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ON THE TENSOR SEMIGROUP OF AFFINE KAC-MOODY LIE ALGEBRAS

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1. Introduction

1.1. **A brief history of the Horn conjecture.** In some sense, the very starting point of this work is the problem of determining the possible spectra of a sum of Hermitian matrices each with known spectrum. Problem of which a brief history reads as follows (see [Ful00, Bri12, Kum15]). In 1912, Herman Weyl [Wey12] gave some necessary conditions on the spectrum of the sum, called Weyl inequalities, which had many applications. In 1962, Alfred Horn [Hor62] gave a list of necessary conditions (called Horn inequalities) and conjectured that it is also sufficient, i.e. solves the problem completely. Horn's conjecture remained open for 36 years until the works of Klyachko [Kly98] and Knutson and Tao [KT99] . Klyachko showed that Horn's conjecture follows from the Saturation conjecture, and Knutson and Tao proved this conjecture.

To state the saturation conjecture, consider the set of triples $(\lambda_1, \lambda_2, \mu)$ of rational dominant weights for the unitary group $U(n)$ such that there is a positive integer N for which the tensor product of the highest weight $U(n)$-representations $V(N\lambda_1)$ and $V(N\lambda_2)$ contains $V(N\mu)$; it is called the *tensor cone* of $U(n)$. One can show that the closure of this cone is exactly the set of spectra of Hermitian matrices A, B, C such that $A+B=C$. The saturation conjecture (proved by Knutson and Tao) says that if $\lambda_1, \lambda_2, \mu$ are integral and the product of $V(N\lambda_1)$ with $V(N\lambda_2)$ contains $V(N\mu)$ for some positive integer N then the product $V(\lambda_1)\otimes V(\lambda_2)$ contains $V(\mu)$.

The tensor cone can be defined for any connected compact Lie group, or equivalently, for any complex connected reductive group, or still equivalently, for any reductive complex Lie algebra. Following Klyachko and Berenstein-Sjamaar [BS00], Belkale-Kumar [BK06] obtained an explicit finite list of inequalities describing the tensor cone of any connected compact Lie group. The Saturation conjecture, on the other hand, is false in this generality, but there exist positive integers d such that if the product $V(N\lambda_1)$ and $V(N\lambda_2)$ contains $V(N\mu)$ for some N and $\lambda_1+\lambda_2-\mu$ belongs to the root lattice then it holds for $N=d$ (and hence for any multiple of d). Such integers are called *saturation factors*, and Belkale, Kumar, Kapovich-Millson and Hong-Shen obtained various upper bounds for the smallest saturation factor for every group. Nevertheless, the precise value is still not known even for classical groups. For example, Kapovich-Millson's conjecture stating that 1 is a saturation factor for any simply laced group is still open.

In 2014 Brown-Kumar [BK06] studied the tensor cone for symmetrizable Kac-Moody algebras, and gave an infinite list of inequalities analogous to Belkale-Kumar inequalities which conjecturally describes the tensor cone. The main contribution of the paper is a new approach which allows to prove the Brown-Kumar conjecture for untwisted affine Lie algebras, which is the most important class of infinite-dimensional Kac-Moody algebras. Another important result is the calculation of upper bounds for saturation factors for such algebras. The proof of the Berown-Kumar conjecture decomposes in several steps, many of which generalize to any symmetrizable Kac-Moody algebra.

1.2. **The tensor cone.** Let A be a symmetrizable irreducible generalized Cartan matrix of size $l+1$. Let $\mathfrak{h} \supset \{\alpha_0^\vee, \dots, \alpha_l^\vee\}$ and $\mathfrak{h}^* \supset \{\alpha_0, \dots, \alpha_l\} =: \Delta$ be a realization of A. We fix an integral form $\mathfrak{h}_\mathbb{Z} \subset \mathfrak{h}$ containing each $\alpha_i^\vee$, such that $\mathfrak{h}_\mathbb{Z}^* := \mathrm{Hom}(\mathfrak{h}_\mathbb{Z}, \mathbb{Z})$ contains Δ and such that $\mathfrak{h}_\mathbb{Z}/\oplus\mathbb{Z}\alpha_i^\vee$ is torsion free. Set $\mathfrak{h}_\mathbb{Q}^* = \mathfrak{h}_\mathbb{Z}^* \otimes \mathbb{Q} \subset \mathfrak{h}^*$, $P_{+,\mathbb{Q}} := \{\lambda \in \mathfrak{h}_\mathbb{Q}^* \,|\, \langle \alpha_i^\vee, \lambda\rangle \geq 0 \quad \forall i\}$, and $P_+ = \mathfrak{h}_\mathbb{Z} \cap P_{+,\mathbb{Q}}$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the associated Kac-Moody Lie algebra with Cartan subalgebra $\mathfrak{h}$. For $\lambda \in P_+$, $V(\lambda)$ denotes the irreducible representation of $\mathfrak{g}$ with highest weight λ. Define the *tensor semigroup* as

$$\Gamma_\mathbb{N}(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_+^3 \,| \qquad V(\mu) \subset V(\lambda_1) \otimes V(\lambda_2)\},$$

and the *tensor cone* as

$$\Gamma(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_{+,\mathbb{Q}}^3 \,|\, \exists N > 1 \qquad V(N\mu) \subset V(N\lambda_1) \otimes V(N\lambda_2)\}.$$

1.3. **The main result.** Let G be the minimal Kac-Moody group as in [Kum02, Section 7.4] and B its standard Borel subgroup. Let $(\varpi_{\alpha_0^\vee}, \dots, \varpi_{\alpha_l^\vee}) \subset \mathfrak{h}_\mathbb{Q}$ be elements dual to the simple roots. Let W be the Weyl group of A. To any simple root α_i, is associated a maximal standard parabolic subgroup P_i, its Weyl group $W_{P_i} \subset W$ and the set W^{P_i} of minimal length representative of elements of W/W_{P_i}. We also consider the partial flag ind-variety G/P_i containing the Schubert varieties $X_w = \overline{BwP_i/P_i}$, for $w \in W^{P_i}$. Let $\{\epsilon_w\}_{w \in W^{P_i}} \subset \mathrm{H}^*(G/P_i, \mathbb{Z})$ be the Schubert basis dual to the basis of the singular homology of G/P_i given by the fundamental classes of X_w. Inspired by the Belkale–Kumar definition [BK06, Section 6] in the finite-dimensional case, Kumar defined in [Kum10] a deformed product $\odot_0$ on $\mathrm{H}^*(G/P_i, \mathbb{Z})$, which is commutative and associative.

Theorem 1. *Assume that $\mathfrak{g}$ is an affine untwisted Kac-Moody Lie algebra with central element c. Let $(\lambda_1, \lambda_2, \mu) \in P_{+,\mathbb{Q}}^3$ be such that $\lambda_1(c) > 0$ and $\lambda_2(c) > 0$.*

Then,

$$(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g})$$

if and only if

$$\mu(c) = \lambda_1(c) + \lambda_2(c), \tag{1}$$

and

$$\langle \mu, v\varpi_{\alpha_i^\vee}\rangle \leq \langle \lambda_1, u_1\varpi_{\alpha_i^\vee}\rangle + \langle \lambda_2, u_2\varpi_{\alpha_i^\vee}\rangle \tag{2}$$

for any $i \in \{0, \dots, l\}$ and any $(u_1, u_2, v) \in (W^{P_i})^3$ such that, in $\mathrm{H}^*(G/P_i, \mathbb{Z})$, ϵ_v occurs with coefficient 1 in the deformed product

$$\epsilon_{u_1} \odot_0 \epsilon_{u_2}.$$

The statement of Theorem 1 is very similar to [BK06, Theorem 22] that describes $\Gamma(\mathfrak{g})$, if $\mathfrak{g}$ is finite-dimensional. In the next subsection, we review shortly, several approaches used in the literature to study $\Gamma(\mathfrak{g})$ and argue why these methods do not apply in the Kac-Moody setting. Then we present our new approach.

1.4. **Various approaches.** Several tolls have already been used to tackle the Horn conjecture and its generalizations. Numerous inequalities (necessary conditions) [Wey12, Lid50, Wie55, TF71, Hor62] were first obtained using the min-max description of eigenvalues and the Rayleigh trace. Next, the introduction of symplectic geometry and moment map techniques allowed to get qualitative convexity results (see [Hec82, Kir84, Sja98]). The interpretation of the problem in terms of tensor product decomposition [Nes84, Appendice] is another decisive step. In 1994, Klyachko's breakthrough was to interpret the Horn inequalities as semistability conditions, allowing him [Kly98] to prove sufficient conditions for a triple of weights to belong to the Horn cone. Next, semistability, and more precisely the Hilbert-Mumford theorem is used in [BS00, Bel01, BK06, Res10] in a crucial way.

For the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, the multiplicities of the tensor product decomposition are the Littlewood-Richardson coefficients. These coefficients are also, the structure constants of the cohomology rings of Grassmannians. This remark allows to transpose the Horn problem in the world of Schubert calculus. Belkale [Bel06] used a very nice interpretation of Horn inequalities. Namely, the non-vanishing of a given Littlewood-Richardson coefficient can be interpreted as the possibility to translate Schubert varieties to get a transverse intersection. Then, some linear map between tangent spaces has to be injective. It turns out that this linear map is block triangular, with rectangular blocks. Here, the Horn inequalities are interpreted as necessary conditions on the size of these blocks to allow the existence of such invertible linear maps.

Still another approach, introduced by Knutson-Tao [KT99], consists in using combinatorial models to express Littlewood-Richardson coefficients like BZ-paterns, honeycombs or puzzles. This approach allowed to prove saturation [KT99], irredundancy [KTW04] and gave recent developments [APS17].

The theory of Bruhat-Tits buildings gives a connection between metric geometry and representation theory of complex semisimple algebraic groups. In this approach, the Horn inequalities are interpreted as triangle inequalities (see [KLM08] or [Kum14, Appendix]).

Another tool used to study the Horn problem is representation theory of quivers. Here, both semi-stability and transversality conditions appear with subtle and fruitful interplays (see [DW00, CBG02, BVW17]).

One can observe that none of these approaches seem useful to show Theorem 1. Indeed, Geometric Invariant Theory does not apply to the action of a loop group on an ind-variety and there is no known extension of the Hilbert-Mumford theorem in this context. The methods of semistability was adapted to the so called multiplicative Horn problem (e.g. the question of describing the possible eigenvalues of the product of two unitary matrices with known spectra) using principal parabolic bundles on curves (see [AW98, BK16, Res13]). For Theorem 1, no such notion of semi-stability seems to be known.

The approach in terms of Schubert calculus relies on a numerical coincidence: the Littlewood-Richardson coefficients encode both tensor product decoposition for

$U(n)$-representations and Schubert calculus for the Grassmannians. Such coincidences are not known beyond the type A.

The use of combinatorial models like honeycomb was used only in type A.

In [GR14], Gaussent-Rousseau defined an avatar of the Bruhat-Tits building for Kac-Moody groups, called masures. It is an intersting open question to see if the masures can be used to prove Theorem 1 or saturation results like Theorem 3 below.

1.5. **Our approach.** We now explain (roughly speaking) our strategy.

Consider the cone $\mathcal{C}(\mathfrak{g})$ defined by equality (1) and inequalities (2). It remains to prove that, up to the assumption "$\lambda_1(c)$ and $\lambda_2(c)$ are positive", the cone $\mathcal{C}(\mathfrak{g})$ is equal to $\Gamma(\mathfrak{g})$. The proof proceeds in five steps.

Step 1. $\Gamma(\mathfrak{g})$ is convex.

This is a well-known consequence of Borel-Weil's theorem (see Lemma 4).

Step 2. The set $\Gamma(\mathfrak{g})$ is contained in $\mathcal{C}(\mathfrak{g})$.

This step is proved in [BK14] and reproved here. The first ingredient is the easy implication in the Hilbert-Mumford's theorem. Indeed "semistable $\Rightarrow$ numerically semistable" is still true for ind-varieties and Kac-Moody groups. In the finite-dimensional case, the second argument is Kleiman's transversality theorem. In [BK14], it is replaced by an argument in K-theory which express the structure constants of $\mathrm{H}^*(G/P_i, \mathbb{Z})$ as the Euler characteristic of sheaves supported by the intersection of three translated Schubert or Birkhoff varieties. Here, we refine this argument by proving a version of Kleiman's theorem that allows to express these structure constants as the cardinality of the intersection of three translated Schubert or Birkhoff varieties.

Step 3. The cone $\mathcal{C}(\mathfrak{g})$ is locally polyhedral.

This is a consequence of Proposition 4 below. We study the inequalities (2) defining $\mathcal{C}(\mathfrak{g})$. In particular, we use some consequences of the non-vanishing of a structure constant of the ring $\mathrm{H}^*(G/P_i, \mathbb{Z})$ (see Lemmas 17 and 18 below and [BK14]).

Step 4. Study of the boundary of $\mathcal{C}(\mathfrak{g})$.

Let $(\lambda_1, \lambda_2, \mu)$ be an integral point in the boundary of $\mathcal{C}(\mathfrak{g})$. Step 3 implies that some inequality (2) has to be an equality for $(\lambda_1, \lambda_2, \mu)$. Then, one can use the following Theorem 2 to describe inductively the multiplicity of $V(\mu)$ in $V(\lambda_1) \otimes V(\lambda_2)$. Let α_i be a simple root and let L_i denote the standard Levi subgroup of P_i. For $w \in W^{P_i}$ and $\lambda \in P_+$, $w^{-1}\lambda$ is a dominant weight for L_i: we denote by $V_{L_i}(w^{-1}\lambda)$ the corresponding irreducible highest weight L_i-representation.

Theorem 2. *Here, $\mathfrak{g}$ is any symmetrizable Kac-Moody Lie algebra and α_i is a simple root. Let $(\lambda_1, \lambda_2, \mu) \in P_+^3$. Let $(u_1, u_2, v) \in (W^{P_i})^3$ such that ϵ_v occurs with coefficient 1 in the ordinary product $\epsilon_{u_1}.\epsilon_{u_2}$. We assume that*

$$\langle \mu, v\varpi_{\alpha_i^\vee}\rangle = \langle \lambda_1, u_1\varpi_{\alpha_i^\vee}\rangle + \langle \lambda_2, u_2\varpi_{\alpha_i^\vee}\rangle. \tag{3}$$

Then the multiplicity of $V(\mu)$ in $V(\lambda_1) \otimes V(\lambda_2)$ is equal to the multiplicity of $V_{L_i}(v^{-1}\mu)$ in $V_{L_i}(u_1^{-1}\lambda_1) \otimes V_{L_i}(u_2^{-1}\lambda_2)$.

Note that Theorem 6 and its corollary in Section 6 are a little bit more general than Theorem 2.

Step 5. Induction.

While there are numerous technical difficulties, the basic idea is simple. By convexity, it is sufficient to prove that the boundary of $\mathcal{C}(\mathfrak{g})$ is contained in $\Gamma(\mathfrak{g})$. Using Step 4, this claim can be proved by induction.

Namely, consider a face $\mathcal{F}$ of codimension one of $\mathcal{C}(\mathfrak{g})$ associated with some structure constant of $\mathrm{H}^*(G/P_i, \mathbb{Z})$ for $\odot_0$ equal to one. We have to prove that $\mathcal{F}$ is contained in $\Gamma(\mathfrak{g})$. By Theorem 2, it remains to prove that the points of $\mathcal{F}$ satisfy the inequalities that characterize the tensor cone $\Gamma(\mathfrak{l}_i)$ of the Lie algebra $\mathfrak{l}_i$ of L_i. Fix such an inequality associated with a structure constant of $\mathrm{H}^*(L_i/(P_j \cap L_i), \mathbb{Z})$ for $\odot_0$ equal to one. Consider the flag ind-varieties:

$$\begin{tikzcd}
L_i/(P_j\cap L_i) \arrow[r] & G/(P_i\cap P_j) \arrow[dl] \arrow[dr] & \\
G/P_i & & G/P_j
\end{tikzcd}$$

Proposition 3 shows a property of multiplicativity for structure constants of the rings $\mathrm{H}^*(G/P, \mathbb{Z})$ that gives us a structure constant of $\mathrm{H}^*(G/(P_i \cap P_j), \mathbb{Z})$ equal to one, for the ordinary product. A crucial point is Theorem 7 that proves that, if the considered inequality of $\Gamma(\mathfrak{l}_i)$ is "useful" then this structure constant of $\mathrm{H}^*(G/(P_i \cap P_j), \mathbb{Z})$ is actually nonzero for $\odot_0$. Then we get a structure constant of $\mathrm{H}^*(G/P_j, \mathbb{Z})$ for $\odot_0$ equal to one. In particular, this gives an inequality of $\mathcal{C}(\mathfrak{g})$ that corresponds to the desired inequality of $\Gamma(\mathfrak{l}_i)$ when restricted to the span of $\mathcal{F}$.

If we prove Theorem 1 only for the untwisted affine case, the general strategy should work more generally. For this reason, we prove some intermediate results for any symmetrizable Kac-Moody Lie algebra. In particular Steps 1, 2 and 4 works with this generality. Proposition 3 of multiplicativity also holds in this context.

1.6. **Saturation factors.** Let Q denote the root lattice of $\mathfrak{g}$. We are now intersted in the *tensor semigroup*. The semigroup $\Gamma_{\mathbb{N}}(\mathfrak{g})$ is not finitely generated when $\mathfrak{g}$ is affine. Despite this, we obtain explicit saturation factors d_0 such that, for any $(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g}) \cap (P_+)^3$ such that $\lambda_1 + \lambda_2 - \mu \in Q$, $V(d_0\mu)$ is a sub-representation of $V(d_0\lambda_1) \otimes V(d_0\lambda_2)$. Observe that the condition $\lambda_1 + \lambda_2 - \mu \in Q$ is necessary to have $V(\mu) \subset V(\lambda_1) \otimes V(\lambda_2)$, because of the action of the center of G.

To describe our saturation factors, we need additional notation. Up to now, $\mathfrak{g}$ is the affine Lie algebra associated with the simple Lie algebra $\dot{\mathfrak{g}}$. Let us define the constant k_s to be the least common multiple of saturation factors of maximal Levi subalgebras of $\mathfrak{g}$. The value of k_s depends on known saturation factors for the finite-dimensional Lie algebras. With the current literature (see Section 10), possible values for k_s are given in the following tabular.

Type of $\mathfrak{g}$	$\tilde{A}_\ell$	$\tilde{B}_\ell(\ell \geq 4)$	$\tilde{C}_\ell(\ell \geq 2)$	$\tilde{D}_\ell(\ell \geq 5)$	$\tilde{D}_4$	$\tilde{B}_\ell(\ell = 2, 3)$
k_s	1	4	2	4	1	2
Type of $\mathfrak{g}$	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$	$\tilde{F}_4$	$\tilde{G}_2$	$\tilde{G}_2$
k_s	36	144	3 600	144	2	3

Let $k_{\dot{\mathfrak{g}}}$ be the least common multiple of coordinates of the highest root $\dot\theta$ written as a combinaison of simple roots. The values of $k_{\dot{\mathfrak{g}}}$ are

Type	A_ℓ	$B_\ell(\ell \geq 2)$	$C_\ell(\ell \geq 3)$	$D_\ell(\ell \geq 5)$	E_6	E_7	E_8	F_4	G_2
$k_{\dot{\mathfrak{g}}}$	1	2	2	2	6	12	60	12	6

Theorem 3. *Let $(\lambda_1, \lambda_2, \mu) \in (P_+)^3$ such that there exists $N > 0$ such that $V(N\mu)$ embeds in $V(N\lambda_1) \otimes V(N\lambda_2)$. We also assume that $\lambda_1 + \lambda_2 - \mu \in Q$.*

Then,

(i) *if $k_s = 1$ then for any integer $d \geq 2$, $V(dk_{\dot{\mathfrak{g}}}\mu)$ embeds in $V(dk_{\dot{\mathfrak{g}}}\lambda_1) \otimes V(dk_{\dot{\mathfrak{g}}}\lambda_2)$;*

(ii) *if $k_s > 1$ then $V(k_{\dot{\mathfrak{g}}}k_s\mu)$ embeds in $V(k_{\dot{\mathfrak{g}}}k_s\lambda_1) \otimes V(k_{\dot{\mathfrak{g}}}k_s\lambda_2)$.*

Observe that, in type A, $k_{\dot{\mathfrak{g}}}k_s = 1$ and Theorem 3 proves that any $d \geq 2$ is a saturation factor. Note that $d = 1$ is not a saturation factor in this case. The case $\tilde{A}_1$ was previously obtained in [BK14].

Let δ denote the fundamental imaginary root. We also obtain the following variation.

Theorem 4. *Let $(\lambda_1, \lambda_2, \mu) \in (P_+)^3$ such that there exists $N > 0$ such that $V(N\mu)$ embeds in $V(N\lambda_1) \otimes V(N\lambda_2)$. We also assume that $\lambda_1 + \lambda_2 - \mu \in Q$.*

Then, for any integer $d \geq 2$, $V(k_{\dot{\mathfrak{g}}}k_s\mu - d\delta)$ embeds in $V(k_{\dot{\mathfrak{g}}}k_s\lambda_1) \otimes V(k_{\dot{\mathfrak{g}}}k_s\lambda_2)$.

In Section 11, we collect some technical lemmas used in the paper.

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Contents

2. Ind-varieties

2.1. **Ind-varieties.** In this section, we collect definitions, notation and properties on ind-varieties. The results are certainly well-known, but we include some proofs by lack of references. They will be applied to ind-varieties derived from flag ind-varieties of Kac-Moody groups.

2.1.1. *The category.* Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of quasi-projective complex varieties given with closed immersions $\iota_n : X_n \to X_{n+1}$. The inductive limit $X = \varinjlim X_n$ is called a *filtered ind-variety.* The Zariski topology on X is defined by setting a subset F closed if $F \cap X_n$ is closed for any $n \in \mathbb{N}$. A continuous map $f : X \longrightarrow Y = \varinjlim Y_n$ between two filtered ind-varieties is a *morphism* if for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $f(X_n) \subset Y_m$ and the restriction $f_{n,m} : X_n \longrightarrow Y_m$ of f is a morphism of quasiprojective varieties. A closed subset Z of X is said to be *finite-dimensional* if there exists $n \in \mathbb{N}$ such that $Z \subset X_n$.

Roughly speaking, an ind-variety is obtained from a filtered ind-variety by forgetting the filtration. Let $X'_n \subset X$ be finite-dimensional closed subsets such that $X = \cup_{n\in\mathbb{N}} X'_n$ and $X'_0 \subset X'_1 \subset \cdots X'_n \subset \cdots$. Then $X' = \varinjlim X'_n$ is a filtered ind-variety. The filtrations $(X_n)_{n\in\mathbb{N}}$ and $(X'_n)_{n\in\mathbb{N}}$ are said to be *equivalent* if the identity maps $X \longrightarrow X'$ and $X' \longrightarrow X$ are morphisms. An ind-variety is a filtered ind-variety endowed with the collection of all the equivalent filtrations or, equivalently, a filtered ind-variety up to isomorphism.

The above definitions are available on any algebraically closed field. Over complex numbers (or more generally an uncountable field) this definition can be simplified thanks to Lemma 1. We will use it repeatedly to change the filtrations of a given ind-variety.

Lemma 1. *Recall that we work over complex numbers. Let $X = \cup_{n\in\mathbb{N}} X_n$ be a filtered ind-variety. Assume, we have a family $(X'_n)_{n\in\mathbb{N}}$ of closed finite-dimensional subsets in X such that $X'_n \subset X'_{n+1}$ and $X = \cup_{n\in\mathbb{N}} X'_n$.*

Then the two filtrations $(X_n)_{n\in\mathbb{N}}$ and $(X'_n)_{n\in\mathbb{N}}$ are equivalent.

Proof. Set $X' = \varinjlim X'_n$. By assumption, for any n there exists m such that $X'_n \subset X_m$. Hence the identity map $\mathrm{Id} : X' \longrightarrow X$ is a morphism.

Conversely, show that $\mathrm{Id} : X \longrightarrow X'$ is a morphism. Let C be an irreducible component of some X_{n_0}. Then $C = \cup_n X'_n \cap C$ and $X'_n \cap C$ is closed in C. Assume that for any n, $X'_n \cap C \neq C$. Then, for any n, $\dim(X'_n \cap C) < \dim C$. Hence C is the union of countably many subvarieties of smaller dimension. This is a contradiction since we are working on the uncountable field of complex numbers: there exists n_C such that $X'_{n_C} \cap C = C$.

Since X_{n_0} has finitely many irreducible components, there exists N_0 such that $X_{n_0} \subset X'_{N_0}$. Then the identity map $X \longrightarrow X'$ is a morphism. □

A *filtration* $X = \cup_n X_n$ of an ind-variety is a collection of closed finite-dimensional subsets X_n such that X is the nondecreasing union of the X_n.

For $x \in X$, the tangent space $T_x X$ of X at x is defined to be $\varinjlim T_x X_n$.

2.2. **Irreducibility.** An ind-variety X is said to be *irreducible* if it is as a topological space for the Zariski topology. Assume a filtration $X = \cup_n X_n$ is given. If the poset of irreducible components of the X_n's is directed for inclusion then X is irreducible. Here, a poset is said to be *directed* if for any two elements x, y there exists z bigger or equal to both x and y. Contrary to what [Sha81, Proposition 1] claims, the converse of this assertion is not true (see [Kam96, Sta12] for examples). Here, the filtered ind-variety X is said to be *ind-irreducible* if the poset of irreducible components of the X_n's is directed for inclusion. The following lemma shows that the ind-irreducibility does not depend on the filtration and can be defined for ind-varieties.

Lemma 2. *Let X be an ind-variety. The following assertions are equivalent:*

(i) *for any filtration $X = \cup_{n\in\mathbb{N}} X_n$, the poset of irreducible components of the X_n's is directed;*
(ii) *there exists a filtration $X = \cup_{n\in\mathbb{N}} X_n$ such that the poset of irreducible components of the X_n's is directed;*
(iii) *there exists a filtration $X = \cup_{n\in\mathbb{N}} X_n$ with X_n irreducible, for any n.*

If X satisfies these properties then X is said to be ind-irreducible.

Proof. We prove $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. The first implication is tautological.

Show $(ii) \Rightarrow (iii)$. The variety X_n having finitely many irreducible components, the assumption implies that

$$\forall n \quad \exists N \text{ and an irreducible component } C_N \text{ of } X_N \text{ such that } X_n \subset C_N.$$

Then one can construct by induction an increasing sequence $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ and irreducible components $C_{\varphi(n)}$ of $X_{\varphi(n)}$ such that

$$\forall n \quad X_{\varphi(n)} \subset C_{\varphi(n+1)} \subset X_{\varphi(n+1)}. \tag{4}$$

Note that the $C_{\varphi(n)}$'s are closed, satisfy $C_{\varphi(n)} \subset C_{\varphi(n+1)}$ and $X = \cup_{n\in\mathbb{N}} C_{\varphi(n)}$. Hence by Lemma 1, they form a filtration of X by irreducible subvarieties.

Show $(iii) \Rightarrow (i)$. Fix a filtration $X = \cup_{n\in\mathbb{N}} X_n$ by irreducible finite-dimensional closed subsets X_n. Let $X = \cup_{n\in\mathbb{N}} X'_n$ be another filtration. Consider two irreducible components C'_{n_1} and C'_{n_2} of some X'_{n_1} and X'_{n_2}. There exist N_1 and N_2 in $\mathbb{N}$ such that $C'_{n_1} \cup C'_{n_2} \subset X_{N_1} \subset X'_{N_2}$. Now, X_{N_1} being irreducible, there exists an irreducible component C'_{N_2} of X'_{N_2} containing X_{N_1}. Hence the poset of irreductible components of the X'_n is directed. □

Examples.

(i) The simplest examples $\mathbb{A}^{(\infty)} = \varinjlim \mathbb{A}^n$, $\mathbb{P}^{(\infty)} = \varinjlim \mathbb{P}^n$ of ind-varieties are ind-irreducible.

(ii) A nonempty open subset of an ind-irreducible ind-variety is ind-irreducible. A product of two ind-irreducible ind-varieties is ind-irreducible.

(iii) Consider a surjective morphism $f : X \longrightarrow Y$ of ind-varieties. If X is ind-irreducible then so is Y. Indeed, let $X = \cup_n X_n$ be a filtration of X by irreducible finite-dimensional subvarieties. Denote by Y_n the closure in Y of $f(X_n)$. Then, by Lemma 1, $Y = \cup_n Y_n$ is a filtration of Y by irreducible subvarieties. Hence Y is ind-irreducible.

(iv) If G is a Kac-Moody group and P is a standard parabolic subgroup then G/P is a projective ind-irreducible ind-variety. Indeed, a filtration of G/P is given by the unions of Schubert varieties of bounded dimension. The Bruhat order being directed (see e.g. [BB05, Proposition 2.2.9]), G/P is ind-irreducible.

(v) The Richardson varieties being irreducible (see [Kum02]), the Birkhoff subvarieties of G/P are ind-irreducible.

(vi) Let G be the minimal Kac-Moody group as defined in [Kum02, Section 7.4]. Then G is an ind-irreducible ind-variety. Indeed, for each real root β, denote by $U_\beta : \mathbb{C} \longrightarrow G$ the radicial subgroup. Consider an infinite word $\underline{w} = \beta_1 \dots \beta_n \dots$ in the real roots of $\mathfrak{g}$ such that any finite word in these roots is a subword of $\underline{w}$. Consider the map

$$\begin{array}{cccc} \theta : & \mathbb{A}^{(\infty)} & \longrightarrow & G \\ & (\tau_i)_{i\in\mathbb{N}} & \longmapsto & \prod_i U_{\beta_i}(\tau_i), \end{array}$$

where the product is made in the order given by $\underline{w}$. Since G is a group-ind-variety and the U_β's are morphisms of ind-varieties, θ is a morphism of ind-varieties. By definition, it is surjective. By Example (iii), G is ind-irreducible.

Another result we need, is the following.

Lemma 3. *Let X be an ind-irreducible ind-variety. Let Ω be a nonempty open subset of X.*

Let $(X_n)_{n\in\mathbb{N}}$ be a collection of closed finite-dimensional subsets of X such that $X_n \subset X_{n+1}$ and $\cup_{n\in\mathbb{N}} X_n$ contains Ω.

Then $X = \cup_{n\in\mathbb{N}} X_n$ is a filtration of X.

Proof. Fix a filtration $X = \cup_{n\in\mathbb{N}} X'_n$ by irreducible subsets of X intersecting Ω.

Let $n_0 \in \mathbb{N}$. Observe that $X'_{n_0} \cap \Omega \subset \cup_{n\in\mathbb{N}}(X'_{n_0} \cap X_n \cap \Omega)$. But $X'_{n_0} \cap X_n \cap \Omega$ is a locally closed subvariety of X'_{n_0} and the sequence $n \mapsto \dim(X'_{n_0} \cap X_n \cap \Omega)$ is nondecreasing.

Assume that $\dim(X'_{n_0} \cap X_n \cap \Omega) < \dim(X'_{n_0} \cap \Omega)$, for any n. Then $X'_{n_0} \cap \Omega$ is the union of countably many strict subvarieties. This is a contration since the base field is uncountable. Hence there exists N such that $\dim(X'_{n_0} \cap X_N \cap \Omega) = \dim(X'_{n_0} \cap \Omega)$.

Then, $X'_{n_0} \cap \Omega$ being irreducible, it is contained in X_N. But X_N is closed and X'_{n_0} is irreducible. Hence X'_{n_0} has to be contained in X_N. In particular, $X = \cup_n X_n$. We conclude using Lemma 1. ☐

2.3. **Line bundles.** Let $X = \cup_{n\in\mathbb{N}} X_n$ be a filtered ind-variety. Denote by $\iota_n : X_n \longrightarrow X$ the inclusion. A *line bundle* $\mathcal{L}$ over X is an ind-variety with a morphism $\pi : \mathcal{L} \longrightarrow X$ such that $\iota_n^*(\mathcal{L})$ is a line bundle over X_n, for any n.

A section of $\mathcal{L}$ is a morphism $\sigma : X \longrightarrow \mathcal{L}$ such that $\pi \circ \sigma = \mathrm{Id}_X$. We denote by $\mathrm{H}^0(X, \mathcal{L})$ the vector space of sections. Given a section σ, the sequence of sections $(\sigma_n = \iota_n^*(\sigma))_{n\in\mathbb{N}}$ satisfies

$$\sigma_{n+1|X_n} = \sigma_n. \tag{5}$$

Conversely, a sequence σ_n of sections of $\iota_n^*(\mathcal{L})$ on X_n satisfying condition (5) induces a well defined section σ of $\mathcal{L}$ such that $\sigma_n = \iota_n^*(\sigma)$ for any n.

3. Using the Borel-Weil Theorem

Using the Borel-Weil theorem, we express the tensor multiplicities as the dimensions of spaces of invariant sections of line bundles. The infinite dimensional setting needs to be careful with duality.

3.1. **Tensor multiplicities.** Recall that $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra. For given λ_1 and λ_2 in P_+, $V(\lambda_1) \otimes V(\lambda_2)$ decomposes as a sum of integrable irreducible highest weight representations (see [Kum02, Corrolary 2.2.7]), with finite multiplicities:

$$V(\lambda_1) \otimes V(\lambda_2) = \bigoplus_{\mu \in P_+} V(\mu)^{\oplus c^{\mu}_{\lambda_1\,\lambda_2}}.$$

Let M be a $\mathfrak{g}$-representation in the category $\mathcal{O}$; under the action of $\mathfrak{h}$, M decomposes as $\oplus_\mu M_\mu$ with finite-dimensional weight spaces M_μ. Set $M^\vee = \oplus_\mu M_\mu^*$: it is a sub-$\mathfrak{g}$-representation of the dual space M^*.

3.2. **Multiplicities as dimensions.** Recall that G is the minimal Kac-Moody group associated with $\mathfrak{g}$. Let B be the standard Borel subgroup of G and B^- be the opposite Borel subgroup. Consider G/B and G/B^- endowed with the usual ind-variety structures. Set $\underline{o} = B/B$ (resp. $\underline{o}^- = B^-/B^-$), the base point of G/B (resp. G/B^-). For $\lambda \in \mathfrak{h}^*_{\mathbb{Z}} = \mathrm{Hom}(T, \mathbb{C}^*) = \mathrm{Hom}(B, \mathbb{C}^*) = \mathrm{Hom}(B^-, \mathbb{C}^*)$, we consider the G-linearized line bundle $\mathcal{L}(\lambda)$ (resp. $\mathcal{L}_-(\lambda)$) on G/B (resp. G/B^-)

such that B (resp. B^-) acts on the fiber over $\underline{o}$ (resp. $\underline{o}^-$) with weight $-\lambda$ (resp. λ). For $\lambda \in P_+$, we have G-equivariant isomorphisms (see [Kum02, Section VIII.3])

$$\begin{array}{lcl} \mathrm{H}^0(G/B, \mathcal{L}(\lambda)) & \simeq & \mathrm{Hom}(V(\lambda), \mathbb{C}), \\ \mathrm{H}^0(G/B^-, \mathcal{L}_-(\lambda)) & \simeq & \mathrm{Hom}(V(\lambda)^\vee, \mathbb{C}). \end{array}$$

Set

$$\mathbb{X} = (G/B^-)^2 \times G/B.$$

A significant part of the following lemma is contained in [BK14, Proof of Theorem 3.2].

Lemma 4. *Let λ_1, λ_2, and μ in P_+. Then the space*

$$\mathrm{H}^0(\mathbb{X}, \mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu))^G$$

of G-invariant sections has dimension $c^\mu_{\lambda_1\,\lambda_2}$. In particular, this dimension is finite.

Proof. Set $\mathcal{L} = \mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu)$. We have the following canonical isomorphisms:

$$\begin{array}{rll} \mathrm{H}^0(\mathbb{X}, \mathcal{L})^G \simeq & \mathrm{Hom}(V(\lambda_1)^\vee \otimes V(\lambda_2)^\vee \otimes V(\mu), \mathbb{C})^G & \\ \simeq & \mathrm{Hom}(V(\mu), (V(\lambda_1)^\vee \otimes V(\lambda_2)^\vee)^*)^G & \\ \simeq & \mathrm{Hom}(V(\mu), (V(\lambda_1)^\vee \otimes V(\lambda_2)^\vee)^\vee)^G & \text{by } \mathfrak{h}\text{-invariance} \\ \simeq & \mathrm{Hom}(V(\mu), V(\lambda_1) \otimes V(\lambda_2))^G & \end{array}$$

Thus this space of invariant sections has dimension $c^\mu_{\lambda_1\,\lambda_2}$. We already mentioned that $c^\mu_{\lambda_1\,\lambda_2}$ is finite. Nevertheless, we prove independently that $\mathrm{H}^0(\mathbb{X}, \mathcal{L})^G$ is finite-dimensional, reproving that $c^\mu_{\lambda_1\,\lambda_2}$ is finite.

Consider the T-equivariant map $\iota : G/B^- \longrightarrow \mathbb{X}, x \longmapsto (\underline{o}^-, x, \underline{o})$. Then $\iota^*(\mathcal{L})$ is a T-linearized line bundle on G/B^-. Consider

$$\iota^* : \mathrm{H}^0(\mathbb{X}, \mathcal{L}) \longrightarrow \mathrm{H}^0(G/B^-, \iota^*(\mathcal{L})).$$

The orbit $G.(\underline{o}^-, \underline{o})$ being dense in $G/B^- \times G/B$, the restriction of ι^* to $\mathrm{H}^0(\mathbb{X}, \mathcal{L})^G$ is injective. Furthermore, the T-equivariance of ι implies that $\iota^*(\mathrm{H}^0(\mathbb{X}, \mathcal{L})^G)$ is contained in $\mathrm{H}^0(G/B^-, \iota^*(\mathcal{L}))^T$. But $\iota^*(\mathcal{L}) \simeq \mathcal{L}_-(\lambda_2) \otimes (\lambda_1 - \mu)$, where $\otimes(\lambda_1 - \mu)$ means that the T-action on $\mathcal{L}_-(\lambda_2)$ induced by the G-action is twisted by the character $\lambda_1 - \mu$ of T. Then

$$\begin{aligned} \mathrm{H}^0(G/B^-, \iota^*(\mathcal{L}))^T &\simeq \mathrm{H}^0(G/B^-, \mathcal{L}_-(\lambda_2))^{(T)_{\mu-\lambda_1}} \\ &\simeq \mathrm{Hom}(V(\lambda_2)^\vee, \mathbb{C})^{(T)_{\mu-\lambda_1}} \\ &\simeq V(\lambda_2)^{(T)_{\mu-\lambda_1}} \end{aligned}$$

Here, if V is a T-representation and χ is a character of T, $V^{(T)_\chi}$ denotes the set of vectors $v \in V$ such that $tv = \chi(t)v$, for any $t \in T$. But $V(\lambda_2)$ belongs to the category $\mathcal{O}$ and the dimension of $V(\lambda_2)^{(T)_{\mu-\lambda_1}}$ is finite. We just proved that ι^* embeds $\mathrm{H}^0(\mathbb{X}, \mathcal{L})^G$ in a finite-dimensional vector space. □

4. Enumerative meaning of structure constants of $\mathrm{H}^*(G/P, \mathbb{Z})$

We now consider the cohomology ring of the flag ind-variety G/P, where P is a standard parabolic subgroup of G. For $v \in W^P$, set $X^P_v = \overline{BvP/P}$. Consider the homology group $\mathrm{H}_*(X,\mathbb{Z}) = \oplus_{v\in W^P}\mathbb{Z}[X_v^P]$. Then $\mathrm{H}^*(X,\mathbb{Z}) \simeq \mathrm{Hom}(\mathrm{H}_*(X,\mathbb{Z}),\mathbb{Z})$ has a "basis" $(\epsilon_u)_{u\in W^P}$ defined by

$$\epsilon_u([X_v^P]) = \delta_v^u, \qquad \forall v \in W^P.$$

For u_1, u_2, and $v \in W^P$, define $n_{u_1u_2}^v \in \mathbb{Z}$ by

$$\epsilon_{u_1}.\epsilon_{u_2} = \sum_{v\in W^P} n_{u_1u_2}^v \epsilon_v.$$

By [KN98], $n_{u_1u_2}^v$ is nonnegative. The aim of this section is to express $n_{u_1u_2}^v$ as the cardinality of an intersection of three subvarieties of X. In the finite dimensional setting, the job is made by Kleiman's transversality theorem.

4.1. **Richardson varieties.** Recall that the neutral component of the automorphism group of a finite-dimensional projective variety Y is a finite-dimensional algebraic group denoted by $\mathrm{Aut}^\circ(Y)$ (see [Ram64]).

Let U denote the commutator subgroup of B.

For $u, v \in W^P$, set $X_P^u = \overline{B^-uP/P}$ and $X_v^P = \overline{BvP/P}$. Set also $\mathring{X}_P^u = B^-uP/P$ and $\mathring{X}_v^P = BvP/P$. Let $\preccurlyeq$ denote the Bruhat order on W^P: $u\preccurlyeq v$ means that $X_u^P \subset X_v^P$. In the following lemma, we collect some well known facts about the Schubert and Richardson varieties. For $w \in W$, $l(w)$ denotes its length.

Lemma 5. *Let $u, v \in W^P$.*

(i) *The variety X_v^P is projective and has dimension $l(v)$. The group $\mathrm{Aut}^\circ(X_v^P)$ is affine.*
(ii) *The image of U in $\mathrm{Aut}^\circ(X_v^P)$ is a unipotent group denoted by U_v.*
(iii) *If $u\preccurlyeq v$ then the intersection $X_v^u := X_P^u \cap X_v^P$ is an irreducible closed normal subvariety (called Richardson variety) of X_v^P of dimension $l(v)-l(u)$. The intersection is empty if $u\preccurlyeq v$ does not hold.*
(iv) *If $u\preccurlyeq v$ then $\mathring{X}_P^u \cap \mathring{X}_v^P$ is a nonempty open subset contained in the smooth locus of X_v^u.*
(v) *If $u\preccurlyeq v$ and $l(v) = l(u)+1$, then the Richardson variety X_v^u is isomorphic to $\mathbb{P}^1$ and $\mathring{X}_P^u \cap \mathring{X}_v^P$ is isomorphic to $\mathbb{C}^*$.*
(vi) *Assume that $u\preccurlyeq v$ and $x \in \mathring{X}_P^u \cap \mathring{X}_v^P$. Then the sequence induced by the inclusions*

$$0 \longrightarrow T_x(\mathring{X}_P^u \cap \mathring{X}_v^P) \longrightarrow T_x\mathring{X}_v^P \longrightarrow \frac{T_xG/P}{T_x\mathring{X}_P^u} \longrightarrow 0$$

is exact.

Proof. The normality of Richardson's varieties is proved in [Kum17, Proposition 6.5]). The last assertion is an easy consequence of [Kum02, Lemma 7.3.10]). The other assertions are banal (see [Kum02]). □

4.2. **Kleiman's lemma.**

Lemma 6. *Let u_1, u_2, and v in W^P such that $u_1\preccurlyeq v$ and $u_2\preccurlyeq v$. Assume that $l(v) = l(u_1) + l(u_2)$.*

For general $h \in U_v$, $X_v^{u_1} \cap hX_v^{u_2} = \mathring{X}_P^{u_1} \cap h(\mathring{X}_P^{u_2} \cap \mathring{X}_v^P)$ is finite and transverse. More precisely, for any $x \in X_v^{u_1} \cap hX_v^{u_2}$ the following map induced by inclusions

$$T_xX_v^P \longrightarrow \frac{T_xG/P}{T_xX_P^{u_1}} \oplus \frac{T_xG/P}{T_x\tilde{h}X_P^{u_2}}$$

is an isomorphism, where $\tilde{h} \in U$ *satisfies* $\tilde{h}_{|X_v^P} = h$.

Proof. We want to apply Kleiman's theorem. Decompose X_v^P in B-orbits: $X_v^P = \cup_{\sigma \preccurlyeq v} \mathring{X}_\sigma^P$.

Fix $\sigma \preccurlyeq v$ such that $\sigma \neq v$. For any $h \in U_v$, we have $X_v^{u_1} \cap hX_v^{u_2} \cap X_\sigma^P = X_\sigma^{u_1} \cap hX_\sigma^{u_2}$.

Assume first that $u_1 \preccurlyeq \sigma$ and $u_2 \preccurlyeq \sigma$. Since $l(\sigma) < l(v)$, $(\dim X_\sigma^P - \dim X_\sigma^{u_1}) + (\dim X_\sigma^P - \dim X_\sigma^{u_2}) > \dim X_\sigma^P$. Kleiman's theorem applied in the U_v-homogeneous space $\mathring{X}_\sigma^P$ shows that $X_v^{u_1} \cap hX_v^{u_2} \cap \mathring{X}_\sigma^P$ is empty for general $h \in U_v$.

Otherwise, $X_\sigma^{u_1}$ or $X_\sigma^{u_2}$ is empty.

The set of $\sigma \in W^P$ such that $\sigma \preccurlyeq v$ being finite, we can conclude that, for general $h \in U_v$, the intersection $X_v^{u_1} \cap hX_v^{u_2} \cap \partial X_v^P$ is empty. Here, $\partial X_v^P = X_v^P - \mathring{X}_v^P$.

Similarly, for general $h \in U_v$, $\partial X_P^{u_1} \cap hX_v^{u_2}$ and $X_P^{u_1} \cap h(\partial X_P^{u_2} \cap X_v^P)$ is empty. Here $\partial X_P^u = X_P^u - \mathring{X}_P^u$. Indeed, only finitely many B^--orbits in $\mathring{X}_P^{u_1}$ intersect X_v^P.

Hence, for general $h \in U_v$, we have $X_v^{u_1} \cap hX_v^{u_2} = \mathring{X}_P^{u_1} \cap h(\mathring{X}_P^{u_2} \cap \mathring{X}_v^P)$.

Now, by Kleiman's theorem in the U_v-homogeneous space $\mathring{X}_v^P$, for general $h \in U_v$, $X_v^{u_1} \cap hX_v^{u_2} = \mathring{X}_P^{u_1} \cap h(\mathring{X}_P^{u_2} \cap \mathring{X}_v^P)$ is finite and for any $x \in X_v^{u_1} \cap hX_v^{u_2}$ the map

$$T_x X_v^P \longrightarrow \frac{T_x X_v^P}{T_x X_v^{u_1}} \oplus \frac{T_x X_v^P}{T_x h X_v^{u_2}}$$

is an isomorphism. The point x belonging to $\mathring{X}_v^P \cap \mathring{X}_P^{u_1}$, Lemma 5 implies that the natural map $\frac{T_x X_v^P}{T_x X_v^{u_1}} \longrightarrow \frac{T_x G/P}{T_x X_P^{u_1}}$ is an isomorphism. Similarly, $\frac{T_{h^{-1}x} X_v^P}{T_{h^{-1}x} X_v^{u_2}} \longrightarrow \frac{T_{h^{-1}x} G/P}{T_{h^{-1}x} X_P^{u_2}}$ is an isomorphism. Since both G/P and X_v^P are $\tilde{h}$-stable, by applying $\tilde{h}$, one deduces that $\frac{T_x X_v^P}{T_x h X_v^{u_2}} \longrightarrow \frac{T_x G/P}{T_x \tilde{h} X_P^{u_2}}$ is an isomorphism. □

Lemma 7. *Let* u_1, u_2, *and* v *in* W^P *such that* $u_1 \preccurlyeq v$ *and* $u_2 \preccurlyeq v$. *Assume that* $l(v) = l(u_1) + l(u_2)$. *Let* $h \in U_v$ *satisfying Lemma 6. Then*

$$\sharp(X_v^{u_1} \cap hX_v^{u_2}) = n_{u_1 u_2}^v.$$

Proof. We first claim that it is sufficient to prove the lemma when $P = B$ is the Borel subgroup. Indeed, consider the projection $\pi : G/B \longrightarrow G/P$ and the associated morphism $\pi^* : \mathrm{H}^*(G/P, \mathbb{Z}) \longrightarrow \mathrm{H}^*(G/B, \mathbb{Z})$ in cohomology. Then, for any $u \in W^P$, $\pi^*(\epsilon_u(G/P)) = \epsilon_u(G/B)$. In particular, $n_{u_1 u_2}^v(G/B) = n_{u_1 u_2}^v(G/P)$, for any u_1, u_2 and v in W^P.

Note that, for any $u \in W^P$, $X_B^u = \pi^{-1}(X_P^u)$ and π maps bijectively $\mathring{X}_u^B$ onto $\mathring{X}_u^P$. Then, for any h in U_v, π maps $X_B^{u_1} \cap h(\mathring{X}_v^B \cap X_B^{u_2})$ bijectively onto $X_P^{u_1} \cap h(\mathring{X}_v^P \cap X_P^{u_2})$. But, if h is as in Lemma 6, $X_P^{u_1} \cap h(\mathring{X}_v^P \cap X_P^{u_2}) = X_P^{u_1} \cap h(X_v^P \cap X_P^{u_2})$. We just checked that the two sides of the equality to be proved do not change when one replaces P by B. Assume now that $P = B$.

Let $\tilde{G}$ be the Kac-Moody group completed along the negative roots (as opposed to completed along the positive roots). Let $\tilde{B}^-$ be the Borel subgroup of $\tilde{G}$. Let $\tilde{X} = \tilde{G}/B$ be the 'thick' flag variety which contains the standard KM-flag variety $X = G/B$. If G is not of finite type, $\tilde{X}$ is an infinite-dimensional non-quasi-compact scheme (cf. [Ka, §4]). For $w \in W$, denote by $\tilde{X}_B^w$ the closure of $\tilde{B}^- w \underline{o}$ in $\tilde{X}$. Observe that $\tilde{X}_B^w \cap X = X_B^w$. Let $K^0(\tilde{X})$ denote the Grothendieck group of coherent $\mathcal{O}_{\tilde{X}}$-modules (see [BK14, §3.5] for details). Similarly, define $K_0(X) := \lim_{n\to\infty} K_0(X_n)$, where $\{X_n\}_{n\geq 1}$ is the filtration of X and $K_0(X_n)$ is the Grothendieck group of coherent sheaves on the projective variety X_n. Then, $\left\{[\mathcal{O}_{X_w^B}]\right\}_{w\in W}$ is a basis of $K_0(X)$ as a $\mathbb{Z}$-module. Define a pairing

$$\langle\,,\,\rangle : K^0(\tilde{X}) \otimes K_0(X) \to \mathbb{Z},\ \langle[\mathcal{S}],[\mathcal{F}]\rangle = \sum_i (-1)^i \chi\big(X_n, \mathcal{T}or_i^{\mathcal{O}_{\tilde{X}}}(\mathcal{S},\mathcal{F})\big),$$

if $\mathcal{S}$ is a coherent sheaf on $\tilde{X}$ and $\mathcal{F}$ is a coherent sheaf on X supported in X_n (for some n), where χ denotes the Euler-Poincaré characteristic. Recall that $\varpi_{\alpha_0},\dots,\varpi_{\alpha_l}$ are characters of T dual of the coroots $\alpha_0^\vee,\dots,\alpha_l^\vee$. Set $\rho = \sum_{i=0}^l \varpi_{\alpha_i}$. Define the sheaf ξ^u on $\tilde{X}$ (see [Kum17, Theorem 10.4]) by

$$\xi^u = \mathcal{L}(-\rho)\mathcal{E}xt^{\ell(u)}_{\mathcal{O}_{\tilde{X}}}(\mathcal{O}_{X_B^u}, \mathcal{O}_{\tilde{X}}) = \mathcal{O}_{\tilde{X}_B^u}(-\partial \tilde{X}_B^u),$$

where $\partial \tilde{X}_B^u = \tilde{X}_B^u - \tilde{B}^- uB/B$. Then, as proved in [Kum17, Proposition 3.5], for any $u, w \in W$,

$$\langle[\xi^u], [\mathcal{O}_{X_w^B}]\rangle = \delta_u^w.$$

Let $\Delta : X \to X \times X$ be the diagonal map. Then, by [Kum17, Proposition 4.1] and [BK14, §3.5], for any $g_1,\, g_2 \in G$

$$n_{u_1\,u_2}^v = \sum_i (-1)^i \chi(\tilde{X}\times\tilde{X}, \mathcal{T}or_i^{\mathcal{O}_{\tilde{X}\times\tilde{X}}}\Big(\xi^{u_1}\boxtimes\xi^{u_2}, (g_1^{-1}, g_2^{-1})\cdot \Delta_*(\mathcal{O}_{X_v^B})\Big)). \tag{6}$$

Let $\tilde{h}$ in U such that $\tilde{h}_{|X_v^P} = h$.

The support of $\mathcal{T}or_i^{\mathcal{O}_{\tilde{X}\times\tilde{X}}}\Big(\xi^{u_1}\boxtimes\xi^{u_2}, (1,\tilde{h}^{-1})\cdot\Delta_*(\mathcal{O}_{X_v^B})\Big)$ is contained in $(\tilde{X}_B^{u_1}\times \tilde{X}_B^{u_2})\cap(1,\tilde{h}^{-1})\Delta(X_v^B)$. The assumptions on $\tilde{h}$ implies that this support is contained in $(\mathring{X}_B^{u_1}\cap\mathring{X}_v^B)\times(\mathring{X}_B^{u_2}\cap\mathring{X}_v^B)$. In particular, this $\mathcal{T}or$-sheaf is equal to

$$\mathcal{T}or_i^{\mathcal{O}_{\tilde{X}\times\tilde{X}}}\Big(\mathcal{O}_{X_B^{u_1}}\boxtimes\mathcal{O}_{X_B^{u_2}}, (1,h^{-1})\cdot\Delta_*(\mathcal{O}_{X_v^B})\Big). \tag{7}$$

The support of the $\mathcal{T}or$-sheave in formula (7) is contained in $u_1\tilde{B}^-\underline{o}\times u_2\tilde{B}^-\underline{o}$. By [KS09, Section 8], there exists an isomorphism $\iota\,:\, u_1\tilde{B}^-\underline{o}\times u_2\tilde{B}^-\underline{o}\longrightarrow \mathbb{A}^\infty = \mathbb{C}^{\mathbb{N}}$ such that $\tilde{B}^-u_1\underline{o}\times\tilde{B}^-u_2\underline{o}$ maps onto $\mathbb{C}^{\mathbb{N}_{\geq l(u_1)+l(u_2)}}$. Here $\mathbb{C}^{\mathbb{N}}$ denote the set of $\mathbb{C}$-valued sequences viewed as $\mathrm{Spec}(\mathbb{C}[T_0,\dots,T_n,\dots])$ and $\mathbb{C}^{\mathbb{N}_{\geq l(u_1)+l(u_2)}}$ is the set of sequences starting with $l(u_1)+l(u_2)$ zeros. We also set $\mathbb{C}^{\mathbb{N}_{\leq m}} := \{(u_k)\in\mathbb{C}^{\mathbb{N}} : u_k = 0\ \forall k>m\}$. Then, there exists $m\geq l(u_1)+l(u_2)$ such that $(u_1\tilde{B}^-\underline{o}\times u_2\tilde{B}^-\underline{o})\cap(X_v^B\times X_v^B)$ is contained in $\mathbb{C}^{\mathbb{N}_{\leq m}}$. Now, for any $i\geq 0$,

$$\mathcal{T}or_i^{\mathcal{O}_{\tilde{X}\times\tilde{X}}}\Big(\mathcal{O}_{\mathring{\tilde{X}}^B_{u_1}}\boxtimes\mathcal{O}_{\mathring{\tilde{X}}^B_{u_2}}, (1,h^{-1})\cdot\Delta_*(\mathcal{O}_{X_v^B})\Big)$$

is the pullback of

$$\mathcal{T}or_i^{\mathcal{O}_{\mathbb{C}^{\mathbb{N}_{\leq m}}}}\Big(\mathcal{O}_{\mathbb{C}^{\mathbb{N}_{\leq m}}\cap\mathbb{C}^{\mathbb{N}_{\geq l(u_1)+l(u_2)}}}, ((1,h^{-1})\cdot(\Delta_*(\mathcal{O}_{X_v^B}))_{|\mathbb{C}^{\mathbb{N}_{\leq m}}}\Big).$$

The intersection $\iota((1,h^{-1})\Delta(X_v^B)\cap(u_1\tilde{B}^-\underline{o}\times u_2\tilde{B}^-\underline{o}))\cap(\mathbb{C}^{\mathbb{N}_{\leq m}}\cap\mathbb{C}^{\mathbb{N}_{\geq l(u_1)+l(u_2)}})$ being transverse in $\mathbb{C}^{\mathbb{N}_{\leq m}}$, these $\mathcal{T}or$-sheaves vanish for $i\geq 1$ and $n_{u_1\,u_2}^v$ is the cardinality of this intersection. The lemma is proved. □

4.3. **The case $n^v_{u_1\,u_2}=1$.**

Lemma 8. *We keep notation and assumptions of Lemma 6. Moreover we assume that $n^v_{u_1,\,u_2}=1$.*

Then there exist a nonempty open subset Ω of U_v and a regular map
$$\psi\,:\,\Omega\longrightarrow \mathring{X}^P_v$$
such that
$$\forall h\in\Omega \qquad X^{u_1}_v\cap hX^{u_2}_v=\{\psi(h)\},$$
and
$$\psi(h)\in \mathring{X}^{u_1}_P\cap h(\mathring{X}^{u_2}_P\cap\mathring{X}^P_v).$$

Proof. Consider
$$\aleph=\{(x,h)\in\mathring{X}^P_v\times U_v\,:\,x\in X^{u_1}_P\cap hX^{u_2}_v\},$$
endowed with the two projections $p_X\,:\,\aleph\longrightarrow X^{u_1}_P\cap\mathring{X}^P_v$ and $p_U\,:\,\aleph\longrightarrow U_v$.

We first prove that $\aleph$ is irreducible. Set $Y_i=X^{u_i}_P\cap\mathring{X}^P_v$, for $i=1,2$ and consider
$$\tilde\aleph=\{(x,h_1,h_2)\in\mathring{X}^P_v\times(U_v)^2\,:\,x\in h_1X^{u_1}_P\cap h_2X^{u_2}_v\}.$$
Let $\kappa\,:\,U_v\longrightarrow\mathring{X}^P_v$, $g\longmapsto gv\underline{o}$ denote the orbit map. Set $\tilde Y_i=\kappa^{-1}(Y_i)$. As a unipotent group, the isotropy of $v\underline{o}$ in U_v is connected. Hence $\tilde Y_i$ is irreducible. Consider the regular map
$$\begin{array}{cccc}\theta\,: & U_v\times\tilde Y_1\times\tilde Y_2 & \longrightarrow & \tilde\aleph\\ & (g,\tilde h_1,\tilde h_2) & \longmapsto & (gv\underline{o},g\tilde h_1^{-1},g\tilde h_2^{-1}).\end{array}$$
One can easily check that θ is well defined and surjective. Thus $\tilde\aleph$ is irreducible, since $U_v\times\tilde Y_1\times\tilde Y_2$ is.

Observe now that $\tilde\aleph$ is stable by the diagonal action of U_v and that $\aleph$ identifies with $\tilde\aleph\cap(U_v\times\{e\}\times U_v)$. It follows that $\tilde\aleph$ is isomorphic to $U_v\times\aleph$. In particular $\aleph$ is irreducible.

By Lemmas 6 and 7, the general fiber of p_U is a singleton. Over the complex numbers, this implies that p_U is birational. Then, a partial converse map ψ of p_U satisfies the lemma (at least, its restriction to an open subset of $h\in U_v$ satisfying Lemma 6). ☐

5. Inequalities for $\Gamma(\mathfrak{g})$

In this section, we reprove [BK14, Theorem 1.1] by similar methods in the goal to introduce some useful notation. Fix once for all, a family of fundamental coweights $\varpi_{\alpha_0^\vee},\dots,\varpi_{\alpha_l^\vee}$ in $\mathfrak{h}_\mathbb{Q}$ such that
$$\langle\varpi_{\alpha_i^\vee},\alpha_j\rangle=\delta_i^j,$$
for any $i,j\in\{0,\dots,l\}$. Similarly fix fundamental weights $\varpi_{\alpha_0},\dots,\varpi_{\alpha_l}$ in $\mathfrak{h}^*$.

Let $\tau\,:\,\mathbb{C}^*\longrightarrow G$ be a morphism of group-ind-varieties. Let $\mathcal{L}$ be a G-linearized line bundle on $\mathbb{X}$ and $x\in\mathbb{X}$. Since $\mathbb{X}$ is ind-projective and the action of $\mathbb{C}^*$ on $\mathbb{X}$ is given by a morphism of ind-varieties, $\lim_{t\to0}\tau(t)x$ exists (i.e. the morphism $\mathbb{C}^*\longrightarrow\mathbb{X}$, $t\longmapsto\tau(t)x$ extends to $\mathbb{C}$). Set $z=\lim_{t\to0}\tau(t)x$. The point z is fixed by $\tau(\mathbb{C}^*)$ and $\tau(\mathbb{C}^*)$ acts linearly on the fiber $\mathcal{L}_z$. There exists $m\in\mathbb{Z}$ such that
$$\forall t\in\mathbb{C}^*\qquad\forall\tilde z\in\mathcal{L}_z\qquad\tau(t)\tilde z=t^m\tilde z.$$

Set $\mu^{\mathcal{L}}(x,\tau)=-m$.

Proposition 1. *(see* [BK14, Theorem 1.1]*) Let P be a standard parabolic subgroup of G. Let α_i be a simple root that does not belong to $\Delta(P)$. Let u_1, u_2, and v in W^P such that $n_{u_1,u_2}^v\neq 0$ in $\mathrm{H}^*(G/P,\mathbb{Z})$.*

If $(\lambda_1,\lambda_2,\mu)\in\Gamma(\mathfrak{g})$ then

$$\langle \mu, v\varpi_{\alpha_i^\vee}\rangle \leq \langle\lambda_1, u_1\varpi_{\alpha_i^\vee}\rangle+\langle\lambda_2, u_2\varpi_{\alpha_i^\vee}\rangle. \tag{8}$$

Proof. Consider $C^+=Pu_1^{-1}\underline{o}^-\times Pu_2^{-1}\underline{o}^-\times Pv^{-1}\underline{o}$. As a locally closed subset of $\mathbb{X}$, it is an ind-variety. Set

$$G\times_P C^+ := \{(gP/P,x)\in G/P\times\mathbb{X}\,:\, g^{-1}x\in C^+\}.$$

As a locally closed subset of $G/P\times\mathbb{X}$, it is an ind-variety. Consider the maps

$$\begin{array}{cccc}\eta: & G\times_P C^+ & \longrightarrow & \mathbb{X}\\ & (gP/P,x) & \longmapsto & x\end{array}$$

and

$$\begin{array}{cccc}p: & G\times_P C^+ & \longrightarrow & G/P\\ & (gP/P,x) & \longmapsto & gP/P.\end{array}$$

Lemma 9. *Let g_1, g_2, and g_3 in G. Then*

$$p(\eta^{-1}(g_1\underline{o}^-,g_2\underline{o}^-,g_3\underline{o}))=g_1\mathring{X}_P^{u_1}\cap g_2\mathring{X}_P^{u_2}\cap g_3\mathring{X}_v^P.$$

Proof. The point $(gP/P,(g_1\underline{o}^-,g_2\underline{o}^-,g_3\underline{o}))$ belongs to the fiber of η if and only if

$$\begin{array}{ll} & (g^{-1}g_1\underline{o}^-,g^{-1}g_2\underline{o}^-,g^{-1}g_3\underline{o}) \text{ belongs to}\\ & Pu_1^{-1}\underline{o}^-\times Pu_2^{-1}\underline{o}^-\times Pv^{-1}\underline{o}\\ \Longleftrightarrow & (g^{-1}g_1,g^{-1}g_2,g^{-1}g_3)\in Pu_1^{-1}B^-\times Pu_2^{-1}B^-\times Pv^{-1}B\\ \Longleftrightarrow & g_1^{-1}g\in B^-u_1P,\ g_2^{-1}g\in B^-u_2P \text{ and } g_3^{-1}g\in BvP\\ \Longleftrightarrow & gP/P\in g_1B^-u_1P/P\cap g_2B^-u_2P/P\cap g_3BvP/P.\end{array}$$

□

Consider the morphism of ind-varieties $r: U\longrightarrow U_v\subset \mathrm{Aut}^\circ(X_v^P)$ given by the action. Set $G\mathring{/}B^-=U\underline{o}^-$. The map $U\longrightarrow G/B$, $u\longmapsto u\underline{o}^-$ being an open immersion, the converse map $p^+: G\mathring{/}B^-\longrightarrow U$ is a morphism satisfying $p^+(x)\underline{o}^-=x$. Similarly, define $G\mathring{/}B=U^-\underline{o}$ and $p^-: G\mathring{/}B\longrightarrow U^-$.

Let $\Omega\subset U_v$ be a nonempty open subset of h's satisfying Lemma 6. Set

$$\Omega_1=\left\{\begin{array}{ll}(x_1,x_2,g_3\underline{o})\in\mathbb{X}: & g_3^{-1}x_1\in G\mathring{/}B^-, g_3^{-1}x_2\in G\mathring{/}B^- \text{ and}\\ & r(p^+(g_3^{-1}x_1)^{-1}p^+(g_3^{-1}x_2))\in\Omega\end{array}\right\}.$$

It is open and nonempty in $\mathbb{X}$. Moreover, for $(x_1,x_2,g_3\underline{o})\in\Omega_1$, Lemma 9 implies that

$$p(\eta^{-1}(x_1,x_2,g_3\underline{o}))=g_3p^+(g_3^{-1}x_1)[\mathring{X}_P^{u_1}\cap h(\mathring{X}_P^{u_2}\cap\mathring{X}_v^P)], \tag{9}$$

where $h=r(p^+(g_3^{-1}x_1)^{-1}p^+(g_3^{-1}x_2))$. By Lemma 7 this fiber is nonempty.

Let $\mathcal{L}$ be the line bundle on $\mathbb{X}$ considered in Lemma 4. Since $(\lambda_1,\lambda_2,\mu)$ belongs to $\Gamma(\mathfrak{g})$, there exists $N>1$ such that $\mathrm{H}^0(\mathbb{X},\mathcal{L}^{\otimes N})^G$ is positive-dimensional. Fix a nonzero $\sigma\in\mathrm{H}^0(\mathbb{X},\mathcal{L}^{\otimes N})^G$. The ind-variety $\mathbb{X}$ being irreducible, Ω_1 has to intersect the nonzero locus of σ: there exists $x\in\Omega_1$ such that $\sigma(x)\neq 0$. The fiber (9) being not empty, Lemma 9 shows that there exists $g \in G$ such that $(gP/P, x)$ belongs to $G \times_P C^+$. Set $y = g^{-1}x$. Since σ is G-invariant, $\sigma(y) \neq 0$.

Let τ be a one parameter subgroup of T belonging to $\oplus_{\alpha_j \notin \Delta(P)} \mathbb{Z}_{>0} \varpi_{\alpha_j^\vee}$. Consider

$$\theta : \mathbb{A}^1 \longrightarrow X,\, t \in \mathbb{C}^* \longmapsto \tau(t)y,\, 0 \longmapsto \lim_{t \to 0} \tau(t)y.$$

Then $\theta^*(\sigma)$ is a nonzero $\mathbb{C}^*$-invariant section of $\theta^*(\mathcal{L})$ on $\mathbb{A}^1$. It follows that $\mu^{\mathcal{L}}(y, \tau) \leq 0$.

Let L denote the standard Levi subgroup of P. Set $C = Lu_1^{-1}\underline{o}^- \times Lu_2^{-1}\underline{o}^- \times Lv^{-1}\underline{o}$. Since $y \in C^+$, $\theta(0)$ belongs to C. Then, a direct computation (using that $\tau(\mathbb{C}^*)$ is central in L) shows that

$$\mu^{\mathcal{L}}(y, \tau) = -\langle \lambda_1, u_1\tau \rangle - \langle \lambda_2, u_2\tau \rangle + \langle \mu, v\tau \rangle \leq 0. \tag{10}$$

Inequality (10) being fulfilled for any sufficiently large $\tau \in \oplus_{\alpha_j \notin \Delta(P)} \mathbb{Z}_{>0} \varpi_{\alpha_j^\vee}$, the inequality of the theorem follows by continuity. □

Remark 5. We use notation of the proposition. Since $n^v_{u_1\, u_2}(G/P) = n^v_{u_1\, u_2}(G/B)$, Proposition 1 implies that inequality 8 is fulfilled for any simple root α_i, even in $\Delta(P)$.

6. Multiplicities on the boundary

We are now interested in triples $(\lambda_1, \lambda_2, \mu)$ for which inequality (8) is an equality and in the corresponding multiplicities $c^\mu_{\lambda_1\, \lambda_2}$. If moreover $n^v_{u_1\, u_2} = 1$, we prove that $c^\mu_{\lambda_1\, \lambda_2}$ is a multiplicity for the tensor product decomposition for some strict Levi subgroup of G.

Theorem 6. *We use notation of Proposition 1 and assume, in addition, that* $n^v_{u_1, u_2} = 1$. *Let L be the standard Levi subgroup of P. Let* $\tau \in \oplus_{\alpha_j \notin \Delta(P)} \mathbb{Z}_{>0} \varpi_{\alpha_j^\vee}$.

Let $(\lambda_1, \lambda_2, \mu) \in (P_+)^3$ *such that*

$$\langle \lambda_1, u_1\tau \rangle + \langle \lambda_2, u_2\tau \rangle = \langle \mu, v\tau \rangle. \tag{11}$$

Consider the line bundle

$$\mathcal{L} = \mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu)$$

on $\mathbb{X}$, *and*

$$C = Lu_1^{-1}\underline{o}^- \times Lu_2^{-1}\underline{o}^- \times Lv^{-1}\underline{o}$$

be the closed subset of $\mathbb{X}$.

Then, the restriction map induces an isomorphism

$$\mathrm{H}^0(\mathbb{X}, \mathcal{L})^G \longrightarrow \mathrm{H}^0(C, \mathcal{L}_{|C})^L. \tag{12}$$

Before proving the theorem, we state a consequence. Set $P_+(L) = \{\bar{\lambda} \in X(T) : \langle \bar{\lambda}, \alpha_i^\vee \rangle \geq 0 \quad \forall \alpha_i \in \Delta(L)\}$. For any $\bar{\lambda} \in P_+(L)$, we have an irreducible L-representation $V(\bar{\lambda})$ of highest weight $\bar{\lambda}$. Let $\bar{c}^{\bar{\mu}}_{\bar{\lambda}_1\, \bar{\lambda}_2}$ denote the multiplicities of the tensor product decomposition for the group L.

Corollary 1. *With the notation and assumptions of the theorem, set* $\bar{\lambda}_1 = u_1^{-1}\lambda_1$, $\bar{\lambda}_2 = u_2^{-1}\lambda_2$ *and* $\bar{\mu} = v^{-1}\mu$. *Then* $\bar{\lambda}_1$, $\bar{\lambda}_2$ *and* $\bar{\mu}$ *belong to* $P_+(L)$ *and*

$$c^\mu_{\lambda_1\, \lambda_2} = \bar{c}^{\bar{\mu}}_{\bar{\lambda}_1\, \bar{\lambda}_2}.$$

Proof. Since $u_i \in W^P$, $u_i^{-1}B^-u_i \cap L = B^- \cap L$. Similarly, $v^{-1}Bv \cap L = B \cap L$. Then, the action of L^3 on C allows to identify C with $\mathbb{X}_L := (L/(B^- \cap L))^2 \times L/(B \cap L)$. But T acts with weight $u_1^{-1}\lambda_1$ on the fiber in $\mathcal{L}_-(\lambda_1)$ over $u_1^{-1}\underline{o}^-$. We deduce that $\mathcal{L}_{|C}$ identifies with $\mathcal{L}_-(\bar\lambda_1) \otimes \mathcal{L}_-(\bar\lambda_2) \otimes \mathcal{L}(\bar\mu)$ on $\mathbb{X}_L$. Since $\mathrm{H}^0(C, \mathcal{L}_{|C})$ is nonzero, $\bar\lambda_1$, $\bar\lambda_2$ and $\bar\mu$ have to be dominant for L. Moreover, the equality of the corollary is obtained by taking dimension in the isomorphism of Theorem 6 using Lemma 4. $\square$

Note that Theorem 2 in the introduction is a particular case of Corollary 1.

Proof of Theorem 6 up to the 5 lemmas below. We use notation of Proposition 1. Let $\bar C^+$ be the closure of C^+ in $\mathbb{X}$. Set

$$G \times_P \bar C^+ := \{(gP/P, x) \in G/P \times \mathbb{X} \,:\, g^{-1}x \in \bar C^+\}.$$

As a closed subset of $G/P \times \mathbb{X}$, it is an ind-variety. Consider the maps

$$\begin{array}{rccc} \bar\eta: & G \times_P \bar C^+ & \longrightarrow & \mathbb{X} \\ & (gP/P, x) & \longmapsto & x \end{array}$$

and

$$\begin{array}{rccc} \bar p: & G \times_P \bar C^+ & \longrightarrow & G/P \\ & (gP/P, x) & \longmapsto & gP/P. \end{array}$$

Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{X}, \mathcal{L})^G \arrow[rr] \arrow[d, "11"', "\bar\eta^*"] & & \mathrm{H}^0(C, \mathcal{L}_{|C})^L \\
\mathrm{H}^0(G \times_P \bar C^+, \bar\eta^*(\mathcal{L}))^G \arrow[r, "12", "\text{rest.}"'] & \mathrm{H}^0(\bar C^+, \mathcal{L}_{|\bar C^+})^P \arrow[r, "13", "\text{rest.}"'] & \mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^P \arrow[u, "\text{rest.}", "14"']
\end{tikzcd}$$

It remains to prove that the top horizontal map is an isomorphism. But, Lemmas 11 to 14 below show that the four other morphisms are isomorphisms. $\square$

Before proving the four mentioned lemmas, we construct a partial converse map to $\bar\eta$.

Lemma 10. *There exists a nonempty open subset Ω_1 of $\mathbb{X}$ such that the restriction of $\bar\eta$ to $\bar\eta^{-1}(\Omega_1)$ is a bijection onto Ω_1 and the converse map ζ is a morphism of ind-varieties, mapping Ω_1 on $G \times_P C^+$.*

Proof. Recall that $r : U \longrightarrow U_v \subset \mathrm{Aut}^\circ(X_v^P)$ denotes the action. By Lemma 8, there exist a nonempty open subset $\Omega \subset U_v$ and a morphism $\psi : \Omega \longrightarrow \mathring{X}_v^P$ such that

$$\forall h \in U_v \qquad \{\psi(h)\} = X_P^{u_1} \cap h(X_P^{u_2} \cap X_v^P).$$

Consider

$$\Omega_2 = \left\{ \begin{array}{c} (x_1, x_2, g_3\underline{o}) \in \mathbb{X} \,: g_3^{-1}x_1 \in \overset{\circ}{G/B^-},\ g_3^{-1}x_2 \in \overset{\circ}{G/B^-}, \text{ and} \\ g_3\underline{o} \in \overset{\circ}{G/B} \end{array} \right\}.$$

Let $(x_1, x_2, x_3) \in \Omega_2$. Write $x_1 = g_1\underline{o}^-$, $x_2 = g_2\underline{o}^-$ and $x_3 = g_3\underline{o}$, with $g_i \in G$.

Observe that

$$\begin{aligned} & g_1X_P^{u_1}\cap g_2X_P^{u_2}\cap g_3X_v^P \\ =& g_1X_P^{u_1}\cap g_2X_P^{u_2}\cap p^-(x_3)X_v^P \\ =& p^-(x_3)\Big((p^-(x_3)^{-1}g_1)X_P^{u_1}\cap(p^-(x_3)^{-1}g_2)X_P^{u_2}\cap X_v^P\Big) \\ =& p^-(x_3)\Big(h_1X_P^{u_1}\cap h_2X_P^{u_2}\cap X_v^P\Big) \\ =& (p^-(x_3)h_1).\left[X_P^{u_1}\cap hX_P^{u_2}\cap X_v^P\right], \end{aligned}$$

where

$$\begin{aligned} h_1 &= p^+(p^-(x_3)^{-1}x_1) \\ h_2 &= p^+(p^-(x_3)^{-1}x_2) \\ h &= h_1^{-1}h_2. \end{aligned}$$

But h belongs to U and

$$g_1X_P^{u_1}\cap g_2X_P^{u_2}\cap g_3X_v^P=(p^-(x_3)h_1).\left[X_P^{u_1}\cap r(h)(X_P^{u_2}\cap X_v^P)\right].$$

Let Ω_1 be the set of $(x_1,x_2,x_3)\in\Omega_2$ such that $r(h)\in\Omega$. It is a nonempty open subset of $\mathbb{X}$. Lemma 8 implies that, for $(x_1,x_2,x_3)\in\Omega_1$,

$$g_1X_P^{u_1}\cap g_2X_P^{u_2}\cap g_3X_v^P=(p^-(x_3)h_1).\{\psi\circ r(h)\}, \tag{13}$$

and

$$(p^-(x_3)h_1).\psi\circ r(h)\in g_1\mathring{X}_P^{u_1}\cap g_2\mathring{X}_P^{u_2}\cap g_3\mathring{X}_v^P.$$

Then the formula

$$(x_1,x_2,x_3)\in\Omega_1\longmapsto((p^-(x_3)h_1).\psi\circ r(h),(x_1,x_2,x_3))$$

defines a morphism ζ from Ω_1 to $G\times_P C^+$ such that $\eta\circ\zeta$ is the identity map of Ω_1.

Finally, Lemma 9 with $\bar\eta$ in place of η and formula (13) show that the fiber of $\bar\eta$ over any point of Ω_1 is a singleton. □

We now go from $\mathbb{X}$ to $G\times_P\bar C^+$.

Lemma 11. *The linear map*

$$\bar\eta^*\,:\,\mathrm{H}^0(\mathbb{X},\mathcal{L})\longrightarrow\mathrm{H}^0(G\times_P\bar C^+,\bar\eta^*(\mathcal{L}))$$

is a G-equivariant isomorphism.

Proof. The image of $\bar\eta$ containing the dense subset Ω_1 of $\mathbb{X}$, $\bar\eta^*$ is injective. Fix a filtration $\mathbb{X}=\cup_{n\in\mathbb{Z}_{\geq0}}\mathbb{X}_n$ such that each $\mathbb{X}_n$ is a product of three finite-dimensional Schubert varieties (i.e. $(B^-\times B^-\times B)$-orbit closures), $\mathbb{X}_n$ intersects Ω_1 and $\mathbb{X}_n\subset\mathbb{X}_{n+1}$. Set $\mathbb{Y}=G\times_P\bar C^+$ and

$$\mathring{\mathbb{Y}}_n:=\bar\eta^{-1}(\mathbb{X}_n\cap\Omega_1)=\zeta(\mathbb{X}_n\cap\Omega_1).$$

Let $\mathbb{Y}_n$ be the closure of $\mathring{\mathbb{Y}}_n$ in $\mathbb{Y}$. Then $\mathbb{Y}_n$ is closed, irreducible, finite-dimensional and projective.

A key point is that the $\mathbb{Y}_n$'s form a filtration of $\mathbb{Y}$. Indeed, as the image of $G\times\bar C^+\longrightarrow G\times_P\bar C^+$, $(g,x)\longmapsto(gP/P,gx)$, $\mathbb{Y}$ is ind-irreducible (see examples of Section 2.2). Moreover, $\cup_{n\in\mathbb{Z}_{\geq0}}\mathring{\mathbb{Y}}_n$ and hence $\cup_{n\in\mathbb{Z}_{\geq0}}\mathbb{Y}_n$ contains the nonempty open subset $\bar\eta^{-1}(\Omega_1)$. Then, Lemma 3 implies that $\mathbb{Y}=\cup_{n\in\mathbb{Z}_{\geq0}}\mathbb{Y}_n$ is a filtration of $\mathbb{Y}$.

We now prove the surjectivity of $\bar\eta^*$. Let $\sigma \in \mathrm{H}^0(G \times_P \bar C^+, \bar\eta^*(\mathcal{L}))$. Consider the restriction $\bar\eta_n : \mathbb{Y}_n \longrightarrow \mathbb{X}_n$ of $\bar\eta$. Then $\bar\eta_n$ is proper, birational and $\mathbb{X}_n$ is normal. Zariski's main theorem implies that the fibers of $\bar\eta_n$ are connected. Moreover

$$\bar\eta_n^* : \mathrm{H}^0(\mathbb{X}_n, \mathcal{L}_{|\mathbb{X}_n}) \longrightarrow \mathrm{H}^0(\mathbb{Y}_n, \bar\eta^*(\mathcal{L})_{|\mathbb{Y}_n})$$

is an isomorphism (see e.g. [Pet04, Chap IV, Corollary 5]).

Let $\tilde\sigma_n \in \mathrm{H}^0(\mathbb{X}_n, \mathcal{L}_{|\mathbb{X}_n})$ such that $\bar\eta_n^*(\tilde\sigma_n) = \sigma_{|\mathbb{Y}_n}$. Since $\Omega_1 \cap \mathbb{X}_n$ is dense in $\mathbb{X}_n$, the restriction of $\tilde\sigma_{n+1}$ to $\mathbb{X}_n$ is equal to $\tilde\sigma_n$. Hence $(\tilde\sigma_n)_{n\in\mathbb{N}}$ is a global section $\tilde\sigma$ of $\mathcal{L}$ on $\mathbb{X}$. Moreover, $\bar\eta^*(\tilde\sigma) = \sigma$. □

We now go from $G \times_P \bar C^+$ to $\bar C^+$.

Lemma 12. *The linear map*

$$\mathrm{H}^0(G \times_P \bar C^+, \bar\eta^*(\mathcal{L}))^G \longrightarrow \mathrm{H}^0(\bar C^+, \mathcal{L}_{|\bar C^+})^P$$

is an isomorphism.

Proof. Embed $\bar C^+$ in $G \times_P \bar C^+$ by mapping $x \in \bar C^+$ to $(P/P, x)$. Since $\bar\eta^*(\mathcal{L})_{|\bar C^+} = \mathcal{L}_{|\bar C^+}$, the map of the lemma is well defined.

Since $\bar C^+$ intersects any G-orbit in $G \times_P \bar C^+$, this map is injective. Let $\sigma \in \mathrm{H}^0(\bar C^+, \mathcal{L}_{|\bar C^+})^P$. Set

$$\begin{array}{cccc} \tilde\sigma : & G \times_P \bar C^+ & \longrightarrow & \bar\eta^*(\mathcal{L}) \\ & (gP/P, x) & \longmapsto & g\sigma(g^{-1}x). \end{array}$$

Note that $\tilde\sigma$ is well defined as a map and G-invariant, since $\bar\eta^*(\mathcal{L})$ is G-linearized and σ is P-invariant. It is regular, since the morphism $G \longrightarrow G/P$ is locally trivial in Zariski topology, because of Birkhoff's decomposition. Hence $\tilde\sigma \in \mathrm{H}^0(G \times_P \bar C^+, \bar\eta^*(\mathcal{L}))^G$ and $\tilde\sigma_{|\bar C^+} = \sigma$. □

We now go from $\bar C^+$ to C^+. Let $\tilde\tau : \mathbb{C}^* \longrightarrow T$ such that $\tilde\tau \in \mathbb{Z}_{>0}\tau$.

Lemma 13. *Recall that $\mu^{\mathcal{L}}(C, \tau) = 0$. Then the restriction map*

$$\mathrm{H}^0(\bar C^+, \mathcal{L}_{|\bar C^+})^{\tilde\tau(\mathbb{C}^*)} \longrightarrow \mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^{\tilde\tau(\mathbb{C}^*)}$$

is an isomorphism.

Proof. Since C^+ is dense in $\bar C^+$, the restriction is injective. For $w \in W$, we recall that

$$X_w^B = \overline{Bw\underline{o}} \qquad X_B^w = \overline{B^-w\underline{o}},$$

and set

$$X_w^{B^-} = \overline{B^-w\underline{o}^-} \qquad X_{B^-}^w = \overline{Bw\underline{o}^-}$$

in such a way $\dim(X_w^B) = \dim(X_w^{B^-}) = l(w)$ and $\mathrm{codim}(X_B^w) = \mathrm{codim}(X_{B^-}^w) = l(w)$. Note also that $\overline{Pu_1^{-1}\underline{o}^-} = X_{B^-}^{u_1^{-1}}$ and $\overline{Pu_2^{-1}\underline{o}^-} = X_{B^-}^{u_2^{-1}}$, whereas $X_{v^{-1}}^B$ is finite-dimensional and closed in $\overline{Pv^{-1}\underline{o}}$.

For $i = 1, 2$, fix an increasing (for Bruhat order) sequence $(w_i^n)_{n\in\mathbb{N}}$ of elements of W^P such that $u_i^{-1} \preccurlyeq w_i^n$ for any n and $\left(X_{w_i^n}^{B^-} \cap X_{B^-}^{u_i^{-1}}\right)_{n\in\mathbb{N}}$ is a filtration of the ind-variety $X_{B^-}^{u_i^{-1}}$. Similarly, fix $(w_3^n)_{n\in\mathbb{N}}$ such that $v^{-1} \preccurlyeq w_3^n \preccurlyeq w_3^{n+1}$ and $\left(X_{w_3^n}^B\right)_{n\in\mathbb{N}}$

is a filtration of the ind-variety $\overline{Pv^{-1}\underline{o}}$. Note that if P has finite type, the sequence $(w_3^n)_{n\in\mathbb{N}}$ can be chosen to be constant. Setting

$$\bar{C}_n^+ := (X_{w_1^n}^{B^-} \cap X_{B^-}^{u_1^{-1}}) \times (X_{w_2^n}^{B^-} \cap X_{B^-}^{u_2^{-1}}) \times X_{w_3^n}^{B},$$

we get a filtration of $\bar{C}^+$. Moreover, $C^+ \cap \bar{C}_n^+$ is open in $\bar{C}_n^+$ and nonempty for any n. Let $\sigma \in \mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^{\tilde{\tau}}$ and let σ_n denote its restriction to $C^+ \cap \bar{C}_n^+$. We have to prove that σ extends to a section $\bar{\sigma}$ on $\bar{C}^+$. It remains to prove that each σ_n extends to a section $\bar{\sigma}_n$ on $\bar{C}_n^+$. Indeed, then $\bar{\sigma} = (\bar{\sigma}_n)_{n\in\mathbb{N}} \in \mathrm{H}^0(\bar{C}^+, \mathcal{L}_{|\bar{C}^+})^{\tilde{\tau}}$ extends σ.

Fix $n \in \mathbb{N}$. By Lemma 5, $\bar{C}_n^+$ is normal. Then, to prove that σ_n extends to $\bar{C}_n^+$, it is sufficient to prove that it has no pole along the divisors of $\bar{C}_n^+ - C^+$. Let D_n be such a divisor. Then, either

(a) $D_n = (X_{w_1^n}^{B^-} \cap X_{B^-}^{\tilde{u}_1^{-1}}) \times (X_{w_2^n}^{B^-} \cap X_{B^-}^{u_2^{-1}}) \times X_{w_3^n}^{B}$, for some $\tilde{u}_1 \in W^P$ such that $u_1^{-1} \preccurlyeq \tilde{u}_1^{-1} \preccurlyeq w_1^n$ and $l(\tilde{u}_1) = l(u_1) + 1$; or

(a') $D_n = (X_{w_1^n}^{B^-} \cap X_{B^-}^{u_1^{-1}}) \times (X_{w_2^n}^{B^-} \cap X_{B^-}^{\tilde{u}_2^{-1}}) \times X_{w_3^n}^{B}$, for some $\tilde{u}_2 \in W^P$ such that $u_2^{-1} \preccurlyeq \tilde{u}_2^{-1} \preccurlyeq w_2^n$ and $l(\tilde{u}_2) = l(u_2) + 1$; or

(b) $D_n = (X_{w_1^n}^{B^-} \cap X_{B^-}^{u_1^{-1}}) \times (X_{w_2^n}^{B^-} \cap X_{B^-}^{u_2^{-1}}) \times X_{\tilde{w}_3}^{B}$ for some $\tilde{w}_3 \in W$ such that $\tilde{w}_3 \preccurlyeq w_3^n$, $l(\tilde{w}_3) = l(w_3^n) - 1$ and $X_{\tilde{w}_3} \subset X_{w_3^n}^{B^-} - Pv^{-1}\underline{o}$.

In each case, we will apply Lemma 11.5 below in an affine neighborhood of D_n in $\bar{C}_n^+$. We first construct such neighborhoods and check the assumptions of Lemma 11.5. We do not consider Case (a') that is similar to the first one. We also skip the power n from w_i^n. Note that the action of $\mathbb{C}^*$ in Lemma 11.5 is given by $\tilde{\tau}$.

Case (a). Set

$$X = (X_{w_1}^{B^-} \cap X_{B^-}^{u_1^{-1}} \cap \tilde{u}_1^{-1} B\underline{o}^-) \times (X_{w_2}^{B^-} \cap \mathring{X}_{B^-}^{u_2^{-1}}) \times \mathring{X}_{w_3}^{B},$$

$D = D_n \cap X$ and $\Omega = X - D$.

The fact that X is T-stable is obvious. Since $\tilde{u}_1^{-1} B\underline{o}^-$, $\mathring{X}_{B^-}^{u_2^{-1}}$ and $\mathring{X}_{w_3}^{B}$ are open in G/B^-, $X_{B^-}^{u_2^{-1}}$ and $X_{w_3}^{B}$ respectively, X is open in $\bar{C}_n^+$. By [Kum02, Lemma 7.3.5], $X_{w_1}^{B^-} \cap \tilde{u}_1^{-1} B\underline{o}^-$ is affine. Then the first factor of X is affine. But, [Kum02, Lemma 7.3.5] also implies that the two other factors are affine, and X is affine.

Moreover, by Lemma 5, X is normal.

Check Assumption (i) *of Lemma 11.5.* It is sufficient to prove it for the first factor. Let $x \in (X_{w_1}^{B^-} \cap X_{B^-}^{u_1^{-1}} \cap \tilde{u}_1^{-1} B\underline{o}^-) - X_{B^-}^{\tilde{u}_1^{-1}}$. Set $y = \lim_{t\to 0} \tilde{\tau}(t)x$. We have to prove that $y \notin (X_{w_1}^{B^-} \cap X_{B^-}^{u_1^{-1}} \cap \tilde{u}_1^{-1} B\underline{o}^-)$.

Let $w \in W$ such that $x \in \mathring{X}_{B^-}^{w}$. Since $x \in X_{B^-}^{u_1^{-1}}$, $u_1^{-1} \preccurlyeq w$. The point x belonging to $\tilde{u}_1^{-1} B\underline{o}^-$, $\tilde{u}_1^{-1}\underline{o}^-$ belongs to $X_{B^-}^{w}$ and $w \preccurlyeq \tilde{u}_1^{-1}$. But $l(\tilde{u}_1) = l(u_1) + 1$, hence $w = \tilde{u}_1^{-1}$ or u_1^{-1}. Now, $x \notin X_{B^-}^{\tilde{u}_1^{-1}}$ and $w = u_1^{-1}$.

In particular, Lemma 11.1 implies that y does not belong to $\tilde{u}_1^{-1} B\underline{o}^-$. The claim is proved.

Check Assumption (ii). We work successively on each factor of X.

Let $x \in \mathring{X}^B_{w_3}$ such that $\lim_{t\to\infty} \tilde\tau(t)x \in \mathring{X}^B_{w_3}$. For any $b \in B$, $\lim_{t\to\infty} \tilde\tau(t)b\tilde\tau(t^{-1})$ exists in B. Then $\lim_{t\to 0} \tilde\tau(t)x \in \mathring{X}^B_{w_3}$. If x is not fixed by $\tilde\tau(\mathbb{C}^*)$, $\overline{\tilde\tau(\mathbb{C}^*)x}$ is isomorphic to $\mathbb{P}^1$. Hence it cannot be contained in the affine variety $\mathring{X}^B_{w_3}$. Hence $y \notin \mathring{X}^B_{w_3}$. Contradiction. It follows that x is fixed by $\tilde\tau(\mathbb{C}^*)$.

Similarly for $x \in X^{B^-}_{w_2} \cap \mathring{X}^{u_2^{-1}}_{B^-}$, if $\lim_{t\to\infty} \tilde\tau(t)x \in \mathring{X}^{u_2^{-1}}_{B^-}$ then x is fixed by $\tilde\tau(\mathbb{C}^*)$.

Now, prove Assumption (ii) for the first factor of X. Let $x_1,\, x_2 \in (X^{B^-}_{w_1} \cap X^{u_1^{-1}}_{B^-} \cap \tilde u_1^{-1}B\underline{o}^-) - X^{\tilde u_1^{-1}}_{B^-}$ such that $y := \lim_{t\to\infty} \tilde\tau(t)x_1 = \lim_{t\to\infty}\tilde\tau(t)x_2$ belongs to the first factor of D.

We already noticed that x_1 and x_2 belong to $\mathring{X}^{u_1^{-1}}_{B^-}$. By assumption, $y \in \tilde u_1^{-1}B\underline{o}^- \cap X^{\tilde u_1^{-1}}_{B^-} = \mathring{X}^{\tilde u_1^{-1}}_{B^-}$. Then Lemma 11.2 below shows that $\tilde\tau(\mathbb{C}^*)x_1 = \tilde\tau(\mathbb{C}^*)x_2$.

Check Assumption (iii). This can be proved factor by factor.

Let $x \in \mathring{X}^B_{w_3}$ (resp. $X^{B^-}_{w_2} \cap \mathring{X}^{u_2^{-1}}_{B^-}$). We already observed that $\lim_{t\to 0}\tilde\tau(t)x$ belongs to $\mathring{X}^B_{w_3}$ (resp. $X^{B^-}_{w_2} \cap \mathring{X}^{u_2^{-1}}_{B^-}$).

Let now $x \in (X^{B^-}_{w_1} \cap X^{\tilde u_1^{-1}}_{B^-} \cap \tilde u_1^{-1}B\underline{o}^-) = (X^{B^-}_{w_1} \cap \mathring{X}^{\tilde u_1^{-1}}_{B^-})$. Hence, we are in the situation of the second factor.

Finally, for any $x \in D$, $\lim_{t\to 0}\tilde\tau(t)x$ belongs to D.

Check Assumption (iv). It is sufficient to consider the first factor. Let $y \in (X^{B^-}_{w_1} \cap X^{\tilde u_1^{-1}}_{B^-} \cap \tilde u_1^{-1}B\underline{o}^-)^{\tilde\tau} = (X^{B^-}_{w_1} \cap \mathring{X}^{\tilde u_1^{-1}}_{B^-})^{\tilde\tau}$. We have to find $x \in (X^{B^-}_{w_1} \cap X^{u_1^{-1}}_{B^-} \cap \tilde u_1^{-1}B\underline{o}^-) - X^{\tilde u_1^{-1}}_{B^-}$ such that $\lim_{t\to\infty}\tilde\tau(t)x = y$.

Assume first that $y = y_0 = \tilde u_1^{-1}\underline{o}^-$. Here $l(\tilde u_1) = l(u_1)+1$ and $X^{B^-}_{\tilde u_1^{-1}} \cap X^{u_1^{-1}}_{B^-}$ is a Richardson variety of dimension one. By Lemma 5, $\mathring{X}^{B^-}_{\tilde u_1^{-1}} \cap \mathring{X}^{u_1^{-1}}_{B^-}$ is dense in $X^{u_1^{-1}}_{\tilde u_1^{-1}}$. Let $x_0 \in \mathring{X}^{B^-}_{\tilde u_1^{-1}} \cap \mathring{X}^{u_1^{-1}}_{B^-}$.

Let $P^{u,-}$ denote the unipotent radical of P in such a way $B^- = P^{u,-}(B^-\cap L)$. But $\tilde u_1 \in W^P$ and $\tilde u_1^{-1}B^-\tilde u_1 \cap L = B^- \cap L$. Hence

$$\mathring{X}^{B^-}_{\tilde u_1^{-1}} = B^-\tilde u_1^{-1}\underline{o}^- = P^{u,-}(B^-\cap L)\tilde u_1^{-1}\underline{o}^- = P^{u,-}\tilde u_1^{-1}\underline{o}^-.$$

In particular, $\lim_{t\to\infty}\tilde\tau(t)x_0 = y_0$.

Since $\tilde u_1^{-1}\preccurlyeq w_1$, $x_0 \in X^{B^-}_{w_1}$. Moreover, $x_0 \in \mathring{X}^{u_1^{-1}}_{B^-}$; thus $x_0 \notin X^{\tilde u_1^{-1}}_{B^-}$. And $x_0 \in \mathring{X}^{B^-}_{\tilde u_1^{-1}} \subset \tilde u^{-1}B\underline{o}^-$. Finally, $x_0 \in (X^{B^-}_{w_1} \cap X^{u_1^{-1}}_{B^-} \cap \tilde u_1^{-1}B\underline{o}^-) - X^{\tilde u_1^{-1}}_{B^-}$.

Now $y \in (\mathring{X}^{\tilde u_1^{-1}}_{B^-})^{\tilde\tau} = (B\cap L)\tilde u_1^{-1}\underline{o}^-$. Let $l \in B\cap L$ such that $y = ly_0$. Set $x = lx_0$. The group $\tau(\mathbb{C}^*)$ being central in L, $\lim_{t\to\infty}\tilde\tau(t)x = y$. Since $l \in L$ and $X^{u_1^{-1}}_{B^-}$ and $X^{u_1^{-1}}$ are L-stable, $x \in X^{u_1^{-1}}_{B^-} - X^{\tilde u_1^{-1}}_{B^-}$. But $\tilde u_1 \in W^P$ and $\tilde u_1^{-1}B\tilde u_1 \cap L = B \cap L$. Hence $x \in \tilde u_1^{-1}B\underline{o}^-$. Recall that $x_0 \in P^{u,-}y_0$. Since l normalizes $P^{u,-}$, this emplies that $x \in P^{u,-}y$. But $X^{B^-}_{w_1}$ is B^--stable, and $x \in X^{B^-}_{w_1}$. Finally, x works.

Check Assumption (v). Observe that, for any $u, v \in W$, if $X^v_{B^-} \cap uB\underline{o}^-$ is not empty then $u\preccurlyeq v$. Thus $X^{u_1^{-1}}_{B^-} \cap \tilde u_1^{-1}B\underline{o}^- = (\mathring{X}^{u_1^{-1}}_{B^-} \cup \mathring{X}^{\tilde u_1^{-1}}_{B^-}) \cap \tilde u_1^{-1}B\underline{o}^-$ and the first factor of Ω is $X_{w_1}^{B^-} \cap \mathring{X}_{B^-}^{u_1^{-1}} \cap \tilde{u}_1^{-1} B\underline{o}^-$. But $\mathring{X}_{B^-}^{u_1^{-1}} = X_{B^-}^{u_1^{-1}} \cap u_1^{-1} B\underline{o}^-$ and the first factor of Ω is

$$X_{w_1}^{B^-} \cap X_{B^-}^{u_1^{-1}} \cap u_1^{-1} B\underline{o}^- \cap \tilde{u}_1^{-1} B\underline{o}^-.$$

The subset $X_{B^-}^{u_1^{-1}}$ being closed, to show that that variety is affine, it is sufficient to prove that $X_{w_1}^{B^-} \cap u_1^{-1} B\underline{o}^- \cap \tilde{u}_1^{-1} B\underline{o}^-$ is. The proof of [Kum02, Lemma 7.3.5], with minor modifications implies this.

Case (b). Set

$$X = (X_{w_1}^{B^-} \cap \mathring{X}_{B^-}^{u_1^{-1}}) \times (X_{w_2}^{B^-} \cap \mathring{X}_{B^-}^{u_2^{-1}}) \times (X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o}),$$

$D = D_n \cap X$ and $\Omega = X - D$.

Lemma 5 and [Kum02, Lemma 7.3.5] imply that X is open in $\bar{C}^+$, T-stable, affine and normal.

Check Assumption (i). It is sufficient to prove it for the third factor. Let $x \in (X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o}) - X_B^{\tilde{w}_3}$. Set $y = \lim_{t\to 0} \tilde{\tau}(t)x$. We have to prove that $y \not\in \tilde{w}_3 B^- \underline{o}$.

Like in Case a-(i), one has $x \in \mathring{X}_{w_3}^{B}$. Hence $y \in (B \cap L).w_3\underline{o}$. But $(\tilde{w}_3 B^- \underline{o})^{\tilde{\tau}} = ((\tilde{w}_3 B^- \tilde{w}_3^{-1}) \cap L)\tilde{w}_3\underline{o}$. In particular, if $y \in \tilde{w}_3 B^- \underline{o}$ then $Pv^{-1}\underline{o} = Pw_3\underline{o} = P\tilde{w}_3\underline{o}$. Contradiction.

Check Assumption (ii). Let $x_1, x_2 \in \Omega$ such that $\lim_{t\to\infty} \tilde{\tau}(t)x_1 = \lim_{t\to\infty} \tilde{\tau}(t)x_2 \in D$. In the proof of Case a - (ii), we proved that the two first factors of each x_i are fixed by $\mathbb{C}^*$. We assume now that $x_1, x_2 \in (X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o}) - X_B^{\tilde{w}_3}$.

Since $X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o}$ is contained in $\mathring{X}_{w_3}^{B} \cup \mathring{X}_{\tilde{w}_3}^{B}$, $x_1, x_2 \in \mathring{X}_{w_3}^{B}$. Similarly $y \in \mathring{X}_{\tilde{w}_3}^{B}$. Then Lemma 11.2 below implies that $\tilde{\tau}(\mathbb{C}^*)x_1 = \tilde{\tau}(\mathbb{C}^*)x_2$.

Check Assumption (iii). We can work on each factor separately. The two first one was treated in Case a-(iii). The last one works since $X_{\tilde{w}_3}^{B} \cap \tilde{w}_3 B^- \underline{o} = \mathring{X}_{\tilde{w}_3}^{B}$.

Check Assumption (iv). We can work in the last factor. Let $y \in (\mathring{X}_{\tilde{w}_3}^{B})^{\tilde{\tau}}$.

Assume first that $y = y_0 = \tilde{w}_3\underline{o}$. Consider the Richardson variety $\mathfrak{l} = X_{w_3}^{B} \cap X_B^{\tilde{w}_3}$ of dimension one. Pick $x_0 \in \mathring{X}_{w_3}^{B} \cap \mathring{X}_B^{\tilde{w}_3}$. Then $x_0 \in X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o}$. Let γ be the character of the action of T on $T_{y_0}\mathfrak{l}$. It is a root of $\mathfrak{g}$. Since $\overline{Py_0}$ does not contain $w_3\underline{o}$, γ is not a root of P. Then γ is a root of $P^{u,-}$. This implies that $\lim_{t\to\infty} \tilde{\tau}(t)x_0 = y_0$.

Now $y \in \mathring{X}_{\tilde{w}_3}^{B}$ and there exists $l \in B \cap L$ such that $y = ly_0$. Consider the curve $l\mathfrak{l}$: it is $\tilde{\tau}(\mathbb{C}^*)$-stable, contained in $X_{w_3}^{B}$ and contains y. Since X is open in $X_{w_3}^{B}$, $\mathfrak{l} \cap X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o}$ is a nonempty open subset of $\mathfrak{l}$. Then $lx \in X$. But $\lim_{t\to\infty} \tilde{\tau}(t)x = y$.

Check Assumption (v). It is sufficient to prove that the third factor Ω_3 of Ω is affine. But

$$\Omega_3 = (X_{w_3}^{B} - X_{\tilde{w}_3}^{B}) \cap \tilde{w}_3 B^- \underline{o} = X_{w_3}^{B} \cap \tilde{w}_3 B^- \underline{o} \cap w_3 B^- \underline{o}$$

Now, the proof of [Kum02, Lemma 7.3.5] implies that Ω_3 (and hence Ω) is affine.

The line bundle on the affine subvarieties. Since $\mathcal{L}_-(\lambda_1)$ is G-linearized and the action of U on $\underline{o}^-$ is free, $\mathcal{L}_-(\lambda_1)$ is trivial as a line bundle on $U\underline{o}^-$. Similarly $\mathcal{L}(\mu)$ is trivial on $U^-\underline{o}$. As a consequence, $\mathcal{L}$ is trivial as a line bundle, on each affine variety X we have considered.

To determine $\mathcal{L}_{|X}$ as a $\mathbb{C}^*$-linearized line bundle, it is sufficient to compute the action of $\mathbb{C}^*$ on the fiber over some $\mathbb{C}^*$-fixed point. Consider

$$\begin{array}{l} x_0 = (u_1^{-1}\underline{o}^-, u_2^{-1}\underline{o}^-, w_3\underline{o}), \\ x_a = (\tilde{u}_1^{-1}\underline{o}^-, u_2^{-1}\underline{o}^-, w_3\underline{o}), \quad \text{and} \\ x_b = (u_1^{-1}\underline{o}^-, u_2^{-1}\underline{o}^-, \tilde{w}_3\underline{o}). \end{array}$$

By assumption, $\mathbb{C}^*$ acts trivially on the fiber $\mathcal{L}_{x_0}$. In Case (a), we have constructed a copy of $\mathbb{P}^1$, containing x_0 and x_a such that for $x \in \mathbb{P}^1 - \{x_0\}$, $\lim_{t\to\infty} \tilde{\tau}(t)x = x_a$. Moreover, $\mathcal{L}_{|\mathbb{P}^1}$ is nonnegative as a line bundle. Now, a computation in $\mathbb{P}^1$ shows that the action of $\mathbb{C}^*$ on $\mathcal{L}_{x_a}$ is given by a nonpositive weight k_a.

Similarly, the action of $\mathbb{C}^*$ on $\mathcal{L}_{x_b}$ is given by a nonpositive weight k_b.

Since $\mathcal{L}$ is trivial on X as a line bundle, we deduce that, for any considered affine variety X, we have

$$\mathrm{H}^0(X, \mathcal{L})^{\tilde{\tau}(\mathbb{C}^*)} \simeq \mathbb{C}[X]^{(k)}, \tag{14}$$

for some nonnegative integer k.

We are now in position to apply Lemma 11.5. By assumption the restriction $(\sigma_n)_{|\Omega}$ belongs to $\mathrm{H}^0(\Omega, \mathcal{L})^{\tilde{\tau}(\mathbb{C}^*)} \simeq \mathbb{C}[\Omega]^{(k)}$. By Lemma 11.5, $\mathbb{C}[\Omega]^{(k)} = \mathbb{C}[X]^{(k)}$. Hence $(\sigma_n)_{|\Omega}$ extends to a regular section on X. In particular, it has no pole along D_n. Then σ_n extends to a regular section on $\bar{C}_n^+$ by normality. This ends the proof of the lemma. ☐

The last step goes from C^+ to C.

Lemma 14. *Recall that $\mu^{\mathcal{L}}(C, \tilde{\tau}) = 0$. Then the restriction map*

$$\mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^{\tilde{\tau}(\mathbb{C}^*)} \longrightarrow \mathrm{H}^0(C, \mathcal{L}_C)^{\tilde{\tau}(\mathbb{C}^*)}$$

is an isomorphism.

Proof. We first prove the injectivity. Let $\sigma \in \mathrm{H}^0(C^+, \mathcal{L}_{|C^+})^{\tilde{\tau}(\mathbb{C}^*)}$ such that $\sigma_{|C} = 0$ and $x \in C^+$. Consider the morphism

$$\begin{array}{rcll} \theta_x : \mathbb{C} & \longrightarrow & C^+ & \\ t & \longmapsto & \tilde{\tau}(t)x & \text{if} \quad t \neq 0, \\ 0 & \longmapsto & \lim_{t\to 0} \tilde{\tau}(t)x. & \end{array}$$

It is $\mathbb{C}^*$-equivariant for the natural actions of $\mathbb{C}^*$. Moreover, $\theta_x^*(\mathcal{L})$ is trivial as $\mathbb{C}^*$-linearized line bundle, since $\mu^{\mathcal{L}}(C, \tilde{\tau}) = 0$. But $\theta_x^*(\sigma)(0) = 0$ and $\theta_x^*(\sigma)$ is $\mathbb{C}^*$-invariant. Thus $\theta_x^*(\sigma) = 0$. In particular $\sigma(x) = 0$.

Consider now the map $\Lambda : C^+ \longrightarrow C$, $x \longmapsto \lim_{t\to 0} \tilde{\tau}(t)x$. We claim that $\Lambda^*(\mathcal{L}_{|C})$ is isomorphic to $\mathcal{L}$.

We can work on each factor of C^+ separately. So assume for proving the claim that $\mathbb{X} = G/B$, $C^+ = Pv^{-1}\underline{o}$ and $C = Lv^{-1}\underline{o}$. Let $\iota_\mu : \mathbb{X} \longrightarrow \mathbb{P}(V(\mu))$ induced by the action of G on the highest weight line of $\mathbb{P}(V(\mu))$. Set $k = \langle\tilde\tau, v^{-1}\mu\rangle$. Recall that $V(\mu)$ has a weight space decomposition $V(\mu) = \oplus_{\chi\in X(T)} V(\mu)_\chi$. Set

$$V(\mu)^k = \bigoplus_{\langle\chi,\tilde\tau\rangle=k} V(\mu)_\chi \qquad V(\mu)^{>k} = \bigoplus_{\langle\chi,\tilde\tau\rangle>k} V(\mu)_\chi.$$

Define

$$\mathcal{E} = \{[v_0 + v_+] \,:\, v_0 \in V(\mu)^k - \{0\} \text{ and } v_+ \in V(\mu)^{>k}\}$$

as a subset of $\mathbb{P}(V(\mu)^k \oplus V(\mu)^{>k})$ and so of $\mathbb{P}(V(\mu))$. Then $\iota_\mu(C)$ is contained in $\mathbb{P}(V(\mu)^k)$ and $\iota_\mu(C^+)$ is contained in $\mathcal{E}$. Moreover, Λ is the restriction of the canonical projection $\mathcal{E} \longrightarrow \mathbb{P}(V(\mu)^k)$. But, $\mathcal{L}(\mu)^*$ is the restriction to $\mathbb{X}$ of the tautological bundle on $\mathbb{P}(V(\mu))$. The claim follows.

Let us prove the surjectivity. Let $\sigma \in \mathrm{H}^0(C, \mathcal{L}_{|C})^{\tilde\tau(\mathbb{C}^*)}$. By the claim, we have to prove that σ extends to a $\mathbb{C}^*$-invariant section of $\Lambda^*(\mathcal{L})$. The morphism

$$\begin{array}{ccc} C^+ & \longrightarrow & \mathcal{L}_{|C} \\ x & \longmapsto & \sigma(\Lambda(x)) \end{array}$$

induces such an extension. $\square$

7. The Belkale-Kumar product

In this section, we purpose a construction of the BK-product $\odot_0$ (see [BK06] if G is finite-dimensional and [Kum10] if G is Kac-Moody) and prove some properties.

7.1. **Preliminaries of linear algebra.** Let V be a complex vector space filtered by linear subspaces

$$\{0\} = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^n \subset \cdots$$

such that $V = \cup_n V^n$. Let U and W be two linear subspaces of V such that U has finite dimension, W has finite codimension and $\dim(U) = \mathrm{codim}(W)$. Consider the linear map

$$\begin{array}{cccc} \Theta : & V & \longrightarrow & V/U \oplus V/W \\ & v & \longmapsto & (v+U, v+W). \end{array}$$

Define the induced filtrations from V to U and V/W:

$$U^n = V^n \cap U \quad \text{and} \quad (V/W)^n = V^n/(W \cap V^n).$$

Set

$$\delta = \sum_{n>0} n\Big(\dim(U^n/U^{n-1}) - \dim((V/W)^n/(V/W)^{n-1})\Big).$$

Lemma 15. *If Θ is an isomorphism then*

$$\dim(U^n) \le \dim(V/W)^n \qquad \forall n \in \mathbb{Z}_{\ge 0}.$$

In particular $\delta \ge 0$.

Proof. Consider

$$\begin{array}{cccc} \bar\Theta : & U & \longrightarrow & V/W \\ & v & \longmapsto & v+W. \end{array}$$

The map Θ being an isomorphism, so is $\bar\Theta$. Moreover $\bar\Theta(U^n) \subset (V/W)^n$. Then the first inequality of the lemma is a consequence of the injectivity of the restriction of $\bar\Theta$ to U^n.

Since U and V/W are finite-dimensional, there exists N such that $U^n = U$ and $(V/W)^n = V/W$ for any $n \geq N$. Then

$$\begin{aligned}\delta &= \sum_{n=1}^{N} n\Big(\dim(U^n/U^{n-1}) - \dim((V/W)^n/(V/W)^{n-1})\Big)\\ &= \sum_{n=1}^{N} n\Big(\dim(U^n) - \dim(U^{n-1}) - \dim((V/W)^n) + \dim((V/W)^{n-1})\Big)\\ &= \sum_{n=0}^{N-1} \dim((V/W)^n) - \dim(U^n),\end{aligned}$$

since $\dim(U) = \dim(V/W)$. In particular, $\delta \geq 0$. □

Consider the graded vector spaces

$$\mathrm{gr}U = \oplus_{n\in\mathbb{Z}_{>0}} U^n/U^{n-1} \quad \text{and} \quad \mathrm{gr}(V/W) = \oplus_{n\in\mathbb{Z}_{>0}} (V/W)^n/(V/W)^{n-1}.$$

The map $\bar{\Theta}$ induces a graded linear map

$$\mathrm{gr}\bar{\Theta}\,:\,\mathrm{gr}U \longrightarrow \mathrm{gr}(V/W).$$

Lemma 16. *Assume that Θ is an isomorphism. The following assertions are equivalent*

(i) $\mathrm{gr}\bar{\Theta}$ *is an isomorphism;*
(ii) $\dim(U^n) = \dim(V/W)^n \qquad \forall n \in \mathbb{Z}_{\geq 0}$;
(iii) $\delta = 0$.

Proof. The second assertion implies the last one by the proof of Lemma 15. If $\mathrm{gr}\bar{\Theta}$ is an isomorphism then for any n, $\dim(U^n) - \dim(U^{n-1}) = \dim(V/W)^n - \dim(V/W)^{n-1}$. The initial subspaces U^0 and $(V/W)^0$ being trivial, the equalities of assertion (ii) follow, by immediate induction.

Assume now that $\delta = 0$. Since $\delta = \sum_{n\geq 0} \dim(V/W)^n - \dim(U^n)$, Lemma 15 shows that $\dim(V/W)^n = \dim(U^n)$, for any n. Then, the injectivity of Θ implies that $\bar{\Theta}$ induces isomorphisms from U^n onto $(V/W)^n$, for any n. Then $\mathrm{gr}\bar{\Theta}$ is an isomorphism. □

7.2. **Definition of the BK product.** Let P be a standard parabolic subgroup of G. Let u_1, u_2, and v in W^P such that $l(v) = l(u_1) + l(u_2)$ and $n^v_{u_1u_2} \neq 0$. Set

$$\mathcal{T} = T_{P/P}G/P$$
$$\mathcal{T}^{u_1} = T_{P/P}{u_1}^{-1}X_P^{u_1} \quad \mathcal{T}^{u_2} = T_{P/P}u_2^{-1}X_P^{u_2} \quad \mathcal{T}_v = T_{P/P}v^{-1}X_v^P.$$

Fix a one parameter subgroup τ of T belonging to $\oplus_{\alpha_j\not\in\Delta(P)}\mathbb{Z}_{>0}\varpi_{\alpha_j^\vee}$. Observe that P acts on $\mathcal{T}$. Under the action of τ, $\mathcal{T}$ decomposes as $\mathcal{T} = \bigoplus_{n\in\mathbb{Z}}\mathcal{T}_n$, where $\mathcal{T}_k = \{\xi \in \mathcal{T}\,:\,\tau(t)\xi = t^k\xi \quad \forall t \in \mathbb{C}^*\}$. Note that $\mathcal{T}_n = \{0\}$ for all $n \geq 0$. Set

$$\mathcal{T}^n = \oplus_{k\leq n}\mathcal{T}_{-k}.$$

Then $(\mathcal{T}^n)_{n\in\mathbb{Z}_{\geq 0}}$ forms a P-stable filtration of $\mathcal{T}$. Moreover $\mathcal{T}^0 = \{0\}$. Consider also the induced filtrations $(\mathcal{T}/\mathcal{T}^{u_1})^n$, $(\mathcal{T}/\mathcal{T}^{u_2})^n$ and $\mathcal{T}_v^n$ on $\mathcal{T}/\mathcal{T}^{u_1}$, $\mathcal{T}/\mathcal{T}^{u_2}$, and $\mathcal{T}_v$. Set

$$\begin{aligned}\delta^v_{u_1\,u_2} = \sum_{n\geq 0} n\Big(&\dim(\mathcal{T}_v^n/\mathcal{T}_v^{n-1}) - \dim(((\mathcal{T}/\mathcal{T}^{u_1})^n)/(\mathcal{T}/\mathcal{T}^{u_1})^{n-1})\\ &- \dim(((\mathcal{T}/\mathcal{T}^{u_2})^n)/(\mathcal{T}/\mathcal{T}^{u_2})^{n-1})\Big).\end{aligned}$$

Lemma 17. *If $n^v_{u_1u_2} \neq 0$ then $\delta^v_{u_1\,u_2} \geq 0$.*

Proof. Consider the map

$$\eta\,:\,G\times_P C^+\longrightarrow \mathbb{X},$$

as in the proof of Proposition 1.

By Lemma 6, there exists $b\in B$ such that $X_v^{u_1}\cap bX_v^{u_2}=\mathring{X}_v^P\cap\mathring{X}_P^{u_1}\cap b\mathring{X}_P^{u_2}$ is transverse and nonempty by Lemma 7. Let $g\in G$ such that gP/P belongs to this intersection. There exist p_1, p_2 and p_3 in P such that

$$(gP/P,(gp_1u_1^{-1}\underline{o}^-,gp_2u_2^{-1}\underline{o}^-,gp_3v^{-1}\underline{o}))$$

belongs to the fiber $\eta^{-1}(\underline{o}^-,b\underline{o}^-,\underline{o})$. Observe that

$$g(p_1u_1^{-1}X_P^{u_1}\cap p_2u_2^{-1}X_P^{u_2}\cap p_3v^{-1}X_v^P)=X_v^{u_1}\cap bX_v^{u_2}$$

is transverse. By Lemma 6, the canonical map

$$p_3\mathcal{T}_v\longrightarrow\frac{\mathcal{T}}{p_1\mathcal{T}^{u_1}}\oplus\frac{\mathcal{T}}{p_2\mathcal{T}^{u_2}} \tag{15}$$

is an isomorphism.

The lemma follows by applying Lemma 15 with $V=\mathcal{T}\oplus\mathcal{T}$, $V^n=\mathcal{T}^n\oplus\mathcal{T}^n$, $W=p_1\mathcal{T}^{u_1}\oplus p_2\mathcal{T}^{u_2}$ and $U\simeq\mathcal{T}_v$ embedded in V by $\xi\mapsto(p_3\xi,p_3\xi)$. ☐

For $w\in W$, we denote by $\Phi_w=w^{-1}\Phi^+\cap\Phi^-$ the inversion set of w. Then, Φ_w consists in $l(w)$ real roots. Recall that $\rho=\sum_{i=0}^l\varpi_{\alpha_i}$.

Lemma 18. *With above notation, we have*

$$\delta_{u_1\,u_2}^v=\langle -v^{-1}\rho+u_1^{-1}\rho+u_2^{-1}\rho-\rho,\tau\rangle.$$

Proof. Since

$$\mathcal{T}/\mathcal{T}^{u_1}\simeq\oplus_{\alpha\in\Phi_{u_1}}\mathfrak{g}_\alpha\quad\mathcal{T}/\mathcal{T}^{u_2}\simeq\oplus_{\alpha\in\Phi_{u_2}}\mathfrak{g}_\alpha\quad\text{and}\quad\mathcal{T}_v\simeq\oplus_{\alpha\in\Phi_v}\mathfrak{g}_\alpha,$$

we have

$$\delta_{u_1\,u_2}^v=-\sum_{\alpha\in\Phi_v}\langle\alpha,\tau\rangle+\sum_{\alpha\in\Phi_{u_1}}\langle\alpha,\tau\rangle+\sum_{\alpha\in\Phi_{u_2}}\langle\alpha,\tau\rangle.$$

But by [Kum02, Lemma 1.3.22], $w^{-1}\rho-\rho=\sum_{\alpha\in\Phi_w}\alpha$ and the lemma follows. ☐

Because of Lemma 18, Lemma 17 can be restated as: by assigning the degree $\langle v^{-1}\rho-\rho,\tau\rangle\in\mathbb{Z}$ to ϵ_v, one obtains a filtration of the cohomology ring $\mathrm{H}^*(G/P,\mathbb{Z})$. Then $\odot_0$ is defined to be the product of the associated graded ring.

In particular, $\odot_0$ satisfies (with obvious identifications):

$$\forall u_1,u_2\in W^P\qquad \epsilon_{u_1}\odot_0\epsilon_{u_2}=\sum_{v\in W^P}{}^{\odot_0}n_{u_1\,u_2}^v\,\epsilon_v,$$

where

$$\begin{aligned}{}^{\odot_0}n_{u_1\,u_2}^v&=0&&\text{if }\delta_{u_1\,u_2}^v\neq 0,\\&=n_{u_1\,u_2}^v&&\text{if }\delta_{u_1\,u_2}^v=0.\end{aligned}$$

Let $Z(L)$ denote the center of L and $Z(L)^\circ$ denote its neutral component. Given an L-representation and a character $\chi\in X(Z(L)^\circ)$ we denote by $V_\chi=\{v\in V\,:\,\forall t\in Z(L)^\circ\quad tv=\chi(t)v\}$ the associated weight space. A priori, the above construction of $\odot_0$ could depend on our choice of an element $\tau\in\oplus_{\alpha_j\not\in\Delta(P)}\mathbb{Z}_{>0}\varpi_{\alpha_j^\vee}$. Actually, it does not depend on:

Lemma 19. *Let u_1, u_2, and v in W^P such that ${}^{\odot_0}n^v_{u_1u_2} \neq 0$. Then, for any $\chi \in X(Z(L)^\circ)$*

$$\dim((\mathcal{T}_v)_\chi) = \dim(\left(\frac{\mathcal{T}}{\mathcal{T}^{u_1}}\right)_\chi) + \dim(\left(\frac{\mathcal{T}}{\mathcal{T}^{u_2}}\right)_\chi).$$

Proof. As in the proof of Lemma 17, choose p_1, p_2, and p_3 in P such that the linear map (15) is an isomorphism. Up to multiplying by p_3^{-1} we assume that p_3 is trivial. Then

$$\bar{\Theta} \,:\, \mathcal{T}_v \longrightarrow \frac{\mathcal{T}}{p_1\mathcal{T}^{u_1}} \oplus \frac{\mathcal{T}}{p_2\mathcal{T}^{u_2}}$$

is an isomorphism. For $i = 1, 2$, write $p_i = g_i^u l_i$ with $g_i^u \in P^u$ and $l_i \in L$.

Let $k \in \mathbb{Z}$ and $\xi \in \mathcal{T}_{-k}$. Then

$$g_i^u \xi \in \xi + \mathcal{T}^{k+1}. \tag{16}$$

Fix bases of $\mathcal{T}_v$, $\frac{\mathcal{T}}{p_1\mathcal{T}^{u_1}}$ and $\frac{\mathcal{T}}{p_2\mathcal{T}^{u_2}}$ adapted to the filtrations. Using (16), one can check that the matrices of $\mathrm{gr}\bar{\Theta}$ and of

$$\tilde{\Theta} \,:\, \mathcal{T}_v \longrightarrow \frac{\mathcal{T}}{l_1\mathcal{T}^{u_1}} \oplus \frac{\mathcal{T}}{l_2\mathcal{T}^{u_2}}$$

coincide. Now Lemma 16 shows that $\delta^v_{u_1\,u_2} = 0$ if and only if $\tilde{\Theta}$ is an isomorphism. But, $\tilde{\Theta}$ being $Z(L)^\circ$-equivariant, we have

$$\dim((\mathcal{T}_v)_\chi) = \dim(\left(\frac{\mathcal{T}}{l_1\mathcal{T}^{u_1}}\right)_\chi) + \dim(\left(\frac{\mathcal{T}}{l_2\mathcal{T}^{u_2}}\right)_\chi),$$

for any χ. Now, $Z(L)^\circ$ being central in L, we have for $i = 1, 2$

$$\dim(\left(\frac{\mathcal{T}}{l_i\mathcal{T}^{u_i}}\right)_\chi) = \dim(\left(\frac{\mathcal{T}}{\mathcal{T}^{u_i}}\right)_\chi).$$

The lemma is proved. □

7.3. **On Levi movability.**

Proposition 2. *We keep notation and assumptions of Section 7.2 and assume in addition that P has finite type. Recall that $n^v_{u_1\,u_2} \neq 0$. If there exist $l_1, l_2, l \in L$ such that*

$$l_1u_1^{-1}\mathring{X}_P^{u_1} \cap l_2u_2^{-1}\mathring{X}_P^{u_2} \cap lv^{-1}\mathring{X}_v^P$$

is finite, then

$$\delta^v_{u_1\,u_2} = 0.$$

Proof. Identify $P^{u,-}$ with an L-stable open subset of G/P. For any $n \geq 2$, consider the normal group $P^{u,-}_{\geq n}$ of $P^{u,-}$ such that $P^{u,-}/P^{u,-}_{\geq n}$ is a finite-dimensional unipotent algebraic group with Lie algebra

$$\bigoplus_{\alpha\in\Phi,\, 1\leq\langle\alpha,\tau\rangle<n} \mathfrak{g}_\alpha$$

(see [Kum02, Lemma 6.1.11]). Let $\pi_n \,:\, P^{u,-} \longrightarrow P^{u,-}/P^{u,-}_{\geq n}$ be the quotient map. Observe that the actions of τ and L commute and that $P^{u,-}_{\geq n}$ is L-stable.

The sets Φ_{u_i} and Φ_v being finite, there exists N such that for any $i = 1, 2$,

$$u_i^{-1}B^-u_i \cap P \supset P^{u,-}_{\geq N}$$

and

$$v^{-1}Bv \cap P^{u,-}_{\geq N} = \{e\}.$$

Then $\pi_N(u_i^{-1}\mathring{X}^{u_i})$ has codimension $l(u_i)$, for $i = 1, 2$; and $\pi_N(v^{-1}\mathring{X}_v^P)$ has dimension $l(v)$. Moreover, π_N maps bijectively $l_1u_1^{-1}\mathring{X}_P^{u_1} \cap l_2u_2^{-1}\mathring{X}_P^{u_2} \cap lv^{-1}\mathring{X}_v^P$ onto

$$\pi_N(l_1u_1^{-1}\mathring{X}_P^{u_1}) \cap \pi_N(l_2u_2^{-1}\mathring{X}_P^{u_2}) \cap \pi_N(lv^{-1}\mathring{X}_v^P). \tag{17}$$

Consider now the exponential map

$$\mathrm{Exp} : \bigoplus_{\alpha\in\Phi,\, 1\leq\langle\alpha,\tau\rangle<n} \mathfrak{g}_\alpha \longrightarrow P^{u,-}/P^{u,-}_{\geq N}.$$

Since $P^{u,-}/P^{u,-}_{\geq N}$ is unipotent, Exp is an isomorphism of varieties. Let $\mathfrak{u}_1$, $\mathfrak{u}_2$ and $\mathfrak{u}_3$ denote the Lie algebras of $\pi_N(l_1u_1^{-1}\mathring{X}_P^{u_1} = l_1u_1^{-1}U^-u_1l_1^{-1} \cap P^{u,-})$, $\pi_N(l_1u_2^{-1}\mathring{X}_P^{u_2})$ and $\pi_N(lv^{-1}\mathring{X}_v^P)$ respectively. These subspaces are stable by the action of τ and decompose as $\mathfrak{u}_i = \oplus_{n<0}\mathfrak{u}_i^n$. The intersection (17) being finite, $\mathfrak{u}_1 \cap \mathfrak{u}_2 \cap \mathfrak{u}_3 = \{0\}$. Since $\dim(\mathfrak{u}_3) = \mathrm{codim}(\mathfrak{u}_1) + \mathrm{codim}(\mathfrak{u}_2)$, it follows that the natural map

$$\mathfrak{u}_3 \longrightarrow \frac{\bigoplus_{\alpha\in\Phi,\,1\leq\langle\alpha,\tau\rangle<n}\mathfrak{g}_\alpha}{\mathfrak{u}_2} \oplus \frac{\bigoplus_{\alpha\in\Phi,\,1\leq\langle\alpha,\tau\rangle<n}\mathfrak{g}_\alpha}{\mathfrak{u}_3}$$

is a τ-equivariant isomorphism. Then for any integer n the τ-eigenspace $\mathfrak{u}_3^n$ has dimension $\mathrm{codim}(\mathfrak{u}_1^n)+\mathrm{codim}(\mathfrak{u}_2^n)$. The actions of L and τ commuting, one deduces that $\delta^v_{u_1\,u_2} = 0$. □

8. Multiplicativity in cohomology

8.1. **The multiplicativity.** Let $B \subset P \subset Q$ be two standard parabolic subgroups of G. Let L^P and L^Q denote the Levi subgroups of P and Q containing T. Then $L^Q \cap P$ is a parabolic subgroup of L^Q and $Q/P = L^Q/(L^Q \cap P)$.

In this section, we study relations between structure constants of $\mathrm{H}^*(G/P,\mathbb{Z})$, $\mathrm{H}^*(G/Q,\mathbb{Z})$ and $\mathrm{H}^*(Q/P,\mathbb{Z})$. To be more precise, we extend results of [Ric12, Res11] from the classical case to the Kac-Moody case.

Let W_Q^P be the set of minimal length representative in W_Q of the classes W_Q/W_P.

Lemma 20. *The map*

$$\begin{array}{ccc} W^Q \times W_Q^P & \longrightarrow & W^P \\ (\bar{w}, \tilde{w}) & \longmapsto & \bar{w}\tilde{w} \end{array}$$

is bijective.

Proof. Recall that (see [Kum02, Exercice 1.3.E]) $W^P = \{w \in W \ :\ w^{-1}\Phi^- \cap \Phi^+(L^P) = \emptyset\}$. We first check that $w = \bar{w}\tilde{w}$ belongs to W^P; this shows that the map of the lemma is well defined. Write

$$w^{-1}\Phi^- \cap \Phi^+(L^P) = w^{-1}\bar{w}(\bar{w}^{-1}\Phi^- \cap \tilde{w}\Phi^+(L^P)).$$

Note that $\Phi^+(L^P) \subset \Phi(L^Q) = \tilde{w}\Phi(L^Q)$ and that $\bar{w}^{-1}\Phi^- \cap \Phi(L^Q) = \Phi^-(L^Q)$ (since $\bar{w} \in W^Q$). Hence

$$\begin{array}{rcl} w^{-1}\Phi^- \cap \Phi^+(L^P) & = & \tilde{w}^{-1}\Big(\bar{w}^{-1}\Phi^- \cap \Phi(L^Q) \cap \tilde{w}\Phi^+(L^P)\Big) \\ & = & \tilde{w}^{-1}\Big(\Phi^-(L^Q) \cap \tilde{w}\Phi^+(L^P)\Big) \\ & = & \tilde{w}^{-1}\Phi^-(L^Q) \cap \Phi^+(L^P). \end{array}$$

This last intersection is empty since $\tilde{w} \in W_Q^P$. Then $w \in W^P$.

Fix now $w \in W^P$. If $w = \bar{w}\tilde{w}$ (with $\bar{w} \in W^Q$ and $\tilde{w} \in W_Q^P$) then $wW_Q = \bar{w}W_Q$ and $\bar{w}$ is necessarily the unique representative of wW_Q in W^Q. Since $\tilde{w} = \bar{w}^{-1}w$, this proves that the map is injective.

Consider now the representative $\bar{w}$ of wW_Q in W^Q and set $\tilde{w} = \bar{w}^{-1}w$. To prove the surjectivity, it remains to prove that $\tilde{w} \in W_Q^P$. The equality $\bar{w}W_Q = wW_Q$ implies that $\tilde{w} \in W_Q$. Moreover

$$\begin{aligned}\tilde{w}^{-1}\Phi^-(L^Q) \cap \Phi^+(L^P) &= w^{-1}\bar{w}\Phi^-(L^Q) \cap \Phi^+(L^P)\\ &\subset w^{-1}\Phi^- \cap \Phi^+(L^P),\end{aligned}$$

since $\bar{w}\Phi^-(L^Q) \subset \Phi^-$ ($\bar{w} \in W^Q$). This last intersection is empty since $w \in W^P$. The lemma is proved. $\square$

Recall that, for $w \in W$

$$X_w^P = \overline{BwP/P} \qquad \mathring{X}_w^P = BwP/P$$
$$X_P^w = \overline{B^-wP/P} \qquad \mathring{X}_P^w = B^-wP/P$$

Lemma 21. *(See* [BK06, Lemma 1]*) Let $w \in W^P$ and $g \in G$.*

(i) *If $g\mathring{X}_w^P$ contains P/P then there exists $p \in P$ such that $g\mathring{X}_w^P = pw^{-1}\mathring{X}_w^P$.*
(ii) *If $g\mathring{X}_P^w$ contains P/P then there exists $p \in P$ such that $g\mathring{X}_P^w = pw^{-1}\mathring{X}_P^w$.*

Proof. Fix a representative $\dot{w}$ of w in $N(T)$. Let $b \in B$ and $p \in P$ such that $gb\dot{w} = p$. Then $g\mathring{X}_w^P = p\dot{w}^{-1}b^{-1}\mathring{X}_w^P = pw^{-1}\mathring{X}_w^P$. The second assertion works similarly. $\square$

Lemma 22. *Let $\tilde{w} \in W_Q^P$ and $\bar{w} \in W^Q$. Set $w = \bar{w}\tilde{w}$. Consider Q/P as a closed subset in G/P. Then*

(i) $\bar{w}^{-1}\mathring{X}_w^P \cap Q/P = (L^Q \cap B)\tilde{w}P/P =: \mathring{X}_{\tilde{w}}^{Q/P}$;
(ii) $\bar{w}^{-1}\mathring{X}_P^w \cap Q/P = (L^Q \cap B^-)\tilde{w}P/P =: \mathring{X}_{Q/P}^{\tilde{w}}$.

Proof. Note that $\bar{w}^{-1}\mathring{X}_w^P \cap Q/P$ is stable by the action of $\bar{w}^{-1}B\bar{w} \cap Q$. Since $\bar{w} \in W^Q$, $\bar{w}^{-1}B\bar{w} \cap Q$ contains $L^Q \cap B$. Moreover each $(L^Q \cap B)$-orbit in Q/P contains a T-fixed point. Hence

$$\bar{w}^{-1}\mathring{X}_w^P \cap Q/P = \bigcup_{x \in (\bar{w}^{-1}\mathring{X}_w^P \cap Q/P)^T} (L^Q \cap B).x.$$

But $(\bar{w}^{-1}\mathring{X}_w^P \cap Q/P)^T \subset \bar{w}^{-1}(\mathring{X}_w^P)^T = \{\tilde{w}P/P\}$. The first assertion of the lemma follows. The second one works similarly. $\square$

We can now prove the main result of this section.

Proposition 3. *Let u_1, u_2, and v in W^P. Write $u_1 = \bar{u}_1\tilde{u}_1$, $u_2 = \bar{u}_2\tilde{u}_2$, and $v = \bar{v}\tilde{v}$ as in Lemma 20. We assume that $l(v) = l(u_1) + l(u_2)$ and $l(\bar{v}) = l(\bar{u}_1) + l(\bar{u}_2)$.*

Consider the structure constants $n_{u_1\,u_2}^v$, $n_{\bar{u}_1\bar{u}_2}^{\bar{v}}$, and $n_{\tilde{u}_1\,\tilde{u}_2}^{\tilde{v}}$ in $\mathrm{H}^*(G/P,\mathbb{Z})$, $\mathrm{H}^*(G/Q,\mathbb{Z})$, and $\mathrm{H}^*(Q/P,\mathbb{Z})$ respectively.

Then

$$n_{u_1\,u_2}^v = n_{\bar{u}_1\bar{u}_2}^{\bar{v}}n_{\tilde{u}_1\,\tilde{u}_2}^{\tilde{v}}.$$

Proof. Since B is irreducible, Lemma 6 implies that there exists $b \in B$ such that

(i) $X_P^{u_1} \cap bX_v^{u_2} = \mathring{X}_P^{u_1} \cap \mathring{X}_v^P \cap b\mathring{X}_P^{u_2}$ is transverse, and
(ii) $X_Q^{\bar{u}_1} \cap bX_{\bar{v}}^{\bar{u}_2} = \mathring{X}_Q^{\bar{u}_1} \cap \mathring{X}_{\bar{v}}^Q \cap b\mathring{X}_Q^{\bar{u}_2}$ is transverse.

By Lemma 7, it remains to determine the cardinality of the intersection (i). We do this by counting in each fiber of the G-equivariant projection $\pi : G/P \longrightarrow G/Q$.

Fix $g \in G$ such that $gQ/Q \in X_Q^{\bar{u}_1} \cap bX_{\bar{v}}^{\bar{u}_2}$. Then $Q/Q \in g^{-1}\mathring{X}_Q^{\bar{u}_1} \cap g^{-1}\mathring{X}_{\bar{v}}^Q \cap g^{-1}b\mathring{X}_Q^{\bar{u}_2}$. As in Lemma 21, there exist q_1, q_2, and q in Q such that $g^{-1} \in q\dot{\bar{v}}^{-1}B$, $g^{-1} \in q_1\dot{\bar{u}}_1^{-1}B^-$, and $g^{-1}b \in q_2\dot{\bar{u}}_2^{-1}B^-$. Let l_1, l_2, and l in Q such that $q_1l_1^{-1}$, $q_2l_2^{-1}$, and ql^{-1} belong to Q^u. Then

$$\begin{aligned} I &:= g^{-1}(\mathring{X}_P^{u_1} \cap b\mathring{X}_P^{u_2} \cap \mathring{X}_v^P \cap \pi^{-1}(gQ/Q)) \\ &= q_1\bar{u}_1^{-1}\mathring{X}_P^{u_1} \cap q_2\bar{u}_2^{-1}\mathring{X}_P^{u_2} \cap q\bar{v}^{-1}\mathring{X}_v^P \cap Q/P \\ &= q\mathring{X}_{\tilde{v}}^{Q/P} \cap q_1\mathring{X}_{Q/P}^{\tilde{u}_1} \cap q_2\mathring{X}_{Q/P}^{\tilde{u}_2} \qquad \text{by Lemma 22} \\ &= l\mathring{X}_{\tilde{v}}^{Q/P} \cap l_1\mathring{X}_{Q/P}^{\tilde{u}_1} \cap l_2\mathring{X}_{Q/P}^{\tilde{u}_2} \end{aligned}$$

The last equality holds since Q^u acts trivially on Q/P. Moreover, since $X_P^{u_1} \cap bX_v^{u_2} = \mathring{X}_P^{u_1} \cap \mathring{X}_v^P \cap b\mathring{X}_P^{u_2}$, we also have

$$I = lX_{\tilde{v}}^{Q/P} \cap l_1X_{Q/P}^{\tilde{u}_1} \cap l_2X_{Q/P}^{\tilde{u}_2}.$$

Claim. The intersection $l\mathring{X}_{\tilde{v}}^{Q/P} \cap l_1\mathring{X}_{Q/P}^{\tilde{u}_1} \cap l_2\mathring{X}_{Q/P}^{\tilde{u}_2}$ is transverse.

Let x be a point in this intersection. By Lemma 6, the map

$$T_xg^{-1}\mathring{X}_v^P \longrightarrow \frac{T_xG/P}{T_xg^{-1}\mathring{X}_P^{u_1}} \oplus \frac{T_xG/P}{T_xg^{-1}b\mathring{X}_P^{u_2}}$$

is an isomorphism. Hence, the natural map

$$T_xg^{-1}\mathring{X}_v^P \cap T_xQ/P \longrightarrow \frac{T_xQ/P}{T_xg^{-1}\mathring{X}_P^{u_1} \cap T_xQ/P} \oplus \frac{T_xQ/P}{T_xg^{-1}b\mathring{X}_P^{u_2} \cap T_xQ/P}$$

is injective. Since $T_xl\mathring{X}_{\tilde{v}}^{Q/P} \subset T_xg^{-1}\mathring{X}_v^P \cap T_xQ/P$ (and similar inclusions hold for u_1 and u_2), we deduce that the natural map

$$T_xl\mathring{X}_{\tilde{v}}^{Q/P} \longrightarrow \frac{T_xQ/P}{T_xl_1\mathring{X}_{Q/P}^{\tilde{u}_1}} \oplus \frac{T_xQ/P}{T_xl_2\mathring{X}_{Q/P}^{\tilde{u}_2}}$$

is injective. The assumption on the length of elements of W_Q^P implies that it is in fact an isomorphism. The claim is proved.

The claim and Lemma 7 imply that the cardinality of I is $n_{\tilde{u}_1\,\tilde{u}_2}^{\tilde{v}}$. This holding for any one of the $n_{\bar{u}_1\,\bar{u}_2}^{\bar{v}}$ points in $X_Q^{\bar{u}_1} \cap bX_{\bar{v}}^{\bar{u}_2}$, we get that $X_P^{u_1} \cap bX_v^{u_2}$ has cardinality $n_{\tilde{u}_1\,\tilde{u}_2}^{\tilde{v}}n_{\bar{u}_1\,\bar{u}_2}^{\bar{v}}$. We conclude by applying Lemma 7 in G/P. □

8.2. **Application to the BK-product.** The following lemma allows to apply Proposition 3 to any nonzero structure constants of the BK-product.

Lemma 23. *Let $P \subset Q$ be two standard parabolic subgroups of G. Let u_1, u_2, and v in W^P such that ${}^{\odot_0}n^v_{u_1u_2} \neq 0$.*
Then $l(\bar{v}) = l(\bar{u}_1) + l(\bar{u}_2)$.

Proof. Let Q^- be the opposite subgroup of Q and $Q^{u,-}$ its "unipotent" subgroup. Let $\mathfrak{q}^{u,-}$ be the Lie algebra of $Q^{u,-}$. Note that

$$l(\bar{v}) = \dim(\mathcal{T}_v \cap \mathfrak{q}^{u,-}) \text{ and } l(\bar{u}_i) = \dim(\frac{\mathfrak{q}^{u,-}}{\mathcal{T}^{u_i} \cap \mathfrak{q}^{u,-}})\ \forall i = 1, 2.$$

There exists a one parameter subgroup τ_Q of $Z(L^Q)$ such that $\mathfrak{q}^{u,-}$ is the sum of the negative weight spaces for τ_Q. Since $Z(L^Q)^\circ$ is contained in $Z(L^P)^\circ$:

$$\mathcal{T}_w \cap \mathfrak{q}^{u,-} = \bigoplus_{\chi \in X(Z(L^P)^\circ)\ \langle \chi, \tau_Q \rangle < 0} (\mathcal{T}_w)_\chi$$

for any $w \in W$. Now the equality of the lemma is a direct consequence of Lemma 19. □

9. The untwisted affine case

9.1. **Notation.** Let $\dot{\mathfrak{g}}$ be a complex finite-dimensional simple Lie algebra with Cartan subalgebra $\dot{\mathfrak{h}}$ and Borel subalgebra $\dot{\mathfrak{b}} \supset \dot{\mathfrak{h}}$. Let $\dot{\alpha}_1, \dots, \dot{\alpha}_l$ denote the simple roots, $\dot{\alpha}_1^\vee, \dots, \dot{\alpha}_l^\vee$ the simple coroots, $\dot{\theta}$ the highest root and $\dot{\theta}^\vee$ the highest coroot. For any simple root $\dot{\alpha}$, we denote by $\varpi_{\dot{\alpha}}$ the corresponding fundamental weight and by $\varpi_{\dot{\alpha}^\vee}$ the corresponding fundamental coweight. Let $\dot{P}_+$ denote the set of dominant integral weights for $\dot{\mathfrak{g}}$. Set $\dot{\rho} = \sum_{i=1}^l \varpi_{\dot{\alpha}_i}$.

Endow $\mathfrak{g} = \dot{\mathfrak{g}} \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ with the usual Lie bracket (see e.g. [Kum02, Chap XIII]). Set $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Define Λ and δ in $\mathfrak{h}^*$ by

$$\begin{array}{l} \delta\ :\ \dot{\mathfrak{h}} \longmapsto 0, c \longmapsto 0, d \longmapsto 1; \\ \Lambda\ :\ \dot{\mathfrak{h}} \longmapsto 0, c \longmapsto 1, d \longmapsto 0. \end{array}$$

We identify $\dot{\mathfrak{h}}^*$ with the orthogonal of $\mathbb{C}c \oplus \mathbb{C}d$ in $\mathfrak{h}^*$ in such a way that $\mathfrak{h}^* = \dot{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda \oplus \mathbb{C}\delta$. The simple roots of $\mathfrak{g}$ are

$$\alpha_0 = \delta - \dot{\theta}, \dot{\alpha}_1, \dots, \dot{\alpha}_l.$$

The simple coroots of $\mathfrak{g}$ are

$$\alpha_0^\vee = c - \dot{\theta}^\vee, \dot{\alpha}_1^\vee, \dots, \dot{\alpha}_l^\vee.$$

For any simple root $\dot{\alpha}_i$ of $\dot{\mathfrak{g}}$, set $\varpi_{\alpha_i} = \varpi_{\dot{\alpha}_i} + \varpi_{\dot{\alpha}_i}(\dot{\theta}^\vee)\Lambda \in \mathfrak{h}^*$. Set $\varpi_{\alpha_0} = \Lambda$. A choice of fundamental weights for $\mathfrak{g}$ is $\varpi_{\alpha_0}, \dots, \varpi_{\alpha_l}$. In particular

$$\rho = \dot{\rho} + \mathscr{h}^\vee \Lambda, \tag{18}$$

where $\mathscr{h}^\vee = 1 + \langle \dot{\rho}, \dot{\theta}^\vee \rangle$ is the dual Coxeter number. Set

$$\mathfrak{h}^*_\mathbb{Z} = \mathbb{Z}\varpi_{\alpha_0} \oplus \dots \oplus \mathbb{Z}\varpi_{\alpha_l} \oplus \mathbb{Z}\delta,$$

and

$$\begin{aligned} P_+ &= \mathbb{Z}_{\geq 0}\varpi_{\alpha_0} \oplus \dots \oplus \mathbb{Z}_{\geq 0}\varpi_{\alpha_l} \oplus \mathbb{Z}\delta, \\ &= \{\dot{\lambda} + \ell\Lambda + b\delta\ :\ \dot{\lambda} \in \dot{P}_+ \text{ and } \langle \dot{\lambda}, \dot{\theta}^\vee \rangle \leq \ell\}. \end{aligned}$$

Denote by $P_{++} = \mathbb{Z}_{>0}\varpi_{\alpha_0} \oplus \cdots \oplus \mathbb{Z}_{>0}\varpi_{\alpha_l} \oplus \mathbb{Z}\delta$, the set of *regular* dominant weights. The chosen fundamental coweights are

$$\varpi_{\alpha_0^\vee} = d \qquad \varpi_{\alpha_i^\vee} = \varpi_{\dot\alpha_i^\vee} + \langle \varpi_{\dot\alpha_i^\vee}, \dot\theta \rangle d.$$

Set $\dot Q^\vee = \oplus_{i=1}^l \mathbb{Z}\dot\alpha_i^\vee$. An element $h \in \dot Q^\vee$ acts on $\mathfrak{h}$ by

$$h \cdot (x + kd + \ell c) = x + kh + kd + \left(\ell - (x,h) - k\frac{(h,h)}{2}\right)c. \tag{19}$$

Then $W = \dot Q^\vee . \dot W$.

9.2. **Essential inequalities and BK-product.** We are now interested in the inequalities (8) of Proposition 1 that are equalities for some regular elements of $\Gamma(\mathfrak{g})$. We prove that such inequalities necessarily appear in Theorem 1.

Theorem 7. *We use notation of Proposition 1 and assume that $n^v_{u_1,u_2} = 1$. Let $\tau \in \oplus_{\alpha_j \notin \Delta(P)} \mathbb{Z}_{>0}\varpi_{\alpha_j^\vee}$. Let $(\lambda_1, \lambda_2, \mu) \in (P_+)^3$ such that*

$$\langle \lambda_1, u_1\tau \rangle + \langle \lambda_2, u_2\tau \rangle = \langle \mu, v\tau \rangle. \tag{20}$$

Assume that μ is regular and that

$$\exists N > 0 \qquad V(N\mu) \subset V(N\lambda_1) \otimes V(N\lambda_2). \tag{21}$$

Then, ϵ_v appears with multiplicity 1 in $\epsilon_{u_1} \odot_0 \epsilon_{u_2}$.

Proof. To prove Theorem 7 we use Proposition 2 and hence have to find l_1, l_2 and l in L such that

$$l_1 u_1^{-1} \mathring{X}_P^{u_1} \cap l_2 u_2^{-1} \mathring{X}_P^{u_2} \cap l v^{-1} \mathring{X}_v^P \qquad \text{is finite.} \tag{22}$$

Consider $C = Lu_1^{-1}\underline{o}^- \times Lu_2^{-1}\underline{o}^- \times Lv^{-1}\underline{o}$, $C^+ = Pu_1^{-1}\underline{o}^- \times Pu_2^{-1}\underline{o}^- \times Pv^{-1}\underline{o}$ and the map

$$\begin{array}{rccc} \eta : & G \times_P C^+ & \longrightarrow & \mathbb{X} \\ & (gP/P, x) & \longmapsto & x. \end{array}$$

By Lemma 9, it remains to prove that there exists a point $\hat x_0$ in C such that the fiber

$$\eta^{-1}(\hat x_0) \qquad \text{is finite.} \tag{23}$$

For $i = 1, 2$, consider the maximal parabolic subgroup Q_i containing B^- such that λ_i extends to Q_i. Set $\underline{\mathbb{X}} = G/Q_1 \times G/Q_2 \times G/B$ and $\pi : \mathbb{X} \longrightarrow \underline{\mathbb{X}}$ the G^3-equivariant projection. Set $\underline{C} = \pi(C)$, $\underline{C}^+ = \pi(C^+)$ and $\underline{\eta} : G \times_P \underline{C}^+ \longrightarrow \underline{\mathbb{X}}$. Rather than demonstrating claim (23), we will show the following stronger statement: there exists x_0 in $\underline{C}$ such that the fiber

$$\underline{\eta}^{-1}(x_0) \qquad \text{is finite.} \tag{24}$$

Indeed, writing $x_0 = (l_1u_1^{-1}Q_1/Q_1, l_2u_2^{-1}Q_2/Q_2, lv^{-1}B/B)$ with $l_1, l_2, l \in L$, the proof of Lemma 9 shows that assertion (24) implies that the intersection

$$l_1u_1^{-1}Q_1u_1P/P \cap l_2u_2^{-1}Q_2u_2P/P \cap lv^{-1}BvP/P$$

is finite. In particular, claims (22) and (23) hold.

Consider the line bundle $\underline{\mathcal{L}} = \mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu)$ on $\underline{\mathbb{X}}$. Recall that, since $\mathfrak{g}$ is affine, L is a finite dimensional reductive group and C is a finite dimensional projective variety. Let $L^{\rm ss}$ be the maximal semi-simple subgroup of L. Now, the proof proceeds in two steps:

Step 1. Up to perturbing the weights $(\lambda_1, \lambda_2, \mu)$, there exists stable points for the action of $L^{\rm ss}$ on $\underline{C}$ relatively to $\underline{\mathcal{L}}$.
Step 2. A general point x_0 in $\underline{C}$ satisfies property (24).

We denote by $C^{\rm ss}(\mathcal{L}_{|C}, L)$ the set of points $x \in C$ such that there exists a L-invariant section σ of some positive power $\mathcal{L}_{|C}^{\otimes N}$ of $\mathcal{L}_{|C}$ such that $\sigma(x) \neq 0$. Note that this definition is the standard one (see [MFK94]) only if $\mathcal{L}_{|C}$ is ample. Similarly define $\underline{C}^{\rm ss}(\underline{\mathcal{L}}_{|\underline{C}}, L)$.

The following lemma is a consequence of Theorem 6. We include here a more direct proof. It allows to take a quotient by $Z(L)^\circ$.

Lemma 24. *With the assumptions of Theorem 7, the set $C^{\rm ss}(\mathcal{L}_{|C}, L)$ is nonempty. In particular, $Z(L)^\circ$ acts trivially on $\mathcal{L}_{|C}$.*

Proof. By Lemma 4, there exist a positive integer N and a nonzero G-invariant section $\sigma \in \mathrm{H}^0(\mathbb{X}, \mathcal{L}^{\otimes N})^G$. By Lemma 10, the image of $\eta : G \times_P C^+ \longrightarrow \mathbb{X}$ contains a nonempty open subset of $\mathbb{X}$. We deduce that there exists $x \in C^+$ such that $\sigma(x) \neq 0$, $\mathbb{X}$ being irreducible and G-invariant,

Set $y = \lim_{t\to 0} \tau(t)x$. Consider the map θ_x defined in the proof of Lemma 14. The vanishing of $\mu^{\mathcal{L}}(C, \tau)$ means that $\theta_x^*(\mathcal{L})$ is trivial as a $\mathbb{C}^*$-linearized line bundle on $\mathbb{C}$. We deduce that $\tilde{y} := \lim_{t\to 0} \tau(t)\sigma(x)$ exists and belongs to $\mathcal{L}_y - \{y\}$. But, σ being G-invariant, $\tilde{y} = \sigma(y)$. In particular y belongs to $C^{\rm ss}(\mathcal{L}_{|C}, L)$.

Since $Z(L)$ acts trivially on C and σ is G-invariant, it fixes $\tilde{y}$. Then $Z(L)$ acts trivially on $\mathcal{L}_{|C}^{\otimes N}$ and $Z(L)^\circ$ acts trivially on $\mathcal{L}_{|C}$. ◻

We can now prove Step 1. Consider the group $X(T)^{Z(L)^\circ}$ of characters χ of T such that $\chi_{|Z(L)^\circ}$ is trivial. By Lemma 24, $\mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu)$ belongs to $\mathrm{Pic}^{L/Z(L)^\circ}(\underline{C})$. Set

$$\begin{array}{rcl} \gamma : X(T)^{Z(L)^\circ} \otimes \mathbb{Q} & \longrightarrow & \mathrm{Pic}^{L/Z(L)^\circ}(\underline{C}) \otimes \mathbb{Q} \\ \mu' & \longmapsto & (\mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu + \mu'))_{|\underline{C}}. \end{array}$$

The set of μ' such that some positive power of $\gamma(\mu')$ is ample and $\underline{C}^{\rm ss}(\gamma(\mu'), L/Z(L)^\circ)$ is nonempty is a convex set denoted by $\mathcal{C}^L(\underline{C})$. But, the image of γ is abundant in the sense of [DH98, Section 4.1]. As a consequence, for μ' general in $\mathcal{C}^L(\underline{C})$, there exist stable points in $\underline{C}$ for $\gamma(\mu')$ and the action of $L/Z(L)^\circ$. Fix such a μ' and N' such that $N'\mu' \in X(T)$. Then, $(N'\lambda_1, N'\lambda_2, N'(\mu+\mu'))$ still satisfies equality (20). But, for any line bundle $\mathcal{M}$ in $\underline{C}$, $\mathrm{H}^0(C, \pi^*(\mathcal{M}))$ is canonically isomorphic to $\mathrm{H}^0(\underline{C}, \mathcal{M})$. We deduce that some posive power of $\mathcal{L}_-(\lambda_1) \otimes \mathcal{L}_-(\lambda_2) \otimes \mathcal{L}(\mu + \mu'))$ on C admits nonzero G-invariant sections. Now, Theorem 6 implies that $(N'\lambda_1, N'\lambda_2, N'(\mu+\mu'))$ satisfies condition (21) for some N.

Replacing $(\lambda_1, \lambda_2, \mu)$ by $(N'\lambda_1, N'\lambda_2, N'(\mu + \mu'))$ if necessary, we may assume that $\underline{C}^{\rm s}(\underline{\mathcal{L}}, L/Z(L)^\circ)$ is nonempty.

Before proving Step 2, we construct a G-invariant map from an open subset $\underline{\mathbb{X}}^{\rm ss}(\underline{\mathcal{L}})$ onto the GIT-quotient $\underline{C}^{\rm ss}(\underline{\mathcal{L}}_{|\underline{C}}, L)/\!/L$.

Since L and $\underline{C}$ are finite-dimensional, the graded algebra $\oplus_k \mathrm{H}^0(\underline{C}, \underline{\mathcal{L}}_{|\underline{C}}^{\otimes k})^L$ is finitely generated. Fix $d > 0$ such that $\mathrm{H}^0(\underline{C}, \underline{\mathcal{L}}_{|\underline{C}}^{\otimes d})^L$ generates $\oplus_k \mathrm{H}^0(\underline{C}, \underline{\mathcal{L}}_{|\underline{C}}^{\otimes dk})^L$.

Let $\sigma_0,\dots,\sigma_D$ be a $\mathbb{C}$-basis of $\mathrm{H}^0(\underline{C},\underline{\mathcal{L}}^{\otimes d}_{|\underline{C}})^L$. By Theorem 6, for any i, σ_i extends to a G-invariant section $\tilde\sigma_i$ on $\underline{\mathbb{X}}$. Set

$$\underline{\mathbb{X}}^{\mathrm{ss}}(\mathcal{L}) := \{x\in\underline{\mathbb{X}}\,:\,\exists i\quad \tilde\sigma_i(x)\neq 0\},$$

and

$$\begin{array}{cccc}\psi: & \underline{\mathbb{X}}^{\mathrm{ss}}(\underline{\mathcal{L}}) & \longrightarrow & \mathbb{CP}^D\\ & x & \longmapsto & [\tilde\sigma_0(x):\cdots:\tilde\sigma_D(x)].\end{array}$$

We want to describe the image of ψ.

Consider $\underline{\eta}\,:\,G\times_P\underline{C}^+\longrightarrow\underline{\mathbb{X}}$. Let $\underline{C}^{+,\,\mathrm{ss}}(\underline{\mathcal{L}},L)$ denote the set of points $y\in\underline{C}^+$ such that $\lim_{t\to 0}\tau(t)y\in\underline{C}^{\mathrm{ss}}(\underline{\mathcal{L}},L)$. Since the $\tilde\sigma_i$'s are G-invariant, $\underline{\eta}(G\times_P\underline{C}^{+,\,\mathrm{ss}}(\underline{\mathcal{L}},L))$ is contained in $\underline{\mathbb{X}}^{\mathrm{ss}}(\mathcal{L})$. By the proof of Proposition 1, this set is dense in $\underline{\mathbb{X}}^{\mathrm{ss}}(\underline{\mathcal{L}})$, and hence in $\underline{\mathbb{X}}$ by irreducibility. But $\psi(\underline{\eta}((gP/P,x)))=\psi(g^{-1}x)$ for any $g\in G$, $x\in\underline{\mathbb{X}}^{\mathrm{ss}}(\underline{\mathcal{L}})$ such that $(gP/P,x)\in G\times_P\underline{C}$. Then $\psi\circ\underline{\eta}(G\times_P\underline{C}^{+,\mathrm{ss}}(\underline{\mathcal{L}},L))$ is contained in $\psi(\underline{C}^{\mathrm{ss}}(\underline{\mathcal{L}},L))=\underline{C}^{\mathrm{ss}}(\underline{\mathcal{L}},L)/\!/L$. The quotient variety $\underline{C}^{\mathrm{ss}}(\underline{\mathcal{L}},L)/\!/L$ being projective, this implies that

$$\psi(\underline{\mathbb{X}}^{\mathrm{ss}}(\underline{\mathcal{L}}))=\underline{C}^{\mathrm{ss}}(\underline{\mathcal{L}},L)/\!/L=\mathrm{Proj}(\oplus_k\mathrm{H}^0(\underline{C},\underline{\mathcal{L}}^{\otimes k}_{|\underline{C}})^L).$$

Fix a general point x_0 in $\underline{C}$. Let $g\in G$ and $y\in\underline{C}^+$ such that $(gP/P,y)\in\underline{\eta}^{-1}(x_0)$. Set $z=\lim_{t\to 0}\tau(t)y\in\underline{C}$. We prove the three following:

Claim a. z belongs to $L.x_0$.

Claim b. y belongs to $P.x_0$.

Claim c. $G_{x_0}P/P$ is finite.

These claims allow to conclude. Indeed, Claim b implies that $g\in G_{x_0}P$. Now Claim c shows that gP/P has finitely many possibilities. Since the restriction of the projection $G\times_P\underline{C}^+\longrightarrow G/P$ to any fiber of $\underline{\eta}$ is injective, we can conclude that $\underline{\eta}^{-1}(x_0)$ is finite and that asserion (24) holds.

Proof of Claim a. Since $\underline{C}^{\mathrm{s}}(\underline{\mathcal{L}},L/Z(L)^\circ)$ is open and nonempty it constains x_0. In particular, x_0 is semistable and there exists i such that $\sigma_i(x_0)\neq 0$.

Consider $\tilde y:=\tilde\sigma_i(y)=g^{-1}\tilde\sigma_i(x_0)\in(\underline{\mathcal{L}}^{\otimes d})_y-\{y\}$. Since $\mu^{\mathcal{L}}(C,\tau)=0$, $\tilde z:=\lim_{t\to 0}\tau(t)\tilde y\in(\mathcal{L}^{\otimes d})_z-\{z\}$. But $\tilde\sigma_i$ being G-invariant, $\tilde\sigma_i(z)=\tilde z$. Hence $z\in\underline{\mathbb{X}}^{\mathrm{ss}}(\underline{\mathcal{L}})\cap\underline{C}=\underline{C}^{\mathrm{ss}}(\mathcal{L})$. Moreover, ψ being G-invariant, we have $\psi(z)=\psi(x_0)$. The point x_0 being stable for the $(L/Z(L)^\circ)$-action, z belongs to the L-orbit of x_0.

Now, Claim b is a direct consequence of Lemma 11.3, since y and z belongs to the G-orbit of x_0.

Proof of Claim c. Since u_2 and w belong to W^P, $u_2^{-1}Q_2u_2$ and $w^{-1}Bw$ contain $B^-\cap L$ and $B\cap L$ respectively. Then $L/(B^-\cap L)\times L/(B\cap L)$ maps onto $Lu_2^{-1}Q_2/Q_2\times Lv^{-1}\underline{o}$. One deduces that the L-orbit of the base point $(u_2^{-1}Q_2/Q_2,v^{-1}\underline{o})$ is dense in $Lu_2^{-1}Q_2/Q_2\times Lv^{-1}\underline{o}$. Then, x_0 being general, there exists $l\in L$ such that $lx_0=(l_1u_1^{-1}Q_1/Q_1,u_2^{-1}Q_2/Q_2,v^{-1}\underline{o})=:x_1$, for some $l_1\in L$.

It is sufficient to prove Claim c, for x_1 in place of x_0. Note that $G_{x_1}\subset u_2^{-1}Q_2u_2\cap v^{-1}Bv$ is finite-dimensional. Moreover, it contains $\tau(\mathbb{C}^*)$. Then, the Lie algebra of G_{x_1} decomposes as $\mathrm{Lie}(G_{x_0})=(\mathrm{Lie}(G_{x_0})\cap\mathrm{Lie}(P^{u,-}))\oplus(\mathrm{Lie}(G_{x_0})\cap\mathrm{Lie}(P))$. Thus Lemma 25 below show that the neutral component $G^\circ_{x_1}$ of G_{x_1} is contained in P. Claim c follows. $\square$

Lemma 25. *Let $x \in \underline{C}^{\rm ss}(\underline{\mathcal{L}}_{|\underline{C}}, L)$. Then $G_x \cap P^{u,-}$ is trivial.*

Proof. Fix $\hat{x} \in C$ such that $\pi(\hat{x}) = x$. By Theorem 6, one can find a G-invariant section σ on $\mathbb{X}$ of some positive power $\mathcal{L}^{\otimes N}$ of $\mathcal{L}$ such that $\sigma(\hat{x}) \neq 0$. But $\mathrm{H}^0(\pi(\mathbb{X}), \underline{\mathcal{L}})$ is isomorphic to $\mathrm{H}^0(\mathbb{X}, \mathcal{L})$ and σ descends to a G-invariant section $\underline{\sigma}$ of $\underline{\mathcal{L}}$ on $\underline{\mathbb{X}}$. The set $\underline{\mathbb{X}}_\sigma = \{y \in \underline{\mathbb{X}} \,:\, \underline{\sigma}(y) \neq 0\}$ is a G-stable affine ind-variety containing x.

Write $x = (l_1 u_1^{-1} Q_1/Q_1, l_2 u_2^{-1} Q_2/Q_2, l v^{-1} \underline{o})$, with l_1, l_2 and l in L. Then $G_x \cap P^{u,-}$ is contained in $l(v^{-1}Bv \cap P^{u,-})l^{-1}$. By [Kum02, Example 6.1.5.b], $v^{-1}Bv \cap P^{u,-}$ is a finite-dimensional unipotent group. In particular, $G_x \cap P^{u,-}$ is connected and it is sufficient to prove that its Lie algebra is trivial.

Assume that there exists a nonzero vector $\xi \in \mathrm{Lie}(G_x \cap P^{u,-})$. Consider a morphism

$$\phi \,:\, \mathrm{SL}_2(\mathbb{C}) \longrightarrow G,$$

such that $T_1\phi(E) = \xi$, given by Proposition 11.1 below. Look the induced $\mathrm{SL}_2(\mathbb{C})$-action on $\underline{\mathbb{X}}$. The unipotent subgroup $U_2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{C})$ fixes the point x. Since $\mathrm{SL}_2(\mathbb{C})/U_2 \simeq \mathbb{C}^2 - \{(0,0)\}$, one gets a regular map

$$\bar{\phi} \,:\, \mathbb{C}^2 - \{(0,0)\} \longrightarrow \underline{\mathbb{X}}.$$

Since $\underline{\mathbb{X}}_\sigma$ is G-stable, the image of $\bar{\phi}$ is contained in $\underline{\mathbb{X}}_\sigma$. But $\underline{\mathbb{X}}_\sigma$ is an affine ind-variety and Arthog's lemma implies that $\bar{\phi}$ extends to a regular map

$$\tilde{\phi} \,:\, \mathbb{C}^2 \longrightarrow \underline{\mathbb{X}}.$$

By density, $\tilde{\phi}$ is $\mathrm{SL}_2(\mathbb{C})$-equivariant. In particular the point $\tilde{\phi}(0,0)$ is fixed by $\mathrm{SL}_2(\mathbb{C})$. This is a contradiction, since $v^{-1}\mathfrak{b}v$ contains no copy of $\mathfrak{sl}_2(\mathbb{C})$. □

9.3. **About $\Gamma(\mathfrak{g})$.** For any $n \in \mathbb{Z}$, $V(n\delta)$ is one-dimensional acted on by the character $n\delta$ of $\mathfrak{g}$. It follows that

$$\Gamma(\mathfrak{g}) = \Gamma_{\rm red}(\mathfrak{g}) + \mathbb{Q}(\delta, 0, \delta) + \mathbb{Q}(0, \delta, \delta),$$

where

$$\Gamma_{\rm red}(\mathfrak{g}) = \{(\lambda_1, \lambda_2, \mu) \in \Gamma(\mathfrak{g}) \,:\, \lambda_1(d) = \lambda_2(d) = 0\}.$$

For any $\lambda \in P_+$, the center $\mathbb{C}c$ of $\mathfrak{g}$ acts on $V(\lambda)$ with weight $\lambda(c) \in \mathbb{Z}$. Then

$$\Gamma_{\rm red}(\mathfrak{g}) \subset \Gamma(\mathfrak{g}) \subset \{(\lambda_1, \lambda_2, \mu) \in (\mathfrak{h}^*_{\mathbb{Q}})^3 \,:\, \mu(c) = \lambda_1(c) + \lambda_2(c)\}.$$

As an application of the GKO construction [GKO85] of representations of Virasoro algebras, Kac-Wakimoto obtained in [KW88] the following properties of the decomposition of $V(\lambda_1) \otimes V(\lambda_2)$.

Lemma 26. *Let λ_1, λ_2 in P_+ such that $\lambda_1(d) = \lambda_2(d) = 0$, $\lambda_1(c) > 0$ and $\lambda_2(c) > 0$. Let $\dot{\mu} \in \dot{P}$ and set $\bar{\mu} := \dot{\mu} + (\lambda_1(c) + \lambda_2(c))\Lambda \in P_+$.*

Then, $\bar{\mu} - \lambda_1 - \lambda_2 \in Q$ if and only if there exists $b \in \mathbb{Z}$ such that $V(\bar{\mu} + b\delta)$ is a sub-representation of $V(\lambda_1) \otimes V(\lambda_2)$.

Moreover, if $\bar{\mu} - \lambda_1 - \lambda_2 \in Q$ then one of the following two assertions holds:

(i) *there exists $b_0 \in \mathbb{Z}$ such that $V(\bar{\mu} + b\delta) \subset V(\lambda_1) \otimes V(\lambda_2)$ if and only if $b \leq b_0$;*

(ii) *there exists $b_0 \in \mathbb{Z}$ such that $V(\bar{\mu} + b\delta) \subset V(\lambda_1) \otimes V(\lambda_2)$ if and only if $b = b_0$ or $b \leq b_0 - 2$.*

Proof. The first assertion is proved in [KW88, p. 194]. The fact that $\{b \in \mathbb{Z} : V(\bar{\mu} + b\delta) \subset V(\lambda_1) \otimes V(\lambda_2)\}$ has an upper bound is proved in [KW88, p. 171]. Let b_0 be the maximum of such $b \in \mathbb{Z}$. It remains to prove that, for all $n \geq 2$, $V(\bar{\mu} + (b_0 - n)\delta)$ is contained in $V(\lambda_1) \otimes V(\lambda_2)$. This is a direct consequence of [KW88, Proof of Proposition 3.2]. See also [BK14, Proposition 4.2]. □

Lemma 26 allows to define

$$b_0(\lambda_1, \lambda_2, \bar{\mu}) = \max\{b \in \mathbb{Z} : V(\bar{\mu} + b\delta) \subset V(\lambda_1) \otimes V(\lambda_2)\}.$$

Remark 8. (i) [KW88, Inequality 2.4.1] implies that

$$b_0(\lambda_1, \lambda_2, \bar{\mu}) \leq \frac{(\dot{\lambda}_1 + 2\dot{\rho}, \dot{\lambda}_1)}{2(\ell_1 + h^\vee)} + \frac{(\dot{\lambda}_2 + 2\dot{\rho}, \dot{\lambda}_2)}{2(\ell_2 + h^\vee)} - \frac{(\dot{\mu} + 2\dot{\rho}, \dot{\mu})}{2(\ell_1 + \ell_2 + h^\vee)}.$$

This inequality is quadratic in $(\lambda_1, \lambda_2, \bar{\mu})$. In this paper, we show stronger linear inequalities.

(ii) If one takes $\ell_1 = 0$ in Lemma 26, one get $\dot{\lambda}_1 = 0$. Set $\lambda_1 = \dot{\lambda}_1 + \ell_1\Lambda = 0$, $\lambda_2 = \dot{\lambda}_2 + \ell_2\Lambda$ and $\mu = \dot{\mu} + \ell_2\Lambda$. We have $V(N0) \otimes V(N\lambda_2) = V(N\lambda_2)$, for any positive integer N. Hence $(0, \lambda_2, \mu)$ belongs to $\Gamma(\mathfrak{g})$ if and only if $\mu = \lambda_2$. In particular, the assumption "ℓ_1 positive" is necessary in Lemma 26. Observe that this implies that $\Gamma(\mathfrak{g})$ is not closed.

Set

$$\Gamma^\circ_{\rm red}(\mathfrak{g}) = \{(\lambda_1, \lambda_2, \mu) \in \Gamma_{\rm red}(\mathfrak{g}) : \lambda_1(c) > 0 \quad \text{and} \quad \lambda_2(c) > 0\},$$

and

$$\mathcal{A} = \{(\lambda_1, \lambda_2, \bar{\mu}) \in (X(\dot{T})_{\mathbb{Q}} \oplus \mathbb{Q}\Lambda)^3 : \begin{array}{l} \lambda_1, \lambda_2 \text{ and } \bar{\mu} \text{ are dominant} \\ \lambda_1(c) > 0,\ \lambda_2(c) > 0 \\ \bar{\mu}(c) = \lambda_1(c) + \lambda_2(c) \end{array} \}.$$

Define a function $\Psi : \mathcal{A} \longrightarrow \mathbb{R}$ by

$$\Psi(\lambda_1, \lambda_2, \bar{\mu}) = \sup_{\substack{N \in \mathbb{Z}_{>0} \text{ s.t.} \\ N\lambda_1, N\lambda_2, N\bar{\mu} \in \mathfrak{h}^*_{\mathbb{Z}} \\ N\bar{\mu} - N\lambda_1 - N\lambda_2 \in Q}} \frac{b_0(N\lambda_1, N\lambda_2, N\bar{\mu})}{N},$$

where b_0 is defined just after Lemma 26. This lemma implies that, for any $(\lambda_1, \lambda_2, \bar{\mu}) \in \mathcal{A}$, $(\lambda_1, \lambda_2, \bar{\mu} + b\delta)$ belongs to the closure of $\Gamma^\circ_{\rm red}(\mathfrak{g})$ in $\mathcal{A} \times \mathbb{R}$ if and only if

$$b \leq \Psi(\lambda_1, \lambda_2, \bar{\mu}).$$

9.4. **A cone defined by inequalities.** Consider the cone $\mathcal{C}$ of points $(\lambda_1, \lambda_2, \mu) \in (\mathfrak{h}^*_{\mathbb{Q}})^3$ such that

(i) $\lambda_1(c) > 0$ and $\lambda_2(c) > 0$;
(ii) λ_1, λ_2, and μ are dominant;
(iii) $\lambda_1(d) = \lambda_2(d) = 0$;
(iv) $\lambda_1(c) + \lambda_2(c) = \mu(c)$;

(v) the inequality

$$\langle \mu, v\varpi_{\alpha_i^\vee}\rangle \leq \langle \lambda_1, u_1\varpi_{\alpha_i^\vee}\rangle + \langle \lambda_2, u_2\varpi_{\alpha_i^\vee}\rangle \tag{25}$$

holds for any $i\in\{0,\dots,l\}$ and any $(u_1,u_2,v)\in W^{P_i}$ such that

$$^{\odot_0}n^v_{u_1u_2} = 1 \qquad \text{in } \mathrm{H}^*(G/P_i,\mathbb{Z}). \tag{26}$$

The aim of this section is to prove Theorem 1 or equivalently that $\Gamma^\circ_{\rm red}(\mathfrak{g}) = \mathcal{C}$ (see Theorem 10 below). We first study the cone $\mathcal{C}$.

9.5. **Realisation of $\mathcal{C}$ as an hypograph.** We just proved that $\Gamma^\circ_{\rm red}(\mathfrak{g})$ is the gypograph of Ψ. We are now proving a similar statement for the cone $\mathcal{C}$.

We endow $\dot{\mathfrak{h}}^*_{\mathbb{R}}$ with a $\dot W$-invariant Euclidean norm $\|\cdot\|$ such that $\|\dot\theta\|^2=2$. For $\mu\in X(\dot T)_{\mathbb{Q}}\oplus\mathbb{Q}\Lambda\oplus\mathbb{Q}\delta$, we denote by $\dot\mu$ (resp. $\bar\mu$) its projection on $X(\dot T)_{\mathbb{Q}}$ (resp. $X(\dot T)_{\mathbb{Q}}\oplus\mathbb{Q}\Lambda$).

Let $\mathcal{I}$ be the set of $(u_1,u_2,v,i)\in(W^{P_i})^3\times\{0,\dots,l\}$ satisfying condition (26). Fix $(u_1,u_2,v,i)\in\mathcal{I}$. Assume first that $i=0$. Let h_1,h_2 and h in $\dot Q^\vee$ such that $u_1W_{P_0}=h_1W_{P_0}$, $u_2W_{P_0}=h_2W_{P_0}$ and $vW_{P_0}=hW_{P_0}$. Define the restricted linear function $\varphi_{(u_1,u_2,v,0)}:\mathcal{A}\longrightarrow\mathbb{Q}$ that maps $(\lambda_1,\lambda_2,\bar\mu)$ to

$$\langle h_1,\dot\lambda_1\rangle + \langle h_2,\dot\lambda_2\rangle - \langle h,\dot\mu\rangle + \frac{\ell_1}{2}(\|h\|^2-\|h_1\|^2) + \frac{\ell_2}{2}(\|h\|^2-\|h_2\|^2), \tag{27}$$

where $\ell_1=\lambda_1(c)$ and $\ell_2=\lambda_2(c)$. Note that $\varpi_{\alpha_0^\vee}=d$ and for $h\in\dot Q^\vee$, by equation (19), we have $h\cdot d = h+d-\frac{(h,h)}{2}c$. For any $(\lambda_1,\lambda_2,\bar\mu)\in\mathcal{A}$, inequality (25) with $i=0$, is fulfilled by $(\lambda_1,\lambda_2,\bar\mu+b\delta)$ if and only if

$$b\leq\varphi_{(u_1,u_2,v,0)}(\lambda_1,\lambda_2,\bar\mu).$$

Assume now that $i\in\{1,\dots,l\}$. Write $u_1=\dot u_1h_1$, $u_2=\dot u_2h_2$ and $v=\dot vh$ with $\dot u_1,\dot u_2,\dot v\in\dot W$ and $h_1,h_2,h\in\dot Q^\vee$. Define the linear function $\varphi_{(u_1,u_2,v,i)}:\mathcal{A}\longrightarrow\mathbb{Q}$ that maps $(\lambda_1,\lambda_2,\bar\mu)$ to

$$\begin{array}{l}\langle \dot u_1(h_1+\frac{\varpi_{\alpha_i^\vee}}{\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle}),\dot\lambda_1\rangle + \langle \dot u_2(h_2+\frac{\varpi_{\alpha_i^\vee}}{\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle}),\dot\lambda_2\rangle - \langle \dot v(h+\frac{\varpi_{\alpha_i^\vee}}{\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle}),\dot\mu\rangle\\ +\frac{\ell_1}{2}(\|h\|^2-\|h_1\|^2+2\frac{(\varpi_{\alpha_i^\vee},h-h_1)}{\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle})\\ +\frac{\ell_2}{2}(\|h\|^2-\|h_2\|^2+2\frac{(\varpi_{\alpha_i^\vee},h-h_2)}{\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle}).\end{array} \tag{28}$$

Recall that $\varpi_{\alpha_i^\vee}=\dot\varpi_{\alpha_i^\vee}+\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle d$. Moreover, for $w=\dot wh\in W$, by equation (19), we have

$$(\dot wh)\cdot\varpi_{\alpha_i^\vee} = \dot w\dot\varpi_{\alpha_i^\vee}+\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle\dot wh+\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle d-\left(\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle\frac{(h,h)}{2}+(\dot\varpi_{\alpha_i^\vee},h)\right)c.$$

Then inequality (25), is fulfilled by $(\lambda_1,\lambda_2,\bar\mu+b\delta)$ if and only if

$$b\leq\varphi_{(u_1,u_2,v,i)}(\lambda_1,\lambda_2,\bar\mu).$$

Define

$$\begin{array}{cccc}\varphi: & \mathcal{A} & \longrightarrow & \mathbb{R}\cup\{-\infty\}\\ & (\lambda_1,\lambda_2,\bar\mu) & \longmapsto & \inf_{(u_1,u_2,v,i)\in\mathcal{I}}\varphi_{(u_1,u_2,v,i)}(\lambda_1,\lambda_2,\bar\mu).\end{array}$$

Then φ is a concave function and $\mathcal{C}$ is the hypograph of φ:

$$\mathcal{C}=\{(\lambda_1,\lambda_2,\bar\mu+b\delta)\,:\,(\lambda_1,\lambda_2,\bar\mu)\in\mathcal{A}\text{ and }b\leq\varphi(\lambda_1,\lambda_2,\bar\mu)\}.$$

9.6. **The convex set $\mathcal{C}$ is locally polyhedral.**

Proposition 4. *Let $\vec{\lambda}_0 \in \mathcal{A}$. Then*

$$\forall M \in \mathbb{R} \quad \exists \text{ an open set } \mathcal{U} \ni \vec{\lambda}_0 \quad \exists \mathcal{J} \subset \mathcal{I} \text{ finite}$$

such that

$$\forall a \in \mathcal{I} - \mathcal{J} \qquad \forall \vec{\lambda} \in \mathcal{U} \qquad \varphi_a(\vec{\lambda}) \geq M.$$

Proof. Let $\vec{\lambda} = (\lambda_1, \lambda_2, \bar{\mu}) \in \mathcal{A}$. Fix $M \in \mathbb{R}$. Let $(u_1, u_2, v, 0) \in \mathcal{I}$. By Proposition 1, the inequality (25) is satisfied by $(\ell\Lambda, 0, \ell\Lambda)$ for any $\ell > 0$. Hence

$$\|h\|^2 - \|h_1\|^2 \geq 0.$$

Similarly, $\|h\|^2 - \|h_2\|^2 \geq 0$. Hence, $\varphi_{(u_1,u_2,v,0)}(\vec{\lambda})$ is greater or equal to

$$\langle h_1, \dot{\lambda}_1 \rangle + \langle h_2, \dot{\lambda}_2 \rangle - \langle h, \dot{\mu} \rangle + \frac{\ell}{2}(2\|h\|^2 - \|h_1\|^2 - \|h_2\|^2), \tag{29}$$

where $\ell = \min(\lambda_1(c), \lambda_2(c))$. Using Lemma 18, one gets $\delta^v_{u_1\,u_2} = \langle \rho, -d - h \cdot d + h_1 \cdot d + h_2 \cdot d \rangle$. Hence

$$\delta^v_{u_1\,u_2} = \langle \dot{\rho}, h_1 + h_2 - h \rangle + \frac{h^\vee}{2}(\|h\|^2 - \|h_1\|^2 - \|h_2\|^2).$$

Lemma 17 implies that $\delta^v_{u_1\,u_2} \geq 0$ and

$$\|h\|^2 - \|h_1\|^2 - \|h_2\|^2 \geq \frac{2}{h^\vee} \langle h - h_1 - h_2, \dot{\rho} \rangle.$$

Then $\varphi_{(u_1,u_2,v,0)}(\vec{\lambda})$ is greater or equal to

$$\frac{\ell}{2}\|h\|^2 - \|h_1\|.\|\dot{\lambda}_1\| - \|h_2\|.\|\dot{\lambda}_2\| - \|h\|.\|\dot{\mu}\| - \frac{2\|\dot{\rho}\|}{h^\vee}(\|h_1\| + \|h_2\| + \|h\|). \tag{30}$$

By construction there exist $\dot{u}_1$, $\dot{u}_2$ and $\dot{v}$ in $\dot{W}$ such that $u_1 = h_1\dot{u}_1$, $u_2 = h_2\dot{u}_2$ and $v = h\dot{v}$. But $l(v) = l(u_1) + l(u_2)$. Then Lemma 11.4 implies that

$$N + \sqrt{2}N\|h\| \geq l(v) \geq l(u_1) \geq K\|h_1\| - N,$$

where $K \in \mathbb{R}^{+,*}$ and $N = \sharp\dot{\Phi}^+$.

We deduce that

$$\max(\|h_1\|, \|h_2\|) \leq \frac{N}{K}(2 + \sqrt{2}\|h\|). \tag{31}$$

The point is that this implies that $\varphi_{(u_1,u_2,v,0)}(\vec{\lambda})$ is greater or equal to $\frac{\ell}{2}\|h\|^2$ minus terms that are linear in $\|h\|$. We deduce that there exist an open neighborhood $\mathcal{U}_0$ of $\vec{\lambda}_0$ and $A_0 \in \mathbb{R}$ such that

$$\forall \vec{\lambda} \in \mathcal{U}_0 \quad \forall a = (u_1, u_2, v, 0) \in \mathcal{I} \qquad l(v) \geq A_0 \Rightarrow \varphi_a(\vec{\lambda}) \geq M.$$

Fix $(u_1, u_2, v, i) \in \mathcal{I}$ with $i > 0$ and consider the associated linear function $\varphi_{(u_1,u_2,v,i)}$. Since $(\ell\Lambda, 0, \ell\Lambda) \in \Gamma(\mathfrak{g})$ for any $\ell > 0$, Proposition 1 implies that

$$\|h\|^2 - \|h_1\|^2 + 2\frac{(\varpi_{\alpha_i^\vee}, h - h_1)}{\langle \dot{\varpi}_{\alpha_i^\vee}, \dot{\theta} \rangle} \geq 0.$$

Set $E := \frac{\|\dot\varpi_{\alpha_i^\vee}\|}{\langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle}$ and $F := \langle\dot\varpi_{\alpha_i^\vee},\dot\theta\rangle$. Then, $\varphi_{(u_1,u_2,v,i)}(\vec\lambda)$ is greater or equal to

$$\begin{array}{c}-\left(\|h_1\|.\|\dot\lambda_1\|+\|h_2\|.\|\dot\lambda_2\|+\|h\|.\|\dot\mu\|\right)-E(\|\dot\lambda_1\|+\|\dot\lambda_2\|+\|\dot\mu\|)\\+\frac{\ell}{2}\left(2\|h\|^2-\|h_1\|^2-\|h_2\|^2-2E(\|h-h_1\|+\|h-h_2\|)\right).\end{array}\tag{32}$$

where $\ell=\min(\lambda_1(c),\lambda_2(c))$. But $\delta^v_{u_1\,u_2}\geq 0$ implies that

$$\begin{array}{rl}\|h\|^2-\|h_1\|^2-\|h_2\|^2\geq & \frac{2}{F\ell^\vee}(1-\langle\dot\rho,-\dot v\dot\varpi_{\alpha_i^\vee}+\dot u_1\dot\varpi_{\alpha_i^\vee}+\dot u_2\dot\varpi_{\alpha_i^\vee}\rangle)\\ & -\frac{2}{\ell^\vee}(\langle\dot\rho,-\dot vh+\dot u_1h_1+\dot u_2h_2\rangle)\\ & -\frac{2}{F}(\dot\varpi_{\alpha_i^\vee},h-h_1-h_2).\end{array}$$

Combining these inequalities with inequality (31) one can get a lower bound for $\varphi_{(u_1,u_2,v,i)}(\vec\lambda)$ equals to $\frac{\ell}{2}\|h\|^2$ minus terms that are linear in $\|h\|$. One can easily deduce that there exist an open neighborhood $\mathcal{U}$ of $\vec\lambda_0$ and $A\in\mathbb{R}$ such that

$$\forall\vec\lambda\in\mathcal{U}\quad\forall a=(u_1,u_2,v,i)\in\mathcal{I}\qquad l(v)\geq A\;\Rightarrow\;\varphi_a(\vec\lambda)\geq M.$$

But there exist only finitely many triples (u_1,u_2,v) with $l(v)<A$ and $l(v)=l(u_1)+l(u_2)$. The proposition follows. □

Remark 9. Proposition 4 is still true with the family of equations corresponding to any parabolic subgroup P, any $\alpha_i\in\Delta-\Delta(P)$ and any coefficient $n^v_{u_1\,u_2}\neq 0$. The same proof works.

We now use Proposition 4 to prove that $\mathcal{C}$ is locally polyhedral. Let $\vec\lambda_0=(\lambda_1,\lambda_2,\bar\mu)$ in $\mathcal{A}$. By Proposition 4, there exists a neighborhood $\mathcal{U}$ of $\vec\lambda_0$ and $\mathcal{J}\subset\mathcal{I}$ finite such that

$$\forall a\in\mathcal{I}-\mathcal{J}\qquad\forall\vec\lambda\in\mathcal{U}\qquad\varphi_a(\vec\lambda)\geq\varphi(\vec\lambda_0)+1.\tag{33}$$

In particular, there exists a_0 in $\mathcal{J}$ and hence in $\mathcal{I}$ such that

$$\varphi(\vec\lambda_0)=\inf_{a\in\mathcal{I}}\varphi_a(\vec\lambda_0)=\min_{a\in\mathcal{J}}\varphi_a(\vec\lambda_0)=\varphi_{a_0}(\vec\lambda_0).$$

By continuity of the function φ_{a_0}, up to changing $\mathcal{U}$ by a smaller neighborhood if necessary, one may assume that

$$\forall\vec\lambda\in\mathcal{U}\qquad\varphi_{a_0}(\vec\lambda)\leq\varphi_{a_0}(\vec\lambda_0)+1.$$

Then, assertion (33) implies that for any $\vec\lambda\in\mathcal{U}$

$$\varphi(\vec\lambda)=\min_{a\in\mathcal{J}}\varphi_a(\vec\lambda).\tag{34}$$

Choose a simplex $\mathcal{S}$ containing $\vec\lambda_0$ in its interior such that $\mathcal{S}\cap\mathcal{A}\subset\mathcal{U}$. Up to replacing $\mathcal{U}$ by $\mathcal{S}\cap\mathcal{A}$, one may assume that $\mathcal{U}$ is a convex polytope. Then formula (34) shows that $\mathcal{C}\cap(\mathcal{U}+\mathbb{Q}(0,0,\delta))$ is a polyhedron.

For any $a\in\mathcal{I}$, we set

$$\mathcal{A}_a=\{\vec\lambda\in\mathcal{A}\,:\,\varphi(\vec\lambda)=\varphi_a(\vec\lambda)\}.$$

The properties of these sets are summarized in the following lemma. The *dimension* of a convex set is defined to be the dimension of the spanned affine space.

Lemma 27. *With above notation,*

(i) *For any $a \in \mathcal{A}$, the set $\mathcal{A}_a$ is convex.*

(ii) *Set $\mathcal{I}_1 = \{(a \in \mathcal{I} \mid \dim(\mathcal{A}_a) = \dim(\mathcal{A})\}$. Then*

$$\mathcal{A} = \cup_{a \in \mathcal{I}_1} \mathcal{A}_a. \tag{35}$$

(iii) *The sets associated with two different elements of $\mathcal{I}_1$ only intersect along their boundaries.*

Proof. The first assertion follows from the linearity of functions $\varphi_{(u_1,u_2,v,i)}$. Since $\mathcal{C}$ is locally polyhedral, it is the union of its codimension one faces. The two last assertions follow. □

The cone $\mathcal{C}$ being locally polyhedral, we have

$$\mathcal{C} = \{(\lambda_1, \lambda_2, \bar\mu + b\delta) : \quad \begin{array}{l} (\lambda_1, \lambda_2, \bar\mu) \in \mathcal{A} \\ b \leq \varphi_a(\lambda_1, \lambda_2, \bar\mu) \qquad \forall a \in \mathcal{I}_1\}. \end{array}$$

Two convex sets $\mathcal{A}_a$ and $\mathcal{A}_{a'}$ are said to be *adjacent* if their intersection has codimension 1 in $\mathcal{A}$.

9.7. **An example of a codimension one face.** Consider the element $(e, e, e, 0) \in \mathcal{I}$. The associated inequality (25) is $b \leq 0$. Moreover, G/P_0 is the affine Grassmannian $\mathcal{G}r_{\dot G}$ and the semi-simple component of the Levi subgroup L_0 is $\dot G$.

Lemma 28. *Let $(\dot\lambda_1 + \ell_1\Lambda, \dot\lambda_2 + \ell_2\Lambda, \dot\mu + (\ell_1 + \ell_2)\Lambda) \in (P_+)^3$. Then $V(\dot\mu + (\ell_1 + \ell_2)\Lambda)$ is contained in $V(\dot\lambda_1 + \ell_1\Lambda) \otimes V(\dot\lambda_2 + \ell_2\Lambda)$ if and only if $V_{\dot G}(\dot\mu)$ is contained in $V_{\dot G}(\dot\lambda_1) \otimes V_{\dot G}(\dot\lambda_2)$.*

In particular, $\mathcal{A}_{(e,e,e,0)}$ has nonempty interior in $\mathcal{A}$. That is $(e, e, e, 0)$ belongs to $\mathcal{I}_1$.

Proof. The first assertion is certainly well known. It can also be obtained as a consequence of Theorem 6. Indeed, in $\mathrm{H}^*(\mathcal{G}r_{\dot G}, \mathbb{Z})$, we have $n^e_{e,e} = 1$. For $\tau = \varpi_{\alpha_0^\vee}$, $(\dot\lambda_1 + \ell_1\Lambda, \dot\lambda_2 + \ell_2\Lambda, \dot\mu + (\ell_1 + \ell_2)\Lambda)$ satisfies equality (11). Corollary 1 shows that the multiplicity of $V(\dot\mu + (\ell_1 + \ell_2)\Lambda)$ in $V(\dot\lambda_1 + \ell_1\Lambda) \otimes V(\dot\lambda_2 + \ell_2\Lambda)$ is equal to those of $V_{\dot G}(\dot\mu)$ in $V_{\dot G}(\dot\lambda_1) \otimes V_{\dot G}(\dot\lambda_2)$.

It is well known (see e.g. [PR13, Theorem 1.4]) that $\Gamma(\dot{\mathfrak{g}})$ has nonempty interior in $(X(\dot T)_\mathbb{Q})^3$. But for any given $(\dot\lambda_1, \dot\lambda_2, \dot\mu) \in \Gamma(\dot{\mathfrak{g}})$, $(\dot\lambda_1 + \ell_1\Lambda, \dot\lambda_2 + \ell_2\Lambda, \dot\mu + (\ell_1 + \ell_2)\Lambda) \in \Gamma(\mathfrak{g})$ for any $\ell_1 \geq \langle \dot\lambda_1, \dot\theta^\vee \rangle$, $\ell_2 \geq \langle \dot\lambda_2, \dot\theta^\vee \rangle$ and $\ell_1 + \ell_2 \geq \langle \dot\mu, \dot\theta^\vee \rangle$. The second assertion follows. □

9.8. **The main result.**

Theorem 10. *With the above notation, we have*

$$\Gamma^\circ_{\mathrm{red}}(\mathfrak{g}) = \mathcal{C}.$$

Proof. The inclusion $\Gamma^\circ_{\mathrm{red}}(\mathfrak{g}) \subset \mathcal{C}$ is a direct consequence of Proposition 1. We have to prove that $\mathcal{C}$ is contained in $\Gamma^\circ_{\mathrm{red}}(\mathfrak{g})$ or in $\Gamma(\mathfrak{g})$; that is, that

$$\forall \vec\lambda \in \mathcal{A} \qquad \varphi(\vec\lambda) \leq \Psi(\vec\lambda). \tag{36}$$

Still equivalently, we have to prove that

$$\forall\vec\lambda\in\mathcal{A}\qquad \vec\lambda+(0,0,\varphi(\vec\lambda)\delta)\in\Gamma(\mathfrak{g}) \tag{37}$$

Let $\vec\lambda=(\lambda_1,\lambda_2,\bar\mu)\in\mathcal{A}$. Since $\mathcal{C}$ is locally polyhedral (see (34)), there exists $a\in\mathcal{I}_1$ such that $\varphi(\vec\lambda)=\varphi_a(\vec\lambda)$. To sum up, it is sufficient to prove that, for any $a\in\mathcal{I}_1$, we have

$$\forall\vec\lambda\in\mathcal{A}_a\qquad \vec\lambda+(0,0,\varphi(\vec\lambda)\delta)\in\Gamma(\mathfrak{g}). \tag{38}$$

Let $\mathcal{I}_1^0$ denote the set of elements in $\mathcal{I}_1$ satisfying condition (38). In order to prove that $\mathcal{I}_1^0=\mathcal{I}_1$, for $a\in\mathcal{I}$, consider the following assumption

$$\text{(H)}\qquad \exists\vec\lambda\in\mathcal{A}_a\text{ regular such that }\vec\lambda+(0,0,\varphi(\vec\lambda)\delta)\in\Gamma(\mathfrak{g}).$$

We prove that $\mathcal{I}_1^0=\mathcal{I}_1$ in three steps:

Claim 1. Any $a\in\mathcal{I}_1$ satisfying assumption (H) belongs to $\mathcal{I}_1^0$.
Claim 2. The element $(e,e,e,0)$ belongs to $\mathcal{I}_1$ and satisfies assumption (H).
Claim 3. If one of two adjacent elements of $\mathcal{I}_1$ satisfies assumption (H) then both satisfy assumption (H).

These claims are sufficient. Indeed, for any $a\in\mathcal{I}_1$, there exists a sequence $a=a_0,\dots,a_n=(e,e,e,0)$ such that $\mathcal{A}_{a_i}$ and $\mathcal{A}_{a_{i+1}}$ are adjacent for any i. By Claim 2, a_n satisfies assumption (H). Thus by an immediate induction and Claim 3, a satisfies (H). Now Claim 1 shows that $a\in\mathcal{I}_1^0$.

Proof of Claim 2. It is a direct consequence of Lemma 28.

Proof of Claim 1 $\Rightarrow$ Claim 3. Let a and a' in $\mathcal{I}_1$ be such that $\mathcal{A}_a$ and $\mathcal{A}_{a'}$ are adjacent along their face $\mathcal{A}'$. Assume that a satisfies assumption (H). Then the interior of $\mathcal{A}'$ is contained in the interior of $\mathcal{A}_a\cup\mathcal{A}_{a'}$. In particular, $\mathcal{A}'$ contains regular weights and a' satisfies assumption (H). Since φ_a and $\varphi_{a'}$ coincide on $\mathcal{A}'$, if one of a and a' satisfies (H) both satisfy it by Claim 1.

Proof of Claim 1. Let $a=(\bar u_1,\bar u_2,\bar v,i)\in\mathcal{I}_1$ satisfying assumption (H). Set $C=L_i\bar u_1^{-1}\underline{o}^-\times L_i\bar u_2^{-1}\underline{o}^-\times L_i\bar v^{-1}\underline{o}$. For $(\lambda_1,\lambda_2,\mu)\in(P_{+,\mathbb{Q}})^3$, denote by $C^{\rm ss}(\lambda_1,\lambda_2,\mu,L_i)$ the set of points in C that are semi-stable for the action of L_i and the restriction of the line bundle $\mathcal{L}$ on $\mathbb{X}$ associated to $(\lambda_1,\lambda_2,\mu)$. Consider

$$\mathcal{C}^{L_i}(C)=\{(\vec\lambda,b)\in\mathcal{A}\times\mathbb{Q}\;:\;C^{\rm ss}(\vec\lambda+(0,0,b\delta),L_i)\neq\emptyset\}.$$

Let $\vec\lambda=(\lambda_1,\lambda_2,\bar\mu)$ in $\mathcal{A}$, b in $\mathbb{Q}$. Set $\mu=\bar\mu+b\delta$. By [BK06], $\mathcal{C}^{L_i}(C)$ is a convex polyhedral cone determined by an explicit finite list of linear inequalities. Namely, $(\vec\lambda,b)$ belongs to $\mathcal{C}^{L_i}(C)$ if and only if

(i) $\bar u_1^{-1}\lambda_1$, $\bar u_2^{-1}\lambda_2$ and $\bar v^{-1}\mu$ are dominant for L_i;
(ii) $Z(L_i)^\circ$ acts trivially on $\mathcal{L}_{|C}$;
(iii) for any $j\in\{0,\dots,l\}-\{i\}$

$$\langle\tilde v\varpi_{\alpha_j^\vee},\bar v^{-1}\mu\rangle\le\langle\tilde u_1\varpi_{\alpha_j^\vee},\bar u_1^{-1}\lambda_1\rangle+\langle\tilde u_2\varpi_{\alpha_j^\vee},\bar u_2^{-1}\lambda_2\rangle, \tag{39}$$

for any $(\tilde u_1,\tilde u_2,\tilde v)\in W_{L_i}^{P_j^i}$ such that $\epsilon_{\tilde v}(L_i/P_j^i)$ appears with coefficient one in $\epsilon_{\tilde u_1}(L_i/P_j^i)\odot_0\epsilon_{\tilde u_2}(L_i/P_j^i)$.

Here P^i_j is the maximal standard parabolic subgroup of L_i associated with j. Note that condition (ii) can be rewritten as

$$\langle \bar{v}\varpi_{\alpha_i^\vee}, \mu\rangle = \langle \bar{u}_1\varpi_{\alpha_i^\vee}, \lambda_1\rangle + \langle \bar{u}_2\varpi_{\alpha_i^\vee}, \lambda_2\rangle. \tag{40}$$

In particular it implies that $b = \varphi_a(\vec{\lambda})$.

On the other hand, Theorem 6 shows that $\vec{\lambda} + (0, 0, b\delta)$ belongs to $\Gamma(\mathfrak{g})$. Thus $b \leq \Psi(\vec{\lambda})$. Finally, the points $(\vec{\lambda}, b)$ of $\mathcal{C}^{L_i}(C)$ satisfy

$$b = \varphi_a(\vec{\lambda}) = \varphi(\vec{\lambda}) = \Psi(\vec{\lambda}) \text{ and } \vec{\lambda} \in \mathcal{A}_a. \tag{41}$$

Conversely, we claim that for any $\vec{\lambda}$ in $\mathcal{A}_a$, $(\vec{\lambda}, \varphi_a(\vec{\lambda}))$ belongs to $\mathcal{C}^{L_i}(C)$. By (41), this claim implies Claim 1 and ends the proof of the theorem.

Consider the set $\overline{\mathcal{C}^{L_i}(C)}$ of $\vec{\lambda} \in \mathcal{A}$ such that $(\vec{\lambda}, \varphi_a(\vec{\lambda}))$ belongs to $\mathcal{C}^{L_i}(C)$. It is the linear projection of $\mathcal{C}^{L_i}(C)$. By assumption (H) and Theorem 6, $\overline{\mathcal{C}^{L_i}(C)}$ intersects the interior of $\mathcal{A}$. Let $\mathfrak{l}_i^{\rm ss}$ be the semisimple part of the Lie algebra $\mathfrak{l}_i$ of L_i. Since $\Gamma(\mathfrak{l}_i^{\rm ss})$ is full dimensional and $\overline{\mathcal{C}^{L_i}(C)}$ intersects the interior of $\mathcal{A}$, one deduces that $\overline{\mathcal{C}^{L_i}(C)}$ is full dimensional in $\mathcal{A}$.

Recall that we want to show that the two full dimensional sub-polyhedron of $\mathcal{A}$, $\overline{\mathcal{C}^{L_i}(C)}$ and $\mathcal{A}_a$ coincide, knowing that $\overline{\mathcal{C}^{L_i}(C)}$ is contained in $\mathcal{A}_a$. The general theory of convex polyhedrons implies that it is sufficient to check the conditions (39) such that the associated face of $\mathcal{C}^{L_i}(C)$ has codimension one and intersects the interior of $\mathcal{A}$. Consider such an inequality associated with $(\tilde{u}_1, \tilde{u}_2, \tilde{v}, j)$ and the four flag varieties

$$\begin{tikzcd}
L_i/(P^i_j) \arrow[r] & G/(P_{i,j}) \arrow[dl] \arrow[dr] & \\
G/P_i & & G/P_j
\end{tikzcd}$$

Here $P_{i,j} = P_i \cap P_j$ and we used that $P^i_j = L_i \cap P_j$.

By Lemma 20, $(u_1 = \bar{u}_1\tilde{u}_1, u_2 = \bar{u}_2\tilde{u}_2, v = \bar{v}\tilde{v}) \in (W^{P_{i,j}})^3$. By Proposition 3, ϵ_v appears with multiplicity one in $\epsilon_{u_1}.\epsilon_{u_2}$, in $\mathrm{H}^*(G/P_{i,j}, \mathbb{Z})$. Set $\tau = \varpi_{\alpha_i^\vee} + \varpi_{\alpha_j^\vee}$. Then, by Proposition 1

$$\langle v\tau, \mu'\rangle \leq \langle u_1\tau, \lambda'_1\rangle + \langle u_2\tau, \lambda'_2\rangle, \tag{42}$$

for any $(\lambda'_1, \lambda'_2, \mu') \in \Gamma(\mathfrak{g})$. Let $\vec{\lambda}''$ be an integral point in the interior of $\mathcal{A}$, in $\mathcal{C}^{L_i}(C)$ and such that inequality (39) is an equality. Then inequality (42) is an equality for $\vec{\lambda}'' + (0, 0, \varphi_{(\bar{u}_1, \bar{u}_2, \bar{v}, i)}(\vec{\lambda}'')\delta)$. The last weight of $\vec{\lambda}''$ being regular, Theorem 7 implies that ϵ_v appears with multiplicity one in $\epsilon_{u_1} \odot_0 \epsilon_{u_2}$.

Consider now the three elements $\bar{u}'_1$, $\bar{u}'_2$ and $\bar{v}'$ in W^{P_j} such that $\bar{u}'_1 W_{P_j} = u_1 W_{P_j}$, $\bar{u}'_2 W_{P_j} = u_2 W_{P_j}$ and $\bar{v}' W_{P_j} = v W_{P_j}$. Lemma 23 and Proposition 3 imply that, in $\mathrm{H}^*(G/P_j, \mathbb{Z})$, $\epsilon_{\bar{v}'}$ appears with multiplicity one in $\epsilon_{\bar{u}'_1} \odot_0 \epsilon_{\bar{u}'_2}$. By definition of $\mathcal{C}$, any point $(\lambda_1, \lambda_2, \mu)$ in it satisfies

$$\langle \mu, \bar{v}'\varpi_{\alpha_j^\vee}\rangle \leq \langle \lambda_1, \bar{u}'_1\varpi_{\alpha_j^\vee}\rangle + \langle \lambda_2, \bar{u}'_2\varpi_{\alpha_j^\vee}\rangle. \tag{43}$$

But, modulo equality (40), inequality (39) is equivalent to inequality (43). Since for any point $\vec{\lambda}$ of $\mathcal{A}_a$, the point $\vec{\lambda} + (0, 0, \varphi_a(\vec{\lambda})\delta)$ satisfies both (40) and (43), it satisfies (39). We conclude that $\mathcal{A}_a$ is contained in $\mathcal{C}^{L_i}(C)$. $\square$

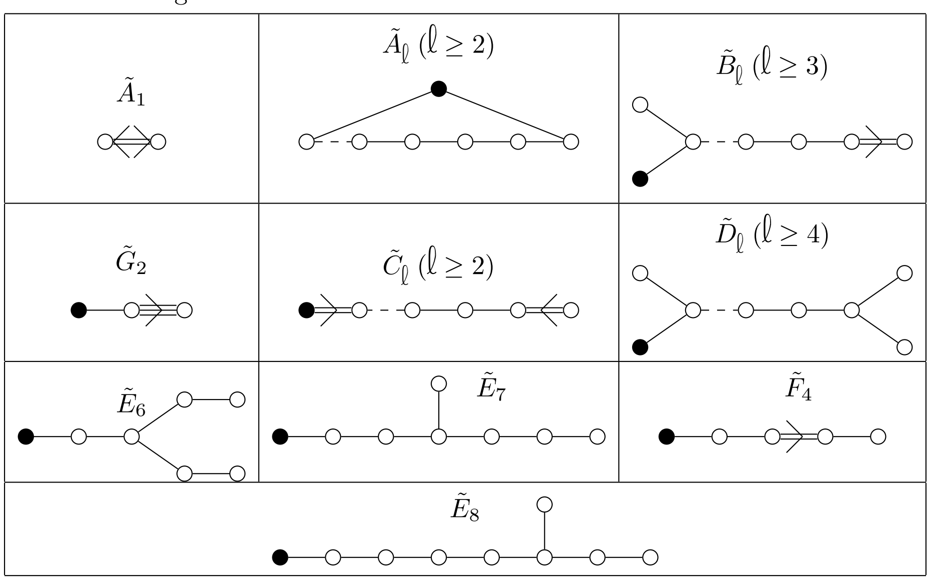
10. Saturation factors

In this section, we prove Theorems 3 and 4 of the introduction. Let us first check the computation of the constants k_s.

In the finite-dimensional case, known saturation factors are collected in the following tabular. These results was obtained in [KT99] for the type A_ℓ, in [KM06] for B_2 and G_2, in [HS15] for type B_ℓ, [BK10] for the type C_ℓ, in [KKM09] for D_4 and in [KM08] for the remaining cases.

Type	A_ℓ	$B_\ell(\ell \geq 3)$	$C_\ell(\ell \geq 2)$	D_4	$D_\ell(\ell \geq 5)$
Saturation factor	1	2	2	1	4
Type	E_6	E_7	E_8	F_4	G_2
Saturation factor	36	144	3 600	144	2, 3

Using these datas, one can easily check the computations of k_s given in the introduction by reading the Dynkin diagrams. Indeed, k_s was defined to be the least common multiple of saturation factors of maximal Levi subalgebras of $\mathfrak{g}$. But these Levi subalgebras are finite dimensional.



Proof of Theorems 3 and 4. Let $(\lambda_1, \lambda_2, \mu) \in (P_+)^3$ such that $\mu - \lambda_1 - \lambda_2 \in Q$ and there exists $N > 0$ such that $(N\lambda_1, N\lambda_2, N\mu) \in \Gamma_{\mathbb{N}}(\mathfrak{g})$. Up to tensoring with $V(\delta)$ one may assume that $\lambda_1(d) = \lambda_2(d) = 0$. Write μ as $\bar{\mu} + n\delta$, with $n \in \mathbb{Z}$ and $\bar{\mu} \in X(\dot{T})$.

Set $b = \varphi(\lambda_1, \lambda_2, \bar{\mu})$. By formula (34), there exists $(u_1, u_2, v, i) \in \mathcal{I}$ such that $b = \varphi_{(u_1,u_2,v,i)}(\lambda_1, \lambda_2, \bar{\mu})$.

We claim that $bk_{\dot{\mathfrak{g}}}$ is an integer. The norm on $\dot{Q}^\vee$ is normalized by $\|\dot{\alpha}^\vee\|^2 = 2$ for a short coroot $\dot{\alpha}^\vee \in \dot{\Phi}^\vee$. Then, case-by-case consideration allows to prove that, for any $h \in \dot{Q}^\vee$, $\frac{\|h\|^2}{2} \in \mathbb{Z}$. Now, formula (27) shows that $b \in \mathbb{Z}$ if $i = 0$. If $i > 0$, formula (28) shows that $k_{\dot{\mathfrak{g}}} b \in \mathbb{Z}$.

Consider $(k_{\dot{\mathfrak{g}}}\lambda_1, k_{\dot{\mathfrak{g}}}\lambda_2, k_{\dot{\mathfrak{g}}}\bar{\mu} + (k_{\dot{\mathfrak{g}}}b)\delta)$. Recall that $Q = \dot{Q} + \mathbb{Z}\delta$ and $\mu - \lambda_1 - \lambda_2 \in Q$. In particular $k_{\dot{\mathfrak{g}}}\bar{\mu} + (k_{\dot{\mathfrak{g}}}b)\delta - k_{\dot{\mathfrak{g}}}\lambda_1 - k_{\dot{\mathfrak{g}}}\lambda_2 \in Q$. For any $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ and $w \in W$,

$\lambda - w\lambda \in Q$. Hence, $v^{-1}k_{\dot{\mathfrak{g}}}\bar{\mu} + (k_{\dot{\mathfrak{g}}}b)\delta - u_1^{-1}k_{\dot{\mathfrak{g}}}\lambda_1 - u_2^{-1}k_{\dot{\mathfrak{g}}}\lambda_2$ also belongs to Q. Since $(N\lambda_1, N\lambda_2, N(\bar{\mu}+b\delta)) \in \Gamma_{\mathbb{N}}(\mathfrak{g})$, Corollary 1 implies that $(Nu_1^{-1}\lambda_1, Nu_2^{-1}\lambda_2, Nv^{-1}(\bar{\mu}+b\delta))$ belongs to $\Gamma_{\mathbb{N}}(\mathfrak{l}_i)$. But k_s is a saturation factor for the group L_i. Then $(u_1^{-1}k_sk_{\dot{\mathfrak{g}}}\lambda_1, u_2^{-1}k_sk_{\dot{\mathfrak{g}}}\lambda_2, v^{-1}k_sk_{\dot{\mathfrak{g}}}(\bar{\mu}+b\delta))$ belongs to $\Gamma_{\mathbb{N}}(\mathfrak{l}_i)$. Corollary 1 implies that $(k_sk_{\dot{\mathfrak{g}}}\lambda_1, k_sk_{\dot{\mathfrak{g}}}\lambda_2, k_sk_{\dot{\mathfrak{g}}}(\bar{\mu}+b\delta))$ belongs to $\Gamma_{\mathbb{N}}(\mathfrak{g})$.

Proposition 1 implies that $n \leq b$. Then $k_{\dot{\mathfrak{g}}}(b-n) \in \mathbb{Z}_{\geq 0}$.

If $b = n$, we already proved that $(k_sk_{\dot{\mathfrak{g}}}\lambda_1, k_sk_{\dot{\mathfrak{g}}}\lambda_2, k_sk_{\dot{\mathfrak{g}}}\mu)$ belongs to $\Gamma_{\mathbb{N}}(\mathfrak{g})$. Theorem 3 is proved in this case.

Moreover, Proposition 1 implies that

$$b_0(k_sk_{\dot{\mathfrak{g}}}\lambda_1, k_sk_{\dot{\mathfrak{g}}}\lambda_2, k_sk_{\dot{\mathfrak{g}}}\bar{\mu}) = k_sk_{\dot{\mathfrak{g}}}b.$$

Then, Lemma 26 implies that $(k_sk_{\dot{\mathfrak{g}}}\lambda_1, k_sk_{\dot{\mathfrak{g}}}\lambda_2, k_sk_{\dot{\mathfrak{g}}}\bar{\mu} + m\delta)$ belongs to $\Gamma_{\mathbb{N}}(\mathfrak{g})$, for any

$$m \leq k_sk_{\dot{\mathfrak{g}}}b - 2. \tag{44}$$

Assume that $k_{\dot{\mathfrak{g}}}(b-n) \in \mathbb{Z}_{>0}$. If $k_s > 1$, $m = k_sk_{\dot{\mathfrak{g}}}n$ satisfies condition (44). Similarly, for any $d > 1$, $k_sk_{\dot{\mathfrak{g}}}n - d$ satisfies condition (44). The theorems follow in these cases.

Assume now that $k_s = 1$ and fix $d > 1$. We may assume that $n \neq b$. Then, the integer

$$b_0(dk_{\dot{\mathfrak{g}}}\lambda_1, dk_{\dot{\mathfrak{g}}}\lambda_2, dk_{\dot{\mathfrak{g}}}\bar{\mu}) = dk_{\dot{\mathfrak{g}}}b.$$

Since $m = dk_{\dot{\mathfrak{g}}}n = d(k_{\dot{\mathfrak{g}}}n - k_{\dot{\mathfrak{g}}}b) + dk_{\dot{\mathfrak{g}}}n$ satisfies $m \leq dk_{\dot{\mathfrak{g}}}b - 2$, Theorem 3 also holds in this case. □

11. Some technical lemmas

In this section we collect some technical results on Birkhoff and Bruhat decompositions, on Geometric Invariant Theory, on affine Kac-Moody groups...

11.1. **Bruhat and Birkhoff decompositions.** In this subsection, G is the minimal Kac-Moody group associated with any symmetrizable GCM. Fix T, W, B and B^- as usually. Let $P \supset B$ be a standard parabolic subgroup with standard Levi subgroup L. Fix a one parameter subgroup τ of T such that for all $\beta \in \Phi$, $\beta \in \Phi(P)$ if and only if $\langle \beta, \tau \rangle \geq 0$.

Lemma 11.1. *Let $u \in W$ and $v \in W^P$ such that $u \neq v$. Let $x \in \mathring{X}_{B^-}^{u^{-1}}$.*

Then $\lim_{t\to 0} \tau(t)x$ does not belong to $v^{-1}B\underline{o}^-$.

Proof. Recall that $(G/B^-)^\tau$ can be decomposed in the two following ways:

$$(G/B^-)^\tau = \sqcup_{w\in W^P} Lw^{-1}\underline{o}^- = \sqcup_{w\in W}(B\cap L)w\underline{o}^-.$$

Moreover $\{w\underline{o}^- \,:\, w \in W\}$ is the set of T-fixed points in G/B^-. Hence, if $Y \subset G/B^-$ is $(B\cap L)-$stable then $Y^\tau = \sqcup_{x\in Y^\tau}(B\cap L)x$. For $Y = \mathring{X}_{B^-}^{u^{-1}} = Bu^{-1}\underline{o}^-$ we get

$$(Bu^{-1}\underline{o}^-)^\tau = (B\cap L)u^{-1}\underline{o}^-.$$

If $v \in W^P$ then $(v^{-1}Bv)\cap L = B\cap L$. Hence, for $Y = v^{-1}B\underline{o}^-$, we get

$$(v^{-1}B\underline{o}^-)^\tau = (B\cap L)v^{-1}\underline{o}^-.$$

Since $\lim_{t\to 0}\tau(t)x$ belongs to $(Bu^{-1}\underline{o}^-)^\tau$, we deduce that it does not belong to $v^{-1}B\underline{o}^-$. □

Lemma 11.2. *Let $u, v \in W$ such that $l(v) = l(u) + 1$.*

(i) *Let* $x_1, x_2 \in \mathring{X}^u_{B^-}$ *such that* $\lim_{t\to\infty} \tau(t)x_1 = \lim_{t\to\infty} \tau(t)x_2$ *belongs to* $\mathring{X}^v_{B^-}$.
Then $\tau(\mathbb{C}^*)x_1 = \tau(\mathbb{C}^*)x_2$.

(ii) *Let* $x_1, x_2 \in \mathring{X}^B_v$ *such that* $\lim_{t\to\infty} \tau(t)x_1 = \lim_{t\to\infty} \tau(t)x_2$ *belongs to* $\mathring{X}^B_u$.
Then $\tau(\mathbb{C}^*)x_1 = \tau(\mathbb{C}^*)x_2$.

Proof. Let us prove the first assertion. Set $y = \lim_{t\to\infty} \tau(t)x_1$. Then $y \in (\mathring{X}^v_{B^-})^\tau = (B\cap L)v\underline{o}^-$. Fix $l \in B\cap L$ such that $y = lv\underline{o}^-$. Note that for any $g^u \in P^{u,-}$, $l' \in L$ and $w \in W$, we have

$$\lim_{t\to\infty} \tau(t)g^u l' w\underline{o}^- = l' w\underline{o}^-.$$

The equalities $Py = Px_1 = Px_2$ allow to find $g_i^u \in P^{u,-}$ such that $x_i = g_i^u y$, for $i = 1, 2$. Then $l^{-1}x_i = (l^{-1}g_i^u l)v\underline{o}^-$ belongs to $\mathring{X}^{B^-}_v$. Since $l \in B$, $l^{-1}x_i$ also belongs to $\mathring{X}^u_{B^-}$. Finally, $l^{-1}x_i$ belongs to $\mathring{X}^u_{B^-} \cap \mathring{X}^{B^-}_v$.

But $l(v) = l(u) + 1$ and $\mathring{X}^u_{B^-} \cap \mathring{X}^{B^-}_v$ is isomorphic to $\mathbb{C}^*$. Since $l^{-1}x_1$ and $l^{-1}x_2$ are not fixed by $\tau(\mathbb{C}^*)$ they belong to the same τ-orbit. The actions of τ and L commuting, one deduces that $\tau(\mathbb{C}^*)x_1 = \tau(\mathbb{C}^*)x_2$.

The second assertion works similarly. Up to translating by an element of $B\cap L$, one may assume that $\lim_{t\to\infty} \tau(t)x_1 = \lim_{t\to\infty} \tau(t)x_2 = u\underline{o}$. Then x_1 and x_2 belong to $\mathring{X}^u_B \cap \mathring{X}^B_v$ that is isomorphic to $\mathbb{C}^*$. □

We are now interested in the Białynicki-Birula cells of G-orbits in $\mathbb{X}$. We prove an anologue of [Res10, Lemma 12] in your infinite dimensional setting.

Lemma 11.3. *Assume that* P *has finite type ie that* L *is finite-dimensional. Let* Q_1, Q_2 *be two parabolic subgroups of* G *containing* B^-. *Consider* $\underline{\mathbb{X}} = G/Q_1 \times G/Q_2 \times G/P$ *with base point* $(\underline{o}_1, \underline{o}_2, \underline{o})$. *Fix* $l \in L$. *Let* u_1, u_2, *and* v *in* W^P. *Set* $x_0 = (lu_1^{-1}\underline{o}_1, u_2^{-1}\underline{o}_2, v^{-1}\underline{o}) \in \underline{\mathbb{X}}$ *and* $\mathcal{O} = G.x_0$.
Then

$$\{x \in \mathcal{O} \,:\, \lim_{t\to 0} \tau(t)x \in L.x_0\} = P.x_0.$$

Proof. Consider first the analogous situation in $G/Q_2 \times G/B$, with its two projections p_1 and p_2 on G/Q_2 and G/B. Set $x_1 = (u_2^{-1}\underline{o}_2, v^{-1}\underline{o})$, $\mathcal{O}_1 = G.x_1$ and $\mathcal{O}_1^0 = L.x_1$. Set also $\mathcal{O}_1^+ = \{x \in \mathcal{O}_1 \,:\, \lim_{t\to 0} \tau(t)x \in L.x_1\}$. We claim that $\mathcal{O}_1^+ = Px_1$.

We have $p_1(\mathcal{O}_1) = G/B^-$ and $p_1(\mathcal{O}_1^0) = Lu_2^{-1}\underline{o}_2$. Moreover,

$$\{x \in G/Q_2 \,:\, \lim_{t\to 0} \tau(t)x \in L.u_2^{-1}\underline{o}\} = P.u_2^{-1}\underline{o}.$$

Since $\mathcal{O}_1^+$ is stable by P, it follows that

$$\mathcal{O}_1^+ = P.\mathcal{I} \text{ where } \mathcal{I} = (\{u_2^{-1}\underline{o}_2\} \times G/B) \cap \mathcal{O}_1^+.$$

Set $x_2 = v^{-1}\underline{o}$. Then, $p_2(\mathcal{I})$ is the set of points $x \in (u_2^{-1}Q_2u_2).x_2$ such that $\lim_{t\to 0} \tau(t)x \in (L\cap u_2^{-1}Q_2u_2)x_2$. In particular $p_2(\mathcal{I})$ is contained in Px_2. The weights of τ acting on $T_{x_2}p_2(\mathcal{I})$ are nonnegative. On the other hand they are contained in $T_{x_2}(u_2^{-1}Q_2u_2)x_2$. It follows that $T_{x_2}p_2(\mathcal{I})$ is contained in $T_{x_2}(P\cap u_2^{-1}Q_2u_2)x_2$. Note that, since P has finite type, $P\cap u_2^{-1}Q_2u_2$ is finite-dimensional. Moreover, the dimension of $\mathcal{I}$ (at x_2) is at most equal to $\dim((P\cap u_2^{-1}Q_2u_2)x_2)$.

It follows that $(P\cap u_2^{-1}Q_2u_2)x_2$ is open in $p_2(\mathcal{I})$. Since $(P\cap u_2^{-1}Q_2u_2)x_2$ contains $(L\cap u_2^{-1}Q_2u_2)x_2$, we deduce that

$$p_2(\mathcal{I})=(P\cap u_2^{-1}Q_2u_2)x_2,$$

and

$$\mathcal{O}_1^+=P.x_1. \tag{45}$$

Consider now

$$\pi_1\,:\,\underline{\mathbb{X}}\longrightarrow G/Q_2\times G/B,\;(x_1,x_2,x_3)\longmapsto(x_2,x_3).$$

Set $\mathcal{O}^+=\{x\in\mathcal{O}\,:\,\lim_{t\to 0}\tau(t)x\in L.x_0\}$ and $\mathcal{O}_0=L.x_1$. Equality (45) and the fact that $\mathcal{O}^+$ is P-stable imply that

$$\mathcal{O}^+=P\Big((G/Q_1\times\{x_1\})\cap\mathcal{O}^+\Big). \tag{46}$$

Note that

$$\begin{array}{ll}(G/Q_1\times\{x_1\})\cap\mathcal{O}=(u_2^{-1}Q_2u_2\cap v^{-1}Bv).x_0 & \text{and}\\ (G/Q_1\times\{x_1\})\cap\mathcal{O}_0=(u_2^{-1}Q_2u_2\cap v^{-1}Bv\cap L).x_0 & \end{array}$$

Then, since $u_2^{-1}Q_2u_2\cap v^{-1}Bv$ is finite-dimensional, [Res10, Lemma 12] shows that

$$(G/Q_1\times\{x_1\})\cap\mathcal{O}^+=(P\cap u_2^{-1}Q_2u_2\cap v^{-1}Bv).x_0.$$

With equality (46) this ends the proof of the lemma. □

11.2. **Affine root systems.** In this subsection, we consider an untwisted affine root system and use the notation of Section 9. Recall in particular, that $\dot{\mathfrak{h}}^*_{\mathbb{R}}$ is endowed with a $\dot W$-invariant Euclidean norm $\|\cdot\|$ such that $\|\dot\theta\|^2=2$.

Lemma 11.4. *Consider the affine Weyl group $W=\dot Q^\vee.\dot W$ and set $N=\sharp\dot\Phi^+$.*

There exists a positive real constant K such that for any $h\in\dot Q^\vee$ and $\dot w\in\dot W$, we have

$$K\|h\|-N\leq l(h\dot w)\leq N+\sqrt{2}N\|h\|.$$

Proof. Set $w=h\dot w$. The length of w is the cardinality of $w^{-1}\Phi^+\cap\Phi^-$. One can deduce (see e.g. [IM65]) that:

$$l(h\dot w)=\sum_{\dot\alpha\in\dot\Phi^+,\,\dot w^{-1}\dot\alpha\in\dot\Phi^+}|\langle h,\dot\alpha\rangle|+\sum_{\dot\alpha\in\dot\Phi^+,\,\dot w^{-1}\dot\alpha\in\dot\Phi^-}|\langle h,\dot\alpha\rangle-1|. \tag{47}$$

The inequality on the right just follows from

$$\begin{array}{l}|\langle h,\dot\alpha\rangle-1|\leq|\langle h,\dot\alpha\rangle|+1\\ |\langle h,\dot\alpha\rangle|\leq\|h\|\|\dot\alpha\|\leq\sqrt{2}\|h\|.\end{array}$$

Moreover,

$$\begin{array}{ll}l(h\dot w) & \geq l(h)-l(\dot w)\\ & \geq\sum_{\dot\alpha\in\dot\Phi^+}|\langle h,\dot\alpha\rangle|-N.\end{array}$$

The set $\dot\Phi^+$ spaning $\dot{\mathfrak{h}}^*_{\mathbb{R}}$, the map $h\mapsto\sum_{\dot\alpha\in\dot\Phi^+}|\langle h,\dot\alpha\rangle|$ is a norm on the real vector space $\dot{\mathfrak{h}}^*_{\mathbb{R}}$. This norm is equivalent to $\|\cdot\|$, and there exists K such that $K\|h\|\leq\sum_{\dot\alpha\in\dot\Phi^+}|\langle h,\dot\alpha\rangle|$. The lemma follows. □

11.3. **The Jacobson-Morozov theorem.** Let $\mathfrak{g}$ be an untwisted affine Kac-Moody Lie algebra and $\mathfrak{p}$ be a standard parabolic subalgebra. Let G be the minimal Kac-Moody group associated with $\mathfrak{g}$ and P be the parabolic subgroup corresponding to $\mathfrak{p}$. Fix a one parameter subgroup τ of T in $\oplus_{\alpha_j\not\in\Delta(P)}\mathbb{Z}_{>0}\varpi_{\alpha_j^\vee}$. Consider the action of τ on $\mathfrak{g}$ and the corresponding weight space decompositions

$$\mathfrak{g}=\oplus_{n\in\mathbb{Z}}\mathfrak{g}_n \qquad \mathfrak{p}=\oplus_{n\in\mathbb{Z}_{\geq 0}}\mathfrak{g}_n.$$

In $\mathfrak{sl}_2(\mathbb{C})$, we denote by (E,H,F) the standard triple

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad F=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

satisfying

$$[E,F]=H\qquad [H,E]=2E\qquad [H,F]=-2F.$$

Proposition 11.1. *Let ξ be a nonzero vector in $\mathfrak{g}_n\cap w\mathfrak{u}^-w^{-1}$ for some positive integer n and $w\in W$.*

Then there exists a morphism $\phi:\mathrm{SL}_2(\mathbb{C})\longrightarrow G$ of group-ind-varieties such that $T_e\phi(E)=\xi$.

Proof. Set $\mathcal{K}=\mathbb{C}((t))=\mathbb{C}[[t]][t^{-1}]$ and $\mathcal{R}=\mathbb{C}[t,t^{-1}]\subset\mathcal{K}$. Consider the Lie algebras $\dot{\mathfrak{g}}\otimes\mathcal{R}$ and $\dot{\mathfrak{g}}\otimes\mathcal{K}$. Recall that $\mathbb{C}d\oplus\dot{\mathfrak{g}}\otimes\mathcal{R}$ is a semi-direct product and that

$$0\longrightarrow\mathbb{C}c\longrightarrow\mathfrak{g}\longrightarrow\mathbb{C}d\oplus\dot{\mathfrak{g}}\otimes\mathcal{R}\longrightarrow 0$$

is a central extension. We first construct an $\mathfrak{sl}_2$-triple in $\dot{\mathfrak{g}}\otimes\mathcal{K}$. Then, we modify it to get one in $\dot{\mathfrak{g}}\otimes\mathcal{R}$. Using $\mathcal{R}$-group schemes we get a morphism from $\mathrm{SL}_2(\mathbb{C})$ to $\dot{G}\otimes\mathcal{R}$ that we finaly raises to G. Here, $\dot{G}\otimes\mathcal{R}$ denotes the set of $\mathcal{R}$-points of $\dot{G}$.

Consider the canonical $\mathbb{C}$-linear embedding $\iota:\dot{\mathfrak{g}}\otimes\mathcal{R}\longrightarrow\mathfrak{g}$. Be careful that it is not an homomorphism of Lie algebras.

Note that the one parameter subgroup τ is equal to $\dot{\tau}+md$ for some one parameter subgroup $\dot{\tau}$ of $\dot{T}$ and some positive integer m. Then, τ acts on $\dot{\mathfrak{g}}\otimes\mathcal{R}$ by $\mathbb{C}$-linear automorphisms and we have the decomposition

$$\dot{\mathfrak{g}}\otimes\mathcal{R}=\oplus_{k\in\mathbb{Z}}(\dot{\mathfrak{g}}\otimes\mathcal{R})_k$$

in τ-eigenspaces. Since each $(\dot{\mathfrak{g}}\otimes\mathcal{R})_k$ is finite-dimensional and m is positive, we have

$$\dot{\mathfrak{g}}\otimes\mathcal{K}=\oplus_{k\in\mathbb{Z}_{<0}}(\dot{\mathfrak{g}}\otimes\mathcal{R})_k\oplus\prod_{k\in\mathbb{Z}_{\geq 0}}(\dot{\mathfrak{g}}\otimes\mathcal{R})_k. \tag{48}$$

Observe that, for any nonzero integer k, $\mathfrak{g}_k=(\dot{\mathfrak{g}}\otimes\mathcal{R})_k$. In particular, ξ belongs to $\dot{\mathfrak{g}}\otimes\mathcal{R}$. We denote by $\bar{\xi}$ (resp. $\tilde{\xi}$) the element ξ viewed as an element of the Lie algebra $\dot{\mathfrak{g}}\otimes\mathcal{R}$ (resp. $\dot{\mathfrak{g}}\otimes\mathcal{K}$).

The space $\mathfrak{g}_n$ being contained in $\mathfrak{u}$, ξ belongs to $\mathfrak{u}\cap w\mathfrak{u}^-w^{-1}$ and by [Kum02, Theorem 10.2.5], $\mathrm{ad}\xi\in\mathrm{End}(\mathfrak{g})$ is locally nilpotent. Being $\mathcal{K}$-linear, $\mathrm{ad}\tilde{\xi}$ is also nilpotent. Applying Jacobson-Morozov's theorem (see e.g. [Bou05, VIII–§11 Proposition 2]) in the Lie algebra $\dot{\mathfrak{g}}\otimes\mathcal{K}$ over the field $\mathcal{K}$ of characteristic zero, we get an $\mathfrak{sl}_2$-triple (X,H,Y) in $\dot{\mathfrak{g}}\otimes\mathcal{K}$ such that $X=\tilde{\xi}$.

Write $Y=\sum_{k\in\mathbb{Z}}Y_k$ according to the decomposition (48). The Lie bracket being graded, we have in $\dot{\mathfrak{g}}\otimes\mathcal{K}$

$$[X,[X,Y_{-n}]]=-2X.$$

Set $H_0 = [X, Y_{-n}]$ and $\mathfrak{n} = \mathrm{Ker}(\mathrm{ad}\tilde{\xi})$. Since X is homogeneous, $\mathfrak{n}$ decomposes as $\oplus_{k\in\mathbb{Z}_{<0}}\mathfrak{n}_k \oplus \prod_{k\in\mathbb{Z}_{\geq 0}} \mathfrak{n}_k$, where $\mathfrak{n}_k = \mathfrak{n}\cap(\dot{\mathfrak{g}}\otimes\mathcal{K})_k$. Note that $[X, Y_{-n}]+2Y_{-n}$ belongs to $\mathfrak{n}_{-n}$. By [Kos59, Corollary 3.4], $\mathrm{ad}H_0 + 2\mathrm{Id}_{\dot{\mathfrak{g}}\otimes\mathcal{K}}$ is injective and stabilizes each $\mathfrak{n}_n$. Moreover, $\mathfrak{n}_{-n}$ is $(\mathrm{ad}H_0 - 2\mathrm{Id}_{\dot{\mathfrak{g}}\otimes\mathcal{K}})$-stable and finite-dimensional as a complex vector space. Then there exists $Y' \in \mathfrak{n}_{-n}$ such that

$$[X, Y_{-n}] + 2Y_{-n} = [X, Y'_{-n}] + 2Y'_{-n}.$$

Then, $(X, H_0, Y_{-n} - Y'_{-n})$ is an $\mathfrak{sl}_2$-triple contained in $(\dot{\mathfrak{g}}\otimes\mathcal{K})_n\times(\dot{\mathfrak{g}}\otimes\mathcal{K})_0\times(\dot{\mathfrak{g}}\otimes\mathcal{K})_{-n}$. In particular, this $\mathfrak{sl}_2$-triple is contained in $\dot{\mathfrak{g}}\otimes\mathcal{R}$. Hence, we get an $\mathcal{R}$-linear Lie algebra homomorphism

$$\phi\,:\,\mathfrak{sl}_2(\mathcal{R}) \longrightarrow \dot{\mathfrak{g}}\otimes\mathcal{R}$$

such that $\phi(E) = \xi$. Since SL_2 is simply connected and $\mathcal{R}$ contains $\mathbb{Q}$, [ABD+66, Exposé XXIV, Proposition 7.3.1] implies that there exists a morphism

$$\Phi\,:\,\mathrm{SL}_2 \longrightarrow \dot{G}$$

of $\mathcal{R}$-group schemes with ϕ as differential map at the identity. In particular, we get a morphism of group-ind-varieties

$$\bar{\Phi}\,:\,\mathrm{SL}_2(\mathbb{C}) \longrightarrow \dot{G}\otimes\mathcal{R}$$

such that $T_e\bar{\Phi}(E) = \xi$.

Consider now the semidirect product $\mathbb{C}^*\ltimes\dot{G}\otimes\mathcal{R}$ associated with the derivation d, and the central extension

$$\{1\} \longrightarrow \mathbb{C}^* \longrightarrow G \xrightarrow{\pi} \mathbb{C}^*\ltimes\dot{G}(\mathcal{R}) \longrightarrow \{1\}.$$

Then $\pi^{-1}(\bar{\Phi}(\mathrm{SL}_2(\mathbb{C})))$ is a central extension of $(P)\mathrm{SL}_2(\mathbb{C})$. Hence, it is isomorphic to either $\mathbb{C}^*\times(P)\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{GL}_2(\mathbb{C})$. In each case, $\bar{\Phi}$ can be lift to a morphism to $\pi^{-1}(\bar{\Phi}(\mathrm{SL}_2(\mathbb{C})))$. This concludes the proof of the proposition. □

11.4. **Geometric Invariant Theory.** For a given $\mathbb{C}^*$-variety X and a given integer k, we denote by $\mathbb{C}[X]^{(k)}$ the set of regular functions f on X such that $(t.f)(x) = f(t^{-1}x) = t^k f(x)$, for any $t\in\mathbb{C}^*$ and $x\in X$.

Lemma 11.5. *Let X be a normal affine $\mathbb{C}^*$-variety and D be a $\mathbb{C}^*$-stable irreducible divisor. Set $\Omega = X - D$. We assume that*

(i) $\forall x\in\Omega \quad \lim_{t\to 0} tx$ *does not exist in X.*
(ii) *For all $x_1, x_2\in\Omega$, if the two limits $\lim_{t\to\infty} tx_1$ and $\lim_{t\to\infty} tx_2$ exist, are equal and belong to D, then the $\mathbb{C}^*$-orbits of x_1 and x_2 are equal.*
(iii) $\forall x\in D \quad \lim_{t\to 0} tx$ *does exist in D.*
(iv) $\forall y\in D^{\mathbb{C}^*} \qquad \exists x\in\Omega \qquad \lim_{t\to\infty} tx = y.$
(v) *Ω is affine.*

Then, for any nonnegative integer k, the restriction map induces an isomorphism $\mathbb{C}[X]^{(k)}\simeq\mathbb{C}[\Omega]^{(k)}$.

Proof. Set $\tilde{X} = X\times\mathbb{C}$, $\tilde{\Omega} = \Omega\times\mathbb{C}$ and $\tilde{D} = D\times\mathbb{C}$. We endow $\tilde{X}$ with an action of $\mathbb{C}^*$ by setting $t.(x,z) = (t.x, tz)$ for any $t\in\mathbb{C}^*$, $x\in X$ and $z\in\mathbb{C}$. Observe that $\mathbb{C}[\tilde{X}]^{\mathbb{C}^*} = \oplus_{k\in\mathbb{N}}\mathbb{C}[X]^{(k)}z^k$ and $\mathbb{C}[\tilde{\Omega}]^{\mathbb{C}^*} = \oplus_{k\in\mathbb{N}}\mathbb{C}[\Omega]^{(k)}z^k$. Then, it is sufficient to prove that $\mathbb{C}[\tilde{X}]^{\mathbb{C}^*} = \mathbb{C}[\tilde{\Omega}]^{\mathbb{C}^*}$.

But, one can easily check that $\tilde{X}$ satisfies all the assumptions of the lemma. As a consequence, it is sufficient to prove it for $k = 0$.

Consider the commutative diagram

$$\begin{tikzcd} \Omega \arrow[r] \arrow[d, "\pi_\Omega"'] & X \arrow[d, "\pi_X"] \\ \Omega/\!/\mathbb{C}^* \arrow[r, "\theta"] & X/\!/\mathbb{C}^*, \end{tikzcd}$$

where $/\!/\mathbb{C}^*$ denotes the GIT-quotient. It remains to prove that θ is an isomorphism.

We first prove the surjectivity of θ. Let $\xi \in X/\!/\mathbb{C}^*$ and $\mathcal{O} \subset X$ be the unique closed $\mathbb{C}^*$-orbit in $\pi_X^{-1}(\xi)$. It $\mathcal{O} \subset \Omega$, it is clear that $\theta(\pi_\Omega(\mathcal{O})) = \xi$. Otherwise $\mathcal{O} \subset D$. The orbit $\mathcal{O}$ being closed, assumption (iii) implies that $\mathcal{O}$ is a fixed point. By assumption (iv), there exists $\mathcal{O}' \subset \Omega$ such that $\overline{\mathcal{O}'} \supset \mathcal{O}$. Then $\theta(\pi_\Omega(\mathcal{O}')) = \xi$. We conclude that θ is surjective.

Let us now prove that θ is injective. Assume that $\xi_1 \neq \xi_2 \in \Omega/\!/\mathbb{C}^*$ satisfy $\theta(\xi_1) = \theta(\xi_2) =: \xi$. Let $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ and $\mathcal{O} \subset X$ be the closed $\mathbb{C}^*$-orbit in $\pi_\Omega^{-1}(\xi_1)$, $\pi_\Omega^{-1}(\xi_2)$ and $\pi_X^{-1}(\xi)$ respectively. Since $\pi_X(\mathcal{O}_1) = \pi_X(\mathcal{O}_2) = \xi$, we have $\mathcal{O} \subset \overline{\mathcal{O}_1} \cap \overline{\mathcal{O}_2}$. In particular, $\mathcal{O}$ is a $\mathbb{C}^*$-fixed point and $\mathcal{O}_1$ and $\mathcal{O}_2$ are one-dimensional. Pick $x_1 \in \mathcal{O}_1$, $x_2 \in \mathcal{O}_2$ and $y \in \mathcal{O}$. By assumption (i), the limit $\lim_{t\to 0} t.x_1$ does not exist. But $y \in \overline{\mathcal{O}_1} - \mathcal{O}_1$, so $\lim_{t\to\infty} t.x_1 = y$. Similarly $\lim_{t\to\infty} t.x_2 = y$. Now, assumption (ii), implies that $\mathcal{O}_1 = \mathcal{O}_2$. Hence θ is injective.

Over the complex numbers, the fact that θ is bijective implies that it is birational. By assumption X and hence $X/\!/\mathbb{C}^*$ are normal. Then Zariski's main theorem (see e.g. [Kum02, Theorem A.11]) implies that θ is an isomorphism. □

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Abstract. The support of the tensor product decomposition of integrable irreducible highest weight representations of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ defines a semigroup of triples of weights. Namely, given λ in the set P_+ of dominant integral weights, $V(\lambda)$ denotes the irreducible representation of $\mathfrak{g}$ with highest weight λ. We are interested in the *tensor semigroup*

$$\Gamma_{\mathbb{N}}(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_+^3 \,|\, V(\mu) \subset V(\lambda_1) \otimes V(\lambda_2)\},$$

and in the *tensor cone* $\Gamma(\mathfrak{g})$ it generates:

$$\Gamma(\mathfrak{g}) := \{(\lambda_1, \lambda_2, \mu) \in P_{+,\mathbb{Q}}^3 \,|\, \exists N \geq 1 \quad V(N\mu) \subset V(N\lambda_1) \otimes V(N\lambda_2)\}.$$

Here, $P_{+,\mathbb{Q}}$ denotes the rational convex cone generated by P_+.

In the special case when $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra, the tensor semigroup is known to be finitely generated and hence the tensor cone to be convex polyhedral. Moreover, the cone $\Gamma(\mathfrak{g})$ is described in [BK06] by an explicit finite list of inequalities.

In general, $\Gamma(\mathfrak{g})$ is nor polyhedral, nor closed. In this article we describe the closure of $\Gamma(\mathfrak{g})$ by an explicit countable family of linear inequalities for any untwisted affine Lie algebra, which is the most important class of infinite-dimensional Kac-Moody algebra. This solves a Brown-Kumar's conjecture [BK14] in this case.

The difference between the tensor cone and the tensor semigroup is measured by the *saturation factors*. Namely, a positive integer d is called a *saturation factor*, if $V(N\lambda_1) \otimes V(N\lambda_2)$ contains $V(N\mu)$ for some positive integer N then $V(d\lambda_1) \otimes V(d\lambda_2)$ contains $V(d\mu)$, assuming that $\mu - \lambda_1 - \lambda_2$ belongs to the root lattice. For $\mathfrak{g} = \mathfrak{sl}_n$, the famous Knutson-Tao theorem [KT99] asserts that $d = 1$ is a saturation factor. More generally, for any simple Lie algebra, explicit saturation factors are known. In the Kac-Moody case, $\Gamma_{\mathbb{N}}(\mathfrak{g})$ is not necessarily finitely generated and hence the existence of such a factor is unclear a priori. Here, we obtain explicit saturation factors for any affine Kac-Moody Lie algebra. For example, in type $\tilde{A}_n$, we prove that any integer $d \geq 2$ is a saturation factor, generalizing the case $\tilde{A}_1$ shown in [BK14].

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