SIGNATURE FOR PIECEWISE CONTINUOUS GROUPS

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Abstract. Let $\widehat{\mathrm{PC}^{\bowtie}}$ be the group of bijections from $[0,1[$ to itself which are continuous outside a finite set. Let $\mathrm{PC}^{\bowtie}$ be its quotient by the subgroup of finitely supported permutations.

We show that the Kapoudjian class of $\mathrm{PC}^{\bowtie}$ vanishes. That is, the quotient map $\widehat{\mathrm{PC}^{\bowtie}} \to \mathrm{PC}^{\bowtie}$ splits modulo the alternating subgroup of even permutations. This is shown by constructing a nonzero group homomorphism, called signature, from $\widehat{\mathrm{PC}^{\bowtie}}$ to $\mathbb{Z}/2\mathbb{Z}$. Then we use this signature to list normal subgroups of every subgroup $\widehat{G}$ of $\widehat{\mathrm{PC}^{\bowtie}}$ which contains $\mathfrak{S}_{\text{fin}}$ and such that G, the projection of $\widehat{G}$ in $\mathrm{PC}^{\bowtie}$, is simple.

1. Introduction

Let X be the right-open and left-closed interval $[0,1[$. We denote by $\mathfrak{S}(X)$ the group of bijections of X to X. This group contains the subgroup composed of all finitely supported permutations is denoted by $\mathfrak{S}_{\text{fin}}$. The classical signature is well-defined on $\mathfrak{S}_{\text{fin}}$ and its kernel, denoted by $\mathfrak{A}_{\text{fin}}$, is the only subgroup of index 2 in $\mathfrak{S}_{\text{fin}}$. An observation, originally due to Vitali [10], is that the signature does not extend to $\mathfrak{S}(X)$.

For every subgroup G of $\mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$, we denote by $\widehat{G}$ its inverse image in $\mathfrak{S}(X)$. The cohomology class of the central extension

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mathfrak{S}_{\text{fin}}/\mathfrak{A}_{\text{fin}} \to \widehat{G}/\mathfrak{A}_{\text{fin}} \to G \to 1$$

is called the Kapoudjian class of G; it belongs to $H^2(G, \mathbb{Z}/2\mathbb{Z})$. It appears in the work of Kapoudjian and Kapoudjian-Sergiescu [6, 7]. The vanishing of this class means that the above exact sequence splits; this means that there exists a group homomorphism from the preimage of G in $\mathfrak{S}(X)$ onto $\mathbb{Z}/2\mathbb{Z}$ which extends the signature on $\mathfrak{S}_{\text{fin}}$ (for more on the Kapoudjian class, see [3, §8.C]). This implies in particular that $\widehat{G}/\mathfrak{A}_{\text{fin}}$ is isomorphic to the direct product $G \times \mathbb{Z}/2\mathbb{Z}$. One can notice that for $G = \mathfrak{S}(X)/\mathfrak{S}_{\text{fin}}$ we have $\widehat{G} = \mathfrak{S}(X)$; in this case the Vitali's observation implies that the Kapoudjian class does not vanish.

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The set of all permutations of X continuous outside a finite set is a subgroup denoted by $\widehat{\mathrm{PC}^{\bowtie}}$. The aim here is to show the following theorem:

Theorem 1.1. *There exists a group homomorphism* $\varepsilon : \widehat{\mathrm{PC}^{\bowtie}} \to \mathbb{Z}/2\mathbb{Z}$ *that extends the classical signature on* $\mathfrak{S}_{\mathrm{fin}}$.

Corollary 1.2. *Let* G *be a subgroup of* $\mathrm{PC}^{\bowtie}$. *Then the Kapoudjian class of* G *is zero.*

This solves a question asked by Y. Cornulier [4, Question 1.15].

The subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ consisting of all permutations of X that are piecewise isometric elements is denoted by $\widehat{\mathrm{IET}^{\bowtie}}$ and the one consisting of all piecewise affine permutations of X is denoted by $\widehat{\mathrm{PAff}^{\bowtie}}$. We also consider for each of these groups the subgroup composed of all piecewise orientation-preserving elements by replacing the symbol " $\bowtie$ " by the symbol " $+$ ".
Let us observe that when $G \subset \mathrm{PC}^{+}$ Corollary 1.2 is trivial. Indeed, in this case G can be lifted inside $\widehat{\mathrm{PC}^{+}}$ itself. However, such a lift does not exist for $\mathrm{PC}^{\bowtie}$ or even $\mathrm{IET}^{\bowtie}$, as was proved in [4].

The idea of proof of Theorem 1.1 is to associate for every $f \in \widehat{\mathrm{PC}^{\bowtie}}$ and every finite partition $\mathcal{P}$ of $[0,1[$ into intervals associated with f, two numbers. The first is the number of interval of $\mathcal{P}$ where f is order-reversing and the second is the signature of a particular finitely supported permutation. The next step is to prove that the sum modulo 2 of this two numbers is independent from the choice of partition. Then we show that it is enough to prove that $\varepsilon|_{\mathrm{IET}^{\bowtie}}$ is a group homomorphism. For this we show that it is additive when we look at the composition of two elements of $\widehat{\mathrm{IET}^{\bowtie}}$ by calculate the value of the signature with a particular partition.

In Section 4, we apply these results to the study of normal subgroups of $\widehat{\mathrm{PC}^{\bowtie}}$ and certain subgroups. More specifically we prove:

Theorem 1.3. *Let* $\widehat{G}$ *be a subgroup of* $\widehat{\mathrm{PC}^{\bowtie}}$ *containing* $\mathfrak{S}_{\mathrm{fin}}$ *and such that its projection* G *in* $\mathrm{PC}^{\bowtie}$ *is simple nonabelian. Then* $\widehat{G}$ *has exactly five normal subgroups given by the list:* $\{\{1\}, \mathfrak{A}_{\mathrm{fin}}, \mathfrak{S}_{\mathrm{fin}}, \mathrm{Ker}(\varepsilon), \widehat{G}\}$.

We denote by $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ the subgroup of $\widehat{\mathrm{IET}^{+}}$ composed of all right-continuous elements. We know that it is naturally isomorphic to IET^{+}. The same is true when we replace IET^{+} by PAff^{+} or PC^{+}. This allows us to use the work of P. Arnoux [2] and the one of N. Guelman and I. Liousse [5] where they prove that $\mathrm{IET}^{\bowtie}$, PC^{+} and PAff^{+} are simple. From this we deduce:

Theorem 1.4. *The groups* $\mathrm{PC}^{\bowtie}$ *and* $\mathrm{PAff}^{\bowtie}$ *are simple.*

This gives us some examples of groups that satisfy the conditions of the Theorem 1.3.

Finally Section 5 is independent and we study some normalizers and in particular we show that the behaviour when we look the group inside $\widehat{\mathrm{PC}^{\bowtie}}$ or $\mathrm{PC}^{\bowtie}$ may not be the same. We denote by $\mathcal{R} \in \mathrm{IET}^{\bowtie}$ the map $x \mapsto 1-x$. Then we define IET^{-} as the coset $\mathcal{R}.\,\mathrm{IET}^{+}$ and PC^{-} as the coset $\mathcal{R}.\,\mathrm{PC}^{+}$. Then the groups $\mathrm{IET}^{\pm} := \mathrm{IET}^{+} \cup \mathrm{IET}^{+}$ and $\mathrm{PC}^{\pm} := \mathrm{PC}^{+} \cup \mathrm{PC}^{-}$ are well-defined.

Proposition 1.5. *The subgroup* $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ *(resp.* $\widehat{\mathrm{PC}^{+}_{\mathrm{rc}}}$*) is its own normalizer in* $\widehat{\mathrm{IET}^{\bowtie}}$ *(resp.* $\widehat{\mathrm{PC}^{+}_{\mathrm{rc}}}$*). The normalizer of* IET^{+} *(*PC^{+} *respectively) in* $\mathrm{IET}^{\bowtie}$ *(*$\mathrm{PC}^{\bowtie}$ *respectively) is* $\mathrm{IET}^{\pm}$ *(*$\mathrm{PC}^{\pm}$ *respectively).*

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2. Preliminaries

For every real interval I we denote by I° its interior in $\mathbb{R}$ and if $I = [0, t[$ we agree that its interior is $]0, t[$.

2.1. Partitions associated.

An important tool to study elements in $\widehat{\mathrm{PC}^{\bowtie}}$ and $\mathrm{PC}^{\bowtie}$ are partitions into intervals of $[0, 1[$. All partitions are assumed to be finite.

Definition 2.1. For every f in $\widehat{\mathrm{PC}^{\bowtie}}$, a finite partition $\mathcal{P}$ into right-open and left-closed intervals of $[0, 1[$ is called *a partition into intervals associated with* f if and only if f is continuous on the interior of every interval of $\mathcal{P}$. We denote by Π_f the set of all partitions into intervals associated with f.
We define also *the arrival partition of* f *associated with* $\mathcal{P}$, denoted $f(\mathcal{P})$, the partition of $[0, 1[$ composed of all right-open and left-closed intervals such that their interior is equal to the image by f of the interior of an interval of $\mathcal{P}$.

Remark 2.2. For every f in $\widehat{\mathrm{PC}^{\bowtie}}$ there exists a unique partition $\mathcal{P}_f^{\min}$ associated with f which has a minimal number of intervals. It is actually minimal in the sense of refinement: Π_f consists precisely of the set of partitions refining $\mathcal{P}_f^{\min}$.

2.2. Decompositions.

We define a family of elements which plays an important role inside our groups:

Definition 2.3. Let I be a non-empty right-open and left-closed subinterval of $[0, 1[$. The element $f \in \widehat{\mathrm{PC}^{\bowtie}}$ which sends the interior of I on itself with slope -1 while fixing the rest of $[0, 1[$ is called the *I-flip*. We define *a flip* as any I-flip for some I.

From the definition we deduce a decomposition inside $\widehat{\mathrm{IET}^{\bowtie}}$ and $\widehat{\mathrm{PC}^{\bowtie}}$.

Proposition 2.4. *Let h be an element of $\widehat{\mathrm{IET}^{\bowtie}}$. There exist $f, g \in \widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ and r, s finite products of flips and σ, τ finitely supported permutations such that $h = r\sigma f = g\tau s$.*

Proof. Let h be an element of $\widehat{\mathrm{IET}^{\bowtie}}$, $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \ldots, I_n\} \in \Pi_h$ (§ 2.1). We denote by $h(\mathcal{P}) := \{J_1, J_2, \ldots, J_n\}$ the arrival partition of h associated with $\mathcal{P}$. Let g be the map that sends I_j° on J_j° by preserving the order and acts as h for every left endpoints of I_j for every $1 \leq j \leq n$. Note that g is bijective and then belongs to $\widehat{\mathrm{IET}^{+}}$. For $1 \leq j \leq n$ let r_j be the J_j-flip if h is order-reversing on I_j otherwise let r_j be the identity. Let r be the product of all r_j, we can notice that r fixes all endpoints of J_j for every $1 \leq j \leq n$. Then it is just a verification to check that $h = rg$. Now as g belongs to $\widehat{\mathrm{IET}^{+}}$ there exists σ in $\mathfrak{S}_n$ such that $g = \sigma f$ with f in $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$.
The other decomposition follows by decomposing h^{-1} under the previous decomposition. □

Proposition 2.5. *For every h in $\widehat{\mathrm{PC}^{\bowtie}}$ there exist ϕ and ψ two order-preserving homeomorphisms of $[0, 1[$ and f, g in $\widehat{\mathrm{IET}^{\bowtie}}$ such that $h = \psi \circ f = g \circ \phi$.*

Proof. Let λ be the Lebesgue measure on $[0, 1[$. Let $h \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P} \in \Pi_h$. Then there exist $\phi, \psi \in \mathrm{Homeo}^{+}([0, 1[)$ such that for every $I \in \mathcal{P}$, $\lambda(\phi(I)) = \lambda(h(I))$ and $\lambda(\psi(h(I))) = \lambda(I)$. Then $h \circ \phi$ and $\psi \circ h$ belongs to $\widehat{\mathrm{IET}^{\bowtie}}$. □

3. Construction of the signature homomorphism

In our case we have $X = [0, 1[$ and $\widehat{\mathrm{PC}^{\bowtie}}$ is a subgroup of $\mathfrak{S}(X)$. We denote here $\mathfrak{S}_{\mathrm{fin}} = \mathfrak{S}_{\mathrm{fin}}(X)$ and $\varepsilon_{\mathrm{fin}}$ the classical signature on $\mathfrak{S}_{\mathrm{fin}}$ taking values in $(\mathbb{Z}/2\mathbb{Z}, +)$.

3.1. Definitions.

Definition 3.1. Let h be an element of $\widehat{\mathrm{PC}^{\bowtie}}$, $n \in \mathbb{N}$ and $\mathcal{P} = \{I_1, I_2, \ldots, I_n\} \in \Pi_h$. For every $1 \leq j \leq n$, let α_j be the left endpoint of I_j and β_j be the left endpoint of $h(I_j^{\circ})$. We define the *default of pseudo right continuity for h about $\mathcal{P}$* denoted $\sigma_{(h,\mathcal{P})}$ as the finitely supported permutation which sends $h(\alpha_j)$ to β_j for every $1 \leq j \leq n$ (this is well-defined because the set of all $h(\alpha_j)$ is equal to the set of all β_j).

Definition 3.2. Let h be an element of $\widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P} \in \Pi_h$. Let k be the number of interval of $\mathcal{P}$ on which h is order-reversing. We called the *flip number of h about $\mathcal{P}$* the number k. We denote it by $R(h, \mathcal{P})$.

Definition 3.3. For $h \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P} \in \Pi_h$, define $\varepsilon(h, \mathcal{P}) \in \mathbb{Z}/2\mathbb{Z}$ as $R(h, \mathcal{P}) + \varepsilon_{\mathrm{fin}}(\sigma_{(h,\mathcal{P})})$ [mod 2]. We define also $\varepsilon(h) = \varepsilon(h, \mathcal{P}_h^{\mathrm{fin}})$.

Proposition 3.4. *For every $\tau \in \mathfrak{S}_{\text{fin}}$ and every $\mathcal{P} \in \Pi_\tau$ we have $\varepsilon(\tau, \mathcal{P}) = \varepsilon_{\text{fin}}(\tau)$.*

Proof. It is clear that for every $\tau \in \mathfrak{S}_{\text{fin}}$ and every partition $\mathcal{P}$ associated with τ we have $R(\tau, \mathcal{P}) = 0$ and $\sigma_{(\tau,\mathcal{P})} = \tau$. □

We deduce that ε extends the classical signature ε_{fin}. Thus we will write ε instead of ε_{fin}.

Proposition 3.5. *Every right-continuous element f of $\widehat{\mathrm{PC}^+}$ satisfies $\varepsilon(f, \mathcal{P}) = 0$ for every $\mathcal{P} \in \Pi_f$.*

Proof. In this case, for every partition $\mathcal{P}$ into intervals associated with f we always have $R(f, \mathcal{P}) = 0$ and $\sigma_{(f,\mathcal{P})} = \mathrm{Id}$. □

3.2. Proof of Theorem 1.1.

In order to prove that ε is a group homomorphism, it is useful to calculate $\varepsilon(h)$ thanks to $\varepsilon(h, \mathcal{P})$ for every $h \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\mathcal{P} \in \Pi_h$.

Lemma 3.6. *For every $h \in \widehat{\mathrm{PC}^{\bowtie}}$ and every $\mathcal{P} \in \Pi_h$ we have $\varepsilon(h) = \varepsilon(h, \mathcal{P})$.*

Proof. Let h and $\mathcal{P}$ be as in the statement. By minimality of $\mathcal{P}_h^{\min}$, in term of refinement, we deduce that there exist $n \in \mathbb{N}$ and $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n \in \Pi_h$ such that:

(1) $\mathcal{P}_1 = \mathcal{P}_h^{\min}$;
(2) $\mathcal{P}_n = \mathcal{P}$;
(3) for every $2 \leq i \leq n$ the partition $\mathcal{P}_i$ is a refinement of the partition $\mathcal{P}_{i-1}$ where only one interval of $\mathcal{P}_{i-1}$ is cut into two.

Hence it is enough to show $\varepsilon(h, \mathcal{Q}) = \varepsilon(h, \mathcal{Q}')$ where $\mathcal{Q}, \mathcal{Q}' \in \Pi_h$ such that there exist consecutive intervals $I, J \in \mathcal{Q}$ with $I \cup J \in \mathcal{Q}'$ and $\mathcal{Q}' \smallsetminus \{I \cup J\} = \mathcal{Q} \smallsetminus \{I, J\}$.

Let α be the left endpoint of I and let x be the right endpoint of I (x is also the left endpoint of J). There are only two cases but in both cases, we know that $\sigma_{(h,\mathcal{Q})} = \sigma_{(h,\mathcal{Q}')}$ except maybe on $h(\alpha)$ and $h(x)$:

(1) The first case is when h is order-preserving on $I \cup J$. Then as $\mathcal{Q} \smallsetminus \{I, J\} = \mathcal{Q}' \smallsetminus \{I \cup J\}$ we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}')$. As h is order-preserving on the interior of $I \cup J$ we know that $\sigma_{(h,\mathcal{Q}')}(h(\alpha))$ is the left endpoint of $h(I \cup J)$ which is the left endpoint of $h(I)$ thus equals to $\sigma_{(h,\mathcal{Q})}(h(\alpha))$. With the same reasoning we deduce that $\sigma_{(h,\mathcal{Q}')}(h(x)) = \sigma_{(h,\mathcal{Q})}(h(x))$ hence $\sigma_{(h,\mathcal{Q})} = \sigma_{(h,\mathcal{Q}')}$. Thus in $\mathbb{Z}/2\mathbb{Z}$ we have $R(h, \mathcal{Q}') + \varepsilon(\sigma_{(h,\mathcal{Q}')}) = R(h, \mathcal{Q}) + \varepsilon(\sigma_{(h,\mathcal{Q})})$.
(2) The second case is when h is order-reversing on $I \cup J$. Then we get $R(h, \mathcal{Q}) = R(h, \mathcal{Q}') + 1$. This time $\sigma_{(h,\mathcal{Q}')}(h(\alpha))$ is still the left endpoint of $h(I \cup J)$ which is the left endpoint of $h(J)$ thus equals to $\sigma_{(h,\mathcal{Q})}(h(x))$. With the same reasoning we deduce that $\sigma_{(h,\mathcal{Q}')}(h(x)) = \sigma_{(h,\mathcal{Q})}(h(\alpha))$. Then by denoting τ the transposition $(h(x)\ \sigma_{(h,\mathcal{Q}')}(h(\alpha)))$, we obtain $\sigma_{(h,\mathcal{Q})} = \tau \circ \sigma_{(h,\mathcal{Q}')}$. We must notice that the transposition is not the identity because $h^{-1}(\sigma_{(h,\mathcal{Q}')}(h(\alpha)))$ is an endpoint of one of the intervals of $\mathcal{Q}'$ and

x is not.

In conclusion in $\mathbb{Z}/2\mathbb{Z}$ we have:

$$R(h,\mathcal{Q}')+\varepsilon(\sigma_{(h,\mathcal{Q}')}) = R(h,\mathcal{Q}')+1+1+\varepsilon(\sigma_{(h,\mathcal{Q}')}) = R(h,\mathcal{Q})+\varepsilon(\sigma_{(h,\mathcal{Q})})$$

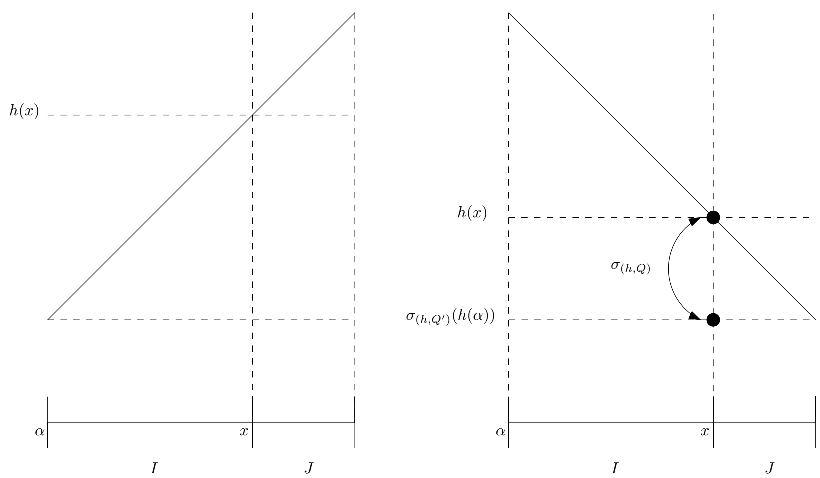

FIGURE 1. Illustrations of the two cases appearing in Lemma 3.6.
On the left we assume h order-preserving on $I \cup J$ and see that $\sigma_{(h,\mathcal{Q})}(h(x)) = \sigma_{(h,\mathcal{Q}')}(h(x))$. On the right we assume h order-reversing on $I \cup J$ and see that $\sigma_{(h,\mathcal{Q})}(h(x)) = (h(x)\ \sigma_{(h,\mathcal{Q}')}(h(\alpha))) \circ \sigma_{(h,\mathcal{Q}')}(h(x))$.

□

If $\phi \in \mathrm{Homeo}^+([0,1[)$, then it follows from Proposition 3.5 that $\varepsilon(\phi)=0$. We improve this, showing that ε is invariant by the action of $\mathrm{Homeo}^+([0,1[)$ on $\widehat{\mathrm{PC}^{\bowtie}}$.

Lemma 3.7. *For every $h \in \widehat{\mathrm{PC}^{\bowtie}}$ and every $\phi \in \mathrm{Homeo}^+([0,1[)$ we have $\varepsilon(h\phi) = \varepsilon(h) = \varepsilon(\phi h)$.*

Proof. Let $h \in \widehat{\mathrm{PC}^{\bowtie}}$ and $\phi \in \mathrm{Homeo}^+([0,1[)$ be as in the statement. Let $n \in \mathbb{N}$ and $\mathcal{P} := \{I_1, I_2, \dots, I_n\} \in \Pi_h$. Then $\mathcal{Q} := \{\phi^{-1}(I_1), \phi^{-1}(I_2), \dots, \phi^{-1}(I_n)\}$ is in $\Pi_{h\phi}$. We know that ϕ is order preserving then for every $1, \leq i \leq n$, $h\phi$ preserves (reverses respectively) the order on $\phi^{-1}(I_i)$ if and only if h preserves (reverses respectively) the order on I_i, so $R(h,\mathcal{P}) = R(h\phi,\mathcal{Q})$. We can notice that the left endpoint of $\phi^{-1}(I_i)$ (denoted by α_i) is send on the left endpoint of I_i (denoted by a_i) by ϕ hence $h(a_i) = h\phi(\alpha_i)$ has to be send on $\sigma_{(h,\mathcal{P})}(h(a_i))$ so $\sigma_{(h\phi,\mathcal{Q})} = \sigma_{(h,\mathcal{P})}$. we deduce that $\varepsilon(h\phi) = \varepsilon(h)$.
The other equality has a similar proof. We denote $h(\mathcal{P})$ the arrival partition of h associated with $\mathcal{P}$. We know that ϕ is continuous thus $h(\mathcal{P})$ is in Π_ϕ and we deduce that $\mathcal{P} \in \Pi_{\phi h}$. Also ϕ is order-preserving then $R(h,\mathcal{P}) = R(\phi h,\mathcal{P}))$. We know that $\sigma_{(\phi,h(\mathcal{P}))} = \mathrm{Id}$ then we can notice that $\phi \circ \sigma_{(h,\mathcal{P})} \circ h$ send the left endpoint of I_i to the left endpoint of $\phi h(I_i^\circ)$. Then $\sigma_{(\phi h,\mathcal{P})} = \phi \sigma_{(h,\mathcal{P})} \phi^{-1}$ and we deduce that $\varepsilon(\sigma_{(\phi h,\mathcal{P})}) = \varepsilon(\sigma_{(h,\mathcal{P})})$. Hence $\varepsilon(\phi h) = \varepsilon(h)$. □

Thanks to Proposition 2.5 it is enough to prove that $\varepsilon|_{\widehat{\mathrm{IET}^{\bowtie}}}$ is a group homomorphism.

Lemma 3.8. *The map* $\varepsilon|_{\widehat{\mathrm{IET}^{\bowtie}}}$ *is a group homomorphism.*

Proof. Let $f, g \in \widehat{\mathrm{IET}^{\bowtie}}$. Let $\mathcal{P} \in \Pi_f$ and $\mathcal{Q} \in \Pi_g$. For every $I \in \mathcal{Q}$ (resp. $J \in \mathcal{P}$) we denote by α_I (resp. β_J) the left endpoint of I (resp. J). Up to refine $\mathcal{P}$ and $\mathcal{Q}$ we can assume that $\mathcal{P} = g(\mathcal{Q})$ thus $g(\{\alpha_I\}_{I\in\mathcal{Q}}) = \{\beta_J\}_{J\in\mathcal{P}}$. Then $Q \in \Pi_{f\circ g}$ and for every $K \in f \circ g(Q)$ we denote by γ_K the left endpoint of K.
In $\mathbb{Z}/2\mathbb{Z}$, we get immediately that $R(f \circ g, Q) = R(g, Q) + R(f, g(Q))$. Now we want to describe the default of pseudo right continuity for $f \circ g$ about $\mathcal{Q}$. We recall that $\sigma_{(f\circ g,\mathcal{Q})}$ is the permutation that sends $f \circ g(\alpha_I)$ on $\gamma_{f\circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Furthermore $\sigma_{(g,\mathcal{Q})}(g(\alpha_I)) = \beta_{g(I)}$ and $\sigma_{(f,g(\mathcal{Q}))}(f(\beta_{g(I)})) = \gamma_{f\circ g(I)}$. Then $\sigma_{(f,g(\mathcal{Q}))}\circ f\circ\sigma_{(g,\mathcal{Q})}\circ g(\alpha_I) = \gamma_{f\circ g(I)}$ and we deduce that the permutation $\sigma_{(f,g(\mathcal{Q}))} \circ f \circ \sigma_{(g,\mathcal{Q})} \circ f^{-1}$ sends $f \circ g(\alpha_I)$ on $\gamma_{f\circ g(I)}$ for every $I \in \mathcal{Q}$ while fixing the rest of $[0, 1[$. Thus $\sigma_{(f\circ g,\mathcal{Q})} = \sigma_{f,g(\mathcal{Q})} \circ f \circ \sigma_{(g,\mathcal{Q})} \circ f^{-1}$. Then $\varepsilon(\sigma_{(f\circ g,\mathcal{Q})}) = \varepsilon(\sigma_{f,g(\mathcal{Q})})+\varepsilon(\sigma_{(g,\mathcal{Q})})$ and we conclude that $\varepsilon(f\circ g) = \varepsilon(f)+\varepsilon(g)$. □

Corollary 3.9. *The map* ε *is a group homomorphism.* □

4. Normal subgroups of $\widehat{\mathrm{PC}^{\bowtie}}$ and some subgroups

Here we present some corollaries of Theorem 1.1. For every group G we denote by $D(G)$ its derived subgroup.

Definition 4.1. For every group H, we define $J_3(H)$ as the subgroup generated by elements of order 3.

Let $\widehat{G}$ be a subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ containing $\mathfrak{S}_{\mathrm{fin}}$. We denote by G its projection on $\mathrm{PC}^{\bowtie}$. We recall that $\mathfrak{A}_{\mathrm{fin}}$ is a normal subgroup of $\widehat{G}$, and has a trivial centraliser. We deduce that for every nontrivial normal subgroup H of $\widehat{G}$ contains $\mathfrak{A}_{\mathrm{fin}}$.

From the short exact sequence:

$$1 \longrightarrow \mathfrak{S}_{\mathrm{fin}} \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

we deduce the next short exact sequence which is a central extension:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widehat{G}/\mathfrak{A}_{\mathrm{fin}} \longrightarrow G \longrightarrow 1.$$

This short exact sequence splits because the signature $\varepsilon_{|\widehat{G}} : \widehat{G} \to \mathbb{Z}/2\mathbb{Z}$ constructed in § 3 is a retraction. Then we deduce that $\widehat{G}/\mathfrak{A}_{\mathrm{fin}}$ is isomorphic to the direct product $\mathbb{Z}/2\mathbb{Z} \times G$.

Corollary 4.2. *The projection* $\widehat{G}_{\mathrm{ab}} \to G_{\mathrm{ab}}$ *extends in an isomorphism* $\widehat{G}_{\mathrm{ab}} \sim G_{\mathrm{ab}} \times \mathbb{Z}/2\mathbb{Z}$. *Furthermore* $D(\widehat{G}) = \mathrm{Ker}(\varepsilon) \cap \widehat{D(G)}$ *is a subgroup of index* 2 *in* $\widehat{D(G)}$. *In particular, if* G *is a perfect group then* $\widehat{G}_{\mathrm{ab}} = \mathbb{Z}/2\mathbb{Z}$.

Corollary 4.3. *Let $\widehat{G}$ be a subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ containing $\mathfrak{S}_{\mathrm{fin}}$ and such that its projection G in $\mathrm{PC}^{\bowtie}$ is simple nonabelian. Then $\widehat{G}$ has exactly 5 normal subgroups given by the list: $\{\{1\}, \mathfrak{A}_{\mathrm{fin}}, \mathfrak{S}_{\mathrm{fin}}, \mathrm{Ker}(\varepsilon), \widehat{G}\}$.*

Proof. Let $\widehat{G}$ as in the statement. First we immediately check that the subgroups in the list are distinct normal subgroups of $\widehat{G}$. In the case of $\mathrm{Ker}(\varepsilon)$, there exists $g \in \widehat{G} \setminus \mathfrak{S}_{\mathrm{fin}}$ thus either $g \in \mathrm{Ker}(\varepsilon) \setminus \mathfrak{S}_{\mathrm{fin}}$ or $\sigma g \in \mathrm{Ker}(\varepsilon) \setminus \mathfrak{S}_{\mathrm{fin}}$ for any transposition σ.
Second let H be a normal subgroup of $\widehat{G}$ distinct from $\{1\}$. Then it contains $\mathfrak{A}_{\mathrm{fin}}$. Also $H/\mathfrak{A}_{\mathrm{fin}}$ is a normal subgroup of $\widehat{G}/\mathfrak{A}_{\mathrm{fin}} \simeq \mathbb{Z}/2\mathbb{Z} \times G$. Furthermore G is simple then there are only four possibilities for $H/\mathfrak{A}_{\mathrm{fin}}$. As two normal subgroups H, K of $\widehat{G}$ containing $\mathfrak{A}_{\mathrm{fin}}$ such that $H/\mathfrak{A}_{\mathrm{fin}} = K/\mathfrak{A}_{\mathrm{fin}}$ are equal, we deduce that $\widehat{G}$ has at most 5 normal subgroups. □

Corollary 4.4. *Let $\widehat{G}$ be a subgroup of $\widehat{\mathrm{PC}^{\bowtie}}$ containing $\mathfrak{S}_{\mathrm{fin}}$ and such that its projection G in $\mathrm{PC}^{\bowtie}$ is simple nonabelian. If there exists an element of order 3 in $G \setminus \mathfrak{A}_{\mathrm{fin}}$ then $J_3(\widehat{G}) = \mathrm{Ker}(\varepsilon) = D(\widehat{G})$.* □

Remark 4.5. In the context of topological-full groups, the group $J_3(G)$ appears naturally (with some mild assumptions) and is denoted by $\mathsf{A}(G)$ by Nekrashevych in [9]. In some case of topological-full groups of minimal groupoids (see [8]) we have the equality $A(G) = D(G)$ thanks to the simplicity of $D(G)$. In spite of the analogy, it is not clear that the corollary can be obtained as particular case of this result.

Remark 4.6. A lot of groups satisfy the conditions of Corollary 4.4. When $\widehat{G}$ contains $\widehat{\mathrm{IET}^{+}}$ there is an element of order 3 in $G \setminus \mathfrak{A}_{\mathrm{fin}}$. We recall that $\mathrm{IET}^{\bowtie}$, PC^{+} and PAff^{+} are simple (see [2, 5]). Thus these groups satisfy the conditions of Corollary 4.4. The next theorem add $\mathrm{PC}^{\bowtie}$ and $\mathrm{PAff}^{\bowtie}$ to the list of examples.

Theorem 4.7. *The groups $\mathrm{PC}^{\bowtie}$ and $\mathrm{PAff}^{\bowtie}$ are simple.*

Lemma 4.8. *The group $\mathrm{IET}^{\bowtie}$ is generated by flips (=images of flips from $\widehat{\mathrm{IET}^{\bowtie}}$).*

Proof. By Proposition 2.4 it is enough to show that IET^{+} is generated by flips. For every consecutive, right-open and left-closed subintervals I and J of $[0, 1[$, we define $R_{I,J}$ the map that exchanges I and J. They are elements of $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ and they formed a generating set. Then their image $r_{I,J}$ in $\mathrm{IET}^{\bowtie}$ is a generating set of IET^{+}. For every right-open and left-closed subinterval I of $[0, 1[$, we define s_I the I-flip. Take I and J be two consecutive, right-open and left-closed subintervals of $]0, 1[$. Then $r_{I,J} = s_I s_J s_{I\cup J}$. □

Proof of Theorem 4.7 (sketched).
Since the argument in [2] could also be adapted, we only provide a sketch.

We work with elements of $\mathrm{PC}^{\bowtie}$; all intervals below are meant modulo finite subsets. Let N be a nontrivial normal subgroup of $\mathrm{PC}^{\bowtie}$ (resp. $\mathrm{PAff}^{\bowtie}$). Let g be a nontrivial element of N. There exists a subinterval I of $[0,1[$ such that:

(1) g is continuous (resp. affine) on I,
(2) $g(I) \cap I = \varnothing$ (modulo finite subsets),
(3) $I \cup g(I) \neq [0,1[$ (modulo finite subsets).

Let f be the I-flip. If g is affine on I then $h = gfg^{-1}f^{-1}$ is the product of the I-flip with the $g(I)$-flip. Observe that h is conjugate to a single flip by a suitable element of IET^{+}. If g is only continuous then h is still of order 2 and it is conjugate in $\mathrm{PC}^{\bowtie}$ to a single flip. Conjugating by elements of PAff^{+}, one obtains that N contains flips of intervals of all possible lengths, and hence contains all flips. Thanks to Lemma 4.8 we know that $\mathrm{IET}^{\bowtie}$ is generated by the set of flips thus N contains $\mathrm{IET}^{\bowtie}$, in particular N intersects PC^{+} (resp. PAff^{+}) nontrivially. By simplicity of PC^{+} (resp. PAff^{+}) we deduce that N contains $\mathrm{PC}^{\bowtie} = \langle \mathrm{PC}^{+}, \mathrm{IET}^{\bowtie} \rangle$ (resp. $\mathrm{PAff}^{\bowtie} = \langle \mathrm{PAff}^{+}, \mathrm{IET}^{\bowtie} \rangle$). □

5. About some Normalizers

Here we show that computing normalizers inside $\widehat{\mathrm{PC}^{\bowtie}}$ and $\mathrm{PC}^{\bowtie}$ may leads to different behaviour. We look the case of PC^{+}, IET^{+} and $\widehat{\mathrm{PC}^{+}_{\mathrm{rc}}}$ and $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$.

Proposition 5.1. *The normalizer of* IET^{+} *in* $\mathrm{IET}^{\bowtie}$ *is reduced to* $\mathrm{IET}^{\pm}$.

Proof. Let $f \in \mathrm{IET}^{+}$ and $g \in \mathrm{IET}^{\pm}$. If $g \in \mathrm{IET}^{+}$ then $gfg^{-1} \in \mathrm{IET}^{+}$. We assume $g \in \mathrm{IET}^{-}$ then $gfg^{-1} = (g \circ \mathcal{R}) \circ (\mathcal{R} \circ f \circ \mathcal{R}) \circ (\mathcal{R} \circ g) \in \mathrm{IET}^{+}$.
For the inclusion from left to right, let $g \in \mathrm{IET}^{\bowtie} \smallsetminus \mathrm{IET}^{\pm}$ and let $\widehat{g}$ be a representative of g in $\widehat{\mathrm{IET}^{\bowtie}}$. Hence we can find I, J, K, L four right-open and left-closed intervals of the same length such that their image by $\widehat{g}$ are intervals and such that $\widehat{g}$ is order-reversing on I and order-preserving on J, K and L. We define $\widehat{f} \in \widehat{\mathrm{IET}^{+}}$ as the element which exchanges $\widehat{g}(I)$ with $\widehat{g}(J)$ and $\widehat{g}(K)$ with $\widehat{g}(L)$ while fixing the rest of $[0,1[$. Then the image f of $\widehat{f}$ in IET^{+} is not trivial and $\widehat{g}\widehat{f}\widehat{g}^{-1} \notin \widehat{\mathrm{IET}^{+}}$ implies $gfg^{-1} \notin \mathrm{IET}^{+}$. □

A similar argument stands for the case of PC thus we obtain:

Proposition 5.2. *The normalizer of* PC^{+} *in* $\mathrm{PC}^{\bowtie}$ *is reduced to* $\mathrm{PC}^{\pm}$. □

We now take a look to inside $\widehat{\mathrm{PC}^{\bowtie}}$:

Proposition 5.3. *The normalizer of* $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ *in* $\widehat{\mathrm{IET}^{\bowtie}}$ *is* $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$.

Proof. Let g be an element of $\widehat{\mathrm{IET}^{\bowtie}}$ which is not the identity. There are two cases:

(1) If $g \in \widehat{\mathrm{IET}^{+}} \setminus \widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ then $g = \sigma g'$ with $\sigma \in \mathfrak{S}_{\mathrm{fin}} \setminus \{\mathrm{Id}\}$ and $g' \in \widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$. Then for every $f \in \widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ we have $gfg^{-1} = \sigma g' f g'^{-1} \sigma^{-1}$. Thus it is enough to treat the case of $\mathfrak{S}_{\mathrm{fin}}$. Let us assume $g \in \mathfrak{S}_{\mathrm{fin}}$ then let x in the support of g. There exist two consecutive right-open and left-closed intervals I and J of the same length such that x is the right endpoint of I (and the left endpoint of J). Up to reduce I and J we can assume that I does not intersect the support of g. Then let $f \in \widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ which exchanges I and J while fixing the rest of $[0, 1[$. Then gfg^{-1} exchanges the interior of I with the interior of J but $gfg^{-1}(x)$ is not equal to $f(x)$ because $f(x)$ is the left endpoint of I and I does not intersect the support of g. Then we deduce that gfg^{-1} is not right-continuous on J.

(2) If $g \in \widehat{\mathrm{IET}^{\bowtie}} \setminus \widehat{\mathrm{IET}^{+}}$. Then we can find two consecutive subinterval I and J where g is continuous and order-reversing on $I \cup J$. Let a be the right endpoint of J. Let f be the element in $\widehat{\mathrm{IET}^{+}_{\mathrm{rc}}}$ which exchanges I and J. Then gfg^{-1} exchanges the interior of $g(J)$ with the interior of $g(I)$. However the left endpoint of $g(J)$ is send by g^{-1} on a which is fixed by f. Then gfg^{-1} fixes the left endpoint of $g(J)$, thus gfg^{-1} is not right-continuous on $g(J)$.

□

A similar argument stands for the case of PC thus we obtain:

Proposition 5.4. *The normalizer of* $\widehat{\mathrm{PC}^{+}_{\mathrm{rc}}}$ *in* $\widehat{\mathrm{PC}^{\bowtie}}$ *is* $\widehat{\mathrm{PC}^{+}_{\mathrm{rc}}}$. □

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