Cours du 7 avril 2020
Model-complete theories.
T is *model-complete* if for all $\mathfrak{M} \subseteq \mathfrak{N}$ models of T one has $\mathfrak{M} \preccurlyeq \mathfrak{N}$.
Proposition 3.30 : If T has QE then T is model-complete.
Theorem 3.32 : T is model-complete iff every formula is equivalent modulo T to an existential formula.
Then φ and $\neg\varphi$ are both equivalent to existential formulas, so also to universal formulas.
Proof. "<=" If every formula is equivalent to an existential formula, clearly T is model-complete. TYPO : "d'où $\mathfrak{N} \models \varphi(\bar{m})$." on line -4.
"=>". Suppose T is model-complete. Note that if $\exists \bar{y}\, \varphi(\bar{x}, \bar{y})$ and $\exists \bar{z}\, \psi(\bar{x}, \bar{z})$ are both existential (φ and ψ are quantifier-free), then the conjunction is equivalent to

$$\exists \bar{y} \exists \bar{z}\, (\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{z}))$$

(renaming the bound variables if necessary so that they are different, and different from all free variables) ; similarly for the disjunction.
First we show that every formula is equivalent to a boolean combination of existential formulas.
For more generality, assume $\mathfrak{M}, \mathfrak{N} \models T$ and all existential $\varphi(\bar{x})$ satisfied by a tuple $\bar{m}$ in $\mathfrak{M}$ are satisfied by a (fixed) tuple $\bar{n}$ in $\mathfrak{N}$. We show first $\mathrm{tp}_{\mathfrak{M}}(\bar{m}) = \mathrm{tp}_{\mathfrak{N}}(\bar{n})$.
Consider the set of formulas Φ as given. A model $\mathfrak{N}'$ of Φ is

- an elementary extension of $\mathfrak{N}$ as it satisfies $\mathrm{Th}(\mathfrak{N}, N)$
- contains a copy $\{c_m^{\mathfrak{N}'} : m \in M\}$ of $\mathfrak{M}$ as a submodel
- satisfies $\bar{c}_{\bar{m}} = \bar{n}$ since $\mathfrak{M} \models \bar{m} = \bar{m}$, so the image of $\bar{m}$ under the embedding $\mathfrak{M} \to \mathfrak{N}'$ is just $\bar{n}$.

  Hence $\mathrm{tp}_{\mathfrak{M}}(\bar{m}) = \mathrm{tp}_{\mathfrak{M}'}(\bar{c}_{\bar{m}}) = \mathrm{tp}_{\mathfrak{N}'}(\bar{c}_{\bar{m}}) = \mathrm{tp}_{\mathfrak{N}'}(\bar{n}) = \mathrm{tp}_{\mathfrak{N}}(\bar{n})$.
  By 3.23 every formula is equivalent to a b.c. of existential ones.
  Secondly, we show that a universal formula is equivalent to an existential one. Then we are done.
  Consider a universal formula $\varphi(\bar{x})$, Φ the set of existential formulas which imply φ modulo T, and ψ their negations.
  Suppose $T \cup \Psi(\bar{c}) \cup \varphi(\bar{c})$ has a model $\mathfrak{M}$. If Φ_0 are the existential formulas satisfied by $\bar{c}^{\mathfrak{M}}$, then $T \cup \Phi_0(\bar{d}) \cup \varphi(\bar{d})$ has no model, as $\bar{c}$ and $\bar{d}$ must have the same type by the first part.
  So there is a finite bit Φ' such that $T \cup \Phi'(\bar{x}) \models \varphi(\bar{x})$. That is $\psi(\bar{x}) = \bigwedge \Phi'(\bar{x})$ is in Φ, a contradiction, as then $\neg\psi \in \Psi$ and $\bar{c} \models \neg\psi(\bar{x})$ by assumption on the model $\mathfrak{M}$.
  Hence the model $\mathfrak{M}$ does not exist. Hence a finite bit Ψ_0 of Ψ such that $T \cup \Psi_0(\bar{c}) \cup \varphi(\bar{c})$ is inconsistent. Then $\bigwedge \Psi_0(\bar{x}) \models_T \neg\varphi(\bar{x})$. Thus

$$\varphi \models_T \bigvee \{\psi : \neg\psi \in \Psi_0\}$$

  But these ψ are in Φ and imply φ. Hence φ and $\bigvee\{\psi : \neg\psi \in \Psi_0\}$ (which is existential) are equivalent. QED
  Chapter 4 : What type are necessarily realized in any model of T ?

- algebraic ones : If a type p contains a formula $\varphi(\bar{x})$ which has only finitely many realizations, then all realizations are in the base model (the theory T will tell us there are only so many of them). Any completion of $\varphi(\bar{x})$, in particular p, must be realized by one of them.
- principal (or isolated) types : If there is a formula $\varphi(\bar{x})$ in p which implies modulo T all other formulas in p. For instance, in an algebraic type p, any formula $\varphi(\bar{x})$ in p with a minimal number of realizations will isolate p. This is not the only possibility. In DLO without endpoints, there is a unique 1-type over $\emptyset$. It is isolated by $x = x$.

Omitting type theorem : If T is countable, then any non-principal type can be omitted in some countable model of T. The proof uses the Baire category theorem ; the question of omitting types is A LOT harder in the uncountable case.
Lemma 4.5 : $\mathrm{tp}(\bar{a}\bar{b}/A)$ is isolated iff $\mathrm{tp}(\bar{a}/A\bar{b})$ and $\mathrm{tp}(\bar{b}/A)$ are.
If $\varphi(\bar{x}, \bar{y})$ isolates $\mathrm{tp}(\bar{a}\bar{b}/A)$, then it is easy to see that $\varphi(\bar{x}, \bar{b})$ isolates $\mathrm{tp}(\bar{a}/A\bar{b})$ and $\exists\bar{x}\varphi(\bar{x}, \bar{y})$ isolates $\mathrm{tp}(\bar{b}/A)$.
Conversely, if $\varphi(x, \bar{b})$ isolates $\mathrm{tp}(\bar{a}/A\bar{b})$ and $\psi(\bar{y})$ isolates $\mathrm{tp}(\bar{b}/A)$, then $\varphi(\bar{x}, \bar{y}) \wedge \psi(\bar{y})$ isolates $\mathrm{tp}(\bar{a}\bar{b}/A)$. QED
$\bar{a}$ = first tuple, $\bar{b}$ = second tuple, $\bar{a}\bar{b}$ = concatenation of the two. $A\bar{b} = A \cup \{\bar{b}\}$
Examples 4.6 for $(\mathbb{Z}, <)$. Unique 1-type, 2-types are given by the distance between x_1 and x_2.
Those of finite distance are isolated, for instance $\exists! y(x_1 < y < x_2)$ means distance 2 and isolates a 2-type. ($\exists!$ = there is a unique). The unique non-isolated 2-type (infinite distance) is not realized in $\mathbb{Z}$, but in every proper elementary extension.
I want to construct small models. For that I use the "Henkinisation" (Proposition 4.7).
Theorem 4.9 (General Omitting Types Theorem). It T is countable, I can omit a meagre set of n-types for all n.
Recall the topology on the set of n-types : basic clopen sets are $[\varphi(\bar{x})] = \{p \in S_n(T) : \varphi \in p\}$. Types are the points ; they are closed. This topological space is compact (compactness theorem) and totally discontinuous.
Meagre = countable union of closed sets of empty interior
co-meagre = compement of meagre = countable intersection of dense open sets.
For a single type p, the set $\{p\}$ is meagre iff p is not isolated, as $\varphi(\bar{x})$ isolates p iff $\{p\} = [\varphi(\bar{x})]$.
So the OTT => non-principal types can be omitted (in a countable language).
But the general OTT allows us to omit several types (even infinitely many) at once.
Recall Baire category theorem : A (countable intersection of) co-meagre sets is dense. True in any locally compact topological space.
Proof : Start with a countable theory T and its Henkinisation T_H using countably many constants C. Consider a closed set p of empty interior in S_n. We want to show that $T_H \setminus [p(\bar{c})]$ is open dense in T_H.
Then $\bigcap_{\bar{c} \in C}(T_H \setminus [p(\bar{c})])$ is still dense, and in particular non-empty. So it contains a completion T' of T_H. If $\mathfrak{N}_C \models T'$, then $C^{\mathfrak{N}_C} \preccurlyeq \mathfrak{N}_C$ omitting p.
To show that $T_H \setminus [p(\bar{c})]$ is open dense in T_H, consider a basic open set $[\varphi'(\bar{c}')]$. We

have to show that $(T_H \setminus [p(\bar{c})]) \cap [\varphi'(\bar{c}')]$ is non-empty. In other words,

$$T_H \cup \{\varphi'(\bar{c}')\} \not\models p(\bar{c}).$$