VANISHING SYMMETRIC KRONECKER COEFFICIENTS

NICOLAS RESSAYRE

Abstract. In this note, we prove the vanishing of infinitely many rectangular symmetric Kronecker coefficients by finding holes in the corresponding semigroup.

Keywords. (Symmetric) Kronecker coefficients, Non saturation, Geometric Complexity Theory, orbit closure of the determinant

1. Introduction

Aim. The aim of this note is to produce infinitely many vanishing symmetric Kronecker coefficient, using geometric methods. It is well known that the set of nonzero symmetric Kronecker coefficients (for partitions of bounded length) forms a finitely generated semigroup S (see Proposition 2 below). Then one can think about two strategies to find vanishing: produce explicit linear inequalities satisfied by the cone generated by S, or find holes in the semigroup. The first strategy is well developed (see e.g. [É92, BS00, Res10, Man15, Fra02, Kly04, Res12, BIH17]). Here we use the second one.

Definitions. Let us first introduce the symmetric Kronecker coefficients. If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_e \geq 0)$ is a partition, we set $|\lambda| = \sum_i \lambda_i$ in such a way λ is a partition of $|\lambda|$. Consider the symmetric group $\mathfrak{S}_N$ on N letters. The irreducible representations of $\mathfrak{S}_N$ are parametrized by the partitions of N, see *e.g.* [Mac95, I. 7]: $[\lambda]$ denotes the representation of $\mathfrak{S}_{|\lambda|}$ corresponding to λ. The Kronecker coefficients $k_{\lambda\,\mu\,\nu}$, depending on three partitions λ, μ, and ν of the same integer N, are defined by

$$[\lambda] \otimes [\mu] = \sum_{\nu} k_{\lambda\,\mu\,\nu}[\nu], \tag{1}$$

and hence encodes the tensor product decomposition for the representations of the symmetric group.

If $\lambda = \mu$, $[\lambda] \otimes [\lambda]$ decomposes as the sum of its symmetric and alternate parts:

$$[\lambda] \otimes [\lambda] = S^2[\lambda] \oplus \Lambda^2[\lambda].$$

In particular, we can define the symmetric and alternate Kronecker coefficients $sk_{\lambda\,\nu}$ and $ak_{\lambda\,\nu}$ by

$$S^2[\lambda] = \sum_{\nu} sk_{\lambda\,\nu}[\nu], \quad \Lambda^2[\lambda] = \sum_{\nu} ak_{\lambda\,\nu}[\nu]. \tag{2}$$

Then

$$k_{\lambda\,\lambda\,\nu} = sk_{\lambda\,\nu} + ak_{\lambda\,\nu}. \tag{3}$$

Results. Fix two positive integers n and δ. The rectangular partition $(\delta,\dots,\delta)$ of $n\delta$ is denoted by δ^n. We can now list our infinite families of vanishing coefficients $sk_{\delta^n\,\nu}$.

Theorem 1. *Let n be a positive integer and $\nu = ab^{n^2-2}c$ be a partition with $a \geq b \geq c \geq 0$ in $\mathbb{N}$. If*

$$\begin{cases} n \equiv 2 \text{ or } 3\ [4]; \\ n \text{ divides } a+c-2b; \\ b \text{ is odd;} \end{cases}$$

then $sk_{\delta^n\,\nu} = 0$ where $\delta = \frac{|\nu|}{n}$.

Let $\nu = ab^{n^2-1}$ be like in Theorem 1 with $b = c$. Then, the corresponding Kronecker coefficient $k_{\delta^n\,\delta^n\,\nu} = k_{(\delta-nb)^n\,(\delta-nb)^n\,a-b} = 1$. Here we have used an invariance property of Kronecker coefficients (see e.g. [BOR09, Lemma 2.1]). In particular, Theorem 1 does not hold for Kronnecker coefficients $k_{\delta^n\delta^n\nu}$.

For $n = 3$, we get more vanishing.

Theorem 2. *Assume that ν is a partition of one of the following form*

(i) $\nu = a^2b^7$ *where $a \geq b$ and*

$$\begin{cases} 3 \text{ divides } a-b; \\ a \text{ is odd.} \end{cases}$$

(ii) $\nu = a^3b^6$ *where $a \geq b$ and*

$$a \text{ is odd.}$$

(iii) $\nu = a^7b^2$ *where $a \geq b$ and*

$$\begin{cases} 3 \text{ divides } a-b; \\ b \text{ is odd.} \end{cases}$$

(iv) $\nu = a^6b^3$ *where $a \geq b$ and*

$$b \text{ is odd.}$$

Then $sk_{\delta^3\,\nu} = 0$, where $\delta = \frac{|\nu|}{3}$.

Comparaison with Kronnecker coefficients. We denote by $l(\lambda)$ the length of the partition λ ie the number of nonzero parts. The set of pairs (λ,ν) such that $sk_{\lambda\,\nu} \neq 0$ and, $l(\lambda) \leq n$ and $l(\nu) \leq n^2$ (for some fixed integer n) is a finitely generated semigroup S_n (see Proposition 2 below). To describe this semigroup, it is natural to describe separately the convex cone generated by it (by inequalities) and the holes. Theorems 1 and 2 determine infinitely many holes, since conditions like "b is odd" are not invariant by scaling.

Since

$$\begin{array}{lll} sk_{\lambda\,\nu} \neq 0 & \Rightarrow & k_{\lambda\,\lambda\,\nu} \neq 0, \quad \text{and} \\ k_{\lambda\,\lambda\,\nu} \neq 0 & \Rightarrow & sk_{2\lambda\,2\nu} \neq 0, \end{array}$$

the cones generated by the semigroups of the symmetric and ordinary Kronecker coefficients are equal. Nevertheless, the paragraph following Theorem 1 show that the semigroups are different. Here, we exploit this difference to get holes in the semigroup S_n.

Motivations. Our original motivation for proving vanishing of rectangular symmetric Kronecker coefficients comes from Valiant's famous determinant versus permanent problem [Val79a] and more precisely from Geometric Complexity Theory

(GCT) [MS01]. Indeed, in [Val79a, Val79b, Val82], Valiant purposed an algebraic analogue of Cook's complexity theory. A polynomial function $P \in \mathbb{C}[X_1, \dots, X_q]$ is called an *affine projection* of the determinant $\det_n$ of size n if there exists an affine linear function $F : \mathbb{C}^q \longrightarrow \mathcal{M}_n(\mathbb{C})$ such that: $P = \det_n \circ F$. The minimal n such that P can be written as $\det_n \circ F$ is called the *determinantal complexity* of P, and denoted by $\mathrm{dc}(P)$. Valiant defined classes VP and VNP analogous to the famous P and NP classes in complexity theory. The *permanent* Perm_m of a matrix $M = (m_{i,j})$ of size $m \times m$ is

$$\mathrm{Perm}_m(M) = \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=1}^{m} m_{i,\sigma(i)},$$

where $\mathfrak{S}_m$ is the permutation group of the set $\{1, \cdots, m\}$. As a polynomial function of $q = m^2$ variables, Perm_m is "VNP-complete" (see [Val79a]). In particular, the main conjecture in Valiant's complexity theory is

Conjecture 1. *The determinantal complexity of* Perm_m *is greater than any polynomial in m, for m big enough.*

Geometric Complexity Theory (GCT) is a program due to Mulmuley and Sohoni (see [MS01, BLMW11]) to attack this conjecture. Set $W = \mathcal{M}_n(\mathbb{C})$ and $G = \mathrm{GL}(W) = \mathrm{GL}_{n^2}(\mathbb{C})$. The group G acts on the space S^nW^* of homogeneous polynomial functions of degree n on W by variable changing. Consider $\det_n \in S^nW^*$ and its orbit $\mathcal{O}_n = G.\det_n$. Consider $P = \det_n \circ F$ for some affine linear function $F : \mathbb{C}^q \longrightarrow \mathcal{M}_n(\mathbb{C})$. Then one constructs $\tilde{P} \in S^nW^*$ (depending on $q+1$ variables, see [MS01] for details) such that $\tilde{P}$ belongs to the closure $\overline{\mathcal{O}_n}$ of $\mathcal{O}_n$. A central question is

Problem 1. *Find methods to decide that a given polynnomial* $\tilde{P}$ *does not belong to* $\overline{\mathcal{O}_n}$.

A related question is

Problem 2. *Find explicit equations for the affine variety* $\overline{\mathcal{O}_n}$.

Let $\mathcal{I}_n \subset \mathbb{C}[S^nW^*]$ be the ideal of functions vanishing on the affine variety $\overline{\mathcal{O}_n}$. It is graded by the degree: $\mathcal{I}_n = \oplus_{\delta \geq 0} \mathcal{I}_n^\delta$. Each $\mathcal{I}_n^\delta$ is a finite dimensional G-module. Let ν be a partition with at most n^2 parts appearing in Theorem 1 or 2. Then the multiplicity of $S_\nu W$ in $\mathbb{C}[\mathcal{O}_n]$ (and hence in $\mathbb{C}[\overline{\mathcal{O}_n}]$) is zero (see Section 2.5). Hence the isotypical component of $S^\bullet S^n W$ of type $S_\nu W$ is contained in $\mathcal{I}_n$. Using a software (see [Wil19] and [Res18]) to compute plethysm coefficients, we obtained for $n = 3$ a lot of partitions ν such that itsisotypical component is nonzero.

For example, there exists a module of dimension 2 842 131 820 027 500 of degree 13 equations in 165 variables vanishing on $\overline{\mathcal{O}_3}$. This module is contained in $S^{13}S^3W$ that has dimension 17 112 638 902 445 186 100. This module was obtained before by C. Ikenmeyer (see [Ike12]) by explicit computer calculation. In Section 6 we give other explicit submodules of $\mathcal{I}_3$. For example we get 16 partitions giving equations of degree at most 20. An interesting question is to understand some of these equations geometrically.

In GCT, a partition ν such that $sk_{\delta^n\nu} = 0$ and $S_\nu W$ embeds in $S^\bullet S^n W$ is called an obstruction. Note that by [BIP19], there is a strong limitation in the power of obstructions in GCT: the obstructions cannot be used to prove lower bounds on $\mathrm{dc}(\mathrm{Perm}_m)$ better than m^{25}. Note than even m^3 would be a highly nontrivial lower bound: the best known lower bound for $\mathrm{dc}(\mathrm{Perm}_m)$ in $\frac{m^2}{2}$ by Mignon-Ressayre [MR04]. Despite the recent negative results of [BIP19], it is still natural to conjecture the following geometric reinforcement of Valiant's conjecture: if $\widetilde{\mathrm{Perm}}_m$ belong to $\overline{\mathcal{O}_{n(m)}}$ then $n(m)$ growths faster than any polynomial in m. Then, to use representation theory to understand the equations of $\overline{\mathcal{O}_n}$ is still an interesting approach.

As already mentioned, usually an efficient way to prove vanishing of multiplicities in invariant theory is to prove linear inequalities necessary for non-vanishing. For the orbit closure of the determinant, Kumar essentially proved in [Kum13, Kum15] that these methods are not relevant for Geometric Complexity Theory. This is an important motivation for this paper where we give lattice conditions implying vanishing.

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2. Geometric realization of the symmetric Kronecker coefficients

The aim of this section is to obtain the coefficients $sk_{\delta^n\,\nu}$ as the dimension of a space of invariant sections of some line bundle.

2.1. Borel-Weil Theorem.

In this subsection, W is any N-dimensional complex vector space. Let P_N^+ denote the set of non-increasing sequences $\nu = (\nu_1 \geq \cdots \geq \nu_N)$ of N integers. Let us recall, how Borel-Weil's theorem allows to obtain $S_\nu W^*$ as a space of sections. Write $\nu = a_1^{m_1} \dots a_s^{m_s}$ with $a_1 > \cdots > a_s$ and $m_1 + \cdots + m_s = n^2$.

Denote by $X_\nu = \mathcal{F}l(m_1, \cdots, m_s; W)$ the flag variety

$$X_\nu = \{(W_1 \subset \cdots \subset W_s = W) : \dim(W_i) = m_1 + \cdots + m_i \quad \forall i\}.$$

Then X_ν can be embedded, using Plücker coordinates, in

$$\mathbb{P}(\Lambda^{m_1} W) \times \cdots \times \mathbb{P}(\Lambda^{m_1+\cdots+m_{s-1}} W).$$

On this product of projective spaces, consider the line bundle

$$\mathcal{L}_\nu = \mathcal{O}(a_1 - a_2) \otimes \cdots \otimes \mathcal{O}(a_{s-1} - a_s) \otimes (\det)^{a_s}. \tag{4}$$

The term $\otimes(\det)^{a_s}$ means that the action of $\hat{G} = \mathrm{GL}(W)$ on $\mathcal{L}_\nu$ is twisted by $(\det)^{a_s}$. The Borel-Weil theorem asserts that

$$\mathrm{H}^0(X_\nu, \mathcal{L}_\nu) = S_\nu W^*, \tag{5}$$

as $\hat{G}$-modules. For short, $\mathcal{F}l(1, 2, \dots, N-1; W)$ is denoted by $\mathcal{F}l(W)$.

2.2. **Schur-Weyl duality.** Fix a partition ν of length at most N and weight $|\nu| = d$. Consider $W^{\otimes d}$ endowed with the natural $\mathfrak{S}_d \times \mathrm{GL}(W)$-action. By Schur-Weyl duality (see [Pro07, Theorem 3.1.4]), it decomposes as

$$W^{\otimes d} = \oplus_{|\lambda|=d, l(\lambda)\leq N}[\lambda] \otimes S_\lambda W. \tag{6}$$

Since $[\nu]$ is self dual as a $\mathfrak{S}_d$-module, we deduce from (6) that

$$S_\nu W = \mathrm{Hom}^{\mathfrak{S}_d}([\nu], W^{\otimes d}), \tag{7}$$

where $\mathrm{Hom}^{\mathfrak{S}_d}$ means the space of $\mathfrak{S}_d$-invariant linear maps. Now, fix two finite dimensional complex vector spaces E and F of dimensions m and n and set $W = E \otimes F$. Let λ, μ and ν be three partitions of the same integer d such that $l(\lambda) \leq m$, $l(\mu) \leq n$ and $l(\nu) \leq mn$. Consider $S_\nu W$ as a $(G = \mathrm{GL}(E) \times \mathrm{GL}(F))$-module and the multiplicity space

$$K^\nu_{\lambda\mu} := \mathrm{Hom}^G(S_\lambda E \otimes S_\mu F, S_\nu W). \tag{8}$$

With (7), we get that $K^\nu_{\lambda\mu}$ is also isomorphic to

$$K^\nu_{\lambda\mu} \simeq \mathrm{Hom}^G\left(\mathrm{Hom}^{\mathfrak{S}_d\times\mathfrak{S}_d}([\lambda]\otimes[\mu], E^{\otimes d}\otimes F^{\otimes d}), \mathrm{Hom}^{\mathfrak{S}_d}([\nu], W^{\otimes d}),\right). \tag{9}$$

Using isomorphism (9) and identifying $E^{\otimes d} \otimes F^{\otimes d}$ with $W^{\otimes d}$, one can define a linear map

$$\begin{array}{rccc} SW: & \mathrm{Hom}^{\mathfrak{S}_d}([\nu], [\lambda]\otimes[\mu]) & \longrightarrow & K^\nu_{\lambda\mu} \\ & f & \longmapsto & g \ \mapsto \ g \circ f \end{array}$$

that is an isomorphism by Schur-Weyl duality. In particular

$$k_{\lambda\mu\nu} = \dim(K^\nu_{\lambda\mu}) \tag{10}$$

is the multiplicity of $S_\lambda E \otimes S_\mu F$ in the $(G = \mathrm{GL}(E) \times \mathrm{GL}(F))$-module $S_\nu W$.

2.3. **Schur-Weyl duality and symmetric Kronecker coefficients.** Assume now that $\dim(E) = \dim(F)$ and fix an isomorphism $\theta : E \longrightarrow F$. Consider the following involutive automorphism of $\mathrm{GL}(E) \times \mathrm{GL}(F)$

$$(g, h) \longmapsto (\theta^{-1} \circ h \circ \theta, \theta \circ g \circ \theta^{-1}), \tag{11}$$

and the associated semidirect product

$$\tilde{G} = \mathbb{Z}/2\mathbb{Z} \ltimes (\mathrm{GL}(E) \times \mathrm{GL}(F)).$$

Denote by τ the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ view as an element of $\tilde{G}$. Define actions of $\tilde{G}$ on $E \oplus F$ and $E \otimes F$ by

$$\begin{array}{ll} (g,h).(e,f) = (g(e), h(e)) & \tau.(e,f) = (\theta^{-1}(f), \theta(e)) \\ (g,h).e\otimes f = g(e)\otimes h(e) & \tau.(e\otimes f) = \theta^{-1}(f)\otimes\theta(e), \end{array}$$

for any $g \in \mathrm{GL}(E)$, $h \in \mathrm{GL}(F)$, $e \in E$ and $f \in F$.

Note that θ induces an isomorphism $S_\lambda E \simeq S_\lambda F$, still denoted by θ. In particular, we get an involution τ on $S_\lambda E \otimes S_\lambda F$. This allows to define a linear action of $\tilde{G}$ on $S_\lambda E \otimes S_\lambda F$.

Proposition 1. *The multiplicity of $S_\lambda E \otimes S_\lambda F$ in $S_\nu W$ as a $\tilde{G}$-module is $sk_{\lambda\nu}$.*

Proof. The action of $\tau \in \tilde{G}$ on $\mathrm{Hom}(S_\lambda E \otimes S_\lambda F, S_\nu W)$ induces an involution (still denoted by τ) on $K^\nu_{\lambda\lambda}$. By definition, the multiplicity of $S_\lambda E \otimes S_\lambda F$ in $S_\nu W$ as a $\tilde{G}$-module is the dimension of the space of fixed points of this involution.

Let $f \in \mathrm{Hom}^{\mathfrak{S}_d}([\nu], [\lambda] \otimes [\lambda])$. Write $f = s + \wedge$ according to the decomposition $[\lambda] \otimes [\lambda] = S^2[\lambda] \oplus \bigwedge^2[\lambda]$. We claim that

$$\tau(SW(f)) = SW(s - \wedge). \tag{12}$$

The claim and the Schur-Weyl isomorphism SW imply the proposition.

Let $g \in \mathrm{Hom}^{\mathfrak{S}_d \times \mathfrak{S}_d}([\lambda] \otimes [\lambda], W^{\otimes d})$. For $v_1, v_2 \in [\lambda]$, one has $(\tau g)(v_1 \otimes v_2) = \tau(g(v_2 \otimes v_1))$. Hence $(\tau g) \circ f = (\tau g) \circ (s + \wedge) = \tau(g \circ s - g \circ \wedge)$. We deduce that

$$\Big(\tau\ SW(f)\Big)(g) = \tau\Big(SW(f)(\tau\, g)\Big) = g \circ s - g \circ \wedge.$$

The claim (12) follows. □

Note that Proposition 1 is also proved in [BLMW11, Section 5] with less details.

2.4. **Semigroup property.** Fix a positive integer n. Denote by S_n the set of pairs of partitions (λ, ν) such that $sk_{\lambda\nu} \neq 0$, $l(\lambda) \leq n$ and $l(\nu) \leq n^2$.

Proposition 2. *As a subset of $\mathbb{Z}^{n+n^2}$, S_n is a finitely generated semigroup.*

Proof. Consider the flag variety $\mathcal{F}l(E) \times \mathcal{F}l(F)$ endowed with its natural G-action. For $(\xi_\bullet, \zeta_\bullet) \in \mathcal{F}l(E) \times \mathcal{F}l(F)$, we set

$$\tau.(\xi_\bullet, \zeta_\bullet) = (\theta^{-1}(\zeta_\bullet), \theta(\xi_\bullet)) \tag{13}$$

and get a $\tilde{G}$-action on $\mathcal{F}l(E) \times \mathcal{F}l(F)$. Given a partition λ of length at most $\dim(E)$, consider the line bundle $\mathcal{L}_\lambda \otimes \mathcal{L}_\lambda$ on $\mathcal{F}l(E) \times \mathcal{F}l(F)$. The G-action on $\mathcal{L}_\lambda \otimes \mathcal{L}_\lambda$ extends to $\tilde{G}$ in such a way

$$\mathrm{H}^0(\mathcal{F}l(E) \times \mathcal{F}l(F), \mathcal{L}_\lambda \otimes \mathcal{L}_\lambda) \simeq S_\lambda E \otimes S_\lambda F$$

is $\tilde{G}$-equivariant. Write $\lambda = (\lambda_1, \dots, \lambda_n)$ and set $\lambda^* = (-\lambda_n, \dots, -\lambda_1)$. Then

$$\mathrm{H}^0(\mathcal{F}l(E) \times \mathcal{F}l(F), \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_{\lambda^*}) \simeq S_\lambda E^* \otimes S_\lambda F^*.$$

Now, set $X = \mathcal{F}l(E) \times \mathcal{F}l(F) \times \mathcal{F}l(E \otimes F)$. Given two partitions (λ, ν) such that $l(\lambda) \leq n$ and $l(\nu) \leq n^2$. We have

$$\mathrm{H}^0(X, \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_\nu) \simeq S_\lambda E^* \otimes S_\lambda F^* \otimes S_\nu(E \otimes F).$$

Then, by Proposition 1, we have

$$sk_{\lambda,\nu} = \dim(\mathrm{H}^0(X, \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_\nu)^{\tilde{G}}).$$

Consider now the Cox ring

$$\mathcal{R} = \bigoplus_{(\lambda,\mu,\nu) \in (P_n^+)^2 \times P_{n^2}^+} \mathrm{H}^0(X, \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_{\mu^*} \otimes \mathcal{L}_\nu).$$

By [ADHL15, Proposition 3.2.3.5], $\mathcal{R}$ is a finitely generated graded ring. Set now

$$\mathcal{R}_\Delta = \bigoplus_{(\lambda,\nu) \in P_n^+ \times P_{n^2}^+} \mathrm{H}^0(X, \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_{\lambda^*} \otimes \mathcal{L}_\nu).$$

By [ADHL15, Proposition 1.1.2.4], $\mathcal{R}_\Delta$ is also finitely generated. Since $\tilde{G}$ is reductive, the invariant ring $\mathcal{R}_\Delta^{\tilde{G}}$ is also a finitely generated graded ring. In particular, its support Σ_n is a finitely generated semigroup. We deduce that

$$S_n = \{(\lambda, \nu) \in \Sigma_n \,|\, \lambda_0 \geq 0 \text{ and } \nu_{n^2} \geq 0\}$$

is a finitely generated semigroup. □

The above proof is an adaptation to a Brion-Knop's argument (see [É92]) that considered connected reductive groups.

2.5. **Isotropy of the determinant.** Fix a positive integer n and a n-dimensional vector space V. We consider the situation as in the last subsection when $E = V$ and $F = V^*$. We also assume that the isomorphism $\theta : V \longrightarrow V^*$ is symmetric. Then $W = E \otimes F$ identifies with $\mathrm{End}(V)$. For $(g, h) \in G = \mathrm{GL}(V) \times \mathrm{GL}(V^*)$ and $u \in \mathrm{End}(V)$ the action is given by the formula

$$(g, h).u = g \circ u \circ {}^t h.$$

Consider the following subgroup of G:

$$SG := \{(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(V^*) \,:\, \det(g) \det(h) = 1\}.$$

Since τ normalizes it (see (11)), one can form the semidirect product $\mathbb{Z}/2\mathbb{Z} \ltimes S(\mathrm{GL}(V) \times \mathrm{GL}(V))$ as a subgroup $S\tilde{G}$ of $\tilde{G}$. By a Frobenius' theorem (see [Fro97]), the stabilizer H of $\det_n$ in $\mathrm{GL}(W)$ is the image of this group in $\mathrm{GL}(W)$. Denote by H° the neutral component of H.

Recall that $\hat{G} = \mathrm{GL}(W)$ and $N = n^2$. By Frobenius' reciprocity theorem,

$$\mathbb{C}[\hat{G}] = \oplus_{\nu \in P_N^+} S_\nu W \otimes S_\nu W^*,$$

as a $(\hat{G} \times \hat{G})$-module. But $\mathcal{O}_n \simeq \hat{G}/H$ and

$$\mathbb{C}[\mathcal{O}_n] = \mathbb{C}[\hat{G}/H] = \oplus_{\nu \in P_N^+} \dim\left((S_\nu W^*)^H\right).S_\nu W. \tag{14}$$

If ζ is a n^{th} root of unity, $(\zeta \mathrm{Id}_V, \mathrm{Id}_{V^*})$ belongs to SG. We deduce that if $(S_\nu W^*)^H$ is nonzero then n divides $|\nu|$. We now assume that n divides $|\nu|$ and set $\delta = \frac{|\nu|}{n}$.

Since $\tilde{G}$ normalizes H it acts on $(S_\nu W^*)^H$. This action is the multiplication by $(\det g. \det h)^{-\delta}$. Hence

$$(S_\nu W^*)^H \simeq (S_\nu W)^H \simeq \mathrm{Hom}^{\tilde{G}}(S_{\delta^n} V \otimes S_{\delta^n} V^*, S_\nu W).$$

In particular

$$sk_{\delta^n \nu} = \dim\left((S_\nu W^*)^H\right),$$

and

$$\mathbb{C}[\mathcal{O}_n] = \oplus sk_{\delta^n \nu} S_\nu W. \tag{15}$$

Using (5), one gets

$$sk_{\delta^n \nu} = \dim\left(\mathrm{H}^0(X_\nu, \mathcal{L}_\nu)^H\right). \tag{16}$$

Fix a basis $\mathcal{B}$ of V such that $\theta(\mathcal{B})$ is the dual bases $\mathcal{B}^*$ of $\mathcal{B}$. Use $\mathcal{B}$ and $\mathcal{B}^*$ to identify $\mathrm{GL}(V)$ and $\mathrm{GL}(V^*)$ with $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{End}(V)$ with $\mathcal{M}_n(\mathbb{C})$. Then the action of $\tilde{G}$ is given by the formulas

$$\begin{aligned}&(A,B).M = AM^tB\\ &\tau.M = {}^tM\end{aligned}$$

3. The key lemma

In this section, H is any affine algebraic group that is the semidirect product of its neutral component H° and a finite subgroup K. Let X be a projective variety acted on by H. Let $\mathcal{L}$ be a H-linearized line bundle on X. We are interested in vanishing criteria for the space $\mathrm{H}^0(X,\mathcal{L})^H$ of H-invariant regular sections.

Regarding Formula (16), we plan to apply such criteria to $X = X_\nu$, $\mathcal{L} = \mathcal{L}_\nu$, H the isotropy group of $\det_n$ and $K = \mathbb{Z}/2\mathbb{Z}$. Nevertheless, it is clearer to state and prove this criterion in its natural context.

Let $Y \subset X^K$ be any irreducible closed subvariety of the set X^K of K-fixed points. Consider the restriction $\mathcal{L}_{|Y}$ of $\mathcal{L}$ to Y. Since K acts trivially on Y, the action of K on $\mathcal{L}_{|Y}$ is given by a character χ of K:

$$k.l = \chi(k)l \qquad \forall k \in K \quad \forall l \in \mathcal{L}_{|Y}. \tag{17}$$

Lemma 1. *Assume that*

(i) *the morphism*

$$\eta\,:\,H \times Y \longrightarrow X, (h,y) \longmapsto hy$$

is dominant;

(ii) *the character χ is non trivial.*

Then

$$\mathrm{H}^0(X,\mathcal{L})^H = \{0\}.$$

Proof. Let $\sigma \in \mathrm{H}^0(X,\mathcal{L})^H$. Show that $\sigma = 0$. The invariance of σ is

$$\forall h \in H,\, x \in X \qquad \sigma(hx) = h\sigma(x).$$

In particular,

$$\forall k \in K,\, y \in Y \qquad \sigma(y) = \chi(k)\sigma(y).$$

Since χ is nontrivial, this implies that the restriction $\sigma_{|Y}$ of σ to Y is zero. Using H-invariance, this implies that σ vanishes on the image of η. Since η is assumed to be dominant, this implies that $\sigma = 0$. □

4. Applications of Lemma 1

We come back to the situation of the orbit of the determinant $\mathcal{O}_n$: V is n-dimensional, $W = \mathrm{End}(V)$, $N = n^2$, $\hat{G} = \mathrm{GL}(W)$ and H is the stabilizer of $\det_n$. Let $\nu \in P_N^+$. Consider the flag variety X_ν and the line bundle $\mathcal{L}_\nu$ defined in Section 2.1.

By formula (16), to obtain Theorems 1 and 2, it is sufficient in each case to find $Y_\nu \subset X_\nu^{\mathbb{Z}/2\mathbb{Z}}$ satisfying Lemma 1. Consider the decomposition $W = \mathcal{S} \oplus \mathcal{A}$ in symmetric and skew-symmetric matrices. A list of working Y_ν's is as follows.

Th 1 For $n \equiv 2$ or 3 $[4]$, $\nu = ab^{n^2-2}c$ where $a \geq b \geq c$, n divides $a+c-2b$ and b odd:

$$X_\nu = \mathcal{F}l(1, n^2-1; W) \qquad \text{and} \qquad Y_\nu = \mathcal{F}l(1, \frac{n(n+1)}{2} - 1; \mathcal{S}).$$

The embedding of Y_ν in X_ν is given by

$$(l_s \subset \mathcal{H}_s) \longmapsto (l_s \subset \mathcal{H}_s \oplus \mathcal{A}).$$

Th 2 (i) For $n = 3$ and $\nu = a^2b^7$ with $a \geq b$, 3 divides $a - b$ and a odd:

$$X_\nu = \mathcal{F}l(2; W) \qquad \text{is a Grassmannian and} \qquad Y_\nu = \mathbb{P}(\mathcal{A}) \times \mathbb{P}(\mathcal{S}).$$

The embedding of Y_ν in X_ν is given by

$$(l_a, l_s) \longmapsto l_a \oplus l_s.$$

Th 2 (ii) For $n = 3$ and $\nu = a^3b^6$ with $a \geq b$ and a odd:

$$X_\nu = \mathcal{F}l(3; W) \qquad \text{is a Grassmannian and} \qquad Y_\nu = \mathbb{P}(\mathcal{A}) \times \mathcal{F}l(2; \mathcal{S}).$$

The embedding of Y_ν in X_ν is given by

$$(l_a, F_s) \longmapsto l_a \oplus F_s.$$

The remaining cases are obtained by the following lemma of duality.

Lemma 2. *Let $\nu = (\nu_1 \geq \cdots \geq \nu_N) \in P_N^+$. Then the multiplicities of $S_\nu W$ and $S_\nu W^*$ in $\mathbb{C}[\mathcal{O}_n]$ are equal. Moreover, $S_\nu W^* = S_{\nu^*} W$, where $\nu^* = (-\nu_N \geq \cdots \geq -\nu_1)$.*

Proof. For the first assertion, by formula (14), it is sufficient to prove that $\dim((S_\nu W)^H) = \dim((S_\nu W^*)^H)$. This equality is satisfied since H is reductive. The last assertion is well known (see e.g. [FH91]). □

Remark 1. If n is even, $X = \mathbb{P}(W)$ and $Y = \mathbb{P}(\mathcal{A})$ then the map η is dominant. Applying Lemma 1 to this pair, one can get vanishing symmetric Kronnecker coefficiients. But these cases can alternatively by obtained applying Theorem 1 with $b = c$.

It is natural to look for other examples of pairs (X, Y) of flag varieties satisfying Lemma 1. Observe that, unfortunately, there is an obvious strong obstruction for an irreducible component of X^τ to work. Namely

$$\dim(X) - \dim(Y) \leq n^2 - 1$$

has to be satisfied. Indeed, the subgroup $\{(P, P) \,:\, P \in \mathrm{GL}_n(\mathbb{C})\}$ is a subgroup of H stabilizing X^τ.

5. Checking assumptions of Lemma 1

In this section, we make a case by case verification of the assumptions of Lemma 1. The numbering refers to that used inn Section 4.

Th 1. *Computation of the character χ.* Fix a line l in $\mathcal{S}$ and consider the point $(l, \mathfrak{sl}_n(\mathbb{C})) \in \mathcal{F}l(1, n^2-1; W)$, where $\mathfrak{sl}_n(\mathbb{C})$ is the space of traceless matrices. The transposition map acts trivially on the fiber in $\mathcal{O}(-1)$ over l in $\mathbb{P}(W)$. The fiber in $\mathcal{O}(-1)$ over $\mathfrak{sl}_n(\mathbb{C})$ in $\mathbb{P}(\wedge^{n^2-1}W)$ identifies with

$$\wedge^{n^2-1}\mathfrak{sl}_n = \wedge^{\frac{n(n+1)}{2}-1}\mathcal{S} \otimes \wedge^{\frac{n(n-1)}{2}}\mathcal{A}.$$

The transposition acts on this fiber by $(-1)^{\frac{n(n-1)}{2}}$. The determinant of the transposition as an element of $\hat{G} = \mathrm{GL}(W)$ is $(-1)^{\frac{n(n-1)}{2}}$. We deduce from equation (4) that the transposition map acts on the fiber over $(l, \mathfrak{sl}_n(\mathbb{C}))$ in $\mathcal{L}_\nu$ with weight

$$(-1)^{((b-c)+c)\frac{n(n-1)}{2}}.$$

The assumption on n implies that $\frac{n(n-1)}{2}$ is odd. Then the character χ of Lemma 1 is $(-1)^b = -1$, since b is odd.

Dominancy of η. Let $l \subset \mathcal{H}$ be a general flag in X_ν. One equation of $\mathcal{H}$ can be written as $\mathrm{tr}(M\square) = 0$ for some matrix M. By genericity, M has full rank and using the action of H, one may assume that $M = I_n$ and so that $\mathcal{H} = \mathfrak{sl}_n(\mathbb{C})$. Let N be a nonzero matrix on l. Then, there exists $P \in \mathrm{GL}_n(\mathbb{C})$ such that $PNP^{-1} \in \mathcal{S}$ (see [HJ13, Theorem 4.4.24]). Since $\mathfrak{sl}_n(\mathbb{C})$ is stable by conjugacy, the H-orbit of $(l, \mathcal{H})$ contains $(\mathbb{C}PNP^{-1}, \mathfrak{sl}_n)$ that belongs to Y_ν. Thus η is dominant.

Th 2 (*i*). *The character χ.* The contribution of $l_a \oplus l_s$ is $(-1)^{a-b}$, that of det is $(-1)^b$. Finally, since a is odd this character is -1.

Notation. Given a point x on a variety X, we denote by T_xX the Zariski tangent space. If x belongs to a subvariety Y, the normal space is defined to be the quotient T_xX/T_xY.

Let $y \in Y_\nu$. Let $\mathcal{N}$ denote the normal space of Y_ν in X_ν at y. Consider the H-orbit $H.y$ of y and its tangent space $T_y(H.y)$ at y.

Claim: if the projection of $T_y(H.y)$ on $\mathcal{N}$ is surjective then η is dominant.

Indeed, the image of the tangent map $T_{(e,y)}\eta$ of η contains $T_y(H.y)$ and T_yY. The assumption implies that $T_{(e,y)}\eta$ is surjective. Since $H \times Y$ is smooth this implies that η is dominant.

We are now looking for an explicit y satisfying the claim. Let (A_1, A_2, A_3) be a base of $\mathcal{A}$ and $(S_1, \dots, S_6)$ be a base of $\mathcal{S}$.

Let $y = \mathrm{Span}(A_1, S_1) \in X_\nu$. Observe that y belongs to Y_ν. The vector space $\mathcal{N}$ identifies with the 7-dimensional vector space

$$\mathrm{Hom}(\mathbb{C}A_1, \mathbb{C}S_2 \oplus \mathbb{C}S_3 \oplus \mathbb{C}S_4 \oplus \mathbb{C}S_5 \oplus \mathbb{C}S_6) \times \mathrm{Hom}(\mathbb{C}S_1, \mathbb{C}A_2 \oplus \mathbb{C}A_3)$$

Let $(E_{ij})_{1\leq i,j\leq 4}$ denote the canonical basis of W. We claim that if

$$\begin{array}{lll} A_1 = E_{12}+E_{13}-E_{23}-E_{21}-E_{31}+E_{32} & S_1 = I_3 & S_2 = E_{12}+E_{21} \\ S_3 = E_{23}+E_{32} & S_5 = E_{11} & S_4 = E_{13}+E_{31} \\ A_2 = E_{23}-E_{32} & S_6 = E_{33} & A_3 = E_{13}-E_{31} \end{array}$$

then y satisfies the assumption of the claim. The details of the computation can be seen in the SageMath program `dimHorb-X2-Y1A1S.sage` on author's webpage [Res18].

Th 2 (ii) The character χ is $(-1)^a = -1$.

To prove that η is dominant, we use the claim of the previous item. A point y that works is

$$y = ([\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}], \mathrm{Span}(I_3, E_{12}+E_{21})).$$

See the SageMath program `dimHorb-X3-Y1A2S.sage` on author's webpage for details.

6. Equations of $\overline{\mathcal{O}_n}$

Let $\nu \in P_N^+$ and $\delta \in \mathbb{N}$ such that $|\nu| = n\delta$. On the one hand, the degree δ component $\mathbb{C}[\overline{\mathcal{O}_n}]_\delta$ of $\mathbb{C}[\overline{\mathcal{O}_n}]$ is a quotient of $\mathbb{C}[S^nW^*]_\delta = S^\delta S^n W$. On the other hand, $\mathbb{C}[\overline{\mathcal{O}_n}]$ is embedded in $\mathbb{C}[\mathcal{O}_n]$. Hence, any partition ν such that

$$\begin{array}{l} \mathrm{mult}(S_\nu W, \mathbb{C}[\mathcal{O}_n]) = 0, \text{ and} \\ \mathrm{mult}(S_\nu W, S^\delta S^n W) \neq 0 \end{array}$$

produces equations for $\overline{\mathcal{O}_n}$. In other words, the isotypical component of S^dS^nW of type ν is contained in the ideal $\mathcal{I}_n$ of $\overline{\mathcal{O}_n}$.

The first multiplicity $\mathrm{mult}(S_\nu W, \mathbb{C}[\mathcal{O}_n])$ is $sk_{\delta^n\,\nu}$. The multiplicity $\mathrm{mult}(S_\nu W, S^\delta S^n W)$ is a plethysm coefficient that we denote by $p_{n\,\nu}$. Evseev-Paget-Wildon obtained [EPW14, Proposition 5.1] a formula that allows to compute $p_{n\,\nu}$ inductively. Moreover, Mark Wildon implemented this algorithm (see [Wil19]). We used this program to check the vanishing or not of several $p_{n\nu}$ with $n = 3, 6$ or 7 like in Theorems 1 and 2.

6.1. **The case $n = 3$.** In "small degrees", we get the following submodules of $\mathcal{I}_3$. The column type refers to the shape of the partition and hence to one case of Theorem 2.

δ	ν	$p_{n\,\nu}$	*type*
12	$(11)^2 2^7$	1	27
13	$(7)^3 3^6$	1	36
15	$(11)^3 2^6$	1	36
15	$9^3 3^6$	1	36
17	$(16)^2 2^7$	2	27
17	$(13)^3 2^6$	1	36
17	$(11)^3 3^6$	5	36
17	$7^6 3^3$	1	63
18	$(13)^2 4^7$	2	27
19	$(11)^2 5^7$	2	27
19	$(15)^3 2^6$	3	36
19	$(13)^3 3^6$	9	36
19	$(11)^3 4^6$	12	36
19	$9^3 5^6$	3	36
19	$8^6 3^3$	1	63
20	$23^2 2^7$	2	27

Remark 2. For the degree 12 partition $(11)^2 2^7$, the corresponding Kronecker coefficient is already zero (so the symmetric one is not really useful). For the degree 13 partition $7^3 3^6$, the Kronecker coefficient is $k_{13^3\,13^3\,7^3 3^6} = k_{4^3\,4^3\,4^3} = 2$ (see e.g. [BOR09, Lemma 2.1] for the first equality and thanks to [BKT] for the second one). So we have a degree 13 equation which cannot be obtained by considering Kronecker coefficients. This equation (among others) was discovered first by C. Ikenmeyer [Ike12].

Theorem 1 applied with $n = 3$ gives no equation of degree at most 20 because the corresponding plethysm coefficients also vanish. The smallest degrees that one can obtain with Theorem 1 are:

δ	ν	$p_{n\,\nu}$	*type*
21	$(13)7^7 1$	3	171
21	$(12)7^7 2$	2	171
21	$(11)7^7 3$	2	171
22	$(17)7^7$	1	17

Note that the last example has length 8.

Here, we just listed very few examples of submodules of $\mathcal{I}_3$, but one can get a lot of them. One can obtain other examples from [Res18]. Here comes some big examples:

δ	ν	$p_{n\,\nu}$	*type*
62	$(65)^2 8^7$	1 614 147	27
51	$(29)^3 (11)^6$	1 907 404 762 420	36
53	$(20)^6 (13)^3$	617 624 065 676	63
55	$(21)^7 9^2$	199 463 016 669	72
41	$(40)(11)^7 6$	5 400 515	171

6.2. **The cases of $n = 6$ and 7.** We already observed that in the case of the first assertion of Theorem 1 the Kronecker coefficient $k_{\delta^n\,\delta^n\,\nu}$ is equal to 1. By

[CRY92, LR04], the only hook-shape partition ν such that $S_\nu W$ is contained in $S^\delta S^n W$ is the obvious one $\nu = n\delta$. In particular, we obtain no equation by applying Theorem 1 with $b = 1$.

Unfortunately, there is no partition $\nu = ab^{n^2-2}c$ as in Theorem 1 with $p_{n\nu} \neq 0$ and

(i) $n = 6$, $b = 3$ and $a \leq 225$;
(ii) $n = 6$, $b = 5$ and $a \leq 148$;
(iii) $n = 6$, $b = 7$ and $a \leq 26$;
(iv) $n = 7$, $b = 3$ and $a \leq 68$;
(v) $n = 7$, $b = 5$ and $a \leq 38$.

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(nr) Institut Camille Jordan (ICJ), UMR CNRS 5208, Université Claude Bernard Lyon I, 43 boulevard du 11 novembre 1918, F - 69622 Villeurbanne cedex