Merzlyakov-type theorems after Sela

Part III

Recall

Theorem (Merzlyakov)

Let $\mathbb{F} \models \forall \bar{x} \exists \bar{y} (\Sigma(\bar{x}, \bar{y}) = 1)$*. Then there exists a retract* $r : G_\Sigma \to \langle \bar{x} \rangle$*.*

Theorem (Extended Merzlyakov)

Let $(\bar{b}_n)_{n<\omega}$ *be a test sequence in* $\mathbb{F}$*. Suppose for each* n *there is* $\bar{c}_n$ *such that* $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$*. Then there exists a retract* $r : G_\Sigma \to \langle \bar{x} \rangle$*.*

- In the first part we saw how to obtain from an infinite sequence of pairwise non-conjugate morphisms from a finitely generated group G to a torsion-free hyperbolic group Γ

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  a non-trivial action of G on a real tree T;
- after moving to the quotient of G by the kernel of the above action, the induced action satisfies some "tameness" hypotheses;
- thus it can be analyzed by Rips' machine in "simpler" components, which are of discrete, axial, surface or exotic type.

Theorem (Extended Merzlyakov)

Let $(\bar{b}_n)_{n<\omega}$ be a test sequence in $\mathbb{F}$. Suppose for each n there is $\bar{c}_n$ such that $\mathbb{F} \models \Sigma(\bar{b}_n, \bar{c}_n) = 1$. Then there exists a retract $r : G_\Sigma \to \langle \bar{x} \rangle$.

Idea of the proof:

- We start with a sequence of morphisms that restrict to a test sequence on the x's and are "very short" with respect to $(\bar{x}, \bar{y})$:

$$(g_n)_{n<\omega} : \langle \bar{x}, \bar{y} \mid \Sigma(\bar{x}, \bar{y}) \rangle \to \mathbb{F}$$

  i.e. morphisms that give the shortest length possible to the sum of the y's;

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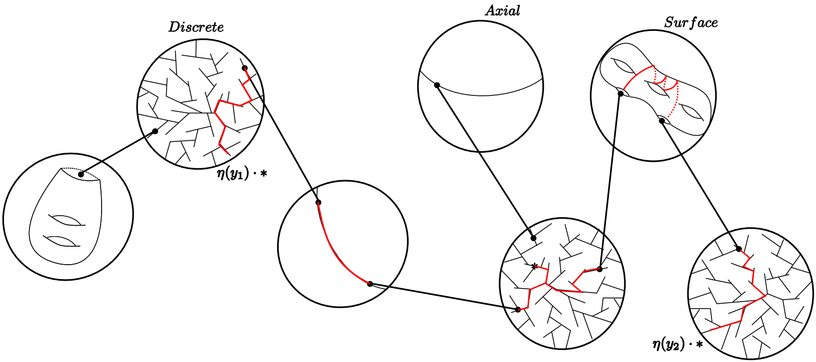
  i.e. morphisms that give the shortest length possible to the sum of the y's;
- we pass to the limit group $L := G_\Sigma / ker\lambda$, where λ is the action of G_Σ on the limit $\mathbb{R}$-tree (obtained by the sequence of morphisms $(g_n)_{n<\omega}$);
- we may assume that L has the following properties:
  - $\langle \bar{x} \rangle$ is a free subgroup of L;
  - L is freely indecomposable with respect to $\langle \bar{x} \rangle$;
  - L admits an action on an $\mathbb{R}$-tree that can be analyzed by Rips' machine.

- we use properties of the test sequences together with the shortening argument to eliminate non-discrete components from the $\mathbb{R}$-tree;



- we end up with an action of L on a simplicial tree and Bass-Serre theory tells us that $L = \langle \bar{x} \rangle$ as we wanted.

Question

- *Can we generalise Merzlyakov's theorem by restricting the universal variables so that they belong to a variety?*
- *if* $\mathbb{F} \models \forall \bar{x}(R(\bar{x}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$, *is it true that there exists a retract* $r : G_\Sigma \twoheadrightarrow G_R$ *(where* $G_R := \langle \bar{x} | R(\bar{x}) \rangle$*)?*

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Theorem

Let $g \geq 2$ *and* $\pi_1(\Sigma_g) = \langle x_1, \ldots, x_{2g} \mid [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] \rangle$ *be the fundamental group of the orientable surface of genus* g*. Let* $\mathbb{F} \models \forall \bar{x}([x_1, x_2] \ldots [x_{2g-1}, x_{2g}] = 1 \rightarrow \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$*. Then there exists a retract* $r : G_\Sigma \rightarrow \pi_1(\Sigma_g)$*.*

Counterexample (Three projective planes)

- Let $3PP := \langle x_1, x_2, x_3 \mid x_1^2 x_2^2 x_3^2 \rangle$;
- (Lyndon) For any $a, b, c \in \mathbb{F}$, if $a^2 b^2 c^2 = 1$ then a, b, c belong to a cyclic subgroup of $\mathbb{F}$;
- $\mathbb{F} \models \forall \bar{x}(x_1^2 x_2^2 x_3^2 = 1 \to (\wedge_{i<j\leq 3}[x_i, x_j] = 1))$;
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Counterexample (Free Abelian groups)

- $\mathbb{F} \models \forall x_1, x_2([x_1, x_2] = 1 \to \exists y(x_1 = y^2 \vee x_2 = y^2 \vee x_1 \cdot x_2 = y^2))$;
- but there is no retract from $\langle x_1, x_2, y \mid [x_1, x_2], y^2 x_1^{-1} \rangle$ to $\langle x_1, x_2 \mid [x_1, x_2] \rangle$;
- neither from $\langle x_1, x_2, y \mid [x_1, x_2], y^2 x_2^{-1} \rangle$;
- nor from $\langle x_1, x_2, y \mid [x_1, x_2], y^2 (x_1 x_2)^{-1} \rangle$.

Theorem

Let $n \geq 2$ *and* $\mathbb{Z}^n := \langle x_1, \ldots, x_n \mid [x_i, x_j] \ \mathit{for} \ \ 1 \leq i < j \leq n \rangle$. *Suppose* $\mathbb{F} \models \forall \bar{x}(\bigwedge_{1 \leq i < j \leq n}[x_i, x_j] = 1 \rightarrow \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$. *Then there exist finitely many free abelian groups* $A^n_1, \ldots, A^n_k$ *that contain* $\mathbb{Z}^n$ *as a finite index subgroup such that:*

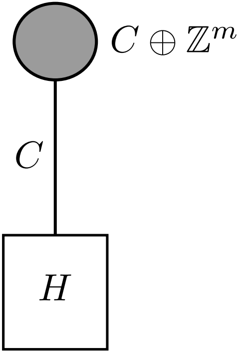
- *for each* $i \leq k$, *there exists a retract* $r_i : G_\Sigma *_{\mathbb{Z}^n} A^n_i \rightarrow A^n_i$;
- *for any* $h : \mathbb{Z}^n \rightarrow \mathbb{F}$ *there exists some* $i \leq k$ *such that* h *extends to a morphism* $h' : A^n_i \rightarrow \mathbb{F}$.

Towers

Towers

Definition (Free Abelian Flat)

Let G be a group and H be a subgroup of G. Then G is a free abelian flat over H if G admits an amalgamated free product splitting $H *_C (C \oplus \mathbb{Z}^m)$ where C is maximal abelian in H.



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Example

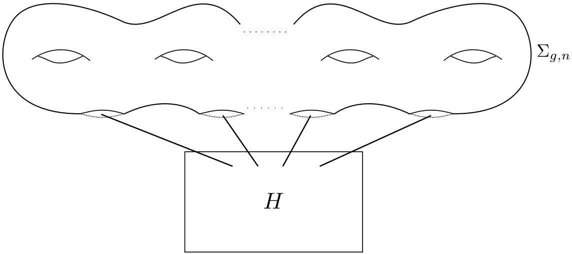
$\mathbb{F}_2 *_{e_1^2 e_2^2 = z} \langle z \rangle \oplus \mathbb{Z}^m$ is a free abelian flat over $\mathbb{F}_2$.

Lemma

If G is a free abelian flat over a limit group, then G is a limit group.

Definition (Surface Flat)

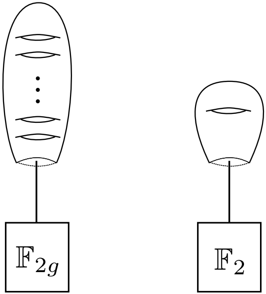
Let G be a group and H be a subgroup of G. Then G is a surface flat over H if G admits a splitting as follows:



- $\Sigma_{g,n}$ is either a punctured torus or $\chi(\Sigma_{g,n}) \leq -2$;
- each edge corresponds to a boundary component and each boundary component is "used";
- there exists a retract $r : G \to H$ that sends the surface group to a non abelian image.

Example

- the fundamental group of the orientable surface of genus 2 is a surface flat over $\mathbb{F}_2$;
- more generally $\pi_1(\Sigma_{2g})$ is a surface flat over $\mathbb{F}_{2g}$.



$$\pi_1(\Sigma_2) := \langle x_1, x_2, e_1, e_2 \mid [x_1, x_2] = [e_1, e_2] \rangle$$

$$r : \pi_1(\Sigma_2) \twoheadrightarrow \mathbb{F}_2 \quad r(x_2) = e_2 \quad r(x_1) = e_1$$

Lemma

If G is a surface flat over a limit group, then G is a limit group.

Definition

A group G has the structure of a tower over a subgroup H if there exists a sequence $G = G^m > G^{m-1} > \ldots > G^0 = H * \mathbb{F}$ such that for each i, $0 \leq i < m$, one of the following holds:

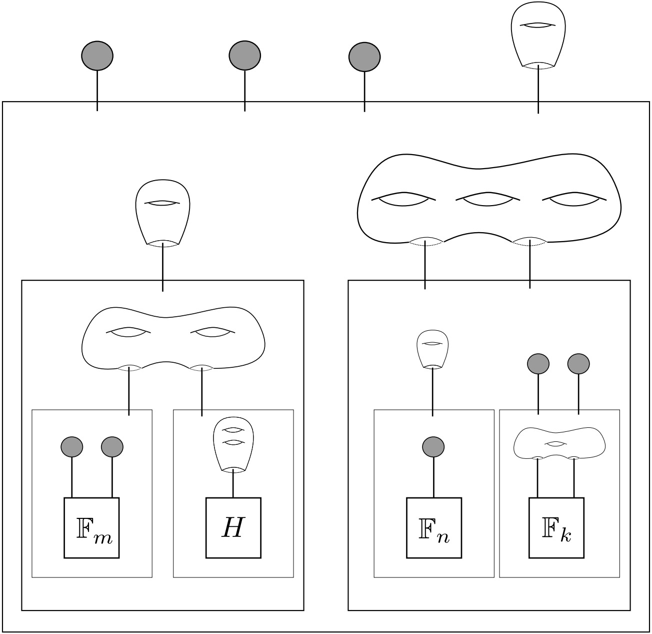
(i) G^{i+1} is a surface flat over G^i;

(ii) G^{i+1} is a free abelian flat over G^i.

Definition (Tower)

Suppose G admits the structure of a tower over H. Then we denote by $T(G, H)$ the following collection of data:

$$\{G, \mathcal{G}(G^m, G^{m-1}), \ldots, \mathcal{G}(G^1, G^0), H\}$$


$\mathbb{F}_m$
H
$\mathbb{F}_n$
$\mathbb{F}_k$

Definition

A tower $T(G, H)$ is called:

- *ω-residually free*, if $H = \{1\}$;
- *hyperbolic*, if no free abelian flat occured;

Remark

- *If G has the structure of an ω-residually free tower then G is a limit group;*
- *But not all limit groups admit the structure of an ω-residually free tower;*
- *(Sela) Let G be finitely generated. Then $G \models T_{fg}$ if and only if G is non abelian and has the structure of a hyperbolic tower over $\{1\}$.*

Definition (Closures of Towers)

Let T be an ω-residually free tower. Then a closure of T is a "tower" that is obtained from T by "enlarging" the free abelian flats that occured in T.

More formally, we replace each free abelian flat $G = H *_C (C \oplus \mathbb{Z}^m)$ of T with $\hat{G} = G *_C (C \oplus \hat{\mathbb{Z}}^m)$, where $\hat{\mathbb{Z}}^m$ is free abelian of rank m and $\mathbb{Z}^m$ is a finite index subgroup of $\hat{\mathbb{Z}}^m$.

Exercise: Show that "enlarging" the free abelian flats is compatible with the tower structure.

Example

Let $\mathbb{Z}^m$ be a finite index subgroup of $\hat{\mathbb{Z}}^m$. The tower $\mathbb{F}_2 *_{e_1^2 e_2^2 = z} \langle z \rangle \oplus \hat{\mathbb{Z}}^m$ is a closure of $\mathbb{F}_2 *_{e_1^2 e_2^2 = z} \langle z \rangle \oplus \mathbb{Z}^m$

Generalized Merzlyakov

Theorem (Sela)

Suppose $G := \langle \bar{x} \mid R(\bar{x}) \rangle$ *has the structure of an* ω*-residually free tower. Let* $\mathbb{F} \models \forall \bar{x}(R(\bar{x}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$*. Then there exist finitely many groups* $G_1, \ldots, G_k$ *corresponding to closures of the* ω*-residually free tower for* G *such that:*

- *for each* $i \leq k$ *there exists a retract* $r_i : G_\Sigma *_G G_i \to G_i$*;*
- *for every morphism* $h : G \to \mathbb{F}$ *there exists some* $i \leq k$ *such that* h *extends to* $h' : G_i \to \mathbb{F}$*.*

Corollary

Suppose $G := \langle \bar{x} \mid R(\bar{x}) \rangle$ *has the structure of a hyperbolic tower over* $\{1\}$*. Let* $\mathbb{F} \models \forall \bar{x}(R(\bar{x}) = 1 \to \exists \bar{y}(\Sigma(\bar{x}, \bar{y}) = 1))$*. Then there exists a retract* $r : G_\Sigma \to G$*.*

Generalized test sequences

The proof is based on the existence of “generalized” test sequences corresponding to ω-residually free towers.

- A test sequence (with respect to $\mathbb{F}_m$) is a sequence $(b_1(n), \ldots, b_m(n))_{n<\omega}$ that satisfies $C'(1/n)$ as $n \to \infty$;

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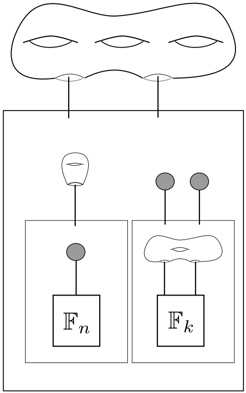
- A test sequence with respect to $\pi_1(\Sigma_g)$ is a sequence that "forces" the limit action to be a free action (of $\pi_1(\Sigma_g)$) of surface type.



- A test sequence with respect to an ω-residually free tower is a sequence constructed from bottom to top giving to each flat a strictly increasing growth rate with respect to the order they appear in the construction;
- Note that any (non-trivial) element of the group G_T corresponding to an ω-residually free tower T has eventually non-trivial image under a test sequence.

Diophantine Envelopes

Theorem (Sela)

Let $\phi(\bar{x}, \bar{a})$ be a first order formula over $\mathbb{F}$. Then there exist finitely many towers over $\mathbb{F}$, $T_1\{\bar{u}, \bar{x}, \bar{a}\}, T_2\{\bar{u}, \bar{x}, \bar{a}\}, \ldots, T_k\{\bar{u}, \bar{x}, \bar{a}\}$, and for each tower T_i there exist finitely many closures $T_i^1\{\bar{v}, \bar{x}, \bar{a}\}, \ldots, T_i^{m_i}\{\bar{v}, \bar{x}, \bar{a}\}$, such that:

(i) *The union of the Diophantine sets corresponding to the towers $T_1, \ldots, T_k$ cover ϕ;*

(ii) *Let $i \leq k$. If $(\bar{u}_n, \bar{x}_n, \bar{a})_{n<\omega}$ is a test sequence with respect to T_i that does not extend to any of the closures $T_i^1, \ldots, T_i^{m_i}$. Then $\mathbb{F} \models \phi(\bar{x}_n, \bar{a})$ for all but finitely many n. Moreover, for each $i \leq k$ such a test sequence exists.*

Definition

A Diophantine envelope for ϕ is a collection of towers and their closures $\{(T_i, T_i^1, \ldots, T_i^{m_i})_{i \leq k}\}$ satisfying the conclusion of the above theorem.

Infinite fields

Infinite fields

Theorem (Perin-Pillay-S.-Tent)

Let $\phi(\bar{x})$ be a first order formula over $\mathbb{F}_n^{eq}$. Suppose $\phi(\mathbb{F}_n^{eq}) \neq \phi(\mathbb{F}_\omega^{eq})$. Then ϕ cannot be given definably the structure of an abelian group.

proof:

the proof in the real case:

- let $\mathbb{F}_n := \langle e_1, \ldots, e_n \rangle$;
- suppose for the sake of contradiction $(\phi(\bar{x}), \odot)$ is an abelian group;
- let $\bar{a}(e_1, \ldots, e_n, e_{n+1}) \in \phi(\mathbb{F}_{n+1}) \setminus \mathbb{F}_n$;
- $\bar{a}(e_1, \ldots, e_n, e_{n+1}) \odot \bar{a}(e_1, \ldots, e_n, e_{n+2}) = \bar{w}(e_1, \ldots, e_n, e_{n+1}, e_{n+2}) \in \mathbb{F}_{n+2} \setminus \mathbb{F}_{n+1}$;
- by the abelianity of $\odot$: $\bar{w}(e_1, \ldots, e_n, e_{n+1}, e_{n+2}) = \bar{w}(e_1, \ldots, e_n, e_{n+2}, e_{n+1})$;
- a contradiction to the normal form theorem for free groups.

Work in progress with Ayala Byron:

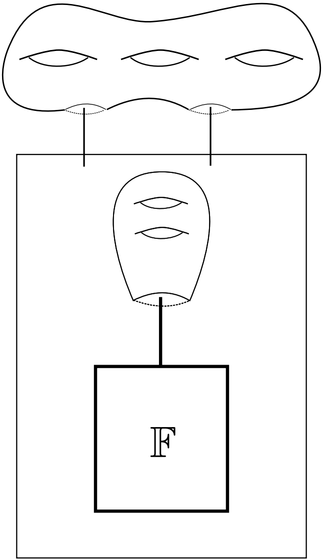
Proposition

Let $\phi(\bar{x})$ be a first order formula over $\mathbb{F}$. Suppose a Diophantine envelope for ϕ contains a hyperbolic tower. Then ϕ cannot be given definably the structure of an abelian group.

proof:

- Suppose not, and let $(\phi, \odot)$ be an abelian group, where $\odot := \psi(\bar{x}, \bar{y}, \bar{z}, \bar{a})$ is a first order formula over $\mathbb{F}$.
- We replace ψ by a "graded" Diophantine envelope. For the sake of clarity we assume that $\psi := \Sigma(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = 1$.
- By our assumptions we have that there exists a hyperbolic tower $T\{\bar{u}, \bar{x}, \bar{a}\}$ over $\mathbb{F}$ in a Diophantine envelope for ϕ.
- Recall that the "projection" of any test sequence with respect to T "lives" eventually in ϕ.
- We consider the "twin tower", $T\#T$, constructed as follows:

Twin Towers



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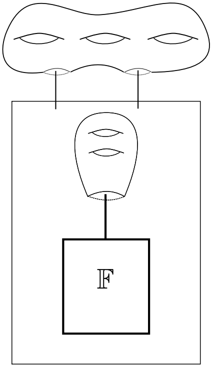

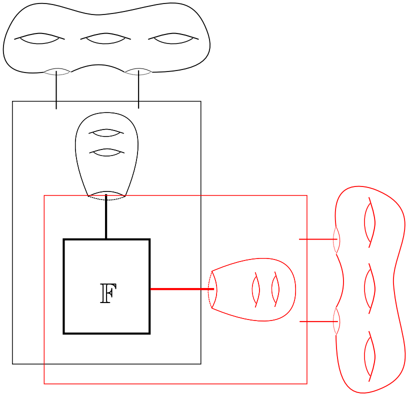

Properties of twin towers

Remark

The group $G_{T\#T}$ corresponding to the the twin tower $T\#T$ is $G_T *_{\mathbb{F}} G_T$.

Fact

Let $T\{\bar{u}, \bar{a}\}$ be a hyperbolic tower over $\mathbb{F}$ and $T\#T\{\bar{u}, \bar{u}', \bar{a}\}$ the corresponding twin tower.

- *Every morphism $h : G_T \to \mathbb{F}$ extends to $(h, h) : G_T *_{\mathbb{F}} G_T \to \mathbb{F}$.*
- *if $(\bar{u}_n, \bar{u}'_n, \bar{a})_{n<\omega}$ is a test sequence with respect to $T\#T$, then:*
  - *both $(\bar{u}_n, \bar{a})_{n<\omega}$, $(\bar{u}'_n, \bar{a})_{n<\omega}$ are test sequences with respect to T;*
  - *$(\bar{u}'_n, \bar{u}_n, \bar{a})_{n<\omega}$ is a test sequence with respect to $T\#T$.*

Proof(continue):

- $(\phi, \odot)$ is an abelian group with $\odot := \Sigma(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = 1$;
- $T\{\bar{u}, \bar{x}, \bar{a}\}$ is a hyperbolic tower in a Diophantine Envelope for ϕ, and $T\#T\{\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a}\}$ its twin tower;
- fix a test sequence with respect to $T\#T$, $(\bar{u}_n, \bar{x}_n, \bar{u}'_n, \bar{y}_n, \bar{a})_{n<\omega}$, then we have that for each n there exists a (unique) $\bar{c}_n$ such that $\mathbb{F} \models \Sigma(\bar{x}_n, \bar{y}_n, \bar{c}_n, \bar{a}) = 1$;
- The hypothesis of the generalized Merzlyakov theorem is true for the hyperbolic tower $T\#T$ and the system of equations $\Sigma(\bar{x}, \bar{y}, \bar{z}, \bar{a}) = 1$;
- Thus, we have a retract $r : G_\Sigma \to G_{T\#T}$.

Lemma

Let $r : G_\Sigma(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{z}, \bar{a}) \to G_{T\#T}(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a})$ *be the retract obtained from the test sequence* $(\bar{u}_n, \bar{x}_n, \bar{u}'_n, \bar{y}_n, \bar{a})_{n<\omega}$. *Then* $r(\bar{z}) \in G_{T\#T} \setminus \mathbb{F}$.

Proof.

- Suppose not, and $r(\bar{z}) = \bar{w}(\bar{a})$;
- then $\bar{x}_n \odot \bar{x}_n = \bar{w}(\bar{a}) = \bar{x}_n \odot \bar{y}_n$ for all but finitely many n;
- thus $\bar{x}_n = \bar{y}_n$ for all but finitely many n;
- this contadicts the difference in growth rate of $(\bar{u}_n, \bar{x}_n)_{n<\omega}$, $(\bar{u}'_n, \bar{y}_n)_{n<\omega}$.

□

- Let $w(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a}) :=$ $\alpha_1(\bar{u}, \bar{x}, \bar{a})\beta_1(\bar{u}', \bar{y}, \bar{a}) \ldots \alpha_m(\bar{u}, \bar{x}, \bar{a})\beta_m(\bar{u}', \bar{y}, \bar{a})\alpha_{m+1}(\bar{u}, \bar{x}, \bar{a})$ be the normal form of some element in $r(\bar{z}) = \bar{w}(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a})$ with respect to the amalgamated free product $G_{T\#T} := G_T *_{\mathbb{F}} G_T$.
- $w(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a}) = w(\bar{u}', \bar{y}, \bar{u}, \bar{x}, \bar{a})$ in $G_{T\#T}$;
  - both $(\bar{u}_n, \bar{x}_n, \bar{u}'_n, \bar{y}_n, \bar{a})_{n<\omega}$, $(\bar{u}'_n, \bar{y}_n, \bar{u}_n, \bar{x}_n, \bar{a})_{n<\omega}$ are test sequences with respect to $T\#T$;
  - thus the product $\bar{x}_n \odot \bar{y}_n$ (resp. $\bar{y}_n \odot \bar{x}_n$) is defined, and since $\Sigma(\bar{x}_n, \bar{y}_n, \bar{w}(\bar{u}_n, \bar{x}_n, \bar{u}'_n, \bar{y}_n, \bar{a}), \bar{a}) = 1$ (resp. $\Sigma(\bar{y}_n, \bar{x}_n, \bar{w}(\bar{u}'_n, \bar{y}_n, \bar{u}_n, \bar{x}_n, \bar{a}), \bar{a}) = 1$), we have that the product is $\bar{w}(\bar{u}_n, \bar{x}_n, \bar{u}'_n, \bar{y}_n, \bar{a})$ (resp. $\bar{w}(\bar{u}'_n, \bar{y}_n, \bar{u}_n, \bar{x}_n, \bar{a})$);
  - but $\odot$ is an abelian group operation, thus $\bar{w}(\bar{u}_n, \bar{x}_n, \bar{u}'_n, \bar{y}_n, \bar{a}) = \bar{w}(\bar{u}'_n, \bar{y}_n, \bar{u}_n, \bar{x}_n, \bar{a})$ for all but finitely many n;
  - now use the fact that a test sequence does not "kill" any non-trivial element of $G_{T\#T}$.

- Let $w(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a}) :=$
  $\alpha_1(\bar{u}, \bar{x}, \bar{a})\beta_1(\bar{u}', \bar{y}, \bar{a}) \ldots \alpha_m(\bar{u}, \bar{x}, \bar{a})\beta_m(\bar{u}', \bar{y}, \bar{a})\alpha_{m+1}(\bar{u}, \bar{x}, \bar{a})$
  be the normal form of some element in $r(\bar{z}) = \bar{w}(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a})$
  with respect to the amalgamated free product
  $G_{T\#T} := G_T *_{\mathbb{F}} G_T$.
- $w(\bar{u}, \bar{x}, \bar{u}', \bar{y}, \bar{a}) = w(\bar{u}', \bar{y}, \bar{u}, \bar{x}, \bar{a})$ in $G_{T\#T}$;
- the normal form theorem
  for amalgamated free products gives the final contradiction, i.e.
  $\alpha_1(\bar{u}, \bar{x}, \bar{a})\beta_1(\bar{u}', \bar{y}, \bar{a}) \ldots \alpha_m(\bar{u}, \bar{x}, \bar{a})\beta_m(\bar{u}', \bar{y}, \bar{a})\alpha_{m+1}(\bar{u}, \bar{x}, \bar{a}) \neq$
  $\alpha_1(\bar{u}', \bar{y}, \bar{a})\beta_1(\bar{u}, \bar{x}, \bar{a}) \ldots \alpha_m(\bar{u}', \bar{y}, \bar{a})\beta_m(\bar{u}, \bar{x}, \bar{a})\alpha_{m+1}(\bar{u}', \bar{y}, \bar{a})$.

Questions & Problems

- *Understand* $Def(T_{fg})$, *e.g. definable/interpretable groups, fields;*
- *Identify regular types;*
- *Characterize the superstable part;*
- *Understand forking independence;*
- *Does* T_{fg} *has nfcp?*
- *What does a saturated model of* T_{fg} *look like?*