Merzlyakov-type theorems after Sela

Part II

Goal

$\mathbb{F}$ is a finitely generated non abelian free group.
$\Sigma(\bar{x}, \bar{y}) \subset_{\text{finite}} \langle \bar{x}, \bar{y} \rangle$.

Theorem (Merzlyakov)

Let $\mathbb{F} \models \forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}) = 1)$. Then there exists a retract $r : G_\Sigma \to \langle \bar{x} \rangle$, where $G_\Sigma := \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle$.

Theorem (Extended Merzlyakov)

Let $(\bar{b}_n)_{n<\omega}$ be a "test sequence" in $\mathbb{F}$. Suppose for each n there is $\bar{c}_n$ such that $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$. Then there exists a retract $r : G_\Sigma \to \langle \bar{x} \rangle$.

Recall

Theorem A

Let $(h_n)_{n<\omega} : G \to \mathbb{F}$ be an infinite sequence of morphisms. Then there exists a sequence of base points $(*_n)_{n<\omega}$ in $X_{\mathbb{F}}$ and a sequence of rescaling constants $(r_n)_{n<\omega} \in \mathbb{R}^+$ such that a subsequence of the induced pseudo-metrics $(d_n/r_n)_{n<\omega}$ converges to a pseudo-metric d which is induced by a non-trivial action of G on a real tree $(T, *)$.

- L is a limit group if it can be obtained as $G/ker\lambda$ where λ is the limit action for a sequence of morphisms $(h_n)_{n<\omega} : G \to \mathbb{F}$;
- L admits an action on a real tree which is non-trivial, super-stable, with trivial tripod stabilizers and abelian arc stabilizers.

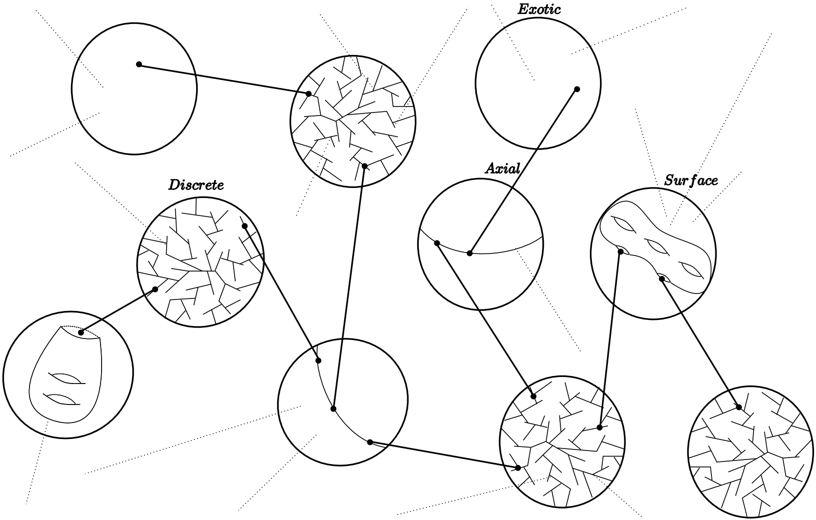
Rips' Machine

Suppose G acts on a real tree T. Then the action is:

- *minimal*, if there is no G-invariant proper subtree;
- *non-trivial*, if there is no globally fixed point;
- *super-stable*, if for any arc I and subarc $J \subset I$ we have that $Stab(J) \neq Stab(I) \Rightarrow Stab(I) = \{1\}$.

Theorem (Rips' Machine)

Let G be a finitely generated group. Suppose G acts non-trivially and minimally on an $\mathbb{R}$-tree T. Moreover, assume that the action is super-stable and tripod stabilizers are trivial. Then the action can be understood in terms of simpler components which are of discrete, axial, surface or exotic type


Exotic
Axial
Surface
Discrete

Lemma (Approximating Sequences)

Assume $(X_{\mathbb{F}}, *_n, d_{X_{\mathbb{F}}})_{n<\omega}$ *"converges" to* $(T, *, d_T)$ *as in Theorem A. Then for any* $x \in T$*, the following hold:*

- *there exists a sequence* $(x_n)_{n<\omega}$ *such that* $\frac{d_{X_{\mathbb{F}}}}{r_n}(x_n, g \cdot *_n) \to d_T(x, g \cdot *)$ *for any* $g \in G$*, we call such a sequence an approximating sequence;*
- *if* $(x_n)_{n<\omega}, (x'_n)_{n<\omega}$ *are two approximating sequences for* $x \in T$*, then* $\frac{d_{X_{\mathbb{F}}}}{r_n}(x_n, x'_n) \to 0$*;*
- *if* $(x_n)_{n<\omega}$ *is an approximating sequence for* x*, then* $(g \cdot x_n)_{n<\omega}$ *is an approximating sequence for* $g \cdot x$*;*
- *if* $(x_n)_{n<\omega}, (y_n)_{n<\omega}$ *are approximating sequences for* x, y *respectively, then* $\frac{d_{X_{\mathbb{F}}}}{r_n}(x_n, y_n) \to d_T(x, y)$*.*

Shortening Argument

Theorem

Suppose G is a non-cyclic finitely generated group. Let $(h_n)_{n<\omega} : G \to \mathbb{F}$ be an infinite sequence of short morphisms. Then either G splits as a non-trivial free product or the action on a real tree T obtained as in Theorem A is not faithful.

Definition

Let S be a finite generating set for G and $h : G \to \mathbb{F}$ be a morphism. Then the *length* of h is

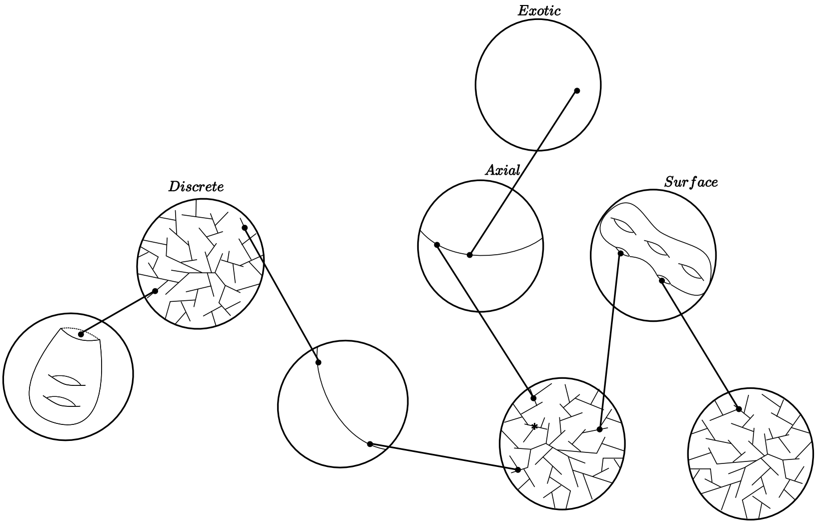
$$\mathfrak{l}(h) := max_{s \in S}\{d_{X_{\mathbb{F}}}(1, h(s) \cdot 1)\}$$

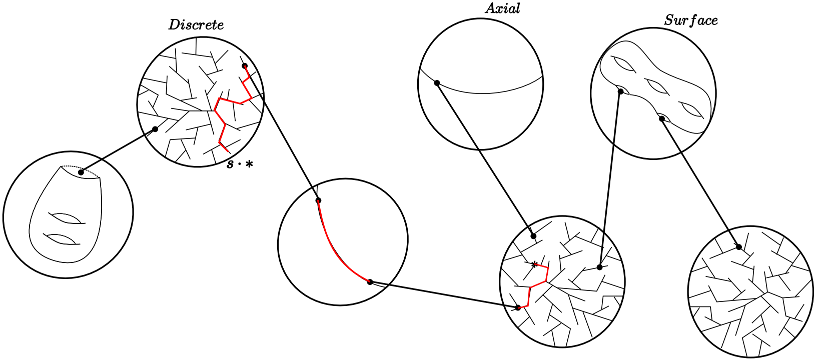
Moreover h is called short if:

$$\mathfrak{l}(h) \leq max_{s \in S}\{d_{X_{\mathbb{F}}}(x, h(\sigma(s)) \cdot x)\}$$

for any $x \in X_{\mathbb{F}}$ and $\sigma \in Aut(G)$

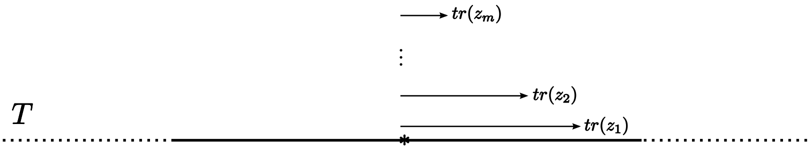
Idea of the proof




Discrete
Axial
Surface
$s \cdot *$
*

Special case I: T is a line

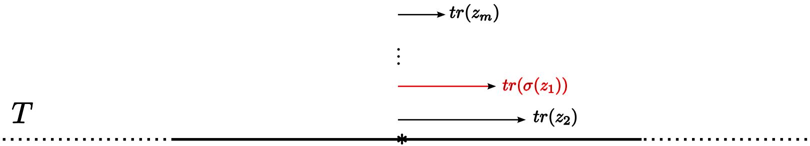
- since $ker\lambda = \{1\}$ we have that G is a limit group;
- in particular G is torsion-free;
- thus $G \hookrightarrow Isom^+(\mathbb{R})$;
- $G \cong \mathbb{Z}^m := \langle z_1, \ldots, z_m \rangle$, with $m > 1$;
- $\{tr(z_1), \ldots, tr(z_m)\}$ forms a linearly independent set;
- without loss of generality $tr(z_1) > tr(z_2) > \ldots > tr(z_m)$.



- without loss of generality $tr(z_1) > tr(z_2) > \ldots > tr(z_m)$;
- there is k such that $tr(z_1) = k \cdot tr(z_2) + u$ and $0 < u < tr(z_2)$;
- let σ be the following automorphism of $\mathbb{Z}^m$:

$$z_1 \mapsto z_1 z_2^{-k}$$

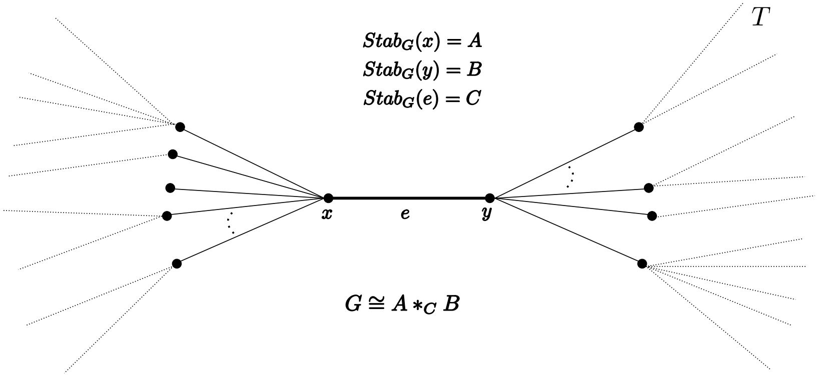
$$z_i \mapsto z_i \text{ for } 2 \leq i \leq m$$



- after finitely many steps we get an automorphism (still denoted) σ such that $d_T(*, \sigma(s) \cdot *) < d_T(*, s \cdot *)$, for every $s \in S$;
- thus $d_{X_{\mathbb{F}}}(*_n, h_n(\sigma(s)) \cdot *_n) < d_{X_{\mathbb{F}}}(*_n, h_n(s) \cdot *_n)$;
- but $*_n = 1$ (exercise), contradicting the shortness of h_n;

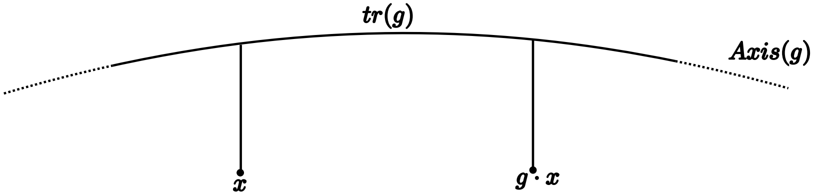
Special case II: Discrete action

- Suppose the action of G on T is discrete;
- we can analyze the action using Bass-Serre theory;

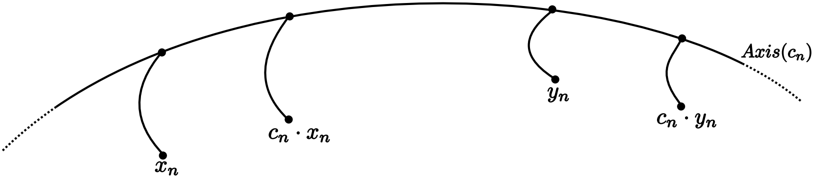


Isometries of $\mathbb{R}$-trees

- Suppose G acts on a real tree T (by isometries);
- let $g \in G$, and $tr(g) := inf_{x \in T}\{d_T(x, g \cdot x)\}$;
- if g fixes a point, then it is called *elliptic*;
- otherwise g is called *hyperbolic* and there is a unique line $L \subset T$ such that g acts on L as translation by $tr(g)$;
- the line L is called the axis of g, moreover if $x \in T$, then $d_T(x, g \cdot x) = tr(g) + 2d_T(x, L)$



- Let $c \in C \setminus \{1\}$;
- $h_n(c) = c_n$ be the (non-trivial) image of c in $\mathbb{F}$, and consider the axis of c_n in $X_{\mathbb{F}}$;
- let $(x_n)_{n<\omega}$ and $(y_n)_{n<\omega}$ be approximating sequences for x, y respectively;



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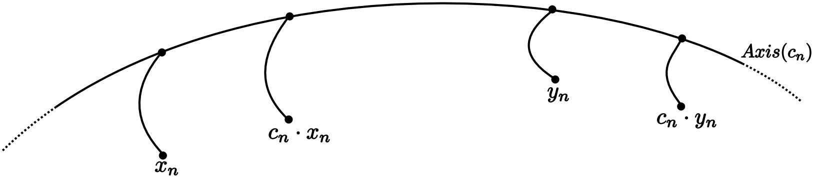

- There exists $(k_n)_{n<\omega} \in \mathbb{Z}$ such that $c_n^{k_n} \cdot x_n$ approximates y;
- (respectively) $c_n^{-k_n} \cdot y_n$ approximates x;

- Consider the Dehn twists of $A *_C B$:

$$\delta_n(g) = \begin{cases} g & \text{if } g \in A \\ c^{-k_n} g c^{k_n} & \text{if } g \in B \end{cases}$$

- let $\hat{d}_n := \frac{d_{X_{\mathbb{F}}}}{r_n}$;
- then we have: $\hat{d}_n(x_n, h_n \circ \delta_n(g) \cdot x_n)) \to 0$ for any $g \in G$;
  - $g = a_0 b_0 a_1 b_1 \ldots a_m b_m a_{m+1}$;
  - $\hat{d}_n(x_n, h_n \circ \delta_n(g) \cdot x_n) \leq \sum \hat{d}_n(x_n, a_i \cdot x_n) + \sum \hat{d}_n(x_n, b_i^{c^{k_n}} \cdot x_n)$;
  - $\hat{d}_n(x_n, a_i \cdot x_n) \to 0$ and $\hat{d}_n(x_n, b_i^{c^{k_n}} \cdot x_n) \to 0$;
- but as before $*_n = 1$, thus $\frac{max_{s \in S} d_{X_{\mathbb{F}}}(1, h_n(s) \cdot 1)}{r_n} = 1$;
- a contradiction to the shortness of h_n.

Extended Merzlyakov theorem

Theorem

Let $(\bar{b}_n)_{n<\omega}$ be a test sequence in $\mathbb{F}$. Suppose for each n there is $\bar{c}_n$ such that $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$. Then there exists a retract $r : G_\Sigma \to \langle \bar{x} \rangle$.

Definition (Test sequence)

An infinite sequence of tuples $(\bar{b}_n)_{n<\omega} \in \mathbb{F}$ is called a *test sequence* if the tuple $(b_1(n), \ldots, b_k(n))$ satisfies $C'(1/n)$ in $\mathbb{F}$, for $n < \omega$.

Recall: Let $\bar{b} := (b_1, \ldots, b_k)$ be a tuple of words in $\mathbb{F}$. A subword w of b_i, for some $i \leq k$, is called a *piece* if it appears in two "different" ways in $\bar{b}$. We say that $\bar{b}$ satisfies $C'(p)$ in $\mathbb{F}$ (for $0 < p < 1$), if for any piece w, if w is a subword of b_i, for some $i \leq k$, then we have that $|w|_{\mathbb{F}} < p\,|b_i|_{\mathbb{F}}$.

Proof(Extended Merzlyakov theorem)

For expositional simplicity of the argument we make the following assumptions:

- the tuples $(b_1(n), \ldots, b_k(n))$ are not singletons, i.e. $k > 1$;
- for any $i < j \leq k$ there are $c_{i,j}, c'_{i,j} \in \mathbb{R}^+$ such that $c_{i,j} < \frac{|b_i(n)|_{\mathbb{F}}}{|b_j(n)|_{\mathbb{F}}} < c'_{i,j}$ for all $n < \omega$.

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Definition (Very Short Morphism)

Let $G := \langle \bar{x}, y_1, \ldots, y_m \rangle$. Then $h : G \to \mathbb{F}$ is called *very short* with respect to $(\bar{x}, \bar{y})$ if for any h' that extends $h \restriction \langle \bar{x} \rangle$ we have that $\Sigma_{i \leq m} |h(y_i)|_{\mathbb{F}} \leq \Sigma_{i \leq m} |h'(y_i)|_{\mathbb{F}}$.

- the notion of a "very short morphism" passes to quotients;
- let $\eta : G \twoheadrightarrow L$ and suppose a very short morphism $g : G \to \mathbb{F}$ factors through η, i.e. $g = h \circ \eta$ with $h : L \to \mathbb{F}$;
- then h is very short with respect to $(\eta(\bar{x}), \eta(\bar{y}))$.

- let $G_\Sigma := \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle$;
- since for each n we have $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$, we obtain a sequence of morphisms $(g_n)_{n<\omega} : G_\Sigma \to \mathbb{F}$;
- we may assume g_n is very short with respect to $(\bar{x}; \bar{y})$, for $n < \omega$;
- consider the limit action $G_\Sigma \curvearrowright^\lambda (T, *)$ of the sequence $(g_n)_{n<\omega}$;
- let $L := G_\Sigma / ker\lambda$ and $\eta : G_\Sigma \twoheadrightarrow L$ be the canonical quotient map.

Claim I: We may assume that $L := G_\Sigma / ker\lambda$ is freely indecomposable with respect to $\eta(\langle \bar{x} \rangle)$.

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Proof of Claim I:

- since $g_n(G_\Sigma)$ is not abelian, we have that T is not isometric to a line (Exercise);
- thus, there is a sequence $(h_n)_{n<\omega} : L \to \mathbb{F}$ such that $g_n = h_n \circ \eta$ for all but finitely many $n < \omega$;
- note that since $(\bar{b}_n)_{n<\omega}$ is a test sequence η is injective on $\langle \bar{x} \rangle$. Thus, we identify $\eta(\bar{x})$ with $\bar{x}$;
- let $L = L_1 * L_2$ be a non-trivial free product with $\langle \bar{x} \rangle \leq L_1$. Continue with L_1 and $h_n \upharpoonright L_1$ after been made very short;

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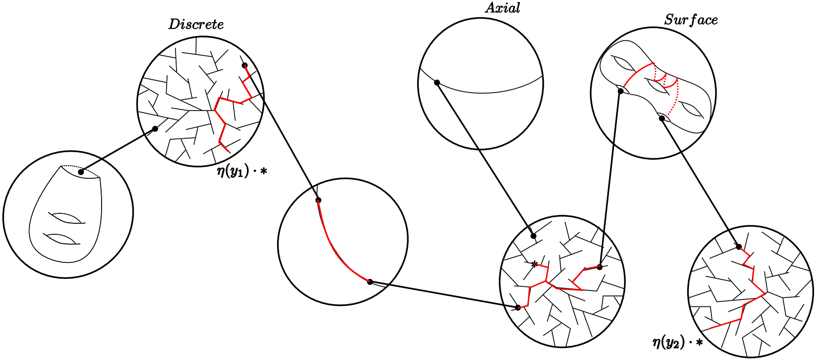
Lemma (DCC for limit groups)

Let $L_1 \twoheadrightarrow L_2 \twoheadrightarrow \ldots \twoheadrightarrow L_m \twoheadrightarrow \ldots$ be a sequence of epimorphisms of limit groups. Then the sequence stabilizes after finitely many steps, i.e. there are only finitely many proper epimorphisms in the sequence.

We are left with:

- $(h_n)_{n<\omega} : L \to \mathbb{F}$ which is very short with respect to $(\bar{x}; \eta(\bar{y}))$;
- $(h_n(\bar{x}))_{n<\omega}$ a test sequence;
- L freely indecomposable with respect to $\langle \bar{x} \rangle$;
- a faithful action of L on T as a limit of the above sequence;
- the action of L on T can be analyzed using Rips' machine.

- The subgroup $\langle \bar{x} \rangle$ does not fix a point;
  - T is covered by translates of the arcs $[*, s \cdot *]$ where $s \in \{\bar{x}, \eta(\bar{y})\}$ (Exercise);
  - and now use the shortening argument.



Minimal G-trees

Recall:
Suppose G acts on a real tree T. Then the action is:

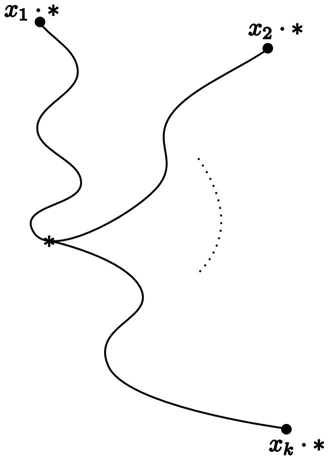
- *non-trivial*, if there is no globally fixed point;
- *minimal*, if there is no G-invariant proper subtree.

Lemma

Let G be finitely generated group. If G acts non-trivially on a real tree T, then T contains a unique minimal G-invariant subtree. It is the union of axes of hyperbolic elements of G.

- Since $\langle \bar{x} \rangle$ does not fix a point, there exists a minimal $\langle \bar{x} \rangle$-invariant subtree of T.

- Let T_{min} be the minimal tree that $\langle \bar{x} \rangle$ acts on. We want to prove that T_{min} lies on the discrete part of T.
- T_{min} is covered by translates of arcs of the form $[*, x_i \cdot *]$ by elements of $\langle \bar{x} \rangle$.



Claim II: Let $I \subseteq [*, x_i \cdot *]$ be a non-trivial arc. Then, for any $g \in L \setminus \{1\}$ and any $j \leq k$, we have that $g.I \cap [*, x_j \cdot *]$ is at most a point.

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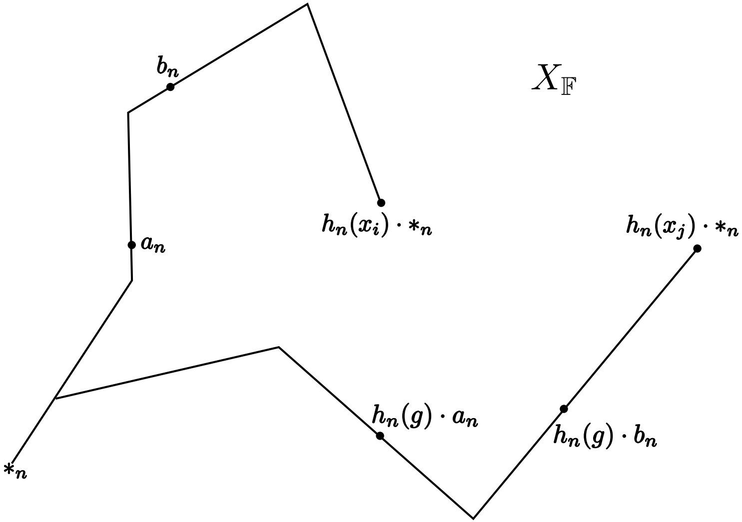

Indecomposable Components

Definition

Suppose G acts on a real tree T. Then a non degenerate tree $Y \subseteq T$ is called *indecomposable* if for every pair of arcs $I, J \subseteq Y$ there is a finite sequence $g_1 \cdot I, \ldots, g_n \cdot I$ which covers J and such that $g_i \cdot I \cap g_{i+1} \cdot I$ is non degenerate.

Fact

Any non discrete component in Rips' decomposition is indecomposable.

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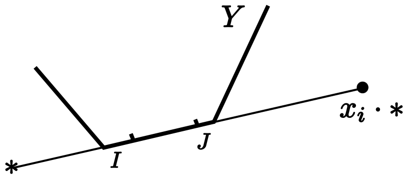

- L acts discretely on T with trivial edge stabilizers;
  - T is covered by translates of arcs of the form $[*, s \cdot *]$, where $s \in \{\bar{x}, \eta(\bar{y})\}$;
  - if Y is a component of axial or surface type, then for some j $[*, \eta(y_j) \cdot *]$ intersects (non-trivially) a translate of Y;
  - thus, we can use the shortening argument to "shorten" $[*, \eta(y_j) \cdot *]$;
  - if e is an edge which is non-trivially stabilized, then for some j $[*, \eta(y_j) \cdot *]$ contains a translate of e;
  - thus, we can again use the shortening argument to "shorten" $[*, \eta(y_j) \cdot *]$ (in the limiting sequence).
- L inherits a splitting from its action on T as $Stab(*) * \langle x_1, \ldots, x_k \rangle$ (Exercise);
- $L = \langle x_1, \ldots, x_k \rangle$.

Thus, $G_\Sigma \twoheadrightarrow L = \langle \bar{x} \rangle$, as we wanted.

Extended Merzlyakov Theorem together with the following:

Theorem (Sela)

Let $\phi(\bar{x}, \bar{y})$ be a Diophantine formula. Then ϕ is an equation (in the sense of Pillay-Srour).

Have been used to prove:

Theorem (Perin-S.)

Let $\phi(\bar{x})$ be a formula over $\mathbb{F}_n$. Suppose $\phi(\mathbb{F}_n) \neq \phi(\mathbb{F}_\omega)$. Then ϕ is not superstable.

Conjecture

Let $\phi(\bar{x})$ be a formula over $\mathbb{F}_n$. Then ϕ is superstable if and only if $\phi(\mathbb{F}_n) = \phi(\mathbb{F}_\omega)$.

Question

- *Can we generalise Merzlyakov's theorem by restricting the universal variables so that they belong to a variety?*
- *if* $\mathbb{F} \models \forall \bar{x}(R(\bar{x}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$, *then there exists a retract* $r : G_\Sigma \twoheadrightarrow G_R$ *(where* $G_R := \langle \bar{x} | R(\bar{x}) \rangle$*)?*

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Theorem

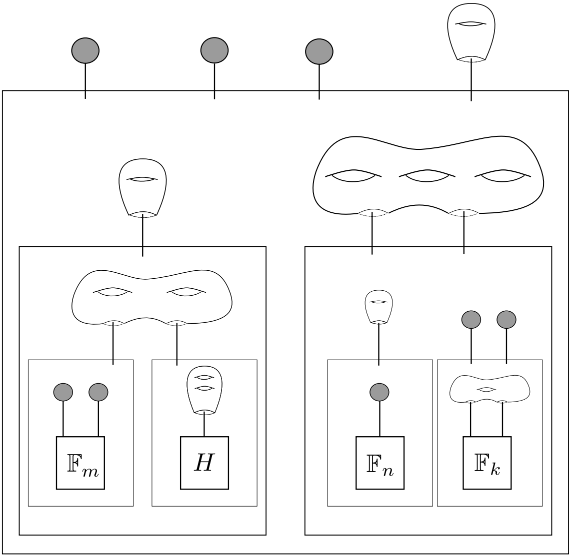
Let $g \geq 2$ *and* $\pi_1(\Sigma_g) = \langle x_1, \ldots, x_{2g} \mid [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] \rangle$ *be the fundamental group of the orientable surface of genus* g*. Let* $\mathbb{F} \models \forall \bar{x}([x_1, x_2] \ldots [x_{2g-1}, x_{2g}] = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$*. Then there exists a retract* $r : G_\Sigma \to \pi_1(\Sigma_g)$*.*

Counterexample (Three projective planes)

- *Let* $3PP := \langle x_1, x_2, x_3 \mid x_1^2 x_2^2 x_3^2 \rangle$*;*
- *(Lyndon) For any* $a, b, c \in \mathbb{F}$*, if* $a^2 b^2 c^2 = 1$ *then* a, b, c *belong to a cyclic subgroup of* $\mathbb{F}$*;*
- $\mathbb{F} \models \forall \bar{x}(x_1^2 x_2^2 x_3^2 = 1 \to (\wedge_{i<j\leq 3}[x_i, x_j] = 1))$*;*
- *But* G_Σ *does not admit a retract to* $3PP$*.*

Counterexample (Free Abelian groups)

- $\mathbb{F} \models \forall x_1, x_2([x_1, x_2] = 1 \to \exists y(x_1 = y^2 \vee x_2 = y^2 \vee x_1 \cdot x_2 = y^2))$*;*
- *but there is no retract from* $\langle x_1, x_2, y \mid [x_1, x_2], y^2 x_1^{-1} \rangle$ *to* $\langle x_1, x_2 \mid [x_1, x_2] \rangle$*;*
- *neither from* $\langle x_1, x_2, y \mid [x_1, x_2], y^2 x_2^{-1} \rangle$*;*
- *nor from* $\langle x_1, x_2, y \mid [x_1, x_2], y^2 (x_1 x_2)^{-1} \rangle$*.*



$\mathbb{F}_m$
H
$\mathbb{F}_n$
$\mathbb{F}_k$