i_0$, the map $\phi_i : \Omega \longrightarrow \operatorname{Gr}_{d_i}(V)$, $(W_1, \cdots, W_n) \longmapsto W_{i-1} + \operatorname{Im}\left(x^{c_\sigma(n+1-i)-1}\right)$ is a morphism. We deduce that

$$\Omega_0 := \{\zeta = (W_1, \cdots, W_n) \in \Omega \ : \ \forall i \leq i_0 \quad W_i \subseteq \phi_i(\zeta)\}$$

is closed in Ω.

But, since i_0 is minimal, Corollary 2.4 and Lemma 3.3 imply that $\xi \in \Omega$. So, ξ belongs to the closure of $\mathcal{F}_{x,\sigma}$ in Ω. Therefore, it belongs to Ω_0. We deduce that $\alpha_{i_0} > c_\sigma(n+1-i_0) - 1$. The theorem follows. □

Corollary 3.5. *We have*

$$\mathcal{F}_{x,\sigma_{min}} = \mathcal{F}_{\sigma_{min}}.$$

In particular $\mathcal{F}_{\sigma_{min}}$ is a smooth subvariety.

4. Robinson-Schensted map

4.1. We now introduce a dual Spaltenstein map. Let V^* be the dual of V. If F is a vector subspace of V, we denote by $F^\perp$ the orthogonal of F in V^*. Consider the isomorphism:

$$\eta\,:\,\mathcal{F}(V) \longrightarrow \mathcal{F}(V^*),\ (V_1,\cdots,V_n) \longmapsto (V_{n-1}^\perp,\cdots,V_1^\perp,V^*).$$

Consider the transposed map ${}^tx \in \mathrm{Hom}(V^*,V^*)$ of x. One easily checks that η induces by restriction an isomorphism from $\mathcal{F}_x$ onto $\mathcal{F}_{{}^tx}$. In particular, it induces a bijection $\tilde\eta\,:\,\mathcal{IF}_x \longrightarrow \mathcal{IF}_{{}^tx}$. By composing the Spaltenstein map of ${}^tx$ with $\tilde\eta$, one obtains a new bijection

$$\mathrm{Spal}^*\,:\,\mathrm{St}_\lambda \longrightarrow \mathcal{IF}_x,\ \sigma \longmapsto \mathcal{F}^*_\sigma.$$

In [5], one can find a combinatorial definition of the Schützenberger involution: $\mathrm{St}_\lambda \longrightarrow \mathrm{St}_\lambda$, $\sigma \longmapsto \sigma^\vee$. We now recall from [5] the following geometric interpretation of this involution: for all $\sigma \in \mathrm{St}_\lambda$, we have

$$\mathcal{F}^*_\sigma = \mathcal{F}_{\sigma^\vee}.$$

4.2. Let $\mathrm{RS}\,:\,\mathrm{St}_\lambda \times \mathrm{St}_\lambda \longrightarrow \mathfrak{S}_n$ denote the Robinson-Schensted map. We now explain the Steinberg geometric interpretation of the Robinson-Schensted map.

Let us start with a geometric interpretation to the permutations. Let $(\xi,\xi') \in \mathcal{F}\times\mathcal{F}$. There exists a unique $w \in \mathfrak{S}_n$ such that there exists a basis $(e_1,\cdots,e_n)$ of V such that ξ (resp. ξ') is the canonical flag associated to $(e_1,\cdots,e_n)$ (resp. $(e_{w(1)},\cdots,e_{w(n)})$). The permutation w is called *the relative position* of ξ and ξ', or of the pair (ξ,ξ'). In fact, by the Bruhat's lemma w determines the $\mathrm{GL}(V)$-orbit of (ξ,ξ').

Let $(C,C') \in \mathcal{IF}_x \times \mathcal{IF}_x$. Since $\mathrm{GL}(V)$ has finitely many orbits in $\mathcal{F}\times\mathcal{F}$, there exits a unique one $\mathcal{O}_{(C,C')}$ such that $\mathcal{O}_{(C,C')} \cap (C\times C')$ is open and dense in $C\times C'$. Let $w_{(C,C')} \in \mathfrak{S}_n$ be the relative position of an element of $\mathcal{O}_{(C,C')}$. Consider the Robinson-Schensted-Steinberg map

$$\mathrm{RSS}\,:\,\mathcal{IF}_x \times \mathcal{IF}_x \longrightarrow \mathfrak{S}_n,\ (C,C') \longmapsto w_{(C,C')}.$$

We can now state the result of Steinberg: for any $(\sigma,\sigma') \in \mathrm{St}_\lambda\times\mathrm{St}_\lambda$, we have

$$\mathrm{RSS}(\mathcal{F}^*_\sigma,\mathcal{F}^*_{\sigma'}) = \mathrm{RS}(\sigma,\sigma').$$

In other words, RS is obtained by composing RSS and the parametrization Spal^* of $\mathcal{IF}_x$.

4.3. We now recall from [6] the definition of a map $\mathrm{Pa} : \mathrm{St}_\lambda \longrightarrow \mathfrak{S}_n$.

Let us fix a flag ξ_0 adapted to x. For any $w \in \mathfrak{S}_n$, we denote by $\mathcal{O}_w$ the set of the flags ξ such that w is the relative position of (ξ_0, ξ). For any $\sigma \in \mathrm{St}_\lambda$, there exists a unique $w_\sigma \in \mathfrak{S}_n$ such that $\mathcal{O}_{w_\sigma} \cap \mathcal{F}_\sigma$ is open and dense in $\mathcal{F}_\sigma$. Obviously, w_σ does not depend on the choice of the adapted flag ξ_0. We set $\mathrm{Pa} : \mathrm{St}_\lambda \longrightarrow \mathfrak{S}_n$, $\sigma \longmapsto w_\sigma$.

We now recall from [6] a combinatorial description of w_σ. Set

$$n_\sigma(i) := \mathrm{card}\{j > i \mid l_\sigma(j) > l_\sigma(i)\}.$$

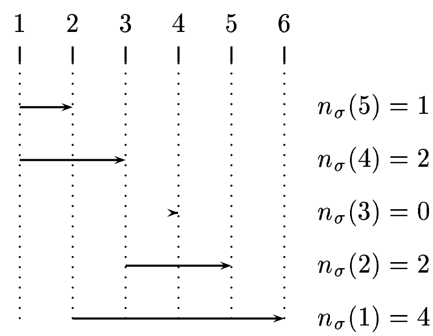
Let c_i denote the increasing cycle of $\mathfrak{S}_n$ of length $n_\sigma(i)$ ending on $n+1-i$. In other words,

$$c_i = s_{n-(i+n_\sigma(i)-1)} \cdots s_{n-(i+1)} s_{n-i}.$$

The main result of [6] is the following formula for w_σ:

$$w_\sigma := c_1 \cdots c_{n-1}. \tag{4.1}$$

We now introduce a diagram useful to understand w_σ. Firstly, we number n vertical lines from 1 to n. Then, the cycle c_i is represented by an horizontal arrow ending at the line $n+1-i$ and of length $n_\sigma(i)$. We draw successively $c_{n-1}, \cdots, c_1$. For example, if $n = 6$, $n_\sigma(5) = 1$, $n_\sigma(4) = 2$, $n_\sigma(3) = 0$, $n_\sigma(2) = 2$ and $n_\sigma(1) = 4$ we obtain:



4.4. Let σ_0 be the unique standard Young tableau such that an adapted flag ξ_0 belongs to $\mathcal{F}_{x,\sigma_0}$. One can describes σ_0 as follows: Firstly, $n, \cdots, n-\lambda_1+1$ are placed such that $c_\sigma(n) = \lambda_1$, $c_\sigma(n-1) = \lambda_1 - 1, \cdots, c_\sigma(n-\lambda_1+1) = 1$. Next, $n-\lambda_1, \cdots, n-\lambda_1-\lambda_2+2$ are placed in the same way among the free cases; and so on. As an example we give σ_0 for the partition $(6, 4, 3, 1)$:

1	3	4	8	13	14
2	6	7	12		
5	10	11			
9					

We can now state a relation between RSS and Pa:

Theorem 4.1. *For all $\sigma \in \mathrm{St}_\lambda$, we have*

$$\mathrm{RSS}(\mathcal{F}_\sigma, \mathcal{F}_{\sigma_0}) = w_\sigma.$$

Proof. Let Ω_x denote the set of the flags adapted to x. Firstly, we notice that Ω_x is one orbit of $Z(x) := \{g \in \mathrm{GL}(V) \,:\, gxg^{-1} = x\}$. Indeed, let ξ and ξ' be two points of Ω_x. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases adapted to x such that ξ and ξ' are respectively the flags associated to $\mathcal{B}$ and $\mathcal{B}'$. Let $g \in \mathrm{GL}(V)$ such that $g(\mathcal{B}) = \mathcal{B}'$. Since the matrices of x in $\mathcal{B}$ and $\mathcal{B}'$ are the same, $g \in Z(x)$. But, $\xi' = g\xi$. Conversely, if $g \in Z(x)$ and $\xi \in \Omega_x$, $g\xi \in \Omega_x$.

Since Ω_x is one orbit, Ω_x is a locally closed subvariety of $\mathcal{F}_x$. We now claim that $\dim \Omega_x = \sum_{i=2}^r (i-1)\lambda_i$.

We make a proof by induction on r. Consider the following locally closed subvariety of $\mathrm{Gr}_{\lambda_1}(V)$:

$$\mathcal{G}_x := \{F \in \mathrm{Gr}_{\lambda_1}(V) \,:\, x(F) \subseteq F \text{ and } x^{\lambda_1 - 1}(F) \neq \{0\}\},$$

and the morphism

$$\begin{array}{cccc} \Theta : & \Omega_x & \longrightarrow & \mathcal{G}_x \\ & (V_1, \cdots, V_n) & \longmapsto & V_{\lambda_1}. \end{array}$$

Let us fix F in $\mathcal{G}_x$. Set $V' = V/F$ and denote $q \,:\, V \longrightarrow V'$ the canonical projection. Let y denote the restriction of x to F and $x' \in \mathrm{Hom}(V', V')$ the linear map induced by x. Let $\xi = (V_1, \cdots, V_n) \in \Theta^{-1}(F)$. Then, for $i = 1, \cdots, \lambda_1$ we have $V_i = \mathrm{Ker}(y^i)$. On the other hand, $(V_{\lambda_1+1}/F, \cdots, V_{n-1}/F)$ is a flag of V' adapted to x'. Conversely, if $(V'_{\lambda_1+1}, \cdots, V'_{n-1})$ is a flag of V' adapted to x' then the flag

$$(\mathrm{Ker}(y), \cdots, \mathrm{Ker}(y^{\lambda_1 - 1}), F, q^{-1}(V'_{\lambda_1+1}), \cdots, q^{-1}(V'_{n-1}))$$

is adapted to x. One easily deduces that $\Theta^{-1}(F)$ is isomorphic to $\Omega_{x'}$. In particular, we have:

$$\dim \Omega_x = \dim \Omega_{x'} + \dim \mathcal{G}_x.$$

But, by induction, $\dim \Omega_{x'} = \sum_{i \geq 3} (i-2)\lambda_i$. It remains to prove that $\dim \mathcal{G}_x = \lambda_2 + \cdots + \lambda_r = n - \lambda_1$.

Consider the set V_0 of the vectors v in V such that $(v, x(v), \cdots, x^{\lambda_1 - 1}(v))$ are linearly independent, and the map $\Gamma \,:\, V_0 \longrightarrow \mathcal{G}_x$ which associates to each $v \in V_0$ the subspace spanned by $(v, x(v), \cdots, x^{\lambda_1 - 1}(v))$. The set V_0 is non empty and open in V. Moreover, Γ is surjective and for all $F \in \mathcal{G}_x$, $\Gamma^{-1}(F)$ is an open subset of F. We deduce that $\dim \mathcal{G}_x = \dim V - \dim F = n - \lambda_1$. The claim follows.

By Proposition 2.2 of [5], $\dim \mathcal{F}_x = \sum (i-1)\lambda_i$. So, we just proved that $\dim \Omega_x = \dim \mathcal{F}_x$. Moreover, we have already noticed that Ω_x is contained in $\mathcal{F}_{x,\sigma_0}$. Finally, Ω_x is open and dense in $\mathcal{F}_{\sigma_0}$.

Let $\sigma \in \mathrm{St}_\lambda$. By definition, for all $\xi \in \mathcal{F}_\sigma \cap \mathcal{C}_{w_\sigma}$ and $\xi_0 \in \Omega_x$, the relative position of ξ and ξ_0 is w_σ. Since $\mathcal{F}_\sigma \cap \mathcal{C}_{w_\sigma}$ and Ω_x are open and dense respectively in $\mathcal{F}_\sigma$ and $\mathcal{F}_{\sigma_0}$, we have $\mathrm{RSS}(\mathcal{F}_\sigma, \mathcal{F}_{\sigma_0}) = w_\sigma$. □

Remark 4.2. We have proved that $\mathcal{F}_{\sigma_0}$ contains a dense orbit of the neutral component $Z(x)^\circ$ of $Z(x)$. By duality, $\mathcal{F}_{\sigma_{\max}}$ has the same property. An interesting question is to determine all the irreducible components $\mathcal{F}_\sigma$ with this property.

5. Robinson-Schensted and Spaltenstein maps

For every irreducible component $\mathcal{F}_\sigma$ of $\mathcal{F}_x$, $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_\sigma$ is an open subset of $\mathcal{F}_\sigma$. On the other hand, Spaltenstein has defined another open subset $\mathcal{F}_{x,\sigma}$ in $\mathcal{F}_\sigma$. Our next theorem compares these two open subsets of $\mathcal{F}_\sigma$:

Theorem 5.1. *For every $\sigma \in \mathrm{St}_\lambda$, we have $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_x \subseteq \mathcal{F}_{x,\sigma}$.*

Before proving Theorem 5.1, we show some lemmas.

5.1. Let σ be a standard Young tableau of shape λ. In this subsection, we collect some useful results about w_σ and $\mathcal{C}_{w_\sigma}$.

Lemma 5.2. *Let $\xi = (V_1, \cdots, V_n) \in \mathcal{C}_{w_\sigma}$. Set $K := \min\{k : V_1 \subseteq F_k\}$.*
Then, we have: $K = \lambda_1 + \cdots + \lambda_{l_\sigma(n)-1} + 1$ and $w_\sigma(1) = K$.

Proof. In this proof, we will say that a flag $(W_1, \cdots, W_n)$ has Property $(*)$ if $\lambda_1 + \cdots + \lambda_{l_\sigma(n)-1} + 1 = \min\{k : W_1 \subseteq F_k\}$. By Corollary 2.3 and since $c_\sigma(n) = \lambda_{l_\sigma(n)}$, generically a point of $\mathcal{F}_{x,\sigma}$ has Property $(*)$. So, there exists a flag in $\mathcal{C}_{w_\sigma} \cap \mathcal{F}_{x,\sigma}$ which has Property $(*)$. By definition of the Bruhat cells, any point of $\mathcal{C}_{w_\sigma}$ has Property $(*)$.

Let us fix a base $\mathcal{B} = (e_1, \cdots, e_n)$ adapted to x such that ξ_0 is the canonical flag associated to $\mathcal{B}$. Then, the canonical flag associated to $(e_{w_\sigma(1)}, \cdots, e_{w_\sigma(n)})$ belongs to $\mathcal{C}_{w_\sigma}$ and has Property $(*)$. One easily deduces that $w_\sigma(1) = K$. □

Lemma 5.3. *Let $\phi : \{1, \cdots, n\} - w_\sigma(1) \longrightarrow \{1, \cdots, n-1\}$ defined by $\phi(k) = k$ if $k < w_\sigma(1)$ and $\phi(k) = k - 1$ if $k > w_\sigma(1)$. Let σ' be the standard Young tableau obtained from σ by deleting the case occupied by n.*
Then, $w_{\sigma'}(i) = \phi(w_\sigma(i+1))$ for all $i = 1, \cdots, n-1$.

Proof. Set $w'(i) = \phi(w_\sigma(i+1))$. We have to prove that $w' = w_{\sigma'}$. Let $\eta = (V_1, \cdots, V_n) \in \mathcal{C}_{w_\sigma} \cap \mathcal{F}_{x,\sigma}$. Consider $\mathcal{F}_{V_1}$ the subvariety of $\mathcal{F}$ consisting of the flags with V_1 as line. The map $\pi : \mathcal{F}_{V_1} \longrightarrow \mathcal{F}(V'), \eta = (V_1, W_2, \cdots, W_n) \longmapsto \eta' = (W_2/V_1, \cdots, W_n/V_1)$ is an isomorphism.

We use notation of Lemma 2.1 with V_1 and set $\xi'_0 = (F'_1, \cdots, F'_{n-1})$. We consider the Schubert decomposition of $\mathcal{F}(V')$ associated to ξ'_0. Since by Lemma 2.1, ξ'_0 is adapted to x', $\mathcal{C}_{w_{\sigma'}} \cap \mathcal{F}_{x',\sigma'}$ is open and dense in $\mathcal{F}_{\sigma'}$.

Since the stabilizer of V_1 and ξ_0 in $\mathrm{Gl}(V)$ acts on $\mathcal{F}(V')$ as the stabilizer of ξ'_0 in $\mathrm{Gl}(V')$ does, $\pi(\mathcal{C}_{w_\sigma} \cap \mathcal{F}_{V_1})$ is a Schubert cell $\mathcal{C}_{w'}$ of $\mathcal{F}(V')$.

Let us fix a non zero vector v_1 in V_1. Let $\mathcal{B} = (e_1, \cdots, e_n)$ be a base of V such that ξ_0 is the canonical flag associated to $\mathcal{B}$. By Lemma 5.2, the canonical flag associates to $(e_1, \cdots, e_{K-1}, v_1, e_{K+1}, \cdots, e_n)$ is also ξ_0: from now on, we assume that $e_K = v_1$.

By Lemma 5.2 again, the canonical flag associated to $(e_{w_\sigma(1)}, \cdots, e_{w_\sigma(n)})$ belongs to $\mathcal{F}_{V_1}$. Moreover, its image by π is the canonical flag associated to $(\varepsilon_{w'(1)}, \cdots, \varepsilon_{w'(n-1)})$ where $(\varepsilon_i = q(e_i)$ if $i \leq K-1$ and $\varepsilon_i = q(e_{i+1})$ if $i \geq K+1)$. Since the canonical flag associated to $(\varepsilon_1, \cdots, \varepsilon_{n-1})$ is ξ'_0, we deduce that $\pi(\mathcal{C}_{w_\sigma} \cap \mathcal{F}_{V_1}) = \mathcal{C}_{w'}$.

On the other hand, by Corollary 2.3, $\pi(\mathcal{F}_{x,\sigma} \cap \mathcal{F}_{V_1}) = \mathcal{F}_{x',\sigma'}$. We deduce that $\mathcal{C}_{w'} \cap \mathcal{F}_{x',\sigma'}$ is dense in $\mathcal{F}_{x',\sigma'}$, and so that $w' = w_{\sigma'}$.

Moreover, by Corollary 2.3, we have $\pi(\mathcal{F}_{V_1} \cap \mathcal{F}_{x,\sigma}) = \mathcal{F}_{x',\sigma'}$. So, $\pi(\mathcal{F}_{V_1} \cap \mathcal{C}_{w_\sigma} \cap \mathcal{F}_{x,\sigma})$ is open in $\mathcal{F}_{x',\sigma'}$ and contains η'. Therefore, $\pi(\mathcal{F}_{V_1} \cap \mathcal{C}_{w_\sigma})$ equals $\mathcal{C}_{w_{\sigma'}}$.

The lemma follows. $\square$

5.2. **Proof of Theorem 5.1.** The proof proceeds by induction on the dimension of V. Let $\sigma \in \mathrm{St}_\lambda$. Let $\xi = (V_1, V_2, \ldots, V_n) \in \mathcal{C}_{w_\sigma} \cap \mathcal{F}_x$. It remains to prove that for all $k = 1, \cdots, n$ we have

$$\left\{ \begin{array}{l} V_k \subseteq V_{k-1} + \mathrm{Im}\left(x^{c_\sigma(n+1-k)-1}\right) \\ V_k \not\subseteq V_{k-1} + \mathrm{Im}\left(x^{c_\sigma(n+1-k)}\right). \end{array} \right. \tag{5.1}$$

Firstly, $V_1 \subseteq F_{w_\sigma(1)} \cap \mathrm{Ker}\, x$. But, by Lemma 5.2 $F_{w_\sigma(1)} \cap \mathrm{Ker}\, x \subseteq \mathrm{Im}\left(x^{c_\sigma(n)-1}\right)$. It follows that $V_1 \subseteq \mathrm{Im}\left(x^{c_\sigma(n)-1}\right)$. On the other hand, $V_1 \not\subseteq F_{w_\sigma(1)-1}$; so $V_1 \not\subseteq \mathrm{Im}\left(x^{c_\sigma(n)}\right)$. So, Relations (5.1) are fulfilled for $k = 1$.

Let $V' = V/V_1$ and x' be as in Lemma 2.1. Obviously, $\xi' = (V_2/V_1, \cdots, V_n/V_1)$ belongs to $\mathcal{F}_{x'}$. Let σ' be the standard Young tableau obtained from σ by deleting the case occupied by n. By Lemma 5.3, $\xi' \in \mathcal{C}_{w'_\sigma}$. So, by induction $\xi' \in \mathcal{F}_{x',\sigma'}$. One easily deduces that Relations (5.1) hold for $k \geq 1$. The theorem is proved. $\square$

6. $\mathbb{P}^1$-FIBRATIONS OF $\mathcal{F}$ AND $\mathcal{F}_x$

6.1. **Notation.** Let k be an integer between 1 and $n-1$. We denote by $\mathcal{F}_k$ the variety consisting of the partial flags $(V_1 \subseteq \cdots \subseteq V_{k-1} \subseteq V_{k+1} \subseteq \cdots \subseteq V_n = V)$ such that $\dim(V_i) = i$ for all i. The group $\mathrm{GL}(V)$ acts naturally on $\mathcal{F}_k$. We denote by $\phi_k : \mathcal{F} \longrightarrow \mathcal{F}_k$ the map which omits the subspace V_k. This map is a $\mathbb{P}^1$-fibration and is $\mathrm{GL}(V)$-equivariant. Let X be a subset of $\mathcal{F}$. The subset $\phi_k^{-1}(\phi_k(X))$ is called the *ϕ_k-saturation of* X and X is said to be *ϕ_k-stable* if it is equal to its ϕ_k-saturation.

6.2. **ϕ_k-stability of $\mathcal{F}_\sigma$.**

Proposition 6.1. *Let σ be a standard Young tableau of shape λ and $2 \leq k \leq n$. Then,*

(1) *$\mathcal{F}_{x,\sigma}$ is ϕ_{n+1-k}-stable if and only if $c_\sigma(k-1) = c_\sigma(k)$;*

(2) *$\mathcal{F}_\sigma$ is ϕ_{n+1-k}-stable if and only if $c_\sigma(k-1) \geq c_\sigma(k)$;*
(3) *If $c_\sigma(k-1) > c_\sigma(k)$ then by switching the places of k and $k-1$ in σ, we obtain a standard Young tableau γ. Moreover, for all $\xi \in \mathcal{F}_{x,\sigma}$, the ϕ_{n+1-k}-saturation of ξ excepted one point which belongs to $\mathcal{F}_{x,\gamma}$ is contained in $\mathcal{F}_{x,\sigma}$.*

Proof. Set $V' := V/V_{n-k}$ and $x' : V' \longrightarrow V'$, $v + V_{n-k} \longmapsto x(v) + V_{n-k}$. By replacing x by x', we may assume that $k = n$.

Firstly, we assume that $c_\sigma(n-1) \geq c_\sigma(n)$. Let $\xi = (V_1, \cdots, V_n) \in \mathcal{F}_{x,\sigma}$. We claim that $V_2 \subseteq \operatorname{Ker} x$.

By Corollary 2.3 we have:

$$x(V_2) \subseteq x(V_1) + x(\operatorname{Im}\left(x^{c_\sigma(n-1)-1}\right)) = \operatorname{Im}\left(x^{c_\sigma(n-1)}\right) \subseteq \operatorname{Im}\left(x^{c_\sigma(n)}\right),$$

since $c_\sigma(n-1) \geq c_\sigma(n)$. But, by Corollary 2.3, $V_1 \not\subseteq \operatorname{Im}\left(x^{c_\sigma(n)}\right)$; and so, $V_1 \cap \operatorname{Im}\left(x^{c_\sigma(n)}\right) = \{0\}$. Since $x(V_2) \subseteq V_1$, the claim follows.

By the claim, the ϕ_1-saturation of $\mathcal{F}_{x,\sigma}$ is contained in $\mathcal{F}_x$. But it is irreducible, and so contained in $\mathcal{F}_\sigma$. Since $\mathcal{F}_\sigma$ is the closure of $\mathcal{F}_{x,\sigma}$, we deduce that $\mathcal{F}_\sigma$ is ϕ_1-stable.

Notice that since $\operatorname{Im}\left(x^{c_\sigma(n-1)-1}\right) \subseteq \operatorname{Im}\left(x^{c_\sigma(n)-1}\right)$, $V_2 \subseteq \operatorname{Im}\left(x^{c_\sigma(n)-1}\right)$.

Let W_1 be a line in V_2. Then, $\zeta := (W_1, V_2, \cdots, V_{n-1}) \in \mathcal{F}_x$. Let γ be the unique standard Young tableau such that $\zeta \in \mathcal{F}_{x,\gamma}$. By Corollary 2.3, we have for all $k = 1, \cdots, n-2$, $c_\sigma(k) = c_\gamma(k)$. Therefore, either $\gamma = \sigma$ or γ is obtained from σ by switching the places of n and $n-1$.

By Corollary 2.3, $V_1 \not\subseteq \operatorname{Im}\left(x^{c_\sigma(n)}\right)$; and so, the dimension of $V_2 \cap \operatorname{Im}\left(x^{c_\sigma(n)}\right)$ equals zero or one. We distinguish these two cases.

Case 1: $V_2 \cap \operatorname{Im}\left(x^{c_\sigma(n)}\right) = \{0\}$.

Since $W_1 \subseteq V_2$, $W_1 \not\subseteq \operatorname{Im}\left(x^{c_\sigma(n)}\right)$, and $W_1 \subseteq \operatorname{Im}\left(x^{c_\sigma(n)-1}\right)$. By Corollary 2.3, we deduce that $c_\gamma(n) = c_\sigma(n)$; and so, that $\gamma = \sigma$. It follows that $\mathcal{F}_{x,\sigma}$ is ϕ_1-stable.

By the assumption on the dimension of $V_2 \cap \operatorname{Im}\left(x^{c_\sigma(n)}\right)$, $V_2 \not\subseteq W_1 + \operatorname{Im}\left(x^{c_\sigma(n)}\right)$. But $V_2 \subseteq \operatorname{Im}\left(x^{c_\sigma(n)-1}\right) \subseteq V_1 + \operatorname{Im}\left(x^{c_\sigma(n)-1}\right)$. We deduce that $c_\gamma(n-1) = c_\sigma(n)$; but $\gamma = \sigma$, and so, $c_\sigma(n-1) = c_\sigma(n)$.

Case 2: $V_2 \cap \operatorname{Im}\left(x^{c_\sigma(n)}\right)$ is a line denoted by l.

If $W_1 = l$, we have $W_1 \subseteq \operatorname{Im}\left(x^{c_\sigma(n)}\right)$; and so, $c_\gamma(n) > c_\sigma(n)$. In particular $\gamma \neq \sigma$, and γ is obtained from σ by switching the places of n and $n-1$. So, $c_\sigma(n-1) > c_\sigma(n)$.

If $W_1 \neq l$, $c_\gamma(n) = c_\sigma(n)$; and so, $\gamma = \sigma$.

To complete the proof of the proposition, it is sufficient to prove that if $c_\sigma(n-1) < c_\sigma(n)$ then $\mathcal{F}_\sigma$ is not ϕ_1-stable. Let v be in $\mathrm{Im}\left(x^{c_\sigma(n)-1}\right) - \mathrm{Im}\left(x^{c_\sigma(n)}\right)$ such that $x(v) = 0$.

Since $x(\mathrm{Im}\left(x^{c_\sigma(n-1)-1}\right)) = \mathrm{Im}\left(x^{c_\sigma(n-1)}\right) \supset \mathrm{Im}\left(x^{c_\sigma(n)-1}\right)$, there exists w in $\mathrm{Im}\left(x^{c_\sigma(n-1)-1}\right)$ such that $x(w) = v$. Set V_1 (resp. V_2) be the vectors spaces spanned by v (resp. v and w). By Corollary 2.4, we can complete (V_1, V_2) in a flag ξ of $\mathcal{F}_{x,\sigma}$. Since $x(V_2) = V_1$, for all $\zeta \in \phi_1^{-1}(\phi_1(\xi)) - \{\xi\}$, $\zeta \notin \mathcal{F}_x$. This completes the proof. □

Proposition 6.1 has the following purely combinatorial corollary:

Corollary 6.2. *Let σ be a standard Young tableau of shape λ and $2 \le k \le n$.*

Then (with notation of Section 4.1), $c_\sigma(k-1) \ge c_\sigma(k)$ if and only if $c_{\sigma^\vee}(n-k) \ge c_{\sigma^\vee}(n+1-k)$.

Proof. By Proposition 6.1, $c_\sigma(k-1) \ge c_\sigma(k)$ if and only if $\mathcal{F}_\sigma$ is ϕ_{n+1-k}-stable which is equivalent to $\eta(\mathcal{F}_\sigma)$ ϕ_k-stable, and so to $\mathcal{F}_{\sigma^\vee}$ ϕ_k-stable. □

Remark 6.3. With notation of Proposition 6.1, Assertion 3 shows that $\mathcal{F}_\gamma$ intersects $\mathcal{F}_\sigma$ in codimension one. It would be interesting to find out all the standard Young tableaux δ such that $\mathcal{F}_\delta$ intersects $\mathcal{F}_\sigma$ in codimension one.

Assertion 2 of Proposition 6.1 deals with the Spaltenstein's parametrization of the irreducible components of $\mathcal{F}_x$ (and was already obtained in [8]), but not with the Spaltenstein's partition by the $\mathcal{F}_{x,\sigma}$. Assertion 3 is more precise but concerns the Spaltenstein's partition. It would be interesting to be able to read on the standard Young tableaux γ and σ, if $\mathcal{F}_\gamma \cap \mathcal{F}_\sigma$ contains an irreducible component C whose $\mathcal{F}_\sigma$ is the ϕ_{n+1-k}-saturation. If $c_\sigma(k-1) \ne c_\sigma(k)$ Proposition 6.1 answers this question. If $c_\sigma(k-1) = c_\sigma(k)$ the two following examples show that the situation is less clear.

Example 6.4. Consider $\sigma = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$. Then, $\mathcal{F}_{x,\sigma} = \mathcal{F}_\sigma$ is ϕ_1-stable. Set $\gamma = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ and $C = \mathcal{F}_\gamma \cap \mathcal{F}_\sigma$. One easily checks that the ϕ_1-saturation of C equals $\mathcal{F}_\sigma$.

Example 6.5. Consider the three standard Young tableaux $\sigma_{\max} := \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}$, $\sigma := \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{4} \\ \cline{1-1} \end{array}$ and $\sigma_{\min} := \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array}$ of shape $\begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1} \multicolumn{1}{|c|}{\ } \\ \cline{1-1} \end{array}$. One easily checks that $\mathcal{F}_{\sigma_{\min}} - (\mathcal{F}_{\sigma_{\max}} \cup \mathcal{F}_\sigma)$ is ϕ_1-stable.

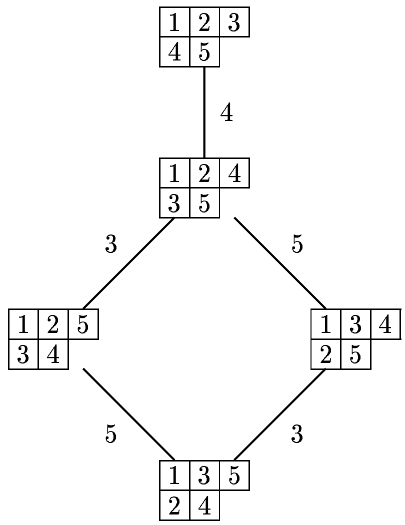
6.3. From now on, we keep our attention on the case $c_\sigma(k-1) > c_\sigma(k)$.

Definition 6.6. *Let $3 \le k \le n$. Two standard Young tableaux of shape λ are said to be k-adjacent if they are obtained one from the other by switching the places of k and $k-1$.*

Definition 6.7. *Let Γ_λ be the oriented graph with vertices set St_λ and the edges labeled by the integers $3, \cdots, n$, where σ is joined to γ by an edge labeled by k if σ and γ are k-adjacent and $\sigma \prec \gamma$.*

The graph Γ_λ is called the adjacency graph of λ.

Example 6.8. The adjacency graph of the partition $(3, 2)$ is:

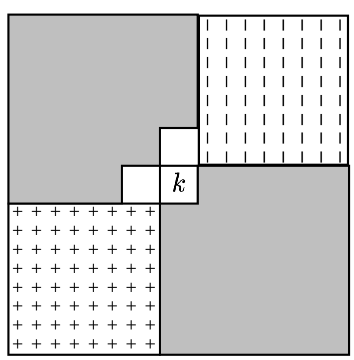


6.4. We can state the first property of Γ_λ:

Proposition 6.9. *Each vertex in Γ_λ is joined to $\sigma_{\max}$ (resp. $\sigma_{\min}$) by an increasing (resp. decreasing) path.*

Before showing Proposition 6.9, we prove two lemmas:

Lemma 6.10. *Let $\sigma \in \mathrm{St}_\lambda$ and $3 \leq k \leq n$. Consider the following partition of the cases of σ:*



Then,

1. *the number $k-1$ cannot belong to a case* $\blacksquare$.
2. *if $k-1$ belongs to a case* $\square$, *σ is k-adjacent to no standard Young tableau.*
3. *if $k-1$ belongs to a case* $\boxplus$, *σ is k-adjacent to a standard Young tableau γ such that $\gamma \prec \sigma$.*
4. *if $k-1$ belongs to a case* $\boxed{|\,|}$, *σ is k-adjacent to a standard Young tableau γ such that $\sigma \prec \gamma$.*

Proof. It is trivial. $\square$

Lemma 6.11. *Let σ be a standard Young tableau and γ be its predecessor for $\prec$. Let i_0 be such that $l_\gamma(i_0) < l_\sigma(i_0)$ and for all $j > i_0$, $l_\gamma(j) = l_\sigma(j)$.*

Then, there exists a standard Young tableau δ^+ (resp. δ^-) which is i_0-adjacent to γ (resp. σ). Moreover, $\sigma \preceq \delta^+$ and $\delta^- \preceq \gamma$.

Proof. By omitting the cases occupied by $i_0+1, \cdots, n$ in each tableau, we may assume that $i_0 = n$. By Figure 1, $n-1$ belongs to the last line of γ. Then, Lemma 6.10 shows that there exists δ^+ n-adjacent to γ such that $\gamma \prec \delta^+$. Since γ is the predecessor of σ, one deduces that $\sigma \preceq \delta^+$. The proof is similar for δ^-. $\square$

Proof of Proposition 6.9. We make the proof for $\sigma_{\max}$. The reader can easily deduce those for $\sigma_{\min}$.

Let $\mathcal{E}$ denote the set of the standard Young tableaux joined to $\sigma_{\max}$ by a increasing path. We have $\sigma_{\max} \in \mathcal{E}$ and assume by absurd that $\mathcal{E} \neq \mathrm{St}_\lambda$. Consider the maximal element γ of $\mathrm{St}_\lambda - \mathcal{E}$ and σ its successor. By Lemma 6.11, there exists δ^+ i_0-adjacent to γ such that $\gamma \prec \delta^+$ (with some $3 \leq i_0 \leq n$). Since $\gamma \prec \delta^+$, δ^+ belongs to $\mathcal{E}$; and so γ belongs to $\mathcal{E}$: contradiction. $\square$

7. Some smooth components of $\mathcal{F}_x$

7.1. **Some homogeneous components of $\mathcal{F}_x$.** In this subsection, we will describe all the irreducible components of $\mathcal{F}_x$ which are homogeneous under a parabolic subgroup of $\mathrm{Gl}(V)$. These components are already classified by Kraft and Hesselink in [4] and [2]. Here, we recover their results and precise the standard Young tableaux corresponding to these components. Let us first introduce some notation.

If λ is a partition of n, we denote by $\hat\lambda = (\hat\lambda_1, \hat\lambda_2, \dots)$ the *dual partition* defined by $\hat\lambda_i := \mathrm{card}\{j \mid \lambda_j \geq i\}$; we can notice that we have $\hat\lambda_1 \geq \hat\lambda_2 \geq \dots$ An *ordered partition* of n is an uple $(\mu_1, \cdots, \mu_r)$ of positive integers whose the sum equal n. If we omit the order on the set of the μ_i's, we obtain the underlying partition of the form $\mu_{i_1} \geq \cdots \geq \mu_{i_r}$. We denote by $\mathcal{O}(\lambda)$ the set of ordered partition with $\hat\lambda$ as underlying partition. In other words, an element of $\mathcal{O}(\lambda)$ is a λ_1-uple of integers whose the components are the $\hat\lambda_i$'s.

Let us fix a base $\mathcal{B} = (e_1, \cdots, e_n)$ of V adapted to x. We number the cases of $Y(\lambda)$ as in $\sigma_{\max}$. In the case number i, we put the vector e_i. Let $\underline{o} = (\mu_1, \cdots, \mu_{\lambda_1}) \in \mathcal{O}(\lambda)$. We define a partial flag $\xi_{\underline{o}} = (\{0\} = W_{\underline{o}}^0 \subset W_{\underline{o}}^1 \subset \cdots \subset W_{\underline{o}}^{\lambda_1} = V)$ by:

- $W_{\underline{o}}^1$ is the subspace spanned by the vectors in the μ_1 first lines and in the first column of $Y(\lambda)$;
- $W_{\underline{o}}^2$ is the subspace spanned by $W_{\underline{o}}^1$ and the vectors in the μ_2 first lines and on the left among the vectors not in $W_{\underline{o}}^1$;
- and so on...

For all $j = 1, \cdots, \lambda_1$, we set $d_j := \sum_{i=1}^{j} \mu_i = \dim W_{\underline{o}}^j$. Consider

$$\mathcal{F}_{\underline{o}} := \{(V_1, \cdots, V_n) \in \mathcal{F} \ : \ \forall j = 1, \cdots, \lambda_1 \ \ V_{d_j} = W_{\underline{o}}^j\}.$$

Notice that $\mathcal{F}_{\underline{o}}$ is homogeneous under the action of the stabilizer in $\mathrm{Gl}(V)$ of $(W_{\underline{o}}^1 \subset \cdots \subset W_{\underline{o}}^{\lambda_1})$.

We now define a standard Young tableau $\sigma_{\underline{o}}$ associated to $\underline{o}$. In the μ_r first lines of the first column of $\sigma_{\underline{o}}$, we put the numbers $1, \cdots, \mu_r$. In the μ_{r-1} first lines and on the left among the not numbered cases, we put the numbers $\mu_r + 1, \cdots, \mu_r + \mu_{r-1}$. And, so on...

Let us give an example of $\xi_{\underline{o}}$ and $\sigma_{\underline{o}}$.

Example 7.1. For the partition $\lambda = (4 \geq 4 \geq 3 \geq 1)$, its dual partition is $(4 \geq 3 \geq 3 \geq 2)$. Then there is exactly 12 elements in $\mathcal{O}(\lambda)$. Set $\underline{o} = (3, 2, 4, 3) \in \mathcal{O}(\lambda)$. We have:

$$\begin{array}{|c|c|c|c|} \hline e_1 & e_2 & e_3 & e_4 \\ \hline e_5 & e_6 & e_7 & e_8 \\ \hline e_9 & e_{10} & e_{11} \\ \cline{1-3} e_{12} \\ \cline{1-1} \end{array} \qquad \sigma_{\underline{o}} = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 8 & 10 \\ \hline 2 & 5 & 9 & 11 \\ \hline 3 & 6 & 12 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

$$\xi_{\underline{o}} = (\{0\} \subset < e_1, e_5, e_9 > \subset < W_{\underline{o}}^1, e_2, e_6 > \subset < W_{\underline{o}}^2, e_3, e_7, e_{10}, e_{12} > \subset W).$$

We can now describe the Richardson components:

Theorem 7.2. *For any $\underline{o} \in \mathcal{O}(\lambda)$, we have $\mathcal{F}_{\underline{o}} = \mathcal{F}_{\sigma_{\underline{o}}}$. Moreover, any irreducible component of $\mathcal{F}_x$ homogeneous under a parabolic subgroup equals $\mathcal{F}_{\sigma_{\underline{o}}}$ for some $\underline{o} \in \mathcal{O}(\lambda)$.*

Proof. Let $\mathcal{F}_k$ denote the flag variety of a vector space of dimension k. Let us fix $\underline{o} \in \mathcal{O}(\lambda)$. Since for all $j = 1, \cdots, \lambda_1$ $x(W_{\underline{o}}^j) = W_{\underline{o}}^{j-1}$, $\mathcal{F}_{\underline{o}}$ is contained in $\mathcal{F}_x$. Since $\mathcal{F}_{\underline{o}}$ is isomorphic to $\mathcal{F}_{\hat{\lambda}_1} \times \cdots \times \mathcal{F}_{\hat{\lambda}_{\lambda_1}}$, it is irreducible and of dimension $\sum_i \hat{\lambda}_i(\hat{\lambda}_i - 1)/2$ which is the dimension of each irreducible component of $\mathcal{F}_x$ (see [8]). We deduce that $\mathcal{F}_{\underline{o}}$ is a Richardson component of $\mathcal{F}_x$.

Conversely, let $\sigma \in \mathrm{St}_\lambda$ be such that $\mathcal{F}_\sigma$ is a Richardson component. There exists a partial flag $(\{0\} = W^0 \subset W^1 \subset \cdots W^s = V)$ such that $\mathcal{F}_\sigma$ is the set of the flags containing the W^j's. Let $d_1, \cdots, d_s$ denote the dimensions of the W^j's. Consider the projection $\pi : \mathcal{F}(V) \longrightarrow \mathbb{P}(V)$, $(V_1, \cdots, V_n) \longmapsto V_1$. By Corollary 2.4, $W^1 = \pi(\mathcal{F}_\sigma) = \mathrm{Im}(x^{c_\sigma(n)-1}) \cap \ker x$. We deduce that $d_1 = l_\sigma(n)$ equals $\hat\lambda_{i_1}$ for some i_1. Consider the endomorphism $x' : V/W^1 \longrightarrow V/W^1$, $v + W^1 \longmapsto x(v) + W^1$. One easily checks that the dual partition of the partition associated to x' is obtained from $\hat\lambda$ by omitting $\hat\lambda_{i_1}$. We deduce by induction on the dimension of V that $\underline{o} := (d_1, \cdots, d_s)$ belongs to $\mathcal{O}(\lambda)$. Moreover, from the equality $W^1 = \mathrm{Im}(x^{c_\sigma(n)-1}) \cap \ker x$ (and by an immediate induction) we deduce that $\mathcal{F}_\sigma = \mathcal{F}_{\underline{o}}$.

We now prove by induction on n that $\sigma_{\underline{o}} = \sigma$. Since $l_\sigma(n) = d_1$, we have $c_{\sigma_{\underline{o}}}(n) = c_\sigma(n)$. Set $\underline{o}' = (d_1 - 1, \cdots, d_s)$. Notice that $\sigma_{\underline{o}'}$ is obtained from $\sigma_{\underline{o}}$ by deleting the case occupied by n. Let σ' be the standard Young tableau obtained from σ by deleting the square occupied by n. Let us fix $V_1 \in (\mathrm{Im}(x^{c_\sigma(n)-1}) - \mathrm{Im}(x^{c_\sigma(n)})) \cap \ker x$. By considering V/V_1, the induction shows that $\sigma_{\underline{o}'} = \sigma'$. The theorem follows. □

Remark 7.3. By the above description we also see that $\mathcal{F}_{\sigma_{\min}}$ is the Richardson component corresponding to ordered partition $(\hat\lambda_1, \cdots, \hat\lambda_{\lambda_1})$. This is an another way to show that $\mathcal{F}_{\sigma_{\min}}$ is smooth. (cf. Corollary 3.5).

7.2. **Some smooth components of $\mathcal{F}_x$.** The main result of this section is

Theorem 7.4. *Let $\underline{o} \in \mathcal{O}(\lambda)$. Let σ a standard Young tableau k-adjacent to $\sigma_{\underline{o}}$ such that $\sigma \prec \sigma_{\underline{o}}$. Then $\mathcal{F}_\sigma$ is smooth.*

Proof. We use notation of Section 7.1. Set $j_0 = c_{\sigma_{\underline{o}}}(k)$ and $k' = d_{j_0} = n + 1 - k$. Since $x(W_{\underline{o}}^{j_0+1}) = W_{\underline{o}}^{j_0}$, $W_{\underline{o}}^{j_0+1} \cap x^{-1}(V_{k'-1})$ is an hyperplane of $W_{\underline{o}}^{j_0+1}$ containing $W_{\underline{o}}^{j_0}$. Consider

$$\mathcal{C} = \{(V_1, \cdots, V_n) \in \mathcal{F}_{\underline{o}} \ : \ V_{k'+1} \subset W_{\underline{o}}^{j_0+1} \cap x^{-1}(V_{k'-1})\}.$$

Notice that the map $V_{k'-1} \longmapsto W_{\underline{o}}^{j_0+1} \cap x^{-1}(V_{k'-1})$ is a morphism between the projective space of the hyperplanes of $W_{\underline{o}}^{j_0}$ containing $W_{\underline{o}}^{j_0-1}$ to the projective space of the hyperplanes of $W_{\underline{o}}^{j_0+1}$ containing $W_{\underline{o}}^{j_0}$. One easily deduces that $\mathcal{C}$ is a smooth irreducible subvariety of $\mathcal{F}_{\underline{o}}$. Moreover, since σ exists, we have $\mu_{j_0+1} > 1$. Therefore, the codimension of $\mathcal{C}$ in $\mathcal{F}_{\underline{o}}$ equals one.

Since for all $(V_1, \cdots, V_n) \in \mathcal{C}$, we have $x(V_{k'+1}) \subset V_{k'-1}$, the $\phi_{k'}$ saturation of $\mathcal{C}$ is an irreducible component $\mathcal{F}_\gamma$ of $\mathcal{F}_x$. Moreover, the restriction of $\phi_{k'}$ to $\mathcal{C}$ is an isomorphism. We deduce that $\mathcal{F}_\gamma$ is smooth.

It remains to prove that $\gamma = \sigma$. Set $\mathcal{D} := \mathcal{F}_{x,\sigma_{\underline{o}}} \cap \mathcal{F}_\sigma$; $\mathcal{D}$ is a closed subvariety of $\mathcal{F}_{x,\sigma_{\underline{o}}}$. By Proposition 6.1, for any $\xi \in \mathcal{F}_{x,\sigma}$ the line $\phi_k^{-1}(\phi_k(\xi))$ intersects $\mathcal{F}_{x,\sigma_{\underline{o}}}$

is exactly one point of $\mathcal{D}$. In particular, $\mathcal{F}_\sigma$ is the closure of the ϕ_k-saturation of $\mathcal{D}$.

Notice that
$$\mathcal{C} = \{\xi \in \mathcal{F}_{\underline{\varrho}} : \phi_k^{-1}(\phi_k(\xi)) \subset \mathcal{F}_x\}.$$
We deduce that $\mathcal{D}$ is contained in $\mathcal{C}$. Therefore, $\mathcal{F}_\sigma$ which is contained in $\mathcal{F}_\gamma$, is the ϕ_k-saturation of $\mathcal{C}$. It follows that $\sigma = \gamma$. □

8. The Robinson-Schensted map and the adjacency graph

8.1. Let Γ_n be the oriented graph with vertices set $\mathfrak{S}_n$ and the edges labeled by the integers $2, \cdots, n$, where w is joined to w' by an edge labeled by k if $w' = ws_{n+1-k}$ and $l(w') = l(w) + 1$. The graph Γ_n is called the *Bruhat graph* of $\mathfrak{S}_n$.

We can now state our

Theorem 8.1 (**Main Theorem**). *By the map* $\mathrm{St}_\lambda \longrightarrow \mathfrak{S}_n$, $\sigma \longmapsto w_\sigma$, *the graph* Γ_λ *identifies with a full subgraph of* Γ_n.

In particular, for all σ *and* γ *in* St_λ, w_σ *and* w_γ *are not joined by an edge labeled by* 2 *in the Bruhat graph.*

Remark 8.2. Since the Robinson-Schensted map is injective, Theorem 4.1 shows that when λ varies among the partitions of n the subgraphs of Γ_n obtained by Theorem 8.1 are pairwise disjoint.

Before showing Theorem 8.1,we prove some useful combinatorial properties about $\mathfrak{S}_n$.

8.2. Consider the set
$$\Delta := \{(n_1, \cdots, n_{n-1}) \in \mathbb{N}^{n-1} \mid 0 \leq n_i \leq n - i\}.$$

Lemma 8.3. *The map* $\pi : \Delta \to \mathfrak{S}_n, (n_i) \mapsto c_1 c_2 \ldots c_{n-1}$, *where*
$$c_i := \begin{cases} s_{n-(i+n_i-1)} \cdots s_{n-(i+1)} s_{n-i} \text{ if } n_i \geq 1 \\ id, \text{ otherwise,} \end{cases}$$
is a bijection.

Proof. Firstly consider the subset $\Delta' := \{(n_i) \in \Delta | \ n_1 = 0\}$, then for every $w \in \pi(\Delta')$ we have $w(n) = n$; by induction the restriction of π to Δ' induces a bijection onto $H = \{w \in \mathfrak{S}_n : w(n) = n\}$. Since $c_1(n) = n - (i + n_i - 1)$, every element $w \in \mathfrak{S}_n$ such that $w(n) = k$ can be uniquely written as $w = c_1 w'$ with $w' \in H$ and $c_1 = s_k s_{k+1} \ldots s_{n-1}$. The lemma follows. □

We now introduce some structure on the set Δ. For $k = 2, \cdots, n-1$, we set $\Delta_k := \{(n_1, \cdots, n_{n-1}) \in \Delta : n_k < n - k\}$ and $\Delta^k := \{(n_1, \cdots, n_{n-1}) \in \Delta : n_{k-1} \neq 0\}$. We define $\tau_k : \Delta^k \longrightarrow \Delta$, $(n_1, \cdots, n_{n-1}) \longmapsto (n'_1, \cdots, n'_{n-1})$ where $n'_i = n_i$ for $i \notin \{k-1, k\}$, $n'_{k-1} = n_k$ and $n'_k = n_{k-1} - 1$. We also define $\rho_k : \Delta_k \longrightarrow \Delta$, $(n_1, \cdots, n_{n-1}) \longmapsto (n'_1, \cdots, n'_{n-1})$ where $n'_i = n_i$ for $i \notin \{k-1, k\}$,

$n'_{k-1} = n_k$ and $n'_k = n_{k-1} + 1$. For $\alpha = (n_1, \cdots, n_{n-1}) \in \Delta$, we define the length $l(\alpha)$ of α by $l(\alpha) = \sum n_i$. Notice that $l(\tau_k(\alpha)) = l(\alpha) - 1$ and $l(\rho_k(\alpha)) = l(\alpha) + 1$.

Via π, we can read the Bruhat order on Δ. We obtain

Lemma 8.4. *Let $\alpha \in \Delta$ and $k \in \{2, \cdots, n\}$. Then, we have:*

(1) *If $n_{k-1} > n_k$ then $\alpha \in \Delta^k$ and $\pi(\tau_k(\alpha)) = \pi(\alpha)s_{n+1-k}$.*
(2) *If $n_{k-1} \leq n_k$ then $\alpha \in \Delta_k$ and $\pi(\rho_k(\alpha)) = \pi(\alpha)s_{n+1-k}$.*

Proof. We use notation of Lemma 8.3 for α. Set $w_1 = c_1 \cdots c_{k-2}$ and $w_2 = c_{k+1} \cdots c_{n-1}$; so, we have: $\pi(\alpha) = w_1 c_{k-1} c_k w_2$. But $w_2(n-k+1) = n-k+1$ and $w_2(n-k+2) = n-k+2$, so w_2 and s_{n-k+1} commutes. Therefore, $\pi(\alpha)s_{n+1-k} = w_1 c_{k-1} c_k s_{n+1-k} w_2$. We compute $c_{k-1} c_k s_{n+1-k}$ with the help of the diagrams of Figure 2.

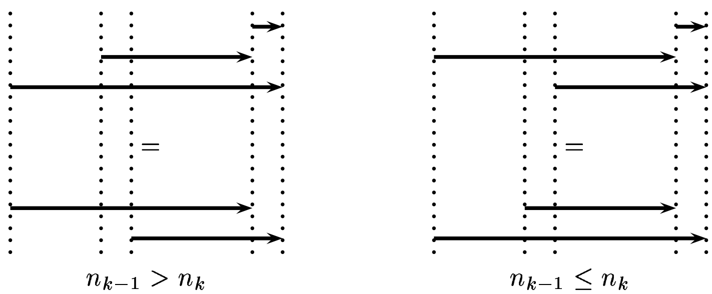


FIGURE 2. A product of cycles

Notice that these two computations can be deduced one from the other. The lemma follows easily. □

Proposition 8.5. *With above notation, for all $\alpha \in \Delta$, we have $l(\alpha) = l(\pi(\alpha))$. In other words, the expression of $\pi(\alpha)$ given in Lemma 8.3 is reduced.*

Proof. Firstly, notice that since the definition of π gives a formula of length $l(\alpha)$ for $\pi(\alpha)$, we have: $l(\pi(\alpha)) \leq l(\alpha)$. By absurd, we assume that the proposition is false. Let us consider an element $\alpha = (n_1, \cdots, n_{n-1}) \in \Delta$ such that $l(\alpha) > l(\pi(\alpha))$ of maximal length. Two cases occurs.

<u>Case A:</u> there exists $k \in \{2, \cdots, n-1\}$ such that $n_k \geq n_{k-1}$.

Since $\alpha \in \Delta_k$, we can set $\beta = \rho_k(\alpha)$. By Lemma 8.4, we have $\pi(\beta) = \pi(\alpha)s_{n-k+1}$. In particular, $l(\pi(\alpha)) = l(\pi(\beta)) \pm 1$. Since $l(\beta) > l(\alpha)$, $l(\pi(\beta)) = l(\beta) = l(\alpha) + 1$. So, $l(\pi(\alpha))$ equals $l(\alpha)$ or $l(\alpha) + 2$. Contradiction.

<u>Case B:</u> for all $k = 2, \cdots n-1$, $n_k < n_{k-1}$.

We have: $n-1 \geq n_1 > n_2 > \cdots > n_{n-1} \geq 0$. So, there exists $k \in \{0, \cdots, n-1\}$ such that $n_i = n-i$ for all $i \leq k-1$ and $n_i = n-i-1$ for all $i \geq k$. We can draw the cycles c_i on Figure 3.

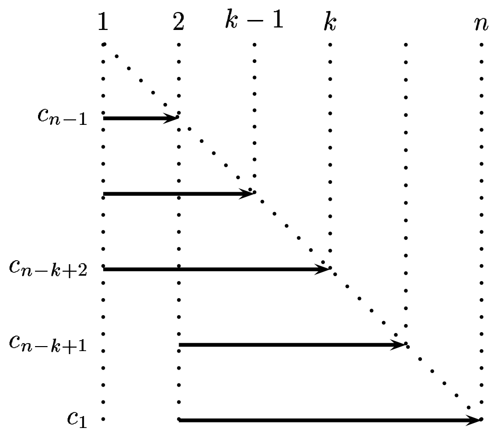


FIGURE 3. Decomposition of $\pi(\alpha)$

One easily reads on the picture the following values of $\pi(\alpha)(i)$:

i	1	$\cdots$	$k-1$	k	$k+1$	$\cdots$	n
$\pi(\alpha)(i)$	n	$\cdots$	$n+2-k$	1	$n+1-k$	$\cdots$	2

In particular, $\{(i<j) : \pi(\alpha)(i) < \pi(\alpha)(j)\} = \{(k,k+1),\cdots,(k,n)\}$. We deduce that $l(\pi(\alpha)) = \frac{n(n-1)}{2} - (n-k)$. On the other hand, $l(\alpha) = \sum_{i=1}^{n-1} n-i-(n-k)$. Contradiction. $\square$

Since the dimension of $\mathcal{C}_w$ equals the length of w, Proposition 8.5 implies

Corollary 8.6. *For any $\sigma \in \mathrm{St}_\lambda$, we have $\dim \mathcal{C}_{w_\sigma} = \sum_i n_\sigma(i)$.*

The partition λ is said to be of hook type if $\lambda_2 = \cdots = \lambda_r = 1$.

Corollary 8.7. *Let σ be a standard Young tableau of shape λ. Then, there exists $w \in \mathfrak{S}_n$ such that $\mathcal{F}_\sigma$ is the closure of $\mathcal{C}_w$ if and only if λ is of hook type and $\sigma = \sigma_{\min}$.*

Proof. If $\mathcal{F}_\sigma$ is the closure of $\mathcal{C}_w$ then $w = w_\sigma$. Now, there exists $w \in \mathfrak{S}_n$ such that $\mathcal{F}_\sigma$ is the closure of $\mathcal{C}_w$ if and only if $\dim(\mathcal{F}_\sigma) = l(w_\sigma)$. Notice that the inequality $\dim(\mathcal{F}_\sigma) \leq l(w_\sigma)$ is obvious. But, by [8], the dimension of $\mathcal{F}_\sigma$ only depends on λ; and, by Propositions 6.9 and 8.5 for any standard Young tableau $\gamma \neq \sigma_{\min}$ of shape λ, we have $l(\sigma_{\min}) < l(\gamma)$. Therefore, if $\dim(\mathcal{F}_\sigma) = l(w_\sigma)$ then $\sigma = \sigma_{\min}$. It remains to prove that $\dim(\mathcal{F}_x) \leq l(w_{\sigma_{\min}})$ if and only if λ is of hook type.

We have: $\dim(\mathcal{F}_x) = \sum_{i=2}^{r}(i-1)\lambda_i = \sum_{i=1}^{r-1} n_{\sigma_{\min}}(i)$. In particular, $\dim(\mathcal{F}_x) \leq l(w_{\sigma_{\min}})$ implies that $n_{\sigma_{\min}}(r+1) = 0$; and so, that λ is of hook type. Conversely, if λ is of hook type, one easily checks that $\sum_{i=1}^{r-1} n_{\sigma_{\min}}(i) = \sum_{i=1}^{n-1} n_{\sigma_{\min}}(i)$. $\square$

8.3. Now, we can prove Theorem 8.1:

Lemma 8.8. *Via the map* Pa, Γ_λ *identifies with a full subgraph of* Γ_n.

Proof. The map $\mathrm{St}_\lambda \longrightarrow \Delta$, $\sigma \longmapsto (n_\sigma(1), \cdots, n_\sigma(n-1))$ is obviously injective. Now, Lemma 8.3 implies that $\mathrm{Pa} : \mathrm{St}_\lambda \longrightarrow \mathfrak{S}_n$ is injective.

Let $\sigma \prec \gamma$ be two k-adjacent standard Young tableaux. One easily checks that $n_\sigma(k-1) > n_\sigma(k)$ and $\tau_k(n_\sigma(1), \cdots, n_\sigma(n-1)) = (n_\gamma(1), \cdots, n_\gamma(n-1))$. So, by Lemma 8.4 $w_\sigma = w_\gamma s_{n+1-k}$. Moreover, by Proposition 8.5 $l(w_\sigma) = l(w_\gamma) + 1$. Finally, via the map Pa, Γ_λ identifies with a subgraph Γ'_λ of Γ_n.

Conversely, consider an edge in Γ_n labeled by k joining w to w' with $l(w') = l(w)+1$. Consider $\alpha = (n_1, \cdots, n_{n-1})$ and $\beta = (n'_1, \cdots, n'_{n-1})$ such that $\pi(\alpha) = w$ and $\pi(\beta) = w'$. By Lemma 8.4, either $\beta = \tau_k(\alpha)$ or $\alpha = \tau_k(\beta)$. Since $l(w') = l(w) + 1$, Proposition 8.5 shows that $\alpha = \tau_k(\beta)$ and $n_{k-1} \leq n_k$.

We now assume that $w = w_\sigma$ for some $\sigma \in \mathrm{St}_\lambda$; and we distinguish three cases:

<u>Case A:</u> $n_{k-1} < n_k$ and $k > 2$.

By Lemma 6.10, there exists a standard Young tableau γ which is k-adjacent to σ. Moreover, the first part of the proof shows that $w_\gamma = w'$. In particular, the considered edge is an edge of Γ'_λ.

<u>Case B:</u> $n_{k-1} = n_k$ and $k > 2$.

We assume that $w' = w_\gamma$ for some $\gamma \in \mathrm{St}_\lambda$. Lemma 8.4 shows that

$$\tau_k(n_\gamma(1), \cdots, n_\gamma(n-1)) = (n_\sigma(1), \cdots, n_\sigma(n-1)).$$

We have to prove that γ and σ are k-adjacent; by the injectivity of the map Pa, it is sufficient to prove that there exists a standard Young tableau k-adjacent to γ. Since $n_\sigma(k-1) = n_\sigma(k)$, by Lemma 6.10 it is sufficient to prove that $k-1$ and k are not in the same line in γ.

Let us assume by absurd that $l_\gamma(k-1) = l_\sigma(k)$. Since for all $i \leq k-2$ $n_\sigma(i) = n_\gamma(i)$, an immediate induction shows that the integers $1, \cdots, k-2$ are in the same case in σ and γ. Notice that $n_\gamma(k-1) = n_\gamma(k)$. Moreover, $n_\sigma(k-1) = n_\gamma(k-1) - 1$ and so $l_\sigma(k-1) > l_\gamma(k-1)$. Since $n_\sigma(k) \geq n_\gamma(k)$, we deduce that $l_\sigma(k) < l_\gamma(k)$. But now, $n_\sigma(k) > n_\gamma(k)$; which is a contradiction.

<u>Case C:</u> $k = 2$.

Since $n_1 = n - \lambda_1$, the condition $n_1 \leq n_2$ implies that $n_2 = n_1$ and $l_\sigma(2) = 1$. In particular, $n'_1 = n_1 - 1 = n - \lambda_1 - 1$. But for all $\gamma \in \mathrm{St}_\lambda$, we have $n_\gamma(1) = n - \lambda_1$. Therefore, w' does not belong to $\mathrm{Pa}(\mathrm{St}_\lambda)$. $\square$

8.4. We now give another description of w_σ. For $i = 2, \cdots, n$, set

$$n^\sigma(i) := \mathrm{card}\{j < i \mid l_\sigma(j) < l_\sigma(i)\}.$$

For any $j = 1, \cdots, n-1$, let c^j denote the decreasing cycle of $\mathfrak{S}_n$ of length $n^\sigma(n+1-j)$ ending on j.

Theorem 8.9. *With above notation, we have*

$$w_\sigma = c^1 \cdots c^{n-1}.$$

Proof. During this proof, we set $w^\sigma = c^1 \cdots c^{n-1}$. Firstly, we prove by induction on n that the theorem holds for $\sigma_{\min}$. Set $\sigma = \sigma_{\min}$.

By Lemma 5.2, $w_\sigma(1) = \lambda_1 + \cdots + \lambda_{l_\sigma(n)-1} + 1 = n^\sigma(n) + 1 = w^\sigma(1)$.

Consider σ' the standard Young tableau obtained from σ by deleting the case occupied by n. One can easily check that for all $i = 2, \cdots, n$,

$$w^\sigma(i) = c^1\left(w^{\sigma'}(i-1) + 1\right).$$

Moreover, by Lemma 5.3 (and using its notation), we have, for all $i = 2, \cdots, n$:

$$w_\sigma(i) = \phi^{-1}\left(w_{\sigma'}(i-1)\right).$$

Therefore, we have to prove that for all $k = 1, \cdots, n-1$, we have $\phi^{-1}(k) = c^1(k+1)$. This follows easily from $n^\sigma(n) + 1 = w_\sigma(1)$.

Let σ and γ be two k-adjacent standard Young tableaux (with some $3 \leq k \leq n$). One can easily check that $w^\sigma = w^\gamma s_{n+1-k}$. Then, by Theorem 8.1 if w^σ satisfies the theorem if and only if w^γ does. Now, the theorem follows from Proposition 6.9 and the first part of this proof. □

Remark 8.10. Theorem 8.9 implies that for any σ we have $\sum_i n^\sigma(i) = \sum_j n_\sigma(j)$. This is obvious, since these two quantities are the cardinality of the set of the $(i < j)$ such that $l_\sigma(i) < l_\sigma(j)$. But, the equality of the theorem does not seem so obvious and keeps mysterious for the authors.

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Université Montpellier II (Mathématiques), Place Eugène Bataillon, 34095 Montpellier Cedex 05, France
E-mail address: pagnon@math.univ-math2.fr

Université Montpellier II (Mathématiques), Place Eugène Bataillon, 34095 Montpellier Cedex 05, France
E-mail address: ressayre@math.univ-math2.fr