Wound algebraic groups and their compactifications

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This serie of lectures deals with algebraic groups defined over an arbitrary field k [18]. We will begin by revisting basics of the theory, e.g. Weil restriction, quotients,... An algebraic k-group G is anisotropic (resp. wound) if it does not carry any k–subgroup isomorphic to $\mathbb{G}_m$ (resp. $\mathbb{G}_m$ or $\mathbb{G}_a$).

If G is reductive, Borel and Tits have shown that the two notions coincide; furthermore if k is perfect this is equivalent for G to admit a projective compactification G^c such that $G(k) = G^c(k)$ [3]. A related (equivalent) condition is that $G(k[[t]]) = G\big(k((t))\big)$ and this is Prasad's viewpoint on the result [38]. We are interested in the generalization of that statement in the following two directions.

1) **The case of an imperfect field.** This includes unipotent groups [7] and pseudo-reductive groups [17]. The main result is Gabber's compactification theorem [24] constructing for an arbitrary G a G-equivariant compactification G^c such that for any separable extension F/k we have $G(F) = G^c(F)$ if and only if G_F is wound.

2) **Group schemes over a ring** A. In the paper [23] we extended the notion of wound group schemes in that setting and this does not involve classification results. More precisely we defined a notion of index and residue for an element in $G(A((t))) \setminus G(A[[t]])$ which connects those elements with subgroup schemes isomorphic to $\mathbb{G}_{a,A}$ or $\mathbb{G}_{m,A}$. In the case of a reductive group G over a field k it provides a kind of stratification of $G\big(k((t))\big)$ related with the theory of affine grassmannians.

Lecture 1, October 24

1. A result by Borel-Tits

We mostly deal with linear algebraic groups defined over a base field k. Such an object G is a smooth affine algebraic k–group or alternatively a smooth closed k–subgroups of some GL_n. Smoothness is automatic for algebraic groups in characteristic zero (Cartier's theorem) but we will also deal with the positive characteristic case.

We remind the reader that Yoneda's lemma shows that an affine k–variety X is determined by its functor of points $h_X : \{k - algebras\} \to \ Sets$, $R \mapsto h_X(R) = X(R) = \mathrm{Hom}_k(k[X], R) = \mathrm{Hom}_k\big(\mathrm{Spec}(R), X\big)$.

An algebraic k–group G is *isotropic* if it contains a k-subgroup isomorphic to the multiplicative k–group G. Equivalently there exists a non-trivial homomorphism $\lambda : \mathbb{G}_m \to G$ (since we can mod out by the kernel). Otherwise the k–group G is said anisotropic.

Example 1.1. Let q be a regular quadratic form on a vector space V of finite dimension ≥ 2. We consider the orthogonal k–group $G = \mathrm{O}(q) \subset \mathrm{GL}(V)$. For each k-algebra R, we have

$$G(R) = \{f \in \mathrm{GL}(V)(R) \mid q_R = q_R \circ f\}.$$

This is a linear algebraic k–group and we claim that G is isotropic if and only if the quadratic form q is isotropic.

If q is isotropic, we have a orthogonal decomposition $q = \mathbb{H} \perp q'$ where $\mathbb{H}$ is the hyperbolic plane, that is, the quadratic form $(x, y) \mapsto xy$. Then G contains $\mathrm{O}(\mathbb{H})$ which contains $\mathbb{G}_m$ (use $\lambda(t).(x, y) = (xt, yt^{-1})$). In other words, G is isotropic.

If G is isotropic, we deal with $\lambda : \mathbb{G}_m \hookrightarrow G \subset \mathrm{GL}(V)$. Let $v \in V$ such that $v \neq \lambda(t).v \in V \otimes_k k[t^{\pm 1}]$. We write $\lambda(t) = \sum_{i=-m}^{n} v_i t^i$ with v_{-m}, v_n non zeros. For example, we have $n \geq 1$ (otherwise change λ by $-\lambda$). We have

$$q(v) = q(\lambda(t).v) = t^{2n} q(v_n) + \text{lower terms}$$

so that $q(v_n) = 0$. Thus q is isotropic.

Theorem 1.2. *(special case of Borel-Tits* [3, th. 8.2]*) Let G be a linear algebraic k–group. We assume that k is of characteristic zero. Then the following are equivalent:*

(i) G is k–wound;

(ii) G admits a projective k–compactification G^c such that $G(k) = G^c(k)$.

The direct sense is easy and does not use the assumption on the characteristic. It is based on [7, §, prop. 3] which involves the following fact.

Lemma 1.3. *Let X be a k–variety and met $U \subset X$ be an open subvariety.Let A be a k-algebra which is a DVR of residue field k. Let $x \in X(A)$ be a point and denote by x_0 its specialization. whose specialization in $X(k)$ Then $x \in U(A) \subset X(A)$ if and only $x_0 \in U(k) \subset X(k)$.*

Proof. The direct implication is obvious. Conversely we assume that $x_0 \in U(A)$. We consider the fiber product $V = U \times_X \mathrm{Spec}(A)$. This is an open subscheme of $\mathrm{Spec}(A)$ which contains its closed point. Thus $V = \mathrm{Spec}(A)$ so that x belongs to $U(A)$. $\square$

We proceed now to the proof of Theorem 1.2, $i) \Longrightarrow (ii)$.

Proof. Let G^c be a projective k–compactification of G. We are given a non-trivial homomorphism $\lambda : \mathbb{G}_m \to G$ or a non-trivial homomorphism $u : \mathbb{G}_a \to G$. Granting to the valuative criterion of properness, it extends to $\widetilde{\lambda} : \mathbb{P}^1_k \to G^c$ (resp. $\widetilde{u} : \mathbb{P}^1_k \to G^c$).

In the first case we consider the limit points $x_0 = \widetilde{\lambda}(0)$ and $x_\infty = \widetilde{\lambda}(\infty)$. If both points belong to $G(k)$, Lemma 1.3 implies that $\widetilde{\lambda}$ factorizes through G which is not possible since G is affine and λ non constant. The argument is similar in the second case. $\square$

The converse uses orbit theory and the theory of Grassmannians and flags.

1.1. **Quotients, Orbits and Stabilizers.** Let G be an algebraic k–group acting on a k–variety (i.e. a separated k–scheme of finite type). For each $x \in X(k)$, we deal with the orbit map $f_x : G \to X$, $g \mapsto g.x$. We denote by $G_x = \{g \in G \mid g.x = x\}$ the stabilizer of x, this is a closed k–subgroup. At this stage there are several viewpoints for orbits.

1) We can consider the set-theoretic orbit O_x and its closure.

2) We can consider the scheme theoretic image I_x of f_x, that is, the smallest closed k-subvariety factorizing f_x. Note that if G is reduced, I_x is reduced as well [42, 056B] and is the reduced induced scheme structure on $\overline{O_x}$ [42, Tag 056B] .

3) We can consider the orbit from the viewpoint of fppf sheaves as the sheafification of the map $R \mapsto f_x(R)$ for R running over the k–algebras. This is nothing but the fppf quotient G/G_x [18, III.3.1.7].

Theorem 1.4. *Assume that G is reduced. Then O_x is a locally closed subset of X and f_x factorizes as follows*

$$G \xrightarrow{f'_x} (O_x)_{red} \xrightarrow{i} X$$

where f'_x is faithfully flat (in particular surjective) and i is an immersion. Furthermore the action of G on X induces an action of G on $(G.x)_{red}$ and $(G.x)_{red}$ represents the fppf quotient G/G_x.

Here $(O_x)_{red}$ stands for the reduced k–subvariety of X whose underlying topological space is O_x. This is a special case of [18, prop. II.5.3.1, prop. III.3.2.1].

Proof. Standard permanence techniques reduce to the case of an algebraically closed field. Clearly we have a factorization $G \xrightarrow{f_x} I_x \xrightarrow{j} X$ of f_x. We need to show then that O_x is an open subset of I_x. According to Chevalley's theorem [27, th. 10.20], O_x is a constructible subset of X. There exists then a dense open k–subscheme U of I_x such that $U \subset O_x$ and using generic flatness we can furthermore assume that the morphism $V = f_x^{-1}(U) \to U$ is flat. Since $V = f_x^{-1}(U) \subset G$ is open, there exists $g_1, \dots, g_n \in G(k)$ such that $G = \bigcup_{i=1,\dots,n} g_i V$. It follows that $O_x = \bigcup_{i=1,\dots,n} g_i U_{top}$ is open in I_x.

Clearly f_x factorizes by $(O_x)_{red} = \bigcup_{i=1,\dots,n} g_i U$ and induces map $G \xrightarrow{f'_x} (O_x)_{red}$ is surjective. By considering the above cover of $(O_x)_{red}$, we see that f'_x is flat by applying the criterion [42, Tag 01U5.(3)]. Thus f'_x is faithfully flat as desired.

Next we appeal to [18, prop. III..3.5.2] (or more generally [40, XVI.2.2]) showing that the fppf quotient G/G_x is representable by a k-variety Q_x so that we have a factorization $G \xrightarrow{q} Q_x \xrightarrow{d} X$ where q is faithfully flat and d a monomorphism. Since f'_x is G_x–invariant, we get a map $Q_x \to (O_x)_{red}$ fitting in a commutative diagram

$$\begin{tikzcd}
G \arrow[r, "q"] \arrow[d, equal] & Q_x \arrow[r, hook, "d"] \arrow[d] & X \arrow[d, equal] \\
G \arrow[r, "f'_x"] & (O_x)_{red} \arrow[r, hook, "i"] & X
\end{tikzcd}$$

Since q and f'_x are faithfully flat, the map $h : Q_x \to (O_x)_{red}$ is faithfully flat according to [20, 2.2.13.(ii)]. It is furthermore a monomorphism so h is an isomorphism according to [20, 17.9.1] Finally $Q_x \cong (O_x)_{red}$ is G-stable. □

Remarks 1.5. (a) If $G(k)$ is Zariski dense in G (which happens for example for GL_n over an infinite field), there is no need to go to the algebraic closure $\overline{k}$.
(b) The end of the argument above is artificial. The proof of the representability of the fppf quotient G/G_x is precisely to show that $G \to (O_x)_{red}$ is the quotient map.

One important result is that a monomorphism of algebraic k–groups $G \to H$ is always a closed immersion [18, prop. II.5.5.1]. If H is affine, Chevalley has shown that H admits a faithful representation V such that G is the stabilizer of a line D of V [18, prop. II.2.3.5]. In other words G is the stabilizer of the point $x = [D]$ for the action of G on the projective space $\mathbb{P}(V)$. Theorem 1.4 shows that the orbit $(H.x)_{red} \subset \mathbb{P}(V)$ represents the fppf quotient H/G. This construction does not depend of the choice of the representation.

Remarks 1.6. (a) If G is normal in H, then G/H comes with a natural structure of algebraic groups. We say that the sequence $1 \to G \to H \to H/G \to 1$ is an exact sequence of algebraic k–groups. Furthermore it can be shown that H/G is an affine k–group [18, th. III.3.5.6].
(b) An avatar of the proof is that H/G is a quasi-projective k–variety.

1.2. **Grassmannians and Flags.** A special case of the preceding discussion is the action of GL_n on the projective space $\mathbb{P}^{n-1}$. The stabilizer of the point $[1:0:\cdots:0]$ is the k–group

$$P_1 = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & * \end{pmatrix}$$

We get an immersion $\mathrm{GL}_n / P_1 \to \mathbb{P}^{n-1}_k$. Since $\mathrm{GL}_n(\overline{k})$ acts transitively on $\mathbb{P}^{n-1}(\overline{k})$, it is surjective so is an isomorphism.

As $\mathbb{P}^{n-1}_k$ is defined by glueing affine space, one can define more generally the Grassmannian variery $\mathrm{Gr}_{n,r}$ for $r = 1, ..., n-1$ [27, prop. 8.14]. For each k–algebra, we have

$$\mathrm{Gr}_{n,r}(R) = \big\{ M \subset R^n \mid M \text{ direct summand locally free of rank } r \ \big\}.$$

The group $\mathrm{GL}_n(k)$ acts transitively on $\mathrm{Gr}_{n,r}(k)$ and the stabilizer of the summand R^r of R^n is

$$P_r = \begin{pmatrix} X_r & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & * \end{pmatrix}$$

where the upper left corner is a matrix of size r. We have then an isomorphism $\mathrm{GL}_n / P_r \xrightarrow{\sim} \mathrm{Gr}_{n,r}$. Clearly $\mathrm{Gr}_{n,r}$ satisfies the valuative criterion of properness. Since $\mathrm{Gr}_{n,r}$ is quasi-projective, it is a projective variety (the classical way is to use Pluecker coordinates).

We could have proceed alternatively as follows. We have a map of k–functors $\mathrm{GL}_n / P_r \to \mathrm{Gr}_{n,r}$ and it is routine to show that it is an isomorphism of fppf sheaves, so that $\mathrm{Gr}_{n,r}$ is representable. Let us proceed like that with the Borel subgroup

$$B_n = \begin{pmatrix} * & * & \cdots & \cdots & * \\ 0 & * & \cdots & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

We denote by $\mathrm{Flag}_n = \mathrm{GL}_n / B_n$. It is called the variety of (complete) flags of k^n since

$$\mathrm{Flag}_n(k) \xrightarrow{\sim} \big\{ \text{complete flags of } k^n \big\}$$

A flag of k^n is a sequence of vector spaces $V_1 \subset V_2 \subset \cdots V_{n-1} \subset k^n$ such that $\dim(V_i) = i$ for $i = 1, \dots, n-1$. Indeed B_n is the stabilizer of the standard flag

$k \subset k^2 \cdots \subset k^{n-1} \subset k^n$. Once again Flag_n is a projective k-variety. For more on this topic, see [18, III.3.2.3].

1.3. **End of proof.** We proceed now to second part of the proof of Theorem 1.2.

Proof. $ii) \Longrightarrow (i)$. We assume that G is k–wound. Let $f : G \to \mathrm{GL}(W)$ be a faithful representation and consider the associated representation $F : G \to \mathrm{GL}(V)$ where $V = W \oplus k$. We observe that F gives rise to a closed embedding $F : G \to \mathrm{PGL}(V)$. In particular $\mathbb{G}_m \cap G = 1$ Let X be the k–variety of complete flags of V. Let $x \in X(k)$ and consider the schematic image Y of the orbit map $G \to X$, $g \mapsto g.x$. Since X is projective, so is Y. We claim that $G \to Y$ is an open immersion and that $G(k) = Y(k)$.

We consider the stabilizer $G_x = \{g \in G \mid g.x = x\}$, it is a closed k–subgroup. Then G_x is a closed k–subgroup of the triangularizable[1] k–group $\mathrm{GL}(V)_x$. It follows that G_x is an extension of a diagonalizable k–group by a unipotent k–group $G_{x,u}$ [18, prop. IV.2.3.3]. In characteristic zero we have a structure theorem for connected unipotent k–groups, they are successive extensions of $\mathbb{G}_a$'s [40, XVII.4.1.3]. The woundness assumption yields that G_x^0 is diagonalizable so that G_x is finite for the same reason. The finite group $\bigcap_{x \in X(k)} G_x$ is included in $\bigcap_{x \in X(k)} \mathrm{GL}(V)_x = \mathbb{G}_m$ which is the center of $\mathrm{GL}(V)$. Since $G \cap \mathbb{G}_m = 1$, it follows that $\bigcap_{x \in X(k)} G_x = 1$. There exists then $x_1, \dots, x_N \in X(k)$ such that $G_{x_1} \cap \cdots \cap G_{x_N} = 1$. We make then G act diagonally on X^N and we have $G_{x_1,\dots,x_N} = 1$.

We denote by Y the schematic closure in X^N of $G.(x_1, \dots, x_N)$. Theorem 1.4 shows that the map $G \to Y$ is an immersion. It is exactly an open immersion since the complement consists in orbits of smaller dimension [5, I.1.8].

It remains to establish that $G(k) = Y(k)$. Let $y \in Y(k) \backslash G(k)$. Then the dimension of the orbit $G.y$ is strictly smaller that the dimension of G so that G_y is of positive dimension. This contradicts the previous argument stating that $(G_y)^0 = 1$. $\square$

Remark 1.7. In the original proof, the authors do not consider the product X^N but we do not understand why it works.

2. Bounded subgroups

Let K be a topological field. For the affine space $\mathbb{A}^n_K$, we have $\mathbb{A}(K) = K^n$ and this vector space comes then with a natural topology. We can make it intrinsecal for any affine K–variety X. We define the K-topology on $X(K)$ as the weakest topology making the applications $f_* : X(K) \to K, x \mapsto f(x)$ continuous for all regular functions $f \in H^0(X, O_X)$. We denote often $X(K)_{top}$ that topological space. Of course it agrees with the previous definition in the case of affine spaces [15, prop. 2.1].

The assignment $X \to X(K)_{top}$ gives rise to a functor from the category of affine K–varieties to the category of topological spaces. This functor has nice properties:

[1] i.e. a k–subgroup of the group of upper triangular matrices.

1) It applies open immersions to open topological embeddings.

2) It applies closed immersions to closed topological embeddings.

3) It is compatible with product of varieties.

The first property permits to extend the construction of $X(K)_{top}$ to an arbitrary K–variety [15, prop. 3.1].

Proposition 2.1. *Assume that K is a locally compact nondiscrete field. Let X be a proper K–variety (e.g. projective). Then $X(K)_{top}$ is a compact topological space.*

Proof. We know that K is $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, a finite extension of $\mathbb{F}_p((t))$. The topology of K aries then from an absolute value $|\ |: K \to \mathbb{R}_{\geq 0}$. We limit ourselves to the projective case, see [15, prop. 4.4] for dealing with the general case. Property 2) above boils down to the case of the projective space $\mathbb{P}^n_K$. The point is that $\mathbb{P}^n(K)$ is the image of the compact ball $\Big\{ x \in K^{n+1} \mid \ \mid x_1 \mid + \dots + \mid x_n \mid = 1 \Big\}$ by a continuous mapping. Thus $\mathbb{P}^n(K)$ is compact. $\square$

Corollary 2.2. *Let K be a locally compact nondiscrete topological field of characteristic zero. Let G be a linear algebraic K–group. Then the following are equivalent:*

(i) G is K–wound;

(ii) $G(K)_{top}$ is compact.

Proof. According to Ostrowki's classification of locally compact fields, K (and $K^\times$) is not compact.
$(ii) \Longrightarrow (i)$. If G contains a (closed) K–subgroup $\mathbb{G}_{m,K}$ (or $\mathbb{G}_{a,K}$), $(K^\times)_{top}$ (resp. K_{top}) being a closed topological subspace of $G(K)_{top}$ is compact as well. This is a contradiction.
$(i) \Longrightarrow (ii)$. Theorem 1.2 provides a projective compactification G^c of G such that $G(K) = G^c(K)$. Since $G^c(K)_{top}$ is compact, so is $G(K)_{top}$. $\square$

It is rassuring that it works for orthogonal groups over the real numbers!

Compactness does not generalize well but boundedness works as follows for affine K–variety if K is a discretely valued field. Given $X \subset \mathbb{A}^n_K$, we say that $X(K)$ is bounded if $X(K)$ is a bounded subset of K^n equipped with the metric $\mid (x_1, \dots, x_n) \mid = \mid x_1 \mid_v + \dots + \mid x_n \mid_v = 2^{-v(x_1)} + \dots + 2^{-v(x_n)}$. That property does not depend of the embedding since it is equivalent to say that for each function $f \in K[X]$, $f(X(K))$ is a bounded subset of K. See [31, §2.2] for details.

3. A refinement of Borel-Tits theorem

Proposition 3.1. *Let G be a smooth affine k–group (k of char. zero) Then the assertions of Theorem 1.2 are furthermore equivalent to*

(iii) $G\big(k[[t]]\big) = G\big(k((t))\big)$;

(iv) the group $G\big(k((t))\big)_{top}$ is bounded.

Proof. We shall prove the implications $(ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$ which is enough since $(i) \Longleftrightarrow (ii)$ in Theorem 1.2.

$(ii) \Longrightarrow (iii)$. If G admits a projective compactification G^c such that $G(k) = G^c(k)$ (that is, assertion (iv) of Theorem 1.2, we claim that $G\big(k[[t]]\big) = G\big(k((t))\big)$. Given a point $g \in G\big(k((t))\big)$, it defines a $g^c \in G^c(k[[t]])$ by means of the valuative criterion of valuation. Since $G(k) = G^c(k)$, Lemma 1.3 shows that $g^c \in G(k[[t]])$ whence $g \in G(k[[t]])$.

$(iv) \Longrightarrow (i)$. If G contains a k–subgroup $\mathbb{G}_m$ or $\mathbb{G}_a$, then $G\big(k((t))\big)$ is not bounded so that $G\big(k[[t]]\big) \subsetneq G\big(k((t))\big)$. It follows that $(iv) \Longrightarrow (i)$ by contraposition.

$(iii) \Longrightarrow (iv)$. We embed G in GL_n. Since $\mathrm{GL}_n(k[[t]]) \subset k[[t]]^{n^2}$ is bounded in $k((t))^{n^2}$, we have clearly the implication $(iii) \Longrightarrow (iv)$. □

Remark 3.2. In the case of a reductive k–group, the equivalence between isotropicity and boundedness is a special case of Bruhat-Tits-Rousseau's theorem. More precisely it follows from the fact that $G\big(k[[t]]\big)_{top}$ is of finite index in a maximal bounded subgroup of $G\big(k((t))\big)_{top}$. Prasad provided a simpler proof of that converse, see [38] or [31, th. 2.2.9].

Lecture 2: Tori, Reductive groups, and Refinements, October 31

Before going back to compactifications of algebraic groups and of their rational points, we discuss a technical important point namely Weil restriction.

4. Weil restriction

We are given the following equation $z^2 = 1 + 2i$ in $\mathbb{C}$. A standard way to solve it is to write $z = x + iy$ with $x, y \in \mathbb{R}$. It provides then two real equations $x^2 - y^2 = 1$ and $xy = 1$. We can abstract this method for affine varieties and for functors.

We are given a finite k-algebra K, for example a finite field extension; we denote by $j : k \to K$ the structural map. Since a K-algebra is a k–algebra, a k-functor in sets F defines a K-functor denoted by F_K and called the scalar extension of F to K. For each K–algebra K', we have $F_K(K') = F(K')$. If X is an affine K-variety, we have $(h_X)_K = h_{X \times_k K}$.

Now we consider a K–functor in sets E and define its Weil restriction to K/k denoted by $R_{K/k}E$ by

$$\Big(\prod_{K/k} E\Big)(R) = E(R \otimes_k K)$$

for each k–algebra R. We note the two following functorial facts.

(I) For a finite k-algebra map $u : K \to L$, we have a natural map

$$u_* : \prod_{K/k} E \to \prod_{L/R} E_L.$$

(II) For each field extension k'/k, there is natural isomorphism of k'–functors

$$\Big(\prod_{K/k} E\Big)_{k'} \xrightarrow{\sim} \prod_{K \otimes_k k'/k'} E_{K \otimes_k k'}.$$

For other functorial properties, see appendix A.5 of [17] and [44, §I.3.12].

At this stage, it is of interest to discuss the example of vector group functors. Let N be a K–module and denote by $S \mapsto W(N)(S) = N \otimes_k K$ the K-functor in commutative groups. According to [36, cor. 2], it is representable by an affine algebraic group if and only if the K-module N is finite locally free; if it is representable it is denoted by $\mathbb{W}(N)$.

We denote by j_*N the scalar restriction of N from K to k [8, §II.1.13]. The k-vector space j_*N is N equipped with the k–module structure induced by the map $j : k \to K$. It satisfies the adjunction property $\mathrm{Hom}_k(M, j_*N) \xrightarrow{\sim} \mathrm{Hom}_K(M \otimes_k K, N)$ for each k-vector space M (*ibid*, §III.5.2).

Lemma 4.1. *(1)* $R_{K/k}W(N) \xrightarrow{\sim} W(j_*N)$.
(2) If N is finite locally free , then the k-functor $R_{K/k}\mathbb{W}(N)$ is representable by the vector group scheme $\mathbb{W}(j_*N)$.

For a more general statement, see [40, I.6.6].

Proof. (1) For each k-algebra R, we have

$$\Big(R_{K/k}W(N)\Big)(R') = N \otimes_K (R \otimes_k K) = (j_*N) \otimes_k R = W(j_*N)(R).$$

(2) The assumptions implies that j_*N is finitely generated over k, hence $\mathbb{W}(j_*N)$ is representable by the vector k–group scheme $\mathbb{W}(j_*N)$. $\square$

Proposition 4.2. *Let Y/K be an affine scheme of finite type. Then the k-functor $R_{K/k}(h_Y)$ is representable by an affine k–variety.*

Again, it is a special case of a much more general statement, see [7, §7.6].

Proof. We see Y as a closed subscheme of some affine space $\mathbb{A}^n_K$, that is given by a system of equations $(P_\alpha)_{\alpha \in I}$ with $P_\alpha \in S[t_1, \dots, t_n]$. Then $R_{K/k}(h_Y)$ is a subfunctor of $\prod_{K/k} \mathbb{W}(K^n) \xrightarrow{\sim} \mathbb{W}(j_*(K^n)) \xrightarrow{\sim} \mathbb{W}(k^{nd})$ by Lemma 4.1. For each I, we write

$$P_\alpha\Big(\sum_{i=1,..,d} y_{1,i}\omega_i, \sum_{i=1,..,d} y_{2,i}\omega_i, \dots, \sum_{i=1,..,d} y_{n,i}\Big) = Q_{\alpha,1}\,\omega_1 + \cdots + Q_{\alpha,r}\,\omega_r$$

with $Q_{\alpha,i} \in k\big[y_{l,i}; i=1,..,d; j=1,...,n\big]$. Then for each k–algebra R, $\Big(R_{K/k}(h_Y)\Big)(R)$ inside R^{nd} is the locus of the zeros of the polynomials $Q_{\alpha,j}$. Hence this functor is representable by an affine k- variety X. $\square$

In conclusion, if H/K is an affine K–group scheme of finite type, then the k–group functor $R_{K/k}(h_H)$ is representable by an affine algebraic k-group. There are basic examples of Weil restrictions.

(a) The case $K = k \times \cdots \times k$ (d times). A K-scheme Y is the data of d varieties Y_1, ..., Y_d. In this case we have $R_{k^d/k}(Y) = Y_1 \times_k \cdots \times_k Y_d$. In other words, the Weil restriction contains the fiber product construction.

(b) The case of a finite separable field extension k'/k (or more generally an étale k-algebra) of degree d. Given an affine algebraic k'-group G'/k', we associate the affine algebraic k–group $G = R_{k'/k}G'$ see [44, §3. 12]. In that case, $R_{k'/k}(G') \times_k k_s \xrightarrow{\sim} G'_{1,k_s} \times_{k_s} \cdots \times_{k_s} G'_{d,k_s}$ where the G'_i are Galois conjugates of G'. In particular, the dimension of G is $[k' : k] \dim_{k'}(G')$; the Weil restriction of a finite algebraic group is a finite group.

(c) The case where $S = k[\epsilon]$ is the ring of dual numbers which is of very different nature. For $Y = Y_0 \times_k k[\epsilon]$ for a k–variety Y_0, the Weil restriction $R_{k[\epsilon]/k}(G)$ is nothing but the tangent bundle of Y_0 [42, Tag 0B28]. For example the quotient k-group $\Big(R_{k[\epsilon]/k}(\mathbb{G}_m)\Big)/\mathbb{G}_m$ is the additive k–group. Also if $p = \mathrm{char}(k) > 0$, $R_{k[\epsilon]/k}\big(\mu_{p,k[\epsilon]}\big)$ is of dimension 1 since $\mathrm{Lie}(\mu_p) = k$.

Remark 4.3. It is natural to ask whether the functor of scalar extension from k to K admits a left adjoint. It is the case and we denote by $\bigsqcup_{S/R} E$ this left adjoint functor, see [18, §I.1.6]. It is called the Grothendieck restriction. If $\rho : k \to K$ is a k–ring section of j, it defines a structure k^ρ of K–ring. We have $\bigsqcup_{K/k} E = \bigsqcup\limits_{\rho: K \to k} E(k^\rho)$. If $E = h_Y$ for an affine K–scheme Y, $\bigsqcup\limits_{K/k} Y$ is representable by the k–scheme Y. This is simply the following k-scheme $Y \to \mathrm{Spec}(K) \xrightarrow{j^*} \mathrm{Spec}(k)$.

5. The case of tori

A k-torus T of rank $r \geq 0$ is a form of the split torus $\mathbb{G}_{m,k}^r$, that is, $T_{\overline{k}} \xrightarrow{\sim} \mathbb{G}_{m,\overline{k}}^r$. There exists a finite Galois extension K/k of group $\Gamma = \mathrm{Gal}(K/k)$ such that $T_K \xrightarrow{\sim} \mathbb{G}_{m,K}^r$, we say that K/k is a splitting extension of T. We remind the reader that the assignement $T \mapsto \widehat{T}(K) = \mathrm{Hom}_{K-gp}(T_K, \mathbb{G}_{m,K})$ is an antiequivalence of categories between the category of k–tori split by K/k and the category of Γ-lattices [40, §X.1] (or [44, §3.4]). The converse map proceeds by Galois descent.

The Galois lattice $\widehat{T} = \widehat{T}(K)$ is called the lattice of characters of T and we can consider its dual $\widehat{T}^0$ defined by $\widehat{T}^0 = \widehat{T}^0(K) = \mathrm{Hom}_{K-gp}(\mathbb{G}_{m,K}, T_K)$ called the cocharacter Galois lattice. The dictionnary exchanges closed immersions (resp. surjective morphisms) with surjective maps (resp. injective maps)

Isotropicity for T can be then rephrased in terms of $\widehat{T}$.

Lemma 5.1. *The following are equivalent.*

(i) T *is isotropic;*

(ii) $\mathrm{Hom}_{k-gp}(\mathbb{G}_{m,k}, T) = (\widehat{T}^0)^\Gamma \neq 0$*;*

(iii) $(\widehat{T}^0 \otimes_{\mathbb{Z}} \mathbb{Q})^\Gamma \neq 0$*;*

(iv)$T \otimes_{\mathbb{Z}} \mathbb{Q})^\Gamma \neq 0$*;*

(v) $\mathrm{Hom}_{k-gp}(T, \mathbb{G}_m) = (\widehat{T})^\Gamma \neq 0$*;*

Proof. The equivalence $(i) \iff (ii)$ is exactly the definition. The equivalence $(ii) \iff (iii)$ (resp. $(iv) \iff (v)$) follows from the fact that $M^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} (M \otimes_{\mathbb{Z}} \mathbb{Q})^\Gamma$ for each Γ-lattice M. Finally the equivalence $(iii) \iff (iv)$ follows of the fact that the category of $\mathbb{Q}[\Gamma]$-modules is semisimple. □

Examples 5.2. Let K/k be a Galois extension of group Γ as above. The dictionnary reads as follows:

$\mathbb{G}_m \quad <---> \quad \mathbb{Z};$

$R_{K/k}(\mathbb{G}_m) \quad <---> \quad \mathbb{Z}[\Gamma];$

$R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m \quad <---> \quad \ker\big(\mathbb{Z}[\Gamma] \xrightarrow{\epsilon} \mathbb{Z}\big);$

$$R^1_{K/k}(\mathbb{G}_m) = \ker\bigl(R_{K/k}(\mathbb{G}_m) \xrightarrow{N_{K/k}} \mathbb{G}_m\bigr) \quad <---> \quad \mathbb{Z}[\Gamma]/\mathbb{Z};$$

The second one is called then induced torus, and the third one the norm one torus associated to K/k. Since $\mathbb{Q} = \mathbb{Q}[\Gamma]^\Gamma$, Lemma 5.1 shows that the tori $R^1_{K/k}(\mathbb{G}_m)$ and $R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m$ are anisotropic.

Remark 5.3. If A is a finite dimensional k–algebra (unital, associative), the unit algebraic k–group $U(A)$ of A is defined by $U(A)(R) = (A \otimes_k R)^\times$ [11, prop. 2.4.2.1]. It is called also the group of invertible elements of A. In particular if K is a commutative k–algebra of finite rank, we see that $R_{K/k}(\mathbb{G}_{m,K}) = U(K)$. Furthermore if K is a k-subalgebra of A, we have a closed immersion $R_{K/k}(\mathbb{G}_{m,K}) = U(K) \subset U(A)$.

The proof of Theorem 1.2 is much easier for tori and is characteristic free [13, lemme 12].

Proof. Let T be a k–torus split by the Galois extension K/k of group Γ. We pick a surjective map $q : \mathbb{Z}[\Gamma]^r \to \widehat{T}$. Since $\widehat{T}$ is anisotropic, we have $q(\mathbb{Z}^r) = 0$, hence a surjective map $(\mathbb{Z}[\Gamma]/\mathbb{Z})^r \to \widehat{T}$. The dictionnary provides a closed immersion

$$T \hookrightarrow \bigl(R^1_{K/k}(\mathbb{G}_m)\bigr)^r.$$

We consider first the case of the norm one torus $E = R^1_{K/k}(\mathbb{G}_m)$. The map $q : E = R^1_{K/k}(\mathbb{G}_m) \to R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m = Q$ is an isogeny (i.e. surjective with finite kernel). Seeing $R_{K/k}(\mathbb{G}_m)$ an open subset of $\mathbb{W}(K) \setminus \{0\}$, we see that k–torus Q admits the compactification $Q^c = \mathbb{P}(K)$ where K is seen as a k-vector space. We have $K^\times/k^\times = Q(k) = Q^c(K)$ so the result holds for Q. We denote by E^c the normalization of Q^c in the finite field extension $k(E)/k(Q)$ [33, §4.1.2]. Then $h : E^c \to Q^c$ is a finite morphism so that E^c is a projective compactification of $E = R^1_{K/k}(\mathbb{G}_m)$. Furthermore the map $E \to h^{-1}(Q)$ is an isomorphism (since E is a normal k–variety) so that the boundary $E^c \to E$ maps on $Q^c \setminus Q$, it has no k–rational points. $\square$

Remark 5.4. The above compactification is T-equivariant but not smooth. The construction of smooth projective compactifications of tori is characteristic free and is based on the theory of toric varieties (involving fans), see [12].

6. Reductive groups

6.1. **Limit subgroups.** Let G be an affine algebraic k–group. Given a homomorphism $\lambda : \mathbb{G}_m \to G$, we can attach the algebraic groups $P_G(\lambda)$, $U_G(\lambda)$ and $C_G(\lambda)$ such that $P_G(\lambda) = U_G(\lambda) \rtimes C_G(\lambda)$, see [17, §2.1] and [41, §13.4].

The representability reduces to the case of $\mathrm{GL}(V)$. In this case we can diagonalize λ, i.e. $V = V_1 \oplus \cdots \oplus V_r$ where $\lambda(t) = \mathrm{diag}(t^{n_1}, \dots, t^{n_r})$ with $n_1 > n_2 > \cdots > n_r$. In this case (see [17, Ex. 2.1.1], we have $C_{\mathrm{GL}(V)}(\lambda) = \mathrm{GL}(V_1) \times \dots \mathrm{GL}(V_r)$, and

$$P_{\mathrm{GL}(V)}(\lambda) = \left[\begin{array}{c|c|c|c} A_{1,1} & A_{1,2} & \dots & A_{1,r} \\ \hline 0 & A_{2,1} & \dots & A_{2,r} \\ \hline \vdots & & & \\ & & & \vdots \\ \hline 0 & \dots & 0 & A_{r,r} \end{array}\right]$$

where $A_{i,j} \in Hom_k(V_j, V_i)$.

Coming back to the general case, if G is smooth, so are the k–groups $P_G(\lambda), U_G(\lambda)$ and $C_G(\lambda)$ [17, Proposition 2.1.8.(3)].

Remarks 6.1. (a) There are other proofs of the smoothness result as special case of representability of attractors for an action of $\mathbb{G}_m$ on a k–variety. See Margaux [35] in the affine case and Drinfeld [19, prop. 1.4.20] in the general case.

(b) A. Mayeux has extended this theory to action of higher diagonalizable groups, this is called *Algebraic Magnetism* [34].

6.2. **Wound reductive groups.** We remind the reader that a linear algebraic k–group G is reductive if it is connected and if $G_{\overline{k}}$ has no non zero unipotent smooth connected normal $\overline{k}$–subgroup. A smooth connected k-subgroup P of G is *parabolic* if the quotient variety G/P is projective.

Proposition 6.2. *Let G be a reductive k–group.*

(1) For each $\lambda : \mathbb{G}_m \to G$, $P_G(\lambda)$ is a k–parabolic subgroup of G.

(2) Let P be a k–parabolic subgroup of G. Then $P = P_G(\lambda)$ for some $\lambda : \mathbb{G}_m \to G$.

In the case of $\mathrm{GL}(V)$, this follows of the above precise description. For the proof of the general case, see [41, Lemma 15.1.2].

Corollary 6.3. *Let G be a reductive k–group. Then the following are equivalent:*

(i) G is isotropic;

(ii) G admits a non-trivial split central k-subtorus or G admits a proper k–parabolic subgroup.

Proof. $(i) \Longrightarrow (ii)$ We are given a non trivial homomorphism $\lambda : \mathbb{G}_m \to G$. If λ is central, then the image of λ is non-trivial split central k-subtorus of G. If λ is non central, then $C_G(\lambda) \subsetneq G$, so that $P_G(\lambda) \subsetneq G$.

$(ii) \Longrightarrow (i)$. If G admits a non-trivial split central k-subtorus, then G is isotropic. If G admits a proper k–parabolic subgroup P, then $P = P_G(\lambda)$ for some non-central $\lambda : \mathbb{G}_m \to G$. Thus G is isotropic. □

Theorem 6.4. *(Borel-Tits) Let G be a reductive k–group, then G is anisotropic if and only if G is k–wound.*

In other words, the statement is that if G contains a k–subgroup isomorphic to $\mathbb{G}_a$, it contains a k–subgroup isomorphic to $\mathbb{G}_m$. This is an advanced fact with historical steps.

Characteristic zero case [3, lemme 8.3]. We consider a k–subgroup $U = \mathbb{G}_a \subset G$. We know that $U_{\overline{k}}$ sits in the unipotent radical of a Borel $\overline{k}$–subgroup of G; it is based on the Borel-Rosenlicht's fixed point theorem for the action on $\mathbb{G}_a$ on the projective variety G/B for B a Borel subgroup of G [5, th. 10.4 and §11]. Then $k.X = \mathrm{Lie}(U)$ is a nilpotent k-subalgebra of $\mathrm{Lie}(G)$. According to the Jacobson-Morozov's theorem [9, ch. 8, §11.2], there exists an embedding $\mathfrak{sl}_2 \subset \mathrm{Lie}(G)$ such that $X \in \mathfrak{sl}_2$. Since $\mathfrak{sl}_2 = [\mathfrak{sl}_2, \mathfrak{sl}_2]$, we can integrate to a homomorphism $f : \mathrm{SL}_2 \to G$, see [5, II, Cor. 7.9], it is actually unique [40, XXIV.7.3.1.(ii)]. Since $\mathrm{Lie}(f)$ is injective, f has finite kernel. Since SL_2 is isotropic, we conclude that G is isotropic.

Perfect field case. The idea is to associate to a smooth connected unipotent k-subgroup U of G a parabolic k–subgroup $P(U)$ such that U is a k–subgroup of the unipotent radical of U. If U is not trivial, we have that $P(U) = P_G(\lambda)$ for a non-central $\lambda : \mathbb{G}_m \to G$ so that G is isotropic. This uses a sequence of groups due to Platonov. We put $N_1 = \bigl(N_G(U)\bigr)_{red}$ and denote by U_1 its unipotent radical, that is, its maximal smooth connected normal unipotent k–subgroup. Then $U \subset U_1$ (and is normal). We continue with $N_{i+1} = \bigl(N_G(U_i)\bigr)_{red}$ and U_{i+1} its unipotent radical. For dimension reasons, the sequence $U = U_0 \subset U_1 \cdots \subset U_n$ stabilizes to some $U \subset G$ which is smooth unipotent connected and one has to establish that $N_G(U)_{red}$ is a parabolic k–subgroup of G [4].

General case. The above argument can be refined to the general case [4, th. 2.5] but also follows from [17, Theorem C.3.8] written in the wider framework of pseudo-reductive groups.

Remark 6.5. We have to be careful with imperfect fields since there are smooth connected unipotent k–groups which are not split. We provide here a non-trivial example of an embedding $U \to G$ of a smooth unipotent connected k–group in an anisotropic semisimple k–group. Let k be field of characteristic $p > 0$ admitting a cyclic field extension l/k of degree p Such an extension is an Artin-Schreier extension $l = k[x](x^p - x - b)$ for some $b \in k$ and the action of the Galois group $\mathbb{Z}/p\mathbb{Z}$ is by $\sigma([x]) = [x + 1]$. We put $K = k((t))$, $L = k((t)) \otimes_k l$ and consider the unital associative K-algebra D generated by L and an element y submitted to the relations $y^p = t$, $yxy^{-1} = x+1$. This is a central simple K–algebra of degree p [26, prop. 2.5.2] which is division (equivalently not split) if and and only if $t \in N_{L/K}(L^\times)$ (*loc. cit.*, Cor.4.7.5). Since the valued field L is an unramified extension of K of degree p, we have $v_t\Bigl(N_{L/K}(L^\times)\Bigr) = p\mathbb{Z}$ so that $t \not\in N_{L/K}(L^\times)$. It turns out that D is a division central K–algebra of degree p and we can consider the k–group $G = \mathrm{PGL}_1(D) = \mathrm{GL}_1(D)/\mathbb{G}_m$ ($\mathrm{GL}_1(D) = U(D)$ with the notation of Remark 5.3). That reductive

K-group is anisotropic. We consider the purely inseparable extension $K' = K(\sqrt[p]{t}) = K(y)$ inside D. In view of Remark 5.3, the map $K' \to D$ gives rise to an embedding $R_{K'/K}(\mathbb{G}_m) \hookrightarrow \mathrm{GL}_1(D)$. By moding out by the diagonal $\mathbb{G}_m$ we obtain an embedding $R_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m \hookrightarrow \mathrm{PGL}_1(D)$. The point is that $U = R_{K'/K}(\mathbb{G}_m)/\mathbb{G}_m$ is unipotent smooth and connected! See [17, Example 1.1.3] for a proof or below in §9.2. This is coherent with the fact that $U(k) = (K')^\times/K^\times$ is a group of p–torsion since $(K')^p \subset K$.

Lecture 3: Resolution of singularities, Gabber's theorem, Unipotent groups, November 7

7. Using resolution of singularities

In this section, the base field k is of characteristic zero. There are several kinds of resolution of singularities which come from Hironaka's fundamental work [29] and refinements [1, 21]. Let us give a short list of statements.

Resolution, Hironaka main theorem I [29, §I.5]**.** Let X be a reduced irreducible k–variety (e.g. X is normal). Then there exists an algebraic subscheme D of X such that

(i) the set of points of D is exactly the singular locus of X, and

(ii) if $f : \widetilde{X} \to X$ is the monoidal transformation of X with center D, then X is smooth.

A stronger form is by using a sequence $f : X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$ of blow-ups with smooth center [29, Main Theorem 1*] (see also [2, th. 1.1]). At each stage, the fibers at a k–point are projective space so that the map $X_n(k) \to X(k)$ is onto. Furthermore Hinonaka also proved the following.

Hironaka main theorem II [29, §I.6]**.** Let $f : X' \to X$ be a birational projective morphism between smooth projective varieties. Let $U \subset X$ be an open subvariety such that $f^{-1}(U) \xrightarrow{\sim} U$. Then f is a composition of blowups with smooth centers disjoint of U.

Once again it follows that the map $X'(k) \to X(k)$ is onto.

Canonical embedded desingularization [1]

Let X be a smooth projective k–variety equipped with a closed reduced irreducible k–subvariety $Y \subset X$. Then there exists a "canonical sequence" $f : X_n \to X_{n-1} \to \cdots \to X$ of blow-ups with smooth centers such that:

i) $f^{-1}(Y_{sm}) \xrightarrow{\sim} Y_{sm}$;

ii) The strict transform Y' of Y is smooth;

iii) $f^{-1}(Y) \setminus Y$ is a normal crossing divisor.

Canonical is stated informally, let us say that if an algebraic k–group acts on $Y \to X$, then its action extends to the whole picture. There is also an equivariant version of the two Hironaka main theorems, see [39].

It applies to compactifications of affine algebraic groups (and homogeneous spaces). Let G be an affine algebraic k–group equipped with the action of an algebraic J. We consider a projective faithful representation $G \rtimes J \hookrightarrow \mathrm{PGL}(V)$ and denote by Y the Zariski closure of G in $\mathbb{P}(\mathrm{End}(V))$. Applying the previous theorem to $Y \subset \mathbb{P}(\mathrm{End}(V))$ provides a smooth projective compactification G^c of G which is $(G \times_k G) \rtimes J$-equivariant and such that $G^c \setminus G$ is a strict crossing divisor.

Remark 7.1. It is a hard job to construct compactifications of a semisimple algebraic group. If G is adjoint, we have the wonderful compactification of de Concini and Procesi [14]. For G simply connected (e.g. SL_n), there are quite complicated constructions (of characteristic free), see Kausz [32] and Huruguen [30].

The embedded desingularization theorem permits to compare equivariant G-compactifications of a smooth G-variety.

Proposition 7.2. *Let G be an algebraic group. Let U be a smooth quasi-projective G–variety and let X_1, X_2 be two smooth projective G–compactifications of U. Then there exists a smooth projective G–compactification X of U with maps $f_i : X \to X_i$ which are a sequence of blow-ups with (G–equivariant) smooth centers such that $f_{i,U} = id_U$.*

Proof. We denote by Y the schematic closure of the diagonal embedding of U in $X_1 \times_k X_2$. We obtain then a birational G-morphism $f : X_3 \to Y$ such that $f^{-1}(U) \xrightarrow{\sim} U$ and which dominates X_2 (resp. X_3). According to main theorem II of Hironaka each map $f_i : X \to X_i$ is a sequence of (G–equivariant) blow-ups with smooth centers disjoint of U. □

We consider now the question of rational points on the boundary of a compactification.

Lemma 7.3. *Let U be a quasi-projective smooth k–variety.*

(1) Let X_1, X_2 be two smooth projective k–compactifications of U. Then $U(k) = X_1(k)$ if and only if $U(k) = X_2(k)$.

(2) Assume that U admits a projective compactification U^c (not necessarily smooth) such that $U(k) = U^c(k)$. Then for each projective compactification X of U, we have $U(k) = X(k)$.

Proof. (1) Proposition 7.2 boils down to the case of a blow-up $f : X_2 \to X_1$ with smooth center C disjoint of U. Clearly if $U(k) = X_1(k)$, we have $U(k) = X_2(k)$. Since the fibers of f at points of $C(k)$ are projective spaces, the map $X_2(k) \setminus U(k) \to X_2(k) \setminus U(k)$ is onto. If $U(k) = X_2(k)$, it follows that $U(k) = X_1(k)$.

(2) We assume that $U(k) = U^c(k)$. By (1) it enough to check that $U(k) = X(k)$ for a specific smooth compactification X of U. We embed U^c in a projective space $Z = \mathbb{P}(V)$. The embedded desingularization theorem provides a sequence of blow-ups $f : Z_n \to Z_{n-1} \to \cdots \to Z$ such that $f^{-1}(U) \xrightarrow{\sim} U$ and such that the strict transform X of U^c is smooth. Since $U(k) = U^c(k)$, we obtain $U(k) = X(k)$ as desired. □

Finally of the same flavour, we have also the following fact.

Equivariant resolution of indeterminacy locus [39]**.** Let G be an algebraic group acting and consider a morphism of smooth quasi-projective G–varieties $f : U \to V$.

Let X (resp. Y) be a smooth projective G–compactification of X (resp. V). Then there exists a G–sequence of blow-ups

$$h : X_n \to X_{n-1} \to \cdots \to X$$

with smooth centers such that $h^{-1}(V) \xrightarrow{\sim} V$ and a (unique) extension $\widetilde{f} : X_n \to Y$ of f.

The given reference is for an algebraically closed field but the naturality of the construction permits to descend [we can also work directly over k].

8. A second refinement of Borel-Tits theorem

The preceding discussion on compactifications has the following consequence.

Corollary 8.1. *Let G be a smooth affine k–group (k of char. zero) and let G^c be a smooth compactification of G. Then the assertions of Theorem 1.2 are furthermore equivalent to each of the following*

(iii) there exists a $(G \times G)$-equivariant smooth projective compactification X of G such that $G(k) = X(k)$;

(iv) for each smooth projective k–compactification X of G we have $G(k) = X(k)$.

9. Gabber's generalization

Theorem 9.1. [24, thm. B].

Let G be an affine algebraic k–group G. Then the following are equivalent:

(I) (rank one subgroups) G does not carry any k–subgroup isomorphic to the additive group $\mathbb{G}_a$ nor the multiplicative group $\mathbb{G}_m$;

(II) (Boundedness property) $G\big(k((t))\big)$ is bounded for the valuation topology.

(III) $G\big(k[[t]]\big) = G\big(k((t))\big)$;

(IV) (No point at infinity) There exists a (projective G-equivariant, extending the left action) projective compactification X of G such that $G(k) = X(k)$.

(V) (No orbit at infinity) There exists a (projective G-equivariant) projective compactification X of G such that $X \setminus G$ has no k-orbit for the left action of G.

This is a simplified version of [24]. Condition (V) is new, it deals with k-orbits as defined by [7, §10.2, def. 4]. If G acts on the k-variety X, a k-subscheme $Z \subset X$ is a k–orbit if there exists a finite field extension k' of k and a point $x' \in X(k')$ such that $Z_{k'} = O_{x'}$, that is the G-orbit of x' as defined in §1.1 (when G is reduced, e. g. smooth); another way to say the same thing is to say that Z is homogeneous under G in the sense of [18, III.3.2]. Of course the orbit of $x \in X(k)$ is a k–orbit but the converse does not hold (think for example to non-trivial G-torsors). The interested reader can investigate whether the compactifications discussed above satisfy the stronger condition (V), we do it below for tori. Once again we have the easy implications $(V) \Longrightarrow (IV) \Longrightarrow (III) \Longrightarrow (II) \Longrightarrow (I)$ so that the implication $(I) \Longrightarrow (V)$ is everything.

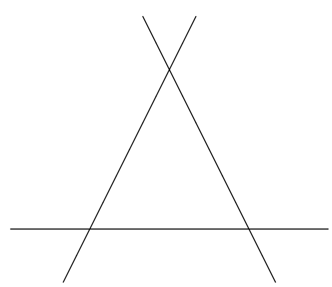
Gabber's result holds for imperfect fields where there are many more kind of smooth algebraic groups, as wound unipotent groups, pseudo-reductive groups. The statement holds also for singular algebraic groups. Gabber proves furthermore than his compactification X satisfies the following property:

For each separable field extension F/k, G_F is wound if and only if $G(F) = X(F)$.

Such a compactification is called a Gabber compactification of G. In the semisimple adjoint case, this is a remarkable fact that the wonderful compactification of de Concini and Procesi is a Gabber compactification. We do not check it in that lecture.

9.1. **Back on tori.** We claim that our compactification T^c of an anisotropic k–torus T satisfies property (V). Once again it reduces to the case of $T = R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m$ for K/k Galois (of group Γ) whose compactification is the projective space $T^c = \mathbb{P}(K) = \big(\mathbb{W}(K) \setminus \{0\}\big)/\mathbb{G}_m$. Using the isomorphism $K \otimes_k \overline{k} \xrightarrow{\sim} \overline{k}^{(\Gamma)}$, $x \otimes y \mapsto (\sigma(x)y)_{\sigma \in \Gamma}$, the $\overline{k}$-orbits of $T_{\overline{k}} \cong \mathbb{G}_{m,\overline{k}}^{(\Gamma)})/\mathbb{G}_{m,\overline{k}}$ in $\mathbb{P}(K)_{\overline{k}} \cong \mathbb{P}(\overline{k}^{(\Gamma)})$ are finitely many, the list is $[a_1 : \dots : a_\sigma : \dots]$ with $a_\sigma = 0$ or 1.

Let Z be a k–orbit for the action of T on $\mathbb{P}(K)$ and let $x \in Z(\overline{k})$. Then the shape of the orbit of x has to be Γ-invariant for the translation action so that $Z \subset T$. For example, for $\Gamma = \mathbb{Z}/3\mathbb{Z}$, $(X \setminus T)_K$ is the K–variety



where the three lines are permuted by Γ.

9.2. **Unipotent groups.** The case of unipotent groups is a special case of Gabber's theorem which is fully handled in [7, ch. 10]. We limit ourself to present a significative example of wound unipotent k–group and of its compactification. Let k be a field of positive characteristic $p > 0$.

A first example is $U = R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m$ for K a purely inseparable extension of degree p^r. This is a smooth connected commutative affine algebraic k–group. We have $U(\overline{k}) = (K \otimes_k \overline{k})^\times/(\overline{k})^\times$. Since $K^{p^r} \subset k$, $(K \otimes_k \overline{k})^{p^r} \subset \overline{k}$ so that $p^r\, U(\overline{k}) = 1$. It follows that all elements of $U(\overline{k})$ are unipotent so that $U_{\overline{k}}$ is unipotent [5, I.4.8] and U is as well [18, IV.2.2.6]. Another way (provided by A. Maffei) is to construct an embedding in some strictly upper triangular k–group. We take as vector space $V = K \otimes_k K$. Then $R_{K/k}(\mathbb{G}_m)$ acts by $L_x \otimes L_x^{-1}$ and we get a faithful representation $U \hookrightarrow \mathrm{GL}(K \otimes_k K)$. Each element of $K^\times/k^\times$ is strictly trigonalizable so that the abelian subgroup $K^\times/k^\times$ of $\mathrm{GL}(V)(k)$ is strictly trigonalizable. Assuming k infinite, it follows that U is strictly trigonalizable in $\mathrm{GL}(V)$ since $U(k)$ is Zariski dense in U.

The k–group U admits the compactification $U = R_{K/k}(\mathbb{G}_m)/\mathbb{G}_m \subset (\mathbb{W}(K) \setminus \{0\})/\mathbb{G}_m = \mathbb{P}(K) = U^c$.

Since $K^\times = K \setminus \{0\}$, we have that $U(k) = U^c(k)$. In particular U is a k–wound group and U^c has no k–point on the boundary.

We claim that $(U^c \setminus U)_{\overline{k}}$ consists in a projective space of dimension $p^r - 2$. We put $R = K \otimes_k \overline{k}$, this is a local Artinian $\overline{k}$–algebra. We have $R = \overline{k} \oplus \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal. In the $\overline{k}$–projective space $\mathbb{P}(R)$, $U_{\overline{k}}$ is the open subspace given by $x_0 \neq 0$ so that $U^c_{\overline{k}} \setminus U_{\overline{k}}$ is isomorphic to the projective space $\mathbb{P}(\mathfrak{m})$.

Assume that $U^c \setminus U$ contains a k–orbit Z. Then Z is absolutely reduced and smooth (since it is a quotient of U on $\overline{k}$). Then Z admits a k_s-rational point [42, Tag 056U] which defines a nonzero nilpotent element of $K \otimes_k k_s$ up to $k_s^\times$. This contradicts the fact that $K \otimes_k k_s$ is a field. Thus the boundary $U^c \setminus U$ has no k-orbit.

Remark 9.2. We consider the special case $K = k(\sqrt[p]{a}) = k[t]/(t^p - a)$ so that $R = \overline{k}[x]/x^p$ with $x = t - \sqrt[p]{a}$. Then $\mathfrak{m} = R[x] = R^\times[x]$ so that $U^c_{\overline{k}} \setminus U_{\overline{k}}$ consists in the orbits $\mathbb{W}(Rx^i)$ for $i = 1, \dots, p-1$.

9.3. **More unipotent groups.** Let

$$f = \sum_{i=1}^{n} \sum_{j=0}^{d_j} a_{i,j} T_i^{p^j} \in k[T_1, \dots, T_n].$$

It is called a p–polynomial and we can consider the unipotent k–subgroup U_f of $\mathbb{G}_a^n$ defined as the kernel of the homomorphism $f : \mathbb{G}_a^n \to \mathbb{G}_a$. Then U_f is smooth if and only if the differential $df_0 : k^n \to k$ is onto, that is, the linear part $\sum_{i=1}^n a_{i,0} T_i$ of f is non zero. The principal part $\widetilde{f}$ is the $\sum_{i=1}^n a_{i,n_i} t_i^{p^{n_i}}$.

Lemma 9.3. *Assume that f is non zero. If $\widetilde{f}$ is anisotropic (that is $\widetilde{f}(x) = 0 \Longrightarrow x = 0$ for all $x \in k^n$), then U_f is k–wound.*

Proof. We assume that U_f contains $\mathbb{G}_a$ (it cannot contains $\mathbb{G}_m$) as k–subgroup. Then there exists $P_1, P_2, \dots, P_r \in k[t^p]$ such that $f\big(P_1(t), \dots, P_n(t)\big) = 0$. We write $P_i(t) = c_i t^{p^{d_i}} + \dots$ and consider the term in of degree $p^d = \mathrm{Max}(p^{n_i + d_i})$ in $f\big(P_1(t), \dots, P_n(t)\big)$. It follows that $\sum_{\substack{i \\ n_i + d_i = d}} c_i t^{p^d} = 0$ so that $\widetilde{f}$ admits a non trivial zero. □

Remark 9.4. The criterion for woundness is more involved, see [16, lemma 1.11].

Already the construction of a compactification of a wound U_f requires an elaborated argument, see [7, §10, prop.11]. We focus on the simplest relevant following example (due to Rosenlicht): we take $n = 2$ and $f(x, y) = x^p - x - ay^p$ for $a \in k \setminus k^p$. Then $G = U_f$ is smooth since $df_0 = -id$ and the principal part $\widetilde{f}(x, y) = x^p - ay^p$ is designed to be anisotropic. The k–group G is then an affine curve. Putting $k' = k(\sqrt[p]{a})$ and $y' = x - \sqrt[p]{a}y$, $G_{f,k'}$ is isomorphic to the k–group of equation $x = (y')^p$ hence to $\mathbb{G}_{a,k'}$. It follows that G is a connected affine curve. There are several ways to compactify G.

(1) We take the hypersurface $X = \{x^p - xz^{p-1} = ay^p\}$ in $\mathbb{P}^2$. The boundary is the point $[\sqrt[p]{a} : 1 : 0]$ so that $G(k) = X(k)$.

(2) We consider the cyclic Galois cover $h : G \to \mathbb{G}_a$, $(x, y) \mapsto y$ and its normalization with respect to $\mathbb{G}_a \subset \mathbb{P}^1_k$. We obtain then a compactification $\widetilde{h} : X \to \mathbb{P}^1_k$. For understanding the boundary, we need to determine $\widetilde{h}^{-1}(\infty)$. We have $k(G) = k(t)[x]/(x^p - x - at^p)$ and consider

$$k(G) \otimes_{k(t)} k((1/t)) = k((1/t))[x]/(x^p - x - at^p) = k((1/t))[w]/(w^p - t^{1-p}w - a)$$

with the change of variable $w = (t)^{-1}x$. The equation $w^p - t^{1-p}w - a$ has no root in $k((1/t))$ so that $L = k(G) \otimes_{k(t)} k((1/t))$ is a finite field cyclic extension of degree p of $k((1/t))$. Furthermore the valuation on $k((1/t))$ extends to a unique valuation on L which is $v_L(x) = \frac{v\left(N_{L/k((1/t))}(x)\right)}{p}$. The valuation of w is zero so that t^{-1} is an uniformizing parameter of L. We denote by B its valuation ring and claim that $B_0 = k[[1/t]][w]/(w^p - t^{1-p}w - a) = B$, that is, w generates B as $k[[1/t]]$-module. Since B_0 and B are free of rank p modules, Nakayama's lemma reduces to show that the k-morphism of vector spaces of dimension p. $B_0/t^{-1}B_0 \to B/t^{-1}B$ is an isomorphism. This clearly holds since $B_0/^{-1}B_0 = k(\sqrt[p]{a})$ is of degree p. We conclude that the unique point of the boundary has residue field $k(\sqrt[p]{a}) = B/t^{-1}B$.

(3) We could use the anti-equivalence of categories between complete regular k–curves and the category of algebraic function fields of transcendence degree 1 over k [43, prop. 4.4.5]). The compactification of (2) is then nothing but the regular completion G^c of the regular curve G.

It is of interest to observe that this completion is regular but not smooth when $p > 2$. A chart at the infinity point is $C = k[1/t, w]/(w^p - t^{1-p}v - a) \cong k[v, w]/(w^p - v^{p-1}w - a)$. Putting $P = w^p - v^{p-1}w - a$, we have $\partial P/\partial w = -v^{p-1}$ and $\partial P/\partial v = -w(p-1)v^{p-2}$. It follows that $(\sqrt[p]{a}, 0)$ is a singular point. Actually all three viewpoints provide the same compactification.

Remark 9.3.1. In the case $p = 2$, the above compactifications are nothing but $\mathbb{P}^1$ (which is smooth) and $G \cong \mathbb{P}^1 \setminus \{\sqrt{a}\}$ as k–scheme.

Lecture 4: Indices and residues, November 14

This last lecture is taken from a paper with Mathieu Florence [23].

Notation and conventions. If $r \in \mathbb{Q}^\times$, the notation $r = m/n$ means that $(m, n) = 1$ with $n \geq 1$. This extends to $0 = 0/1$.

We denote by $k^u = k[u]$ the ring of k-polynomials in the indeterminate u. For each ring A we denote by $A[[t]]$ the ring of power series and define $A(\!(t)\!) = A[[t]][x]/(1-tx)$. For each non-negative integer $n \geq 1$, we define $A[[t^{1/n}]] = A[[t]][y]/(y^n - t)$ and $A(\!(t^{1/n})\!) = A[[t]][x, y]/(y^n - t, 1 - xy)$. We have natural maps $A[[t^{1/n}]] \to A[[t^{1/mn}]]$ and $A(\!(t^{1/n})\!) \to A(\!(k^{1/mn})\!)$ for $m \geq 1$.

If $r = m/n \in \mathbb{Q}_{\geq 0}$, we put $k_r = k^u[[t^{1/n}]]$ and $\mathcal{K}_r = k^u(\!(t^{1/n})\!)$. We have a specialization homomorphism $j : k_r \to k^u$.

For each $r = m/n \in \mathbb{Q}_{\geq 0}$, the assignment $t \to t(1 + ut^r)$ defines ring homomorphisms $\sigma_r : k^u[[t]] \to k_r$; if $r > 0$, its extends to $\sigma_r : k^u(\!(t)\!) \to \mathcal{K}_r$.

Inverting $\lambda := 1 + u$, we come now to analogues $k^{u,+} = k[u][z]/(1 - (1 + u)z)) = k[\lambda, \lambda^{-1}]$. We have the variants $k_r^+ = k^{u,+}[[t^{1/n}]]$; $\mathcal{K}_r^{u,+} = k^{u,+}(\!(t^{1/n})\!)$, $\sigma_r : k^{u,+}(\!(t)\!) \to \mathcal{K}_r^+$, $t \mapsto t(1 + ut^r)$, and the specialization $j^+ : k_r^{u,+} \to k^{u,+}$ for all $r \in \mathbb{Q}_{\geq 0}$.

10. The ramification index

Let G be an affine algebraic k-group equipped with a closed embedding $\rho : G \to \mathrm{SL}_{N,k}$.

Proposition 10.1. *Let $g \in G\big(k(\!(t)\!)\big) \setminus G\big(k[[t]]\big)$.*
(1) *The set*

$$\Sigma(g) = \Big\{ r \in \mathbb{Q}_{>0} \mid g^{-1}\sigma_r(g) \in G(k_r) \Big\}$$

is non-empty and let $r(g)$ be its lower bound in $\mathbb{R}$. Then $r(g) \in \mathbb{Q}_{\geq 0}$ and $\Sigma(g) = \mathbb{Q}_{>0} \cap [r(g), +\infty[$.
(2) *Assume that $r(g) > 0$. Then $j\big(g^{-1}\,\sigma_{r(g)}(g)\big)$ belongs to $G(k^u) \setminus G(k)$.*
(3) *Assume that $r(g) = 0$. Then $g^{-1}\sigma_0(g) \in G(k^{u,+}[[t]])$ and $j\big(g^{-1}\,\sigma_0(g)\big)$ belongs to $G(k^{u,+}) \setminus G(k)$.*

Proof. (1) Clearly the statement reduces to the case of SL_N. Our assumption implies that $g = t^{-d}\underline{\mathrm{g}}$ with $d \geq 1$ and $\underline{\mathrm{g}} \in \mathrm{M}_N(k[[t]]) \setminus t\mathrm{M}_N(k[[t]])$. The number $-d$ is called the gauge in t of the matrix $g \in \mathrm{M}_N(k(\!(t)\!))$ and is denoted[2] by $V_t(g)$. It follows that $\det(\underline{\mathrm{g}}) = t^{Nd}$. For $r \in \mathbb{Q}_{>0}$, we have

$$g^{-1}\sigma_r(g) = \frac{t^d}{t^d(1 + ut^r)^d}\ \underline{\mathrm{g}}^{-1}\,\sigma_r(\underline{\mathrm{g}}) = (1 + ut^r)^{-d}\ \underline{\mathrm{g}}^{-1}\,\sigma_r(\underline{\mathrm{g}}). \tag{10.1}$$

[2] The gauge is not additive but is supra-additive, i.e. $V_t(g_1 g_2) \geq V_t(g_1) + V_t(g_2)$.

We write $\underline{\mathrm{g}} = \big(P_{i,j}\big)_{i,j=1,..,N}$ with $P_{i,j} \in k[[t]]$ and denote by $\Delta_{i,j} \in k[[t]]$ the minor of index (i,j) of $\underline{\mathrm{g}}$. We have $\underline{\mathrm{g}}^{-1} = \Big(t^{-Nd}\,\Delta_{i,j}\Big)_{i,j=1,..,N}$ so that the (i,j)–coefficient $C_{i,j,r}$ of $\underline{\mathrm{g}}^{-1}\,\sigma_r(\underline{\mathrm{g}})$ is

$$C_{i,j,r} = t^{-Nd}\sum_{k=1}^{N}\Delta_{i,k}(t)\,P_{k,j}(t(1+ut^r)) \in k_r. \tag{10.2}$$

When $u=0$, $C_{i,j,r}$ specializes on $\delta_{i,j}$ so that

$$C_{i,j,r} = \delta_{i,j} \,+\, t^{-Nd}\sum_{k=1}^{N}\Delta_{i,k}(t)\,\big(P_{k,j}(t(1+ut^r)) - P_{k,j}(t)\big). \tag{10.3}$$

We consider the identity

$$\sum_{k=1}^{N}\Delta_{i,k}(t)\,\big(P_{k,j}(t(1+\epsilon)) - P_{k,j}(t)\big) = \sum_{a\geq 0,\, b\geq 1} c_{i,j}^{a,b}\,t^a\,\epsilon^b \tag{10.4}$$

with $c_{i,j}^{a,b} \in k$. Taking $\epsilon = ut^r$, we get

$$C_{i,j,r} = \delta_{i,j} \,+\, t^{-Nd}\sum_{a\geq 0,\, b\geq 1} c_{i,j}^{a,b}\,t^{a+rb}\,u^b \tag{10.5}$$

We consider the sets $\mathrm{supp}(i,j) = \big\{(a,b) \mid c_{i,j}^{a,b} \neq 0\,\big\}$ and $\mathrm{supp}(g) = \bigcup\limits_{(i,j)} \mathrm{supp}(i,j)$.

Claim 10.2. $\mathrm{supp}(g) \neq \emptyset$.

If $\mathrm{supp}(g) = \emptyset$, then $\underline{\mathrm{g}} = \sigma_r(\underline{\mathrm{g}})$ and all coefficients of $\underline{\mathrm{g}}$ belong to k which contradicts the fact $\det(\underline{\mathrm{g}}) = t^{Nd}$. The Claim is established and enables us to define the function

$$f_g(r) = \mathrm{Inf}\Big\{-Nd + a + rb \mid (a,b) \in \mathrm{supp}(g)\Big\}.$$

We have $f_g(r) \geq r + f_g(0)$ so that f admits positive values and $\Sigma(g)$ is not empty. Since $1+ut^r \in k_r^\times$, the set $\Sigma(g)$ consists in the positive rational numbers r such that $\underline{\mathrm{g}}^{-1}\,\sigma_r(\underline{\mathrm{g}})$ belongs to $\mathrm{GL}_N(k_r)$. We get that

$$\Sigma(g) = \big\{r \in \mathbb{Q}_{>0} \mid f_g(r) \geq 0\big\}.$$

By definition of $r(g)$, we have

$$r(g) = \mathrm{Inf}\big\{r \in \mathbb{Q}_{>0} \mid f_g(r) \geq 0\big\} \in \mathbb{R}_{\geq 0}.$$

If $f_g(0) \geq 0$, then $r(g) = 0$. If $f_g(0) < 0$, then there exists a,b such that $-Nd + a + r(g)b = 0$ whence $r(g) \in \mathbb{Q}_{>0}$. In both cases, we have $\Sigma(g) = \mathbb{Q}_{>0} \cap [r(g), +\infty[$.

(2) Along the proof of (1), we have seen that then there exists a,b such that $-Nd + a + rb = 0$ and $c_{i,j}^{a,b} \neq 0$. Formula (10.5) shows that $j\big(g^{-1}\,\sigma_r(g)\big) \notin \mathrm{SL}_N(k)$.

(3) Once again, it is enough to consider the case of $\mathrm{SL}_{N,k}$. We recall the notation $k_1 = k^u[[t]]$ and $\mathcal{K}_1 = \mathcal{K}^u((t))$. If $r(g) = 0$, we have $a - Nd \geq 0$ for each a occurring in formula (10.5) (more precisely such that $c_{i,j}^{a,b} \neq 0$ for some b). The point is that the computation of (1) works also for $r = 0$. It follows that

$$\underline{\mathrm{g}}^{-1}\sigma_0(\underline{\mathrm{g}}) - I_N \in u\mathrm{M}_N(k_1)$$

so that $\underline{\mathrm{g}}^{-1}\sigma_0(\underline{\mathrm{g}}) \in \mathrm{M}_N\big(k_1\big)$. Taking into account the identity (10.5) for $r = 0$, we get (10.6)

$$g^{-1}\sigma_0(g) = \lambda^{-d}\underline{\mathrm{g}}^{-1}\sigma_0(\underline{\mathrm{g}}) = \lambda^{-d}[I_N + uM_0(u) + tM_1(u) + \dots] \in \mathrm{M}_N\big(k[\lambda, \lambda^{-1}][[t]]\big).$$

with $M_i(u) \in \mathrm{M}_N(k[u])$ for $i = 1, 2, ...$ We have $j\big(g^{-1}\,\sigma_0(g)\big) = \lambda^{-d}\big(I_N + uM_0(u)\big)$. Assume that $j\big(g^{-1}\,\sigma_0(g)\big) = M \in \mathrm{SL}_N(k)$. The formula (10.6) above reads

$$g^{-1}\sigma_0(g) = M + \lambda^{-d}\Big(tM_1(u) + t^2M_2(u) + \dots\Big) \in \mathrm{M}_N\big(k[\lambda, \lambda^{-1}][[t]]\big). \tag{10.7}$$

We consider the subring $B = \Big(k[\lambda][[t]]\Big)[\lambda^{-1}]$ of $k[\lambda, \lambda^{-1}][[t]]$ and its analogue $\mathcal{B} = \Big(k[\lambda]((t))\Big)[\lambda^{-1}] \subset k[\lambda, \lambda^{-1}]((t))$. The map $\sigma_0 : k((t)) \to k[\lambda, \lambda^{-1}]((t))$, $t \mapsto \lambda t$, factorizes through $\mathcal{B}$. It follows that the equation (10.7) holds in $\mathrm{M}_N(\mathcal{B})$, i.e.

$$g^{-1}g(\lambda t) = M + \lambda^{-d}[tM_1(u) + t^2M_2(u) + \dots] \in \mathrm{M}_N\big(\mathcal{B}\big). \tag{10.8}$$

In other words we have

$$g^{-1}g(\lambda t) = M + \lambda^{-d}[tM_1(\lambda - 1) + t^2M_2(\lambda - 1) + \dots] \in \mathrm{M}_N\big(\mathcal{B}\big). \tag{10.9}$$

The homomorphism $\varphi : k[\lambda]((t)) \to k((t))$, $\sum_{i \geq -L} P_i(\lambda)t^i \mapsto \sum_{i \geq -L} P_i(t)t^{di}$ extends uniquely to a homomorphism $\varphi : \mathcal{B} \to k((t))$. Specializing the equation (10.8) by φ yields that $g^{-1}(t^d)g\big(t^{d+1}\big) \in \mathrm{M}_N\big(k[[t]]\big)$ hence

$$g(t^{1+d}) = g(t^d)Q \ \text{ with } \ Q \in \mathrm{M}_N\big(k[[t]]\big).$$

It follows that $V_t(g(t^{1+d})) = V_t(g(t^d)Q) \geq V_t(g(t^d)) + V_t(Q) \geq V_t(g(t^d))$, so that $-d(d{+}1) \geq -d^2$, this is a contradiction. We conclude that $j\big(g^{-1}\,\sigma_0(g)\big) \not\in \mathrm{SL}_N(k)$. □

Remark 10.3. By inspection of the proof, we see that

$$r(g) = \mathrm{Inf}\Big\{r \in \mathbb{Q}_{>0} \mid g^{-1}\sigma_r(g) \in G\big(k(u)[[t]]\big)\Big\}.$$

Definition 10.4. Let $g \in G\big(k((t))\big)$. If $g \not\in G\big(k[[t]]\big)$, we define the *ramification index* $r(g)$ as in Proposition 10.1. If $g \in G\big(k[[t]]\big)$, we define $r(g) = -1$.

It is straightforward to check that the index does not depend of the choice of the representation ρ.

Lemma 10.5. *The function $g \to r(g)$ is right $G(k[[t]])$–invariant (resp. left $G(k)$-invariant) and is insensitive to any base change $k \hookrightarrow F$.*

Proof. It readily follows of the definition of $r(g)$. □

11. The residue

Let G and $g \in G(k((t)))$ as in Proposition 10.1. If $r(g) > 0$, we define the residue $\mathrm{res}(g)$ as the image of $g^{-1}\sigma_{r(g)}(g)$ by the homomorphism $j_* : G(k_r) \to G(k^u) = G(k[u])$. We see it as a k–map $\mathrm{res}(g) : \mathbb{G}_{a,k} = \mathrm{Spec}(k[u]) \to G$ and will use sometimes the notation $\mathrm{res}(g)(u)$.

If $r(g) = 0$, we define the residue $\mathrm{res}(g)$ as the image of $g^{-1}\sigma_0(g)$ by the homomorphism $j_* : G(k_0^+) \to G(k^{u,+}) = G(k[u, \frac{1}{1+u}])$. Putting $\lambda = 1 + u$, we have $k^{u,+} = k[\lambda, \lambda^{-1}]$ so that we see the residue as an A–map $\mathrm{res}(g) : \mathbb{G}_{m,k} = \mathrm{Spec}(k[\lambda, \lambda^{-1}]) \to G$. Similarly we use sometimes the notation $\mathrm{res}(g)(\lambda)$.

If $r(g) = -1$, i.e. $g \in G(k[[t]])$, we put $\mathrm{res}(g) = 1 \in G(k^u)$. Again this does not depend of the choice of a representation.

Examples 11.1. (1) If $G = \mathbb{G}_{m,k}$ and $g = \frac{1}{t^d}$, we have

$$g^{-1}\,\sigma_0(g) = (1+u)^d = \lambda^d.$$

In this case we have $r(g) = 0$ and $\mathrm{res}(g)(\lambda) = \lambda^d$.
(2) If $G = \mathbb{G}_{a,k}$ and $g = \frac{1}{t^d}$ with $d \in \mathbb{Z}_{\geq 1}$ invertible in k, we have

$$g^{-1}\,\sigma_r(g) = \frac{-1}{t^d} + \frac{1}{t^d(1+u\,t^r)^d} = \frac{-d\,u\,t^r + \dots}{t^d(1+u\,t^r)^d}$$

In this case we have $r(g) = d$ and $\mathrm{res}(g)(u) = -d\,u$.
(3) If $G = \mathbb{G}_{a,k}$ and $g = \frac{1}{t^p}$ with A of characteristic $p > 0$ we have

$$g^{-1}\,\sigma_r(g) = \frac{-1}{t^p} + \frac{1}{t^p(1+ut^r)^p} = -\frac{u^p\,t^{rp}}{t^p(1+ut^r)^p.}$$

In this case we have $r(g) = 1$ and $\mathrm{res}(g)(u) = -u^p$.

Example 11.2. We consider the case $G = \mathrm{GL}_2$, and the element $g = \begin{pmatrix} t^a & P(t) \\ 0 & t^d \end{pmatrix}$ with $P(t) \in k[t]$ and $a, d \in \mathbb{Z}$. Putting $\epsilon = ut^r$, we have

$$g^{-1}\sigma_r(g) = t^{-a-d}\begin{pmatrix} t^d & -P(t) \\ 0 & t^a \end{pmatrix}\begin{pmatrix} t^a(1+\epsilon)^a & P(t(1+\epsilon)) \\ 0 & t^d(1+\epsilon)^d \end{pmatrix} = \begin{pmatrix} (1+\epsilon)^a & f \\ 0 & (1+\epsilon)^d \end{pmatrix}$$

with

$$f = t^{-a-d}\Big(t^d P(t(1+\epsilon)) - t^d(1+\epsilon)^d P(t)\Big) = t^{-a}\Big(P(t(1+\epsilon)) - (1+\epsilon)^d P(t)\Big).$$

(a) We take $a, d \geq 1$, $P(t) = 1$ and assume than d is invertible in k. In this case, $P(t(1+\epsilon)) - (1+\epsilon)^d P(t) = 1 - (1+ut^r)^d$ so that

$$g^{-1}\sigma_r(g) = \begin{pmatrix} (1+ut^r)^a & -d\,t^{-a+r}u + \dots \\ 0 & (1+ut^r)^d \end{pmatrix}$$

It follows that $r(g) = a$ and that $\mathrm{res}(g) = \begin{pmatrix} 1 & -d\,u \\ 0 & 1 \end{pmatrix}$.

(b) Assume that A is an $\mathbb{F}_p$–algebra and take $d = p^s$ $(s \geq 1)$ and $P(t) = t^{p^{ms}}$ with $m \geq 2$. Then

$$f = t^{-a}\Big(t^{p^{ms}}(1+\epsilon^{p^{ms}})-(1+\epsilon^{p^s})t^{p^{ms}}\Big) = t^{-a}\Big(t^{p^{ms}}(ut^r)^{p^{ms}}-t^{p^{ms}}(ut^r)^{p^s}\Big) = -t^{-a+p^{ms}+rp^s}(u)^{p^s}+\dots.$$

If $a > p^{ms}$, we have $-a + p^{ms} + r\,p^s = 0$ so that $r = \frac{a}{p^s} + p^{m(s-1)}$. In particular r can belong to $\mathbb{Z}[\frac{1}{p}] \setminus \mathbb{Z}$.

(c) For the multiplicative indeterminate λ, we compute also

$$g^{-1}g(\lambda t) = t^{-a-d}\begin{pmatrix} t^d & -P(t) \\ 0 & t^a \end{pmatrix}\begin{pmatrix} \lambda^a\, t^a & P(\lambda t) \\ 0 & \lambda^d\, t^d \end{pmatrix} = \begin{pmatrix} \lambda^a & f \\ 0 & \lambda^d \end{pmatrix}$$

with

$$f = t^{-a-d}\Big(t^d P(\lambda t) - \lambda^d t^d P(t)\Big) = t^{-a}\Big(P(\lambda t) - \lambda^d P(t)\Big).$$

If $a \leq -1$, we have $r(g) = 0$ and $\operatorname{res}(g) = \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^d \end{pmatrix}$. Furthermore for $g' = g\operatorname{res}(g)(t^{-1})$, we have

$$g'^{-1}\, g'(\lambda t) = \begin{pmatrix} t^a & 0 \\ 0 & t^d \end{pmatrix}\begin{pmatrix} \lambda^a & f \\ 0 & \lambda^d \end{pmatrix}\begin{pmatrix} \lambda^{-a}t^{-a} & 0 \\ 0 & \lambda^{-d}t^{-d} \end{pmatrix} = \begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix}$$

with $f_1 = t^{a-d}\lambda^{-d} f = \lambda^{-d}t^{-d}\Big(P(\lambda t) - \lambda^d P(t)\Big)$. For $a = -1$, $d = 1$ and $P(t) = 1$, we see that $g'^{-1}g'(\lambda t)$ does not belong to $\mathrm{GL}_2\big(k[\lambda, \lambda^{-1}][[t]]\big)$.

Similarly for $g'' = \operatorname{res}(g)(t^{-1})g$, we have

$$g''^{-1}\, g''(\lambda t) = t^{-a-d}\begin{pmatrix} t^d & -P(t) \\ 0 & t^a \end{pmatrix}\begin{pmatrix} \lambda^{-a} & 0 \\ 0 & \lambda^{-d} \end{pmatrix}\begin{pmatrix} \lambda^a\, t^a & P(\lambda t) \\ 0 & \lambda^d\, t^d \end{pmatrix} = \begin{pmatrix} 1 & f_2 \\ 0 & 1 \end{pmatrix}$$

with $f_2 = t^{-a}\Big[\lambda^{-a}P(\lambda t) - P(t)\Big]$. So for $a \leq -1$, we see that $g''^{-1}\, g''(\lambda t)$ belongs to $\mathrm{GL}_2\big(k[[t]]\big)$.

For a group Γ, we recall the notation ${}^\sigma\tau = \sigma\tau\sigma^{-1}$ and $\tau^\sigma = \sigma^{-1}\tau\sigma$ for $\sigma, \tau \in \Gamma$.

Lemma 11.3. *Let $g \in G(k(\!(t)\!))$.*

(1) Let $g_1 \in G(k)$ and $g_2 \in G(k[[t]])$. Then $\operatorname{res}(g_1 g g_2) = \operatorname{res}(g)^{\overline{g_2}}$ where $\overline{g_2}$ stands for the specialization of g_2 in $G(k)$. In particular we have $\operatorname{res}({}^{g_1}g) = {}^{\overline{g}_1}\operatorname{res}(g)$.

(2) Let $f : G \to H$ be a proper homomorphism between affine algebraic k–groups. We have $\operatorname{res}(f(g)) = \operatorname{res}(g)$.

(3) Let d be a non–negative integer and consider the map $\phi_d : k(\!(t)\!) \to k(\!(T)\!)$ defined by $\phi_d(t) = T^d$. We consider the map $\phi_{d,*} : G\big(k(\!(t)\!)\big) \to G\big(k(\!(T)\!)\big)$.

(i) If d is not a zero divisor in k, we have $\operatorname{res}\big(\phi_{d,*}(g)\big)(u) = \operatorname{res}(g)(du)$ if $r > 0$, or $\operatorname{res}\big(\phi_{d,*}(g)\big)(\lambda) = \operatorname{res}(g)(\lambda^d)$ if $r = 0$.

(ii) If k is of characteristic $p > 0$ and $d = p^e$, we have $\operatorname{res}\big(\phi_d(g)\big) = \operatorname{res}(g)(u^{p^e})$.

Proof. We write $r = r(g) = m/n$.
(1) Since $\sigma_r(g_1) = g_1$, we have $(g_1 g g_2)^{-1}\,\sigma_r(g_1 g g_2) = g_2^{-1}(g^{-1}\sigma_r(g))\,\sigma_r(g_2) = g_2^{-1}(g^{-1}\sigma_r(g))\sigma_r(g_2)$. When we specialize at $t=0$, we get $\mathrm{res}(g_1 g g_2) = \mathrm{res}(g)^{\overline{g_2}}$. Assertions (2) and (3) follow of Lemma 10.5.
(4) We continue the proof of Lemma 10.5.(5). We have four cases to verify.
Case (i), $r>0$. We have $r\big(\phi_{d,*}(g)\big) = d\,r(g) = \frac{dm}{n}$. The change $T \mapsto T(1+uT^r)$ induces $t = T^d \mapsto T^d(1+uT^r)^d = \tau(t) = t(1 + d\,u\,t^{\frac{m}{nd}} + \dots)$. It follows that

$$\phi_{d,*}(g^{-1}\,\tau(g)) = \phi_{d,*}(g)^{-1}\,\sigma_{T,\frac{m}{nd}}\big(\phi_{d,*}(g)\big) \in G(k^u[[T]])$$

Proposition 10.1.(4) yields that $j(g^{-1}\,\tau(g)) = j(g^{-1}\,\sigma_{t,r}(g)) = d\ \mathrm{res}(g) \in G(k^u)$.
Case (i), $r=0$. The change $T \mapsto \lambda T$ induces $t = T^d \mapsto \lambda^d t$. It follows that

$$\phi_{d,*}\big(g^{-1}\,g(\lambda t)\big) = \phi_{d,*}(g)^{-1}\,g(\lambda^d T)\,\mathrm{res}(g)(\lambda^d)\,(1+\epsilon) \in G(k[\lambda^{\pm 1}[[T]])$$

with $j(1+\epsilon) = 1$. We conclude that $\mathrm{res}\big(\phi_{d,*}(g)\big)(\lambda) = \mathrm{res}(g)(\lambda^d)$.
Case (ii), $r>0$. We have $r\big(\phi_{d,*}(g)\big) = r(g)$ and consider the base change $t = T^{p^e} \mapsto T^{p^e}(1+uT^r)^{p^e} = \tau'(t) = t(1 + u^{p^e} t^{m/n} + \dots)$. It follows that

$$\phi_{d,*}(g^{-1}\,\tau'(g)) = \phi_{d,*}(g)^{-1}\,\sigma_{T,r}\big(\phi_{d,*}(g)\big) \in G(k^u[[T]]). \tag{11.1}$$

We put $v = u^{p^e}$ and consider $\sigma_r^v : A^v[[t]] \to A^v[[t^{1/n}]]$. By using Proposition 10.1.(4) and the functoriality of the construction $k^v \to k^u$, we have $j(g^{-1}\,\tau'(g)) = j(g^{-1}\,\sigma_r^v(g)) = \mathrm{res}(g)(v) = \mathrm{res}(g)(u^{p^e}) \in G(k^u)$. By specializing formula (11.1) at $T=0$, we get $\mathrm{res}(\phi_{d,*}(g)\big)(u) = \mathrm{res}(g)(u^{p^e})$.
Case (ii), $r=0$. It is similar. □

Theorem 11.4. (1) *If* $r(g)>0$*, then* $\mathrm{res}(g)$ *is non-trivial homomorphism* $\mathbb{G}_{a,k} \to G$.
(2) *If* $r(g)=0$*, then* $\mathrm{res}(g)$ *is a non-trivial homomorphism* $\mathbb{G}_{m,k} \to G$.

Proof. We can continue to work with SL_N. We write $r = r(g) = m/n$.
(1) We assume firstly that n is invertible in k. By developing the serie $(1+ut^r)^{1/n}$ in k_r, we can extend $\sigma_r : k^u[[t]] \to k_r$ to $\widetilde{\sigma}_r : k_r \to k_r$. The trick is to use the rings $k^{v_1,v_2} = k[v_1,v_2]$, $k^{v_1,v_2}[[t]]$ and $k^{v_1,v_2}[[t^{1/n}]]$ and to define morphisms $\tau_i : k^{v_1,v_2}[[t]] \to k^{v_1,v_2}[[t^{1/n}]]$ $(i=1,2,3)$ by $t \mapsto t(1+v_1t^r)$, $t(1+v_2t^r)$, $t\big(1+(v_1+v_2)t^r\big)$ respectively. These morphisms extend to morphisms $\widetilde{\tau}_i : k^{v_1,v_2}[[t^{1/n}]] \to k^{v_1,v_2}[[t^{1/n}]]$ for $i=1,2,3$. We have the cocycle relation

$$g^{-1}\,(\widetilde{\tau}_1\widetilde{\tau}_2)(g) = g^{-1}\,\widetilde{\tau}_1(g)\,\widetilde{\tau}_1\big(g^{-1}\,\widetilde{\tau}_2(g)\big) \tag{11.2}$$

inside $G\big(k^{v_1,v_2}(\!(t^{1/n})\!)\big)$. By using functoriality properties (Lemmas 10.5.(1) and 11.3.(2)) we have $g^{-1}\,\widetilde{\tau}_i(g) \in G\big(k^{v_1,v_2}[[t^{1/n}]]\big)$ with specialization $\mathrm{res}(g)(v_i)$. It follows that

$$g^{-1}\,(\widetilde{\tau}_1\widetilde{\tau}_2)(g) = \mathrm{res}(g)(v_1)\ \mathrm{res}(g)(v_2) \tag{11.3}$$

inside $G\big(k^{v_1,v_2}[[t^{1/n}]]\big)$ modulo the kernel of $G\big(k^{v_1,v_2}[[t^{1/n}]]\big) \to G\big(k^{v_1,v_2}\big)$. On the other hand $\widetilde{\tau}_1\widetilde{\tau}_2(t) = t\big(1 + (v_1+v_2)t^r + \text{upper terms}\big)$. Proposition 10.1.(3) applied to the ring $k[v_1]$ and $u = v_1 + v_2$ shows that $g^{-1}\,(\widetilde{\tau}_1\widetilde{\tau}_2)(g) = \mathrm{res}(g)(v_1+v_2)$ inside $G\big(k^{v_1,v_2}[[t^{1/n}]]\big)$ modulo the kernel of $G\big(k^{v_1,v_2}[[t^{1/n}]]\big) \to G\big(k^{v_1,v_2}\big)$. We conclude that $\mathrm{res}(g)(v_1+v_2) = \mathrm{res}(g)(v_1) \times \mathrm{res}(g)(v_2)$. Thus $\mathrm{res}(g)$ is an A–group homomorphism.

We explain now the refinement to the case $n = q\,n'$ when A is of characteristic $p > 0$ and $(n',p) = 1$ and $q = p^e$. We consider $k^u[[t]] \xrightarrow{\sigma_r} k^u[[t^{1/n}]] \to k[u^{1/q}][[t^{1/nq}]]$ to $\widetilde{\sigma}_r : k^u[[t^{1/n}]] \to k[u^{1/q}][[t^{1/nq}]]$ by mapping $t^{1/n}$ to the series $(1+u^{1/q}t^{r/q})^{1/n'}$.

We extend then similarly the morphisms $k^{v_1^q,v_2^q}[[t]] \xrightarrow{\tau_i} k^{v_1^q,v_2^q}[[t^{1/n}]] \to k^{v_1,v_2}[[t^{1/nq}]]$ $(i=1,2,3)$ defined by $t \mapsto t(1+v_1^q t^r)$, $t(1+v_2^q t^r)$, $t\big(1+(v_1^q+v_2^q)t^r\big)$ in $\widetilde{\tau}_i : k^{v_1^q,v_2^q}[[t^{1/n}]] \to k^{v_1,v_2}[[t^{1/nq}]]$ for $i = 1,2,3$. The cocycle condition reads then

$$g^{-1}\,(\widetilde{\tau}_1\tau_2)(g) = g^{-1}\,\tau_1(g)\,\widetilde{\tau}_1\big(g^{-1}\,\tau_2(g)\big).$$

The same method yields $\mathrm{res}(g)(v_1^q+v_2^q) = \mathrm{res}(g)(v_1^q) \times \mathrm{res}(g)(v_2^q)$. Thus $\mathrm{res}(g)$ is an k–group morphism.

(2) We have seen that $\mathrm{res}(g) \in G(k[\lambda^{\pm 1}]) \setminus G(k)$ in Lemma 11.3.(4). We consider the ring $k[\lambda_1^{\pm 1},\lambda_2^{\pm 1}]$ and the $k[\lambda_1,\lambda_2]$-automorphisms ρ_i of $k[\lambda_1,\lambda_2]((t))[\lambda_1^{-1},\lambda_2^{-1}]$ defined respectively by $\rho_1(t) = \lambda_1 t$, $\rho_2(t) = \lambda_2 t$, and $\rho_3(t) = \lambda_1\lambda_2 t$. Since $\rho_3 = \rho_2 \circ \rho_1$, the cocycle relation $g^{-1}\,(\rho_1\rho_2)(g) = g^{-1}\,\rho_1(g)\,\rho_1\big(g^{-1}\,\rho_2(g)\big)$ in $G\big(k[\lambda_1,\lambda_2]((t))[\lambda_1^{-1},\lambda_2^{-1}]\big)$ yields $\mathrm{res}(g)(\lambda_1\lambda_2) = \mathrm{res}(g)(\lambda_1)\ \mathrm{res}(g)(\lambda_2)$ as desired. □

This provides some control on the indices.

Corollary 11.5. *Let $d \geq 1$ be an integer such that $t^d g \in M_N(k[[t]])$.*

(1) *If k is of characteristic zero, then $r(g) \in \mathbb{Z}$ and $r(g) \leq Nd$.*
(2) *If k is of characteristic $p >$, then $r(g) \in \mathbb{Z}[\frac{1}{p}]$ and there exists $s \geq 0$ such that $p^s r(g) \in \mathbb{Z}$ and $p^s r(g) \leq Nd$.*

Proof. (1) and (2). If $r(g) = 0$ the statements are clear so that we can assume that $r(g) > 0$. We use now the decomposition $\mathrm{SL}_N\big(k((t))\big) = B_N\big(k((t))\big)\ \mathrm{SL}_N\big(k[[t]]\big)$ where B_N stands for the k–subgroup of upper triangular matrices [10, 4.4.3]. Lemma 11.3.(1) permits to assume that $g \in B_N\big(k((t))\big)$. Coming back in the proof of Proposition 10.1, we consider the coefficients of $g^{-1}\sigma_r(g)$

$$D_{i,j,r} = (1+ut^r)^{-d}\Big(\delta_{i,j} + t^{-Nd} \sum_{a\geq 0,\, b\geq 1} c_{i,j}^{a,b}\, t^{a+rb}\, u^b\Big). \tag{11.4}$$

We have $D_{i,j,r} = 0$ if $j < i$. We consider the non-empty set

$$\Upsilon(g) = \Big\{(i,j,a,b) \ \mid\ -Nd + a + rb = 0 \ \text{ and } \ c_{i,j}^{a,b} \neq 0\Big\}.$$

It follows that the (i,j)-entry of $\mathrm{res}(g) \in B_N(k[u])$ is

$$c_{i,j} = \delta_{i,j} + \sum_{(i,j,a,b)\in\Upsilon(g)} c_{i,j}^{a,b}\, u^b \tag{11.5}$$

It follows that $c_{i,i} = 1$ for each $i = 1,..,N$, that is $\mathrm{res}(g) \in U_N(k[u])$ where U_N stands for the unipotent radical of B_N. Let (i,j,a,b) in $\Upsilon(g)$ such that $i+j$ is minimal. Since $\mathrm{res}(g)$ is a group homomorphism, it follows that $u \mapsto c_{i,j}^{a,b} u^b$ is a group homomorphism.

Case of characteristic zero. In this case we have $b = 1$. The equation $-Nd+a+rb = 0$ yields that $r \in \mathbb{Z}$ and that $r = Nd - a \leq Nd$.

Case of characteristic $p > 0$. It follows that b is a p–power, i.e. $b = p^s$. Thus $r \in \mathbb{Z}[\frac{1}{p}]$ and $p^s r = Nd - a \leq Nd$. □

12. Applications to torsors.

Let G be an affine algebraic group as before. Let X be a G-torsor, that is, an affine k–variety equipped with a right G–action satisfying the two following conditions:

(i) the action map $X \times_k G \to X \times_k X$, $(x, g) \mapsto (x, x.g)$ is an isomorphism;

(ii) $X(\overline{k}) \neq \emptyset$.

The first condition expresses the simple transitivity of the action. The k-variety G equipped with the right action is called the trivial torsor; each point $x \in X(k)$ gives rise to an isomorphism $G \xrightarrow{\sim} X$, $g \mapsto x.g$, of G–torsors. We denote by $H^1(k, G)$ the set of isomorphism classes of G–torsors; it is pointed by the class of the trivial G–torsor.

Example 12.1. Let $T = R^1_{K/k}(\mathbb{G}_m)$ be the normic torus for a Galois extension K/k. For each $c \in k^\times$, T acts on the right on the k–variety

$$X_c = \Big\{ N_{K/k}(y) = c \Big\} \subset R_{K/k.}(\mathbb{G}_m).$$

This is a T–torsor which is trivial if and only if c is a norm for K/k. In this case, it is known that $H^1(k, T) \cong k^\times / N_{K/k}(K^\times)$.

Proposition 12.2. *Assume that G is k–wound. Let X be a G-torsor. Then $X(k[[t]]) = X(k((t)))$. In particular, if $X(k((t))) \neq \emptyset$, we have $X(k) \neq \emptyset$*

Proof. We reason by sake of contradiction by picking $x \in X(k((t))) \setminus X(k[[t]])$. For $r = m/n \in \mathbb{Q}_{>0}$, we consider the point $\sigma_r(x) \in X(\mathcal{K}_r)$. Then $\sigma_r(x) = x.g_r$ for a unique $g_r \in G(\mathcal{K}_r)$. After extending the scalars to a finite field extension, X is isomorphic to G so that the procedure of ramification index and residue applies (remember that it is insensible to change of fields). We can then define $r(x) \in \mathbb{Q}_{\geq 0}$ and a non trivial homomorphism $\mathrm{res}(x) : H \to G$ where $H = \mathbb{G}_a$ or $\mathbb{G}_m$. This is a contradiction. □

Remarks 12.3. (a) If G is reductive anisotropic, this result is due to Bruhat-Tits by means of euclidean buildings, see [28, Theorem 5] for proof.

(b) In the special case of Example 12.1, the index and residue tecnique provide a complicated way to show that a scalar $c \in k^\times$ is a norm for L/k if it is a norm for $L((t))/k((t))$!

This step is crucial for obtaining the general case.

Theorem 12.4. [23, th. 5.4] *Let X be a G-torsor. If $X(k((t))) \neq \emptyset$, then we have $X(k) \neq \emptyset$.*

In other words, the map of pointed sets $H^1(k, G) \to H^1(k((t)), G)$ has trivial kernel. In the reductive case, this is also due from Bruhat-Tits using a classical "dévissage" from the anisotropic case, see [28, prop. 11]. In the general case, the proof involves the theory of pseudo-reductive groups [17].

A consequence of the result is when admits a smooth compactification X^c; in this case Theorem 12.4 implies that $X(k)$ is not empty if and only if $X^c(k)$ is not empty [22, lemme 5.5].

More generally, Theorem 12.4 is known for homogeneous spaces over a perfect field k (M. Florence, [22]). This is an open question in the imperfect case.

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