Reconstruction de formes en grandes dimensions

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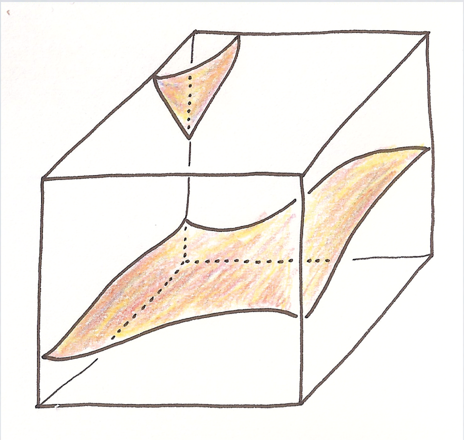
Conférence Mathématiques et Grandes Dimensions
de la théorie aux développements industriels

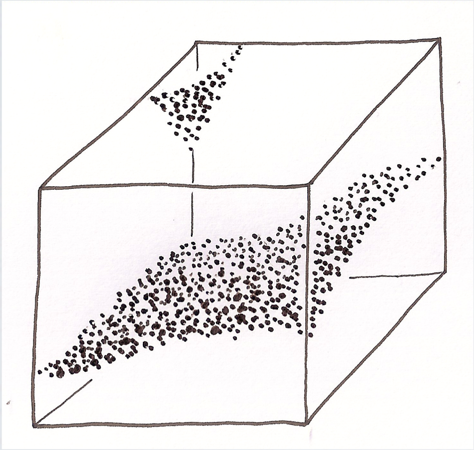
10 décembre 2012
Lyon

Shape



Approximation

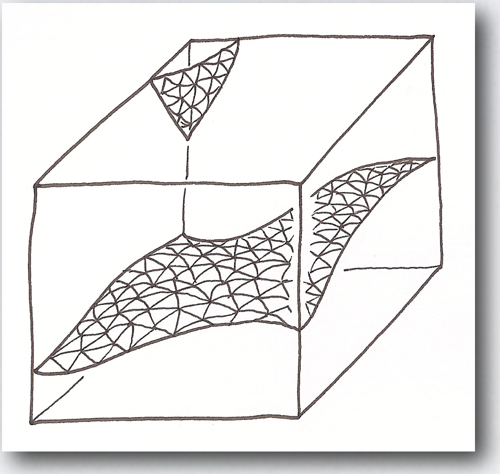
n points



Input

Reconstruction →

Simplicial complex



Output

Processing →

Betti numbers

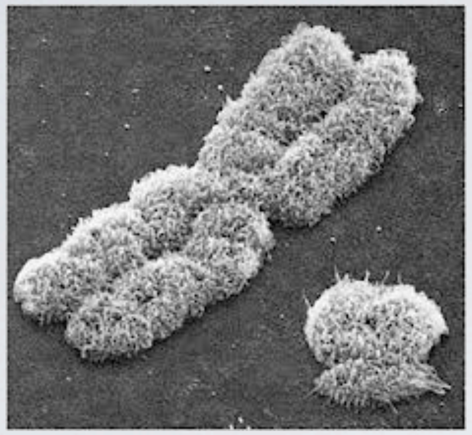
Volume

Medial axis

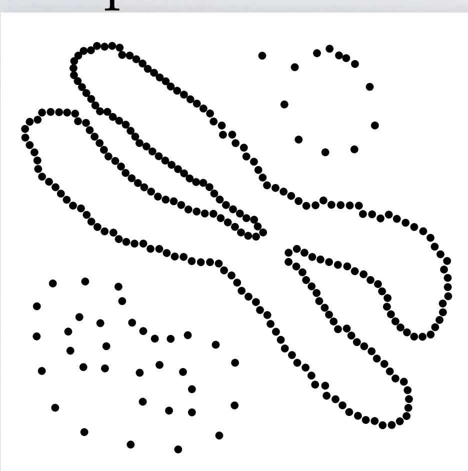
Signatures

. . .

in 2D



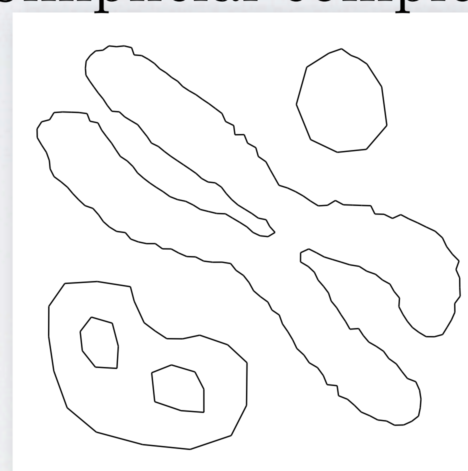
n points in $\mathbb{R}^2$



Input

RECONSTRUCTION →

Simplicial complex

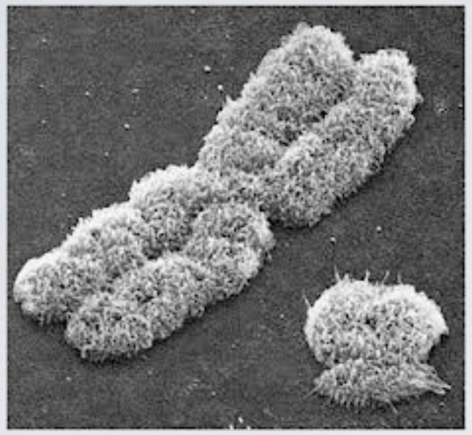


Output

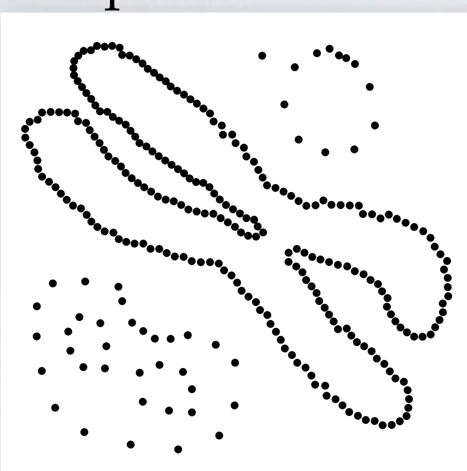
PROCESSING →

Medial axis

in 2D

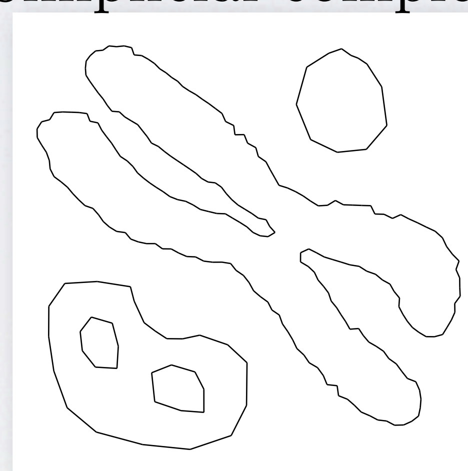


n points in $\mathbb{R}^2$



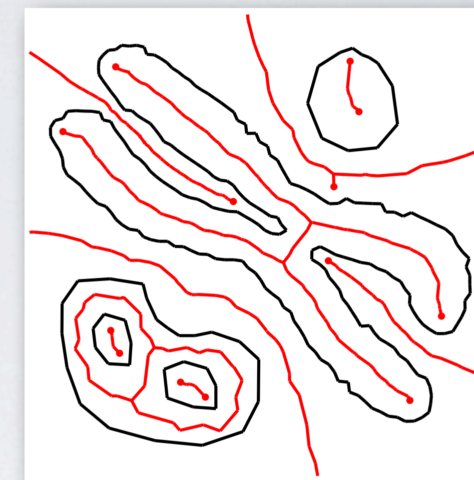
Reconstruction →

Simplicial complex



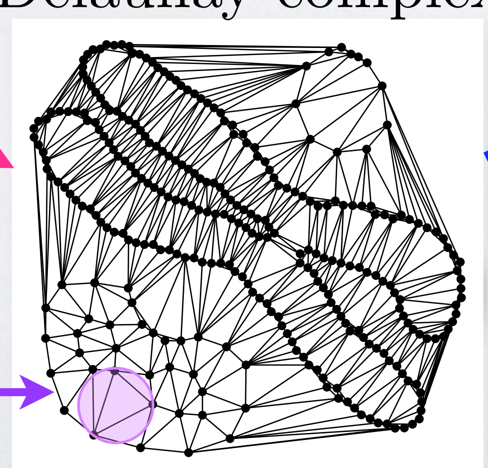
Processing →

Medial axis



Building

Delaunay complex



Empty circle property →

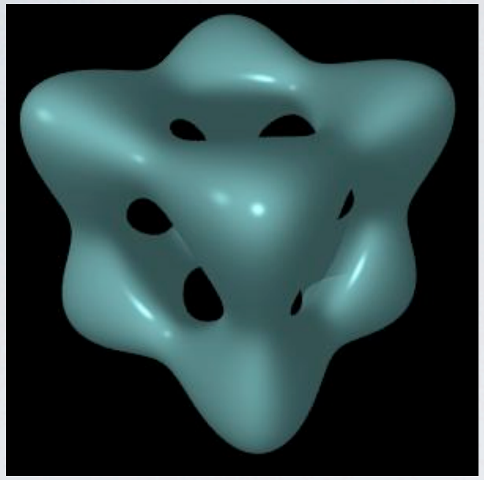
(1995 – 2005) Heuristics
(Crust, Power crust, Co-cone, Wrap, ...)

✶ In $\mathbb{R}^2$, has size $O(n)$

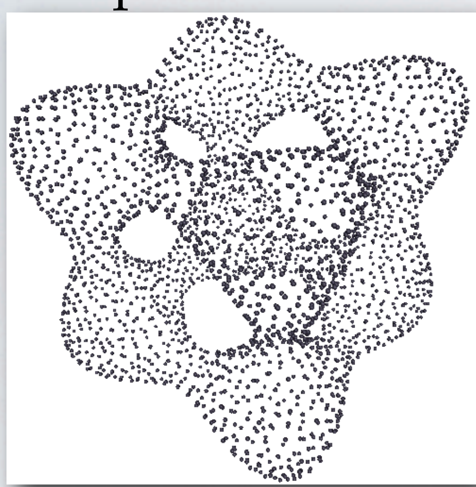
CGAL

Delaunay of 10M points in 2D ≈ 10 s

in 3D

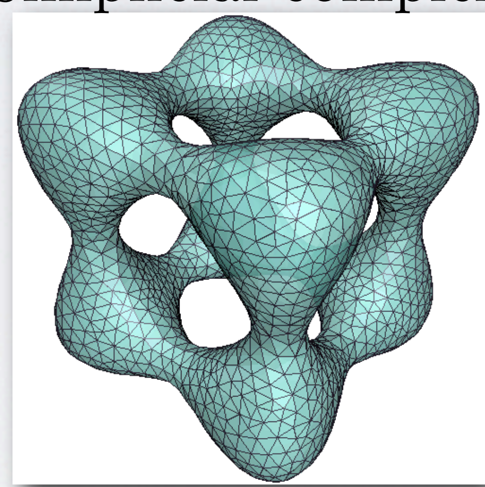


n points in $\mathbb{R}^3$



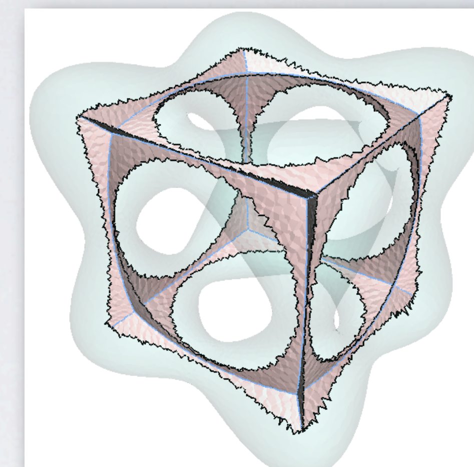
RECONSTRUCTION

Simplicial complex



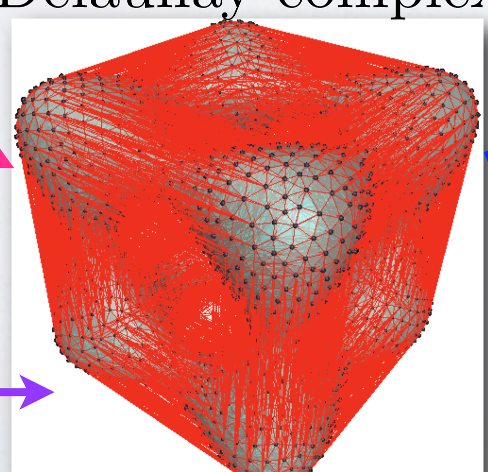
PROCESSING

Medial axis



BUILDING

Delaunay complex



Empty sphere property

(1995 – 2005)

(Crust, Power crust, Co-cone, Wrap, . . .)

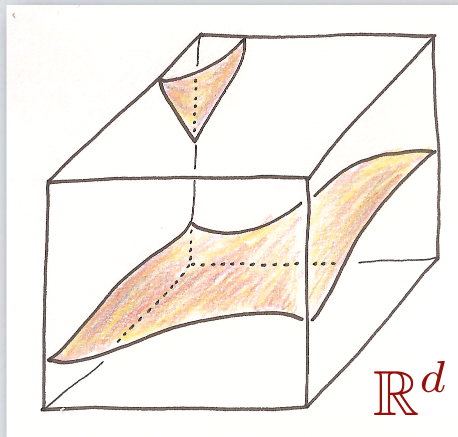
- ✳ In $\mathbb{R}^3$, has size $O(n^2)$
- ✳ In practice, has size $O(n)$

CGAL

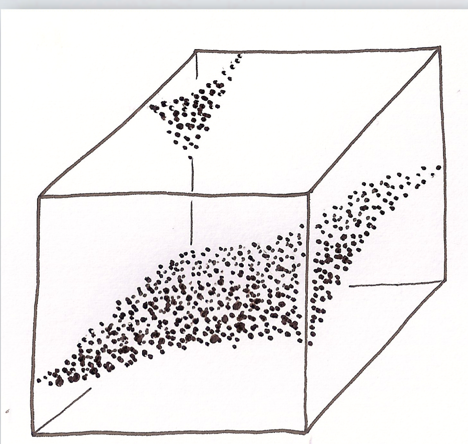
Delaunay of 10M points in 3D ≈ 80 s

in dD

Shape

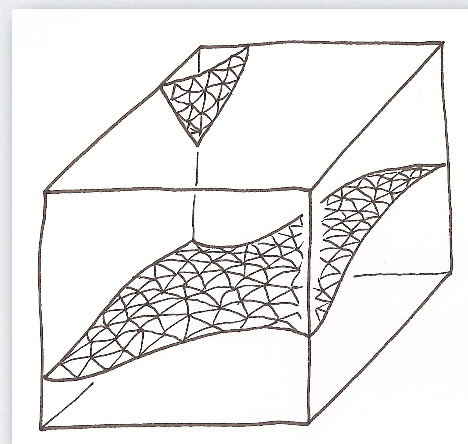


n points in $\mathbb{R}^d$



RECONSTRUCTION →

Simplicial complex



PROCESSING →

Betti numbers

Volume

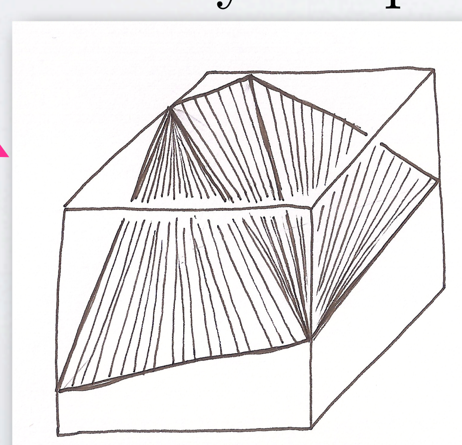
Medial axis

Signatures

. . .

BUILDING

Delaunay complex

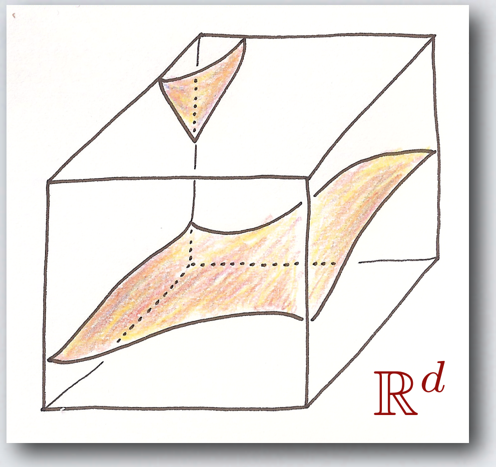


curse of dimensionality

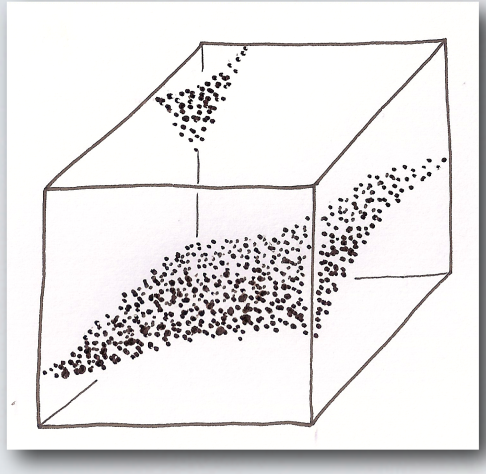
- ✶ In $\mathbb{R}^d$, has size $O(n^{\lceil d/2 \rceil})$
- ✶ The bound is tight (and achieved for points that sample curves).

in dD

Shape

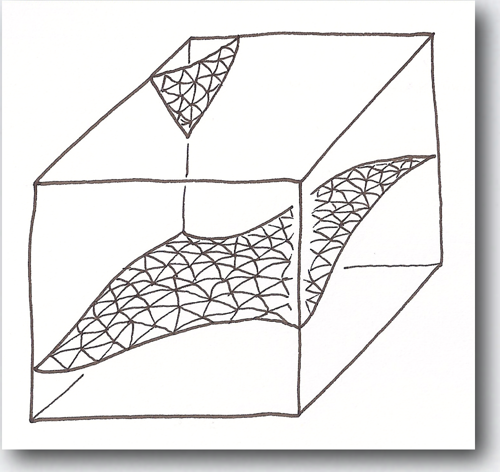


n points in $\mathbb{R}^d$



RECONSTRUCTION →

Simplicial complex



PROCESSING →

Betti numbers

Volume

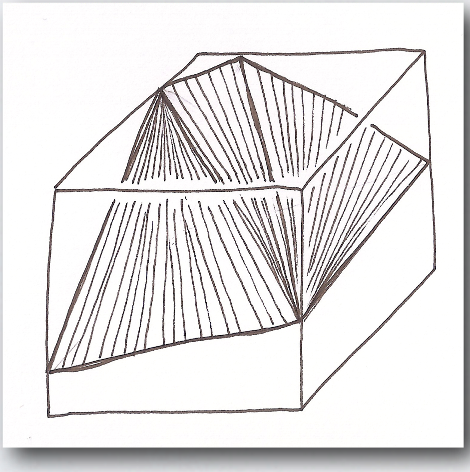
Medial axis

Signatures

...

~~BUILDING~~

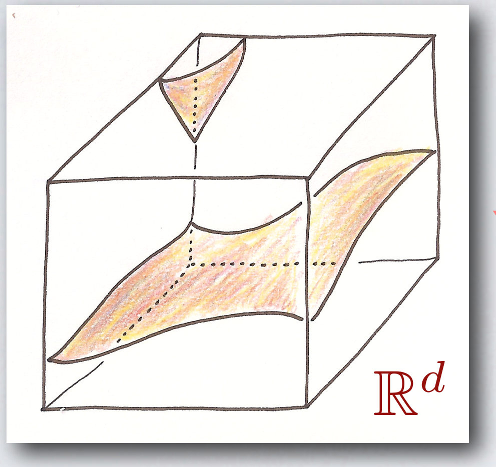
Delaunay complex



How to reconstruct without Delaunay?

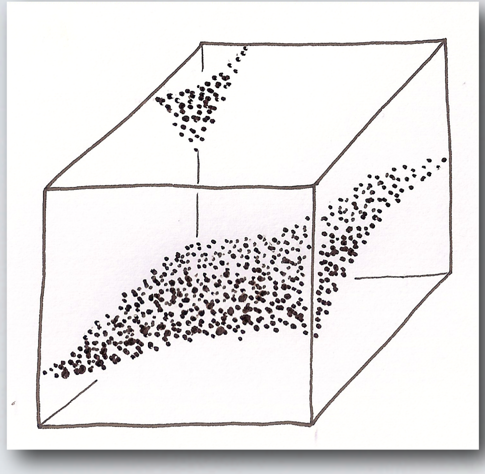
in dD

Shape



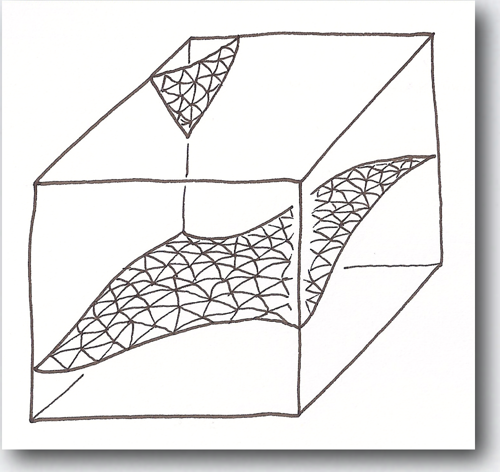
Guaranties on the result?

n points in $\mathbb{R}^d$



RECONSTRUCTION

Simplicial complex



PROCESSING

Betti numbers

Volume

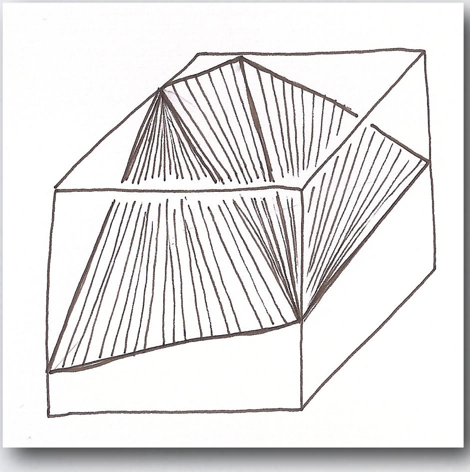
Medial axis

Signatures

. . .

~~BUILDING~~

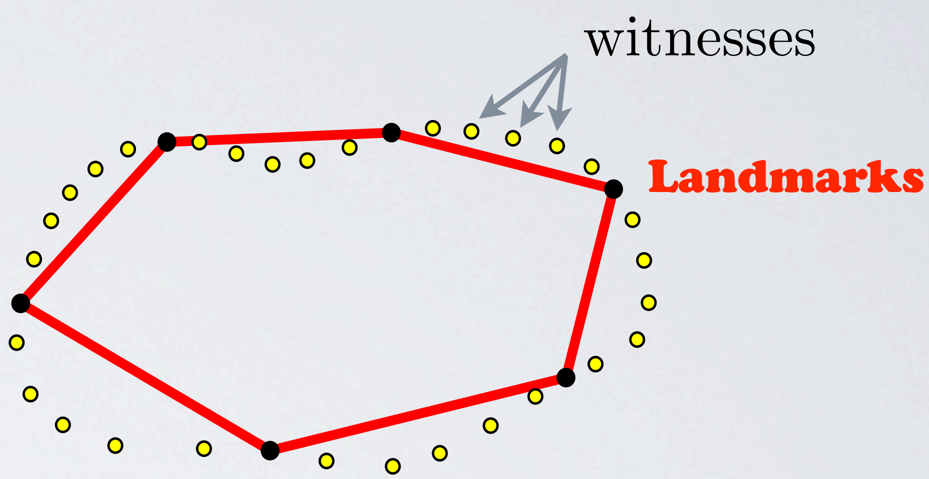
Delaunay complex



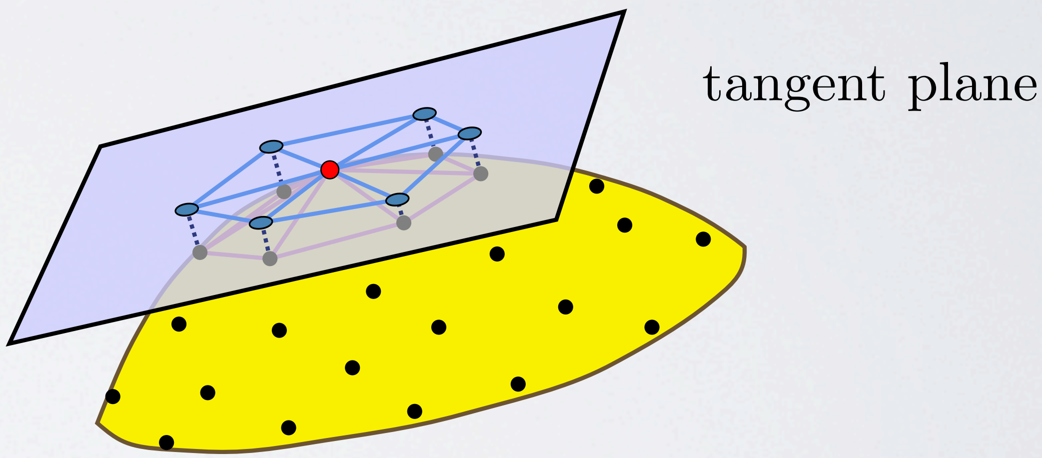
How to reconstruct without Delaunay?

How to reconstruct without building the whole Delaunay complex?

- weak Delaunay triangulation
  [V. de Silva 2008]

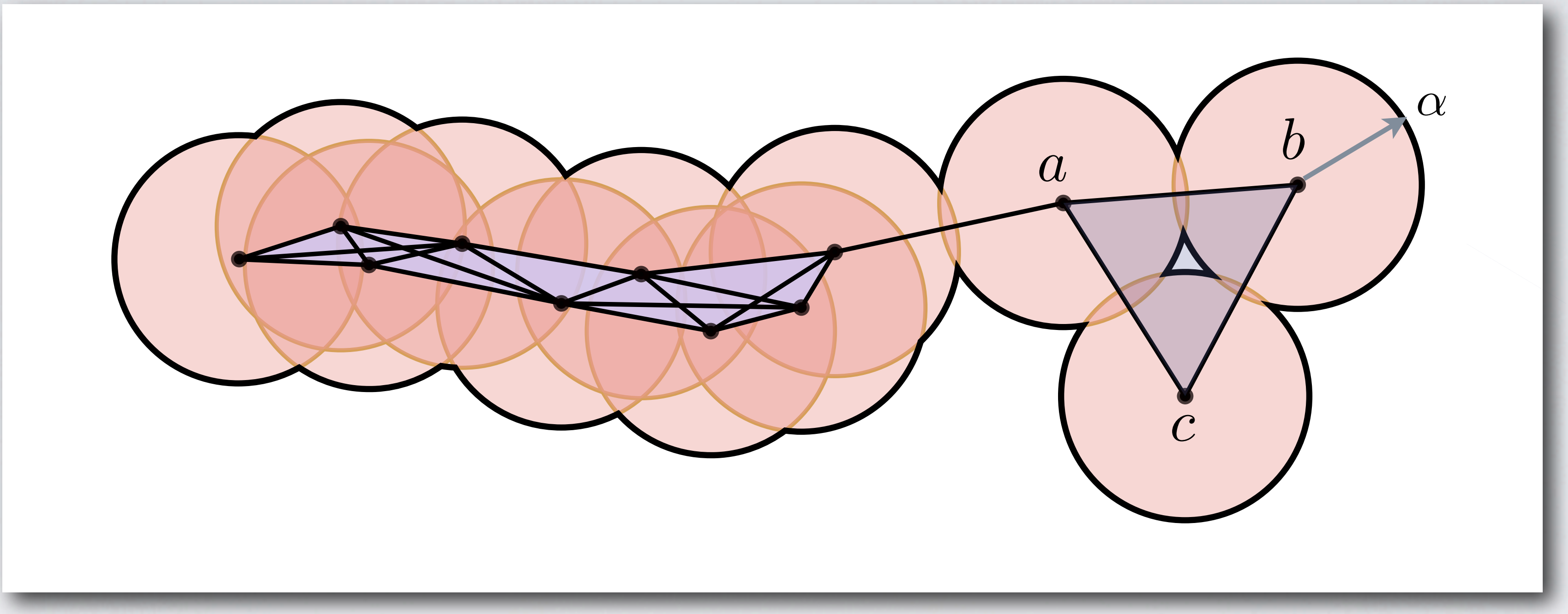


- tangential Delaunay complexes
  [J. D. Boissonnat & A. Ghosh 2010]



- Rips complexes
  our approach with André Lieutier and David Salinas

Rips complexes

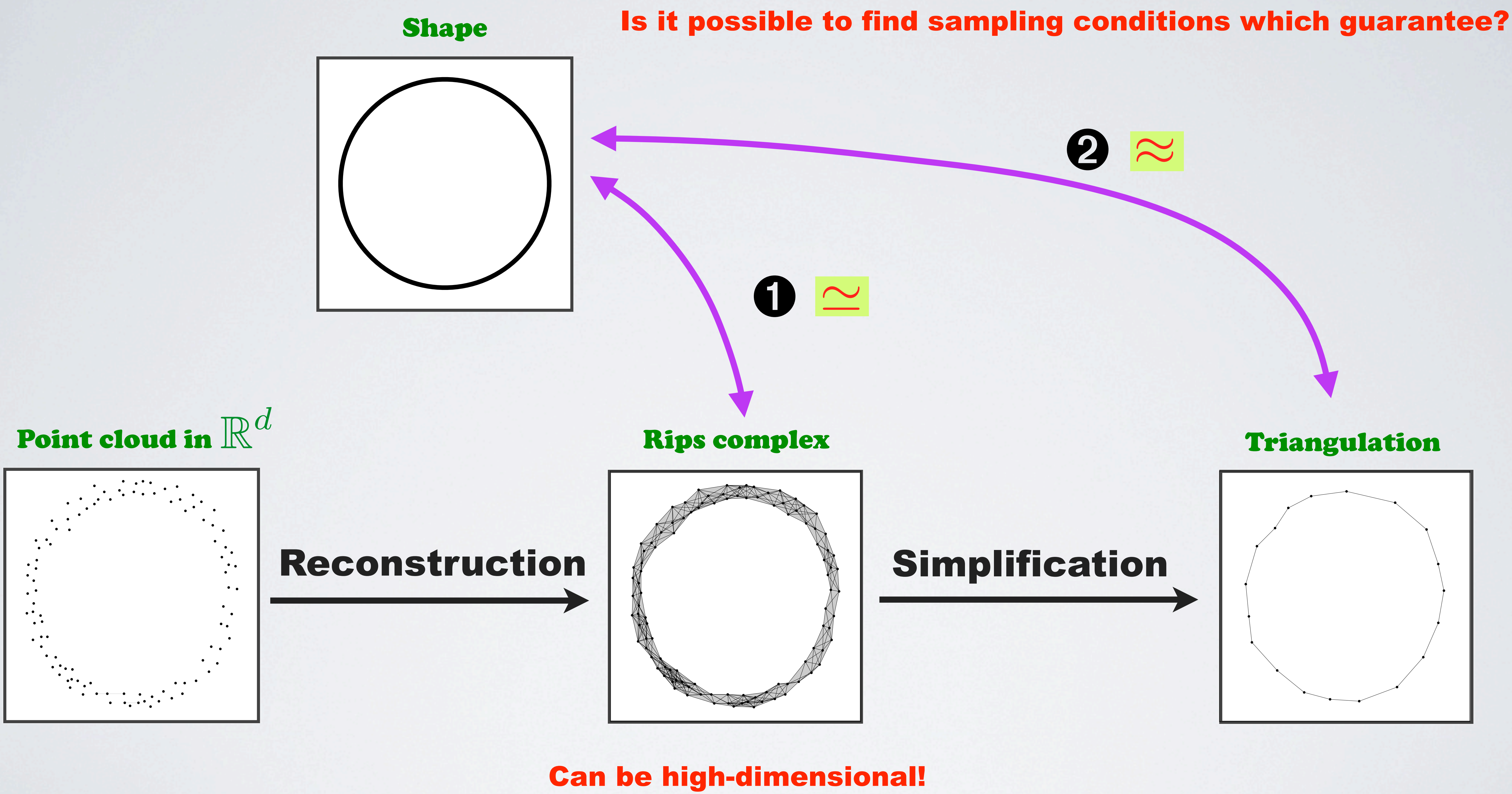


$$\mathrm{Rips}(P,\alpha) = \{\sigma \subset P \mid \mathrm{Diameter}(\sigma) \leq 2\alpha\}$$

$$\mathrm{Rips}(P,\alpha) \supset \mathrm{Cech}(P,\alpha)$$

- proximity graph G_α connects every pair of points within 2α
- $\mathrm{Rips}(P,\alpha) = \mathrm{Flag}\, G_\alpha$ [Flag G = largest complex whose 1-skeleton is G]
- compressed form of storage through the 1-skeleton
- easy to compute

Overview


Shape
Is it possible to find sampling conditions which guarantee?
❷ ≈
❶ ≃
Point cloud in $\mathbb{R}^d$
Reconstruction
Rips complex
Simplification
Triangulation
Can be high-dimensional!

Simplification by iteratively applying elementary operations

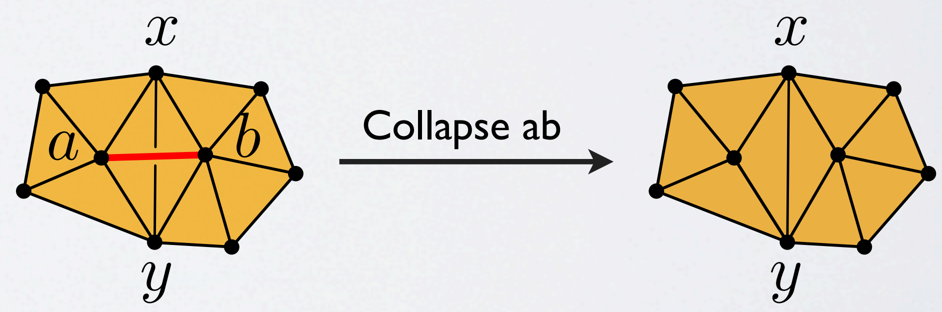
- Edge contraction $ab \mapsto c$

  - Identifies vertices a and b to vertex c
  - Preserves homotopy type if $\mathrm{Lk}(ab) = \mathrm{Lk}(a) \cap \mathrm{Lk}(b)$
  - The result may not be a flag complex anymore . . .

  $\Longrightarrow$ data structure = (1-skeleton, blocker set)

  σ blocker of K iff $\dim \sigma \geq 2$, $\forall \tau \subsetneq \sigma, \tau \in K$ and $\sigma \notin K$

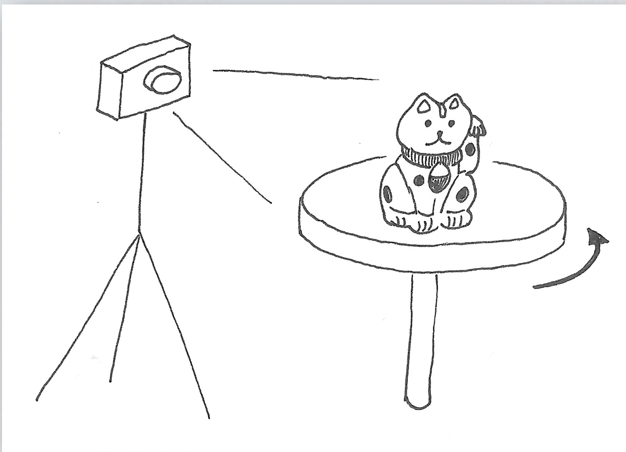
- Collapse of a simplex σ



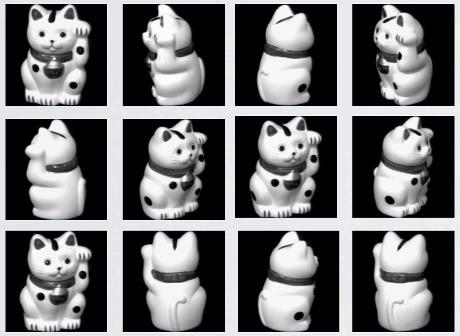
  - Removes σ and its cofaces
  - Preserves homotopy type if $\mathrm{Lk}(\sigma)$ is a cone
  - The result is a flag complex if σ a vertex or an edge

Example

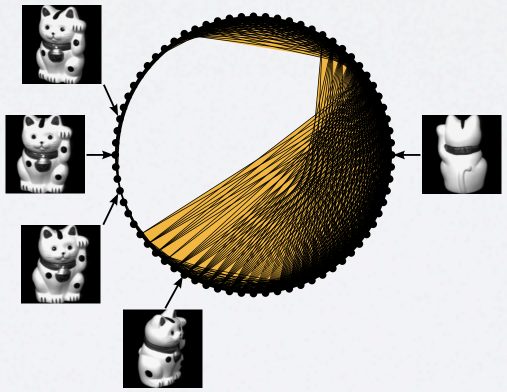
Physical system



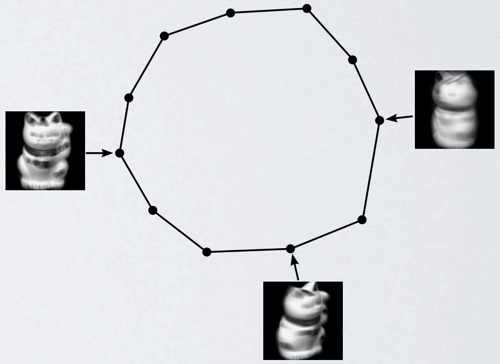
Point cloud in $\mathbb{R}^{128^2}$



Rips complex



Polygonal curve



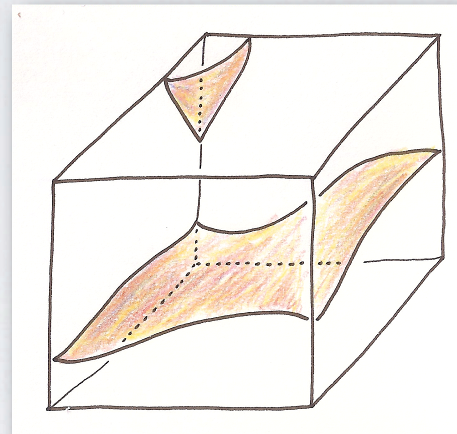
Correct homotopy type

Correct intrinsic dimension

Is high-dimensional!

Reconstruction theorems

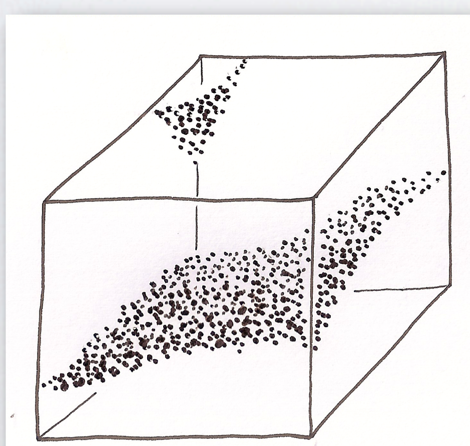
A



Sampling conditions: $\Longrightarrow$ $\simeq$

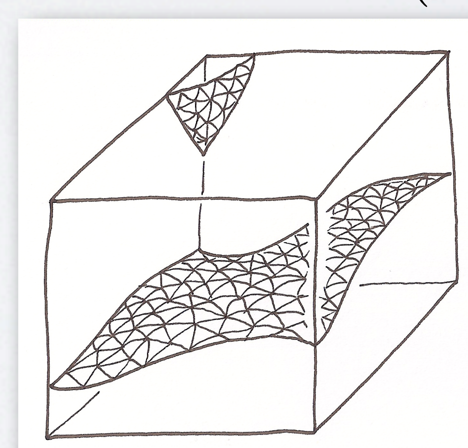
$$d_H(A, P) < \lambda\, \text{feature\_size}(A)$$

P



Input

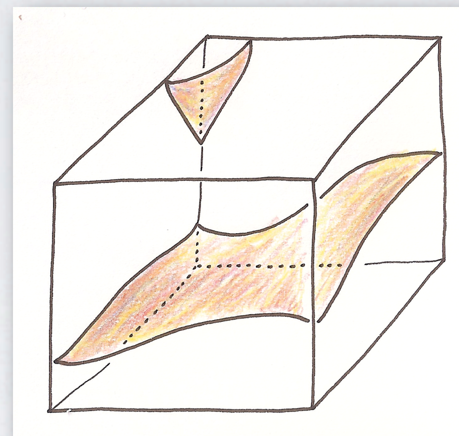
$\text{Reconstruction}(P, \alpha)$



Output

Reconstruction theorems

A

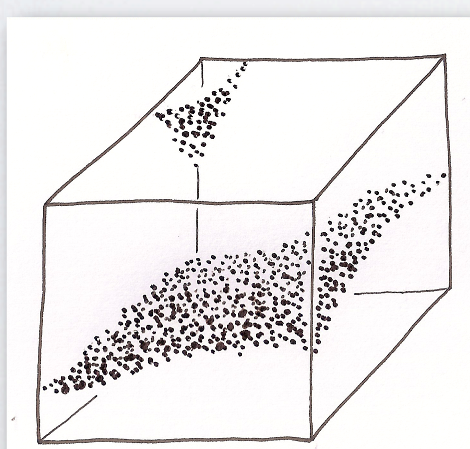


Sampling conditions:

$d_H(A, P) < \lambda\,\text{feature\_size}(A)$

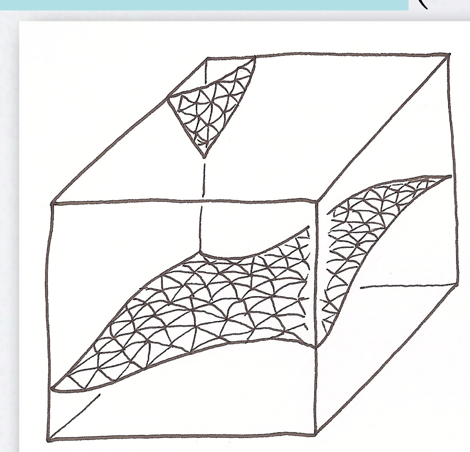
$\Longrightarrow$ $\simeq$

P



Input

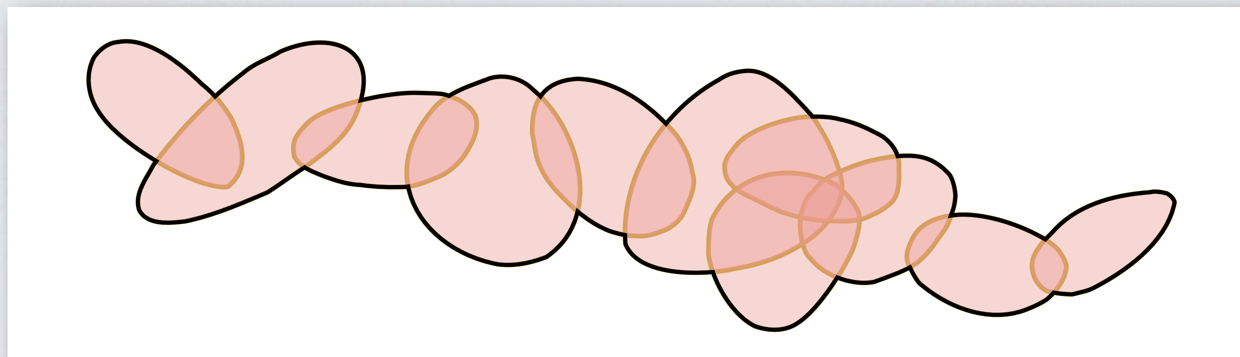
$\text{Reconstruction}(P, \alpha)$



Output

Nerve

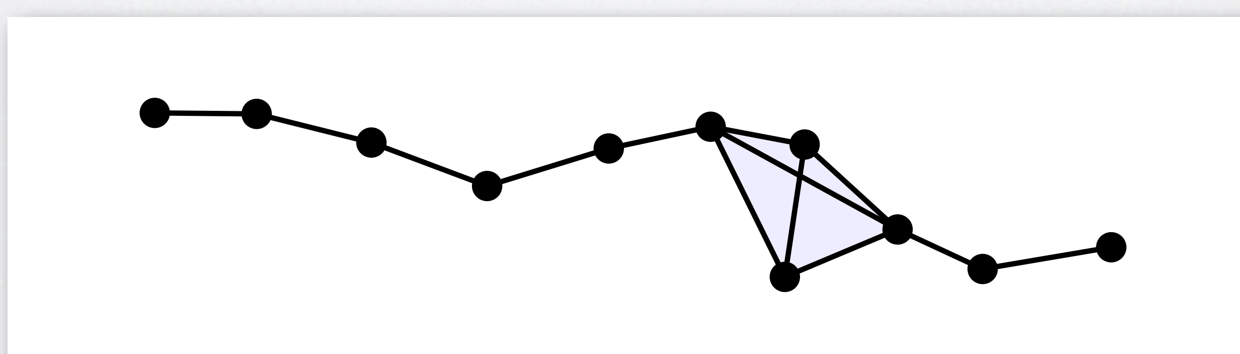
$\bigcup \mathcal{C}$, where $\mathcal{C}$ finite collection of sets



If sets in $\mathcal{C}$ are convex

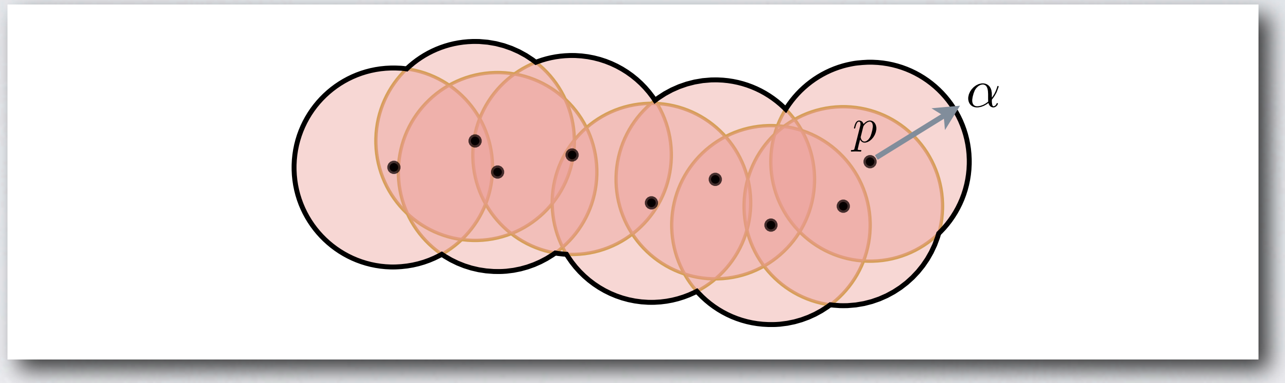
$\simeq$ **Nerve Lemma.**

$\text{Nerve}\,\mathcal{C} = \{\eta \subset \mathcal{C} \mid \bigcap \eta \neq \emptyset\}$



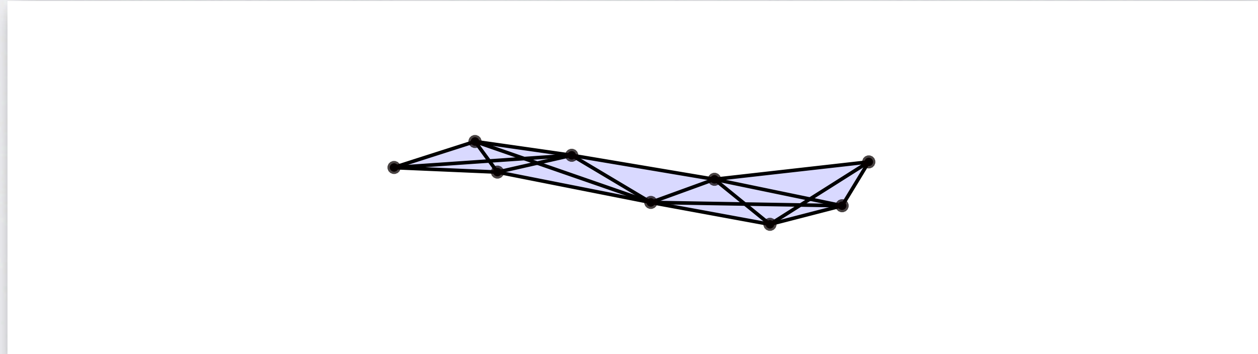
Cech complex

$$P^\alpha = \bigcup_{p \in P} B(p, \alpha)$$ α-offset of P



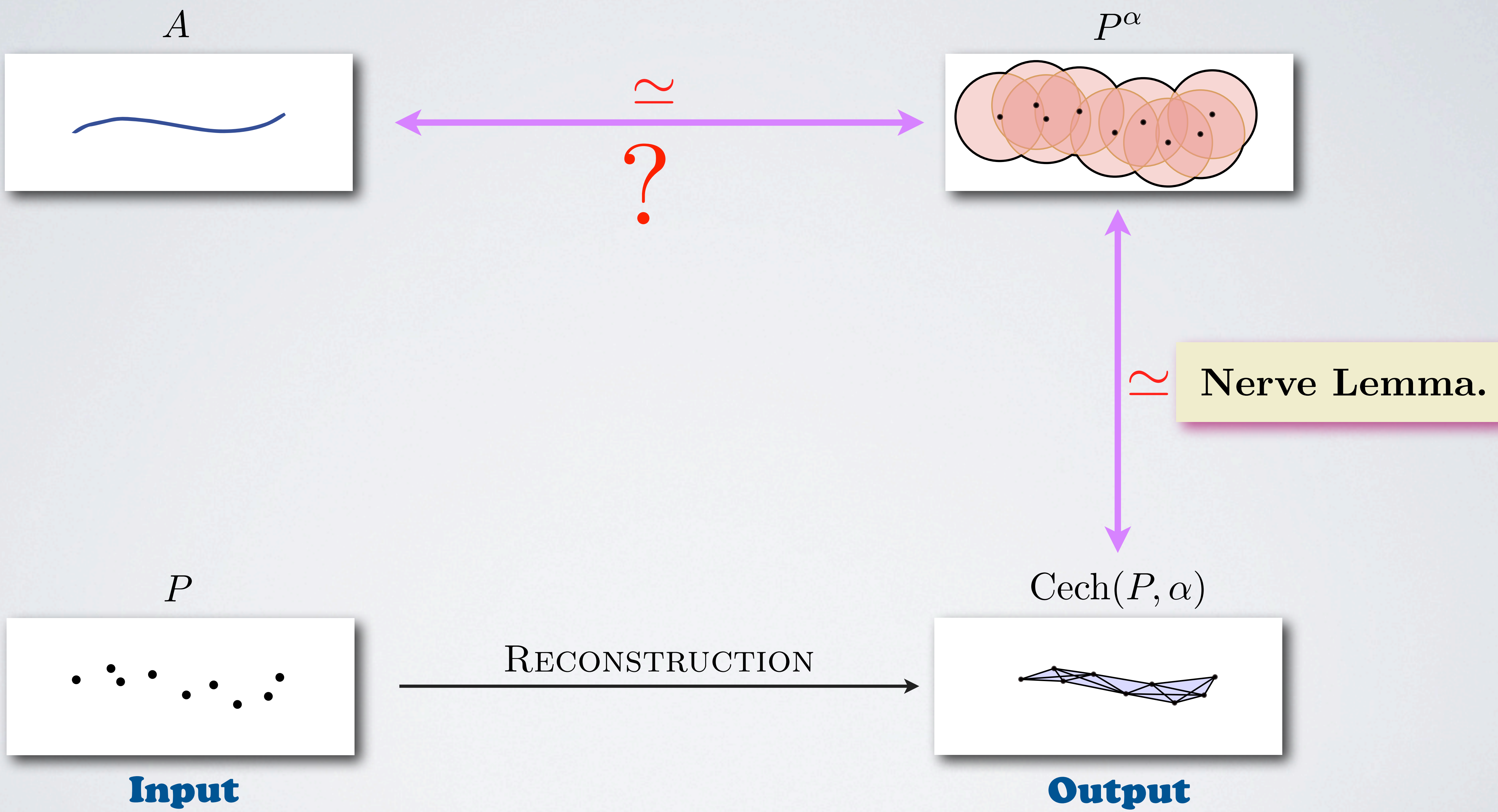
$\simeq$ **Nerve Lemma.**

$$\mathrm{Cech}(P, \alpha) = \mathrm{Nerve}\{B(p, \alpha) \mid p \in P\}$$

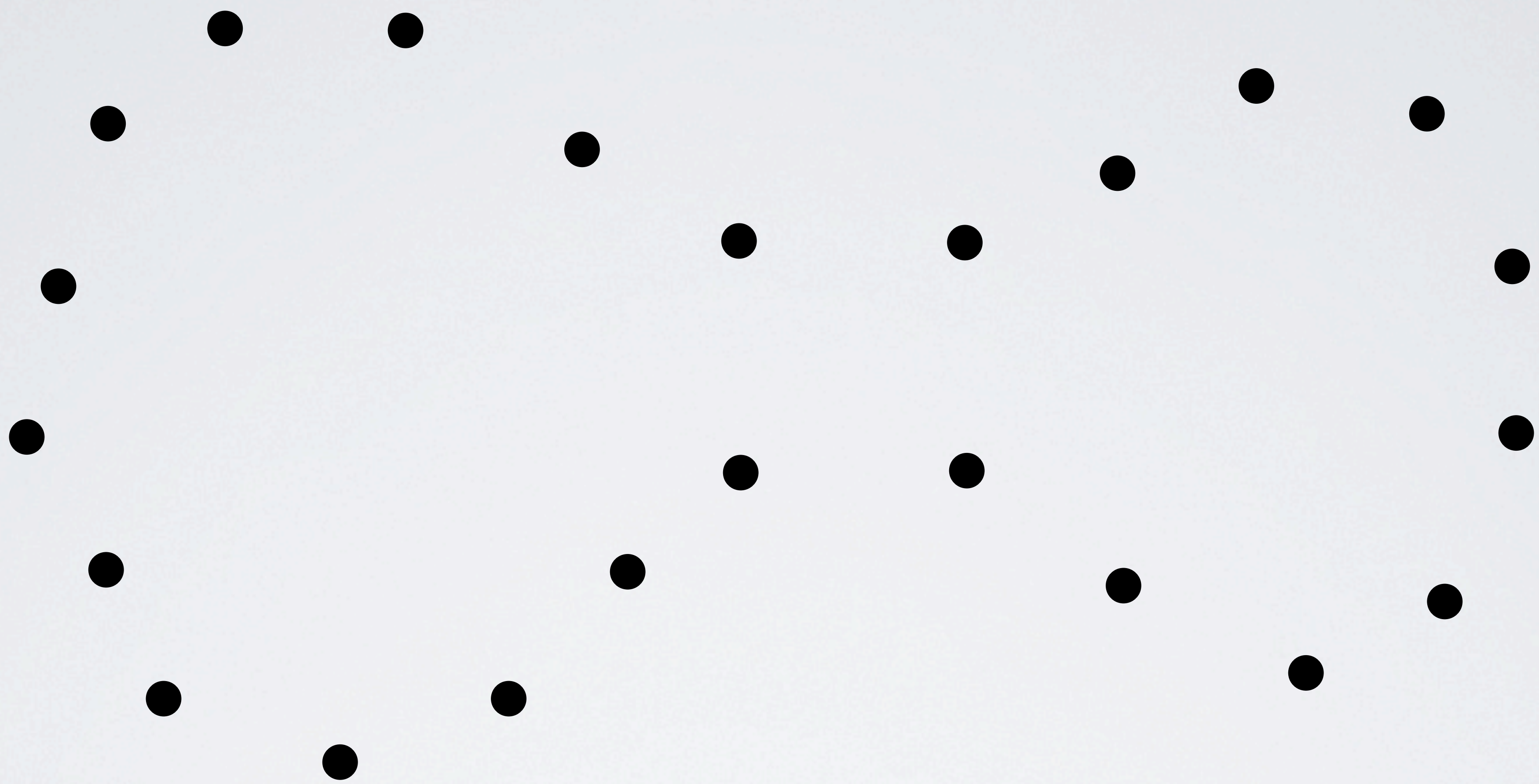


Can be
high-dimensional!
&
expensive to compute

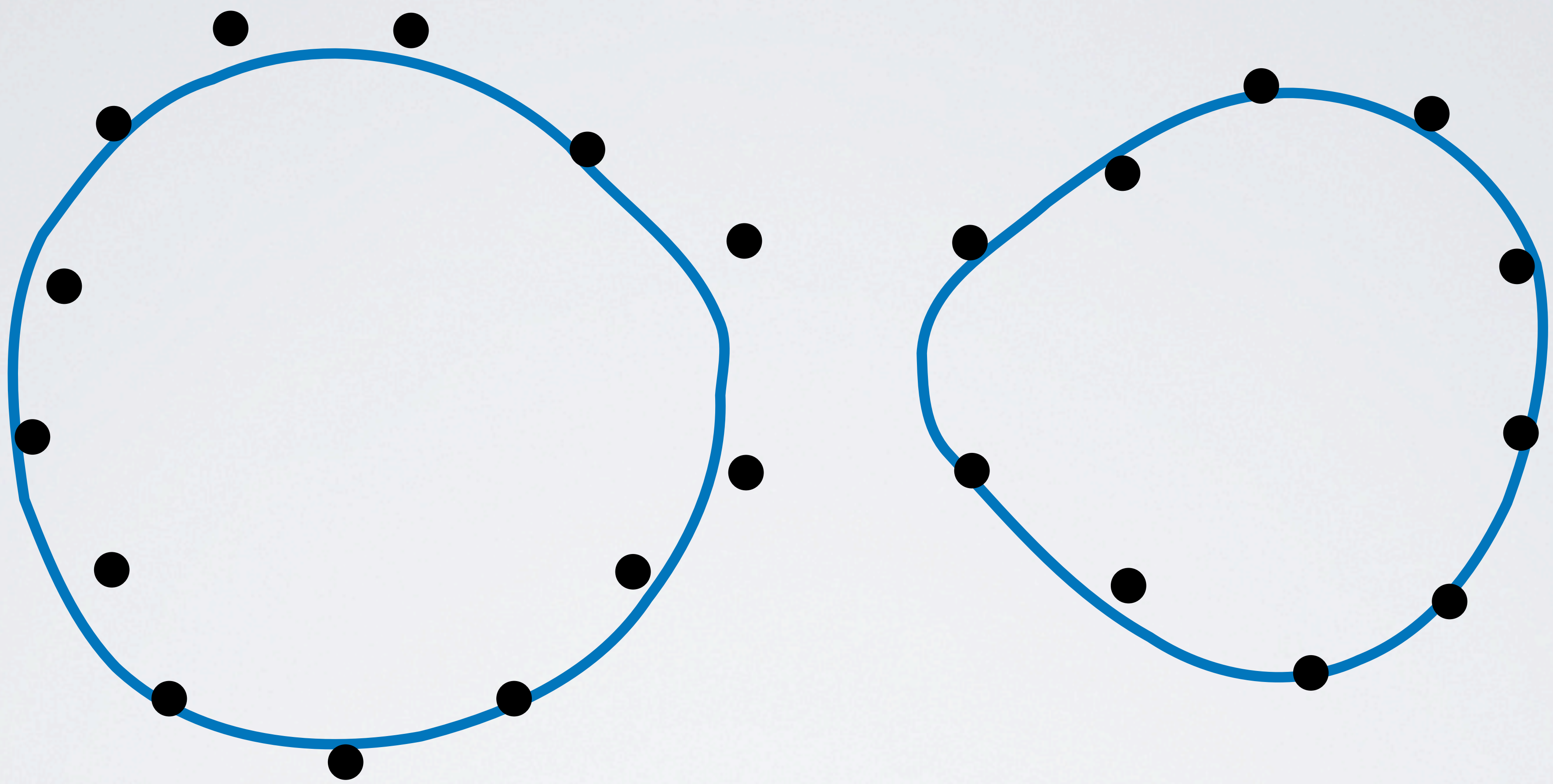
Cech complex



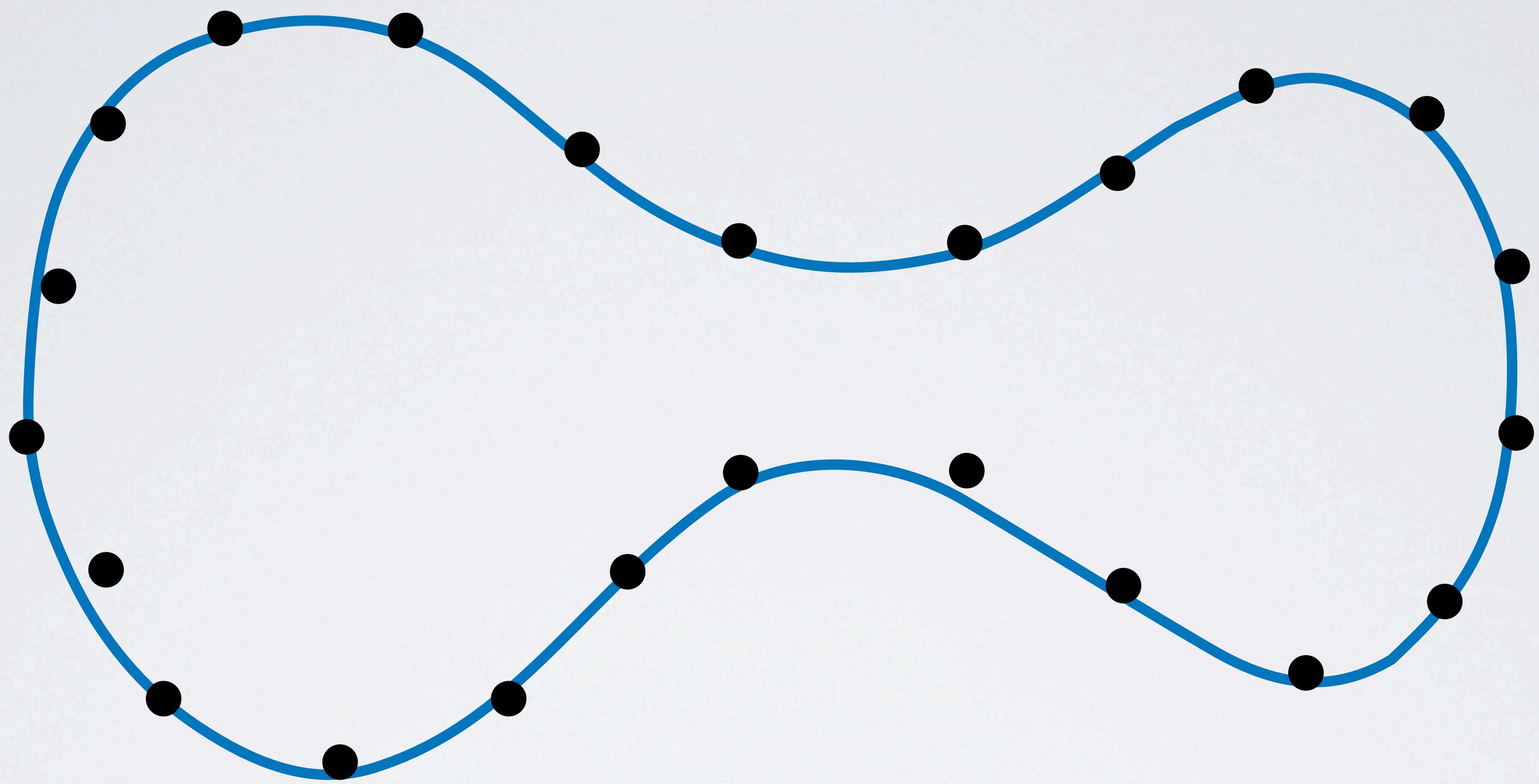
Shapes and Reach



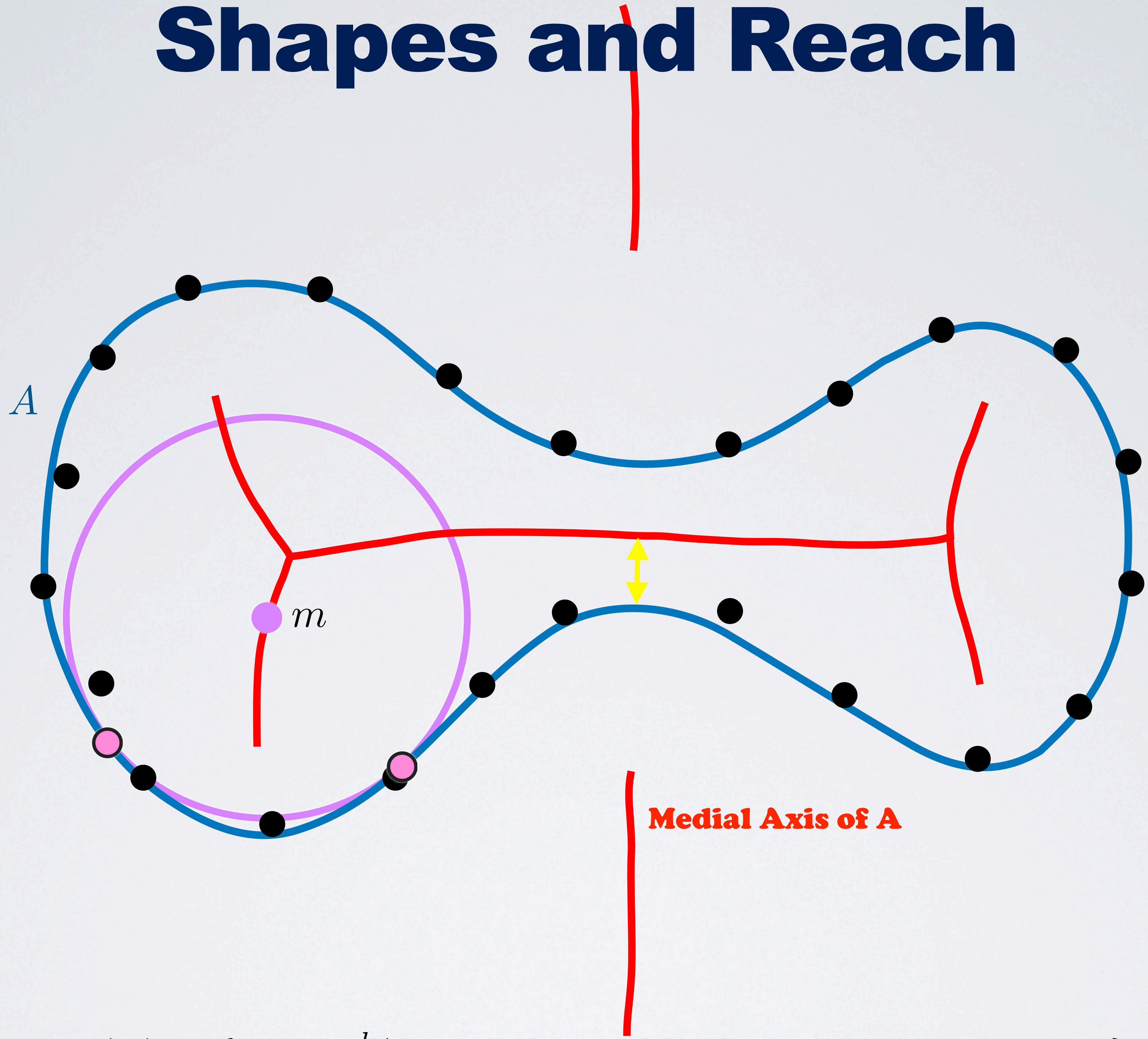
Shapes and Reach



Shapes and Reach



Shapes and Reach

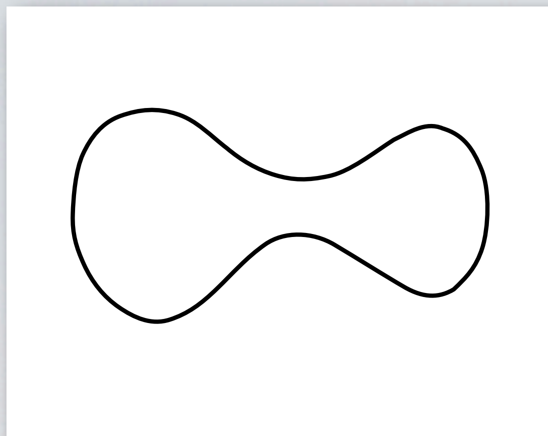


$$\mathrm{MedialAxis}(A) = \{m \in \mathbb{R}^d \mid \ m \text{ has at least two closest points in } A\ \}$$

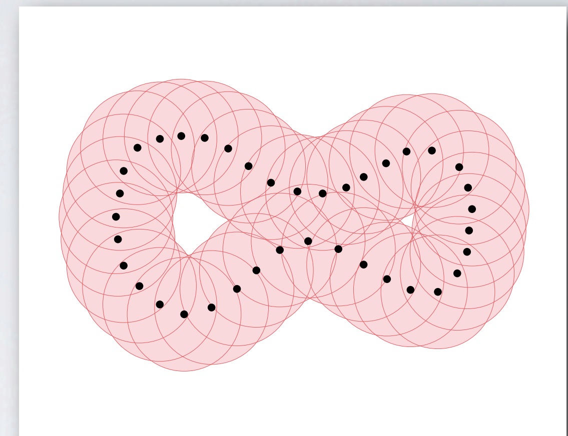
$$\text{Reach } A = d(A, \mathrm{MedialAxis}(A))$$

Cech complex

A



P^{α}



[Niyogi Smale Weinberger 2004]

deformation retracts to

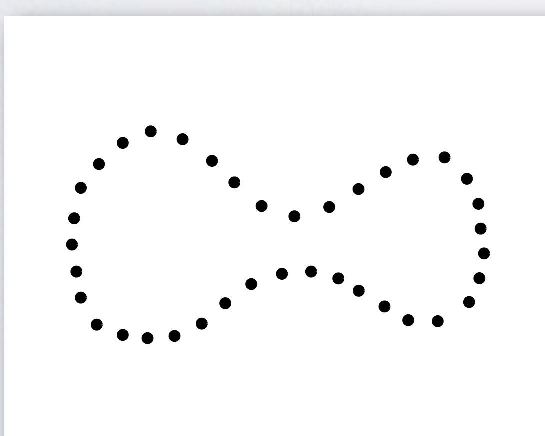
if

$d_H(A, P) \leq \varepsilon < (3 - \sqrt{8})\,\text{Reach}\,A$

$\alpha = (2 + \sqrt{2})\varepsilon$

$\simeq$ **Nerve Lemma.**

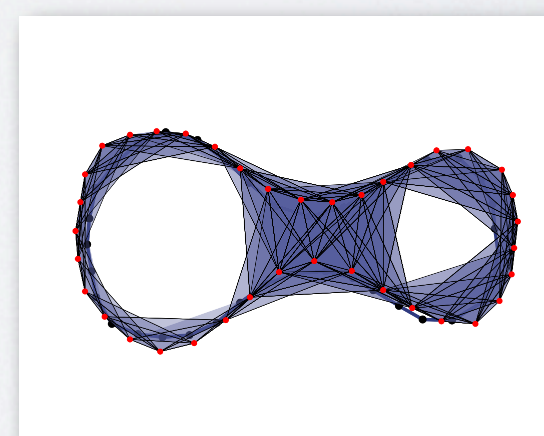
P



Input

RECONSTRUCTION

$\text{Cech}(P, \alpha)$



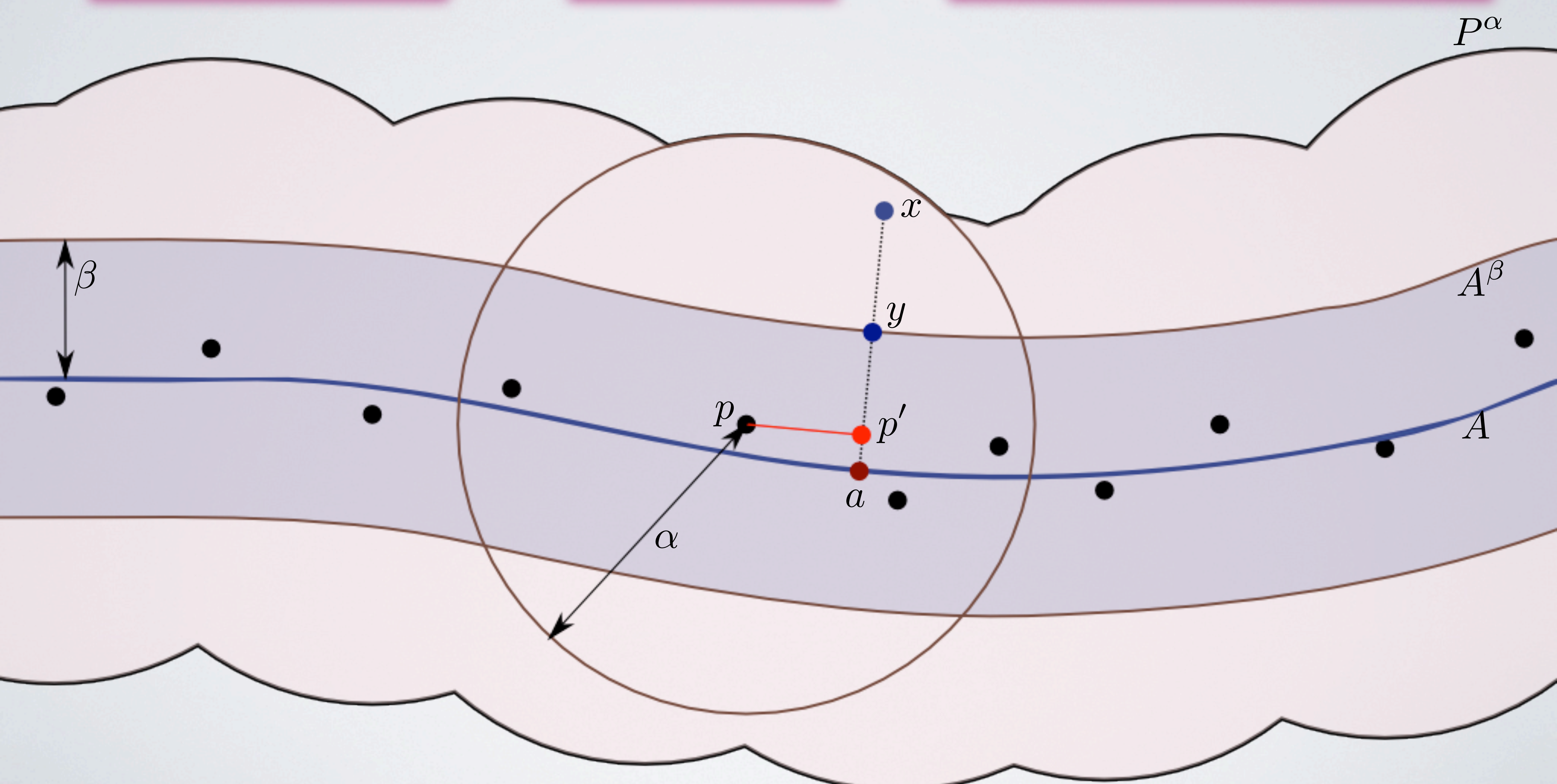
Output

$R = \text{Reach } A$

$\beta = \sqrt{R - (R-\varepsilon)^2 - \alpha^2}$

Short proof

$\left.\begin{array}{l}\varepsilon < (3-\sqrt{8})R \\ \alpha = (2+\sqrt{2})\varepsilon\end{array}\right\} \implies \left.\begin{array}{l}\alpha < R - \varepsilon \\ \beta < \alpha - \varepsilon\end{array}\right\} \implies$ P^α deformation retracts to A^β

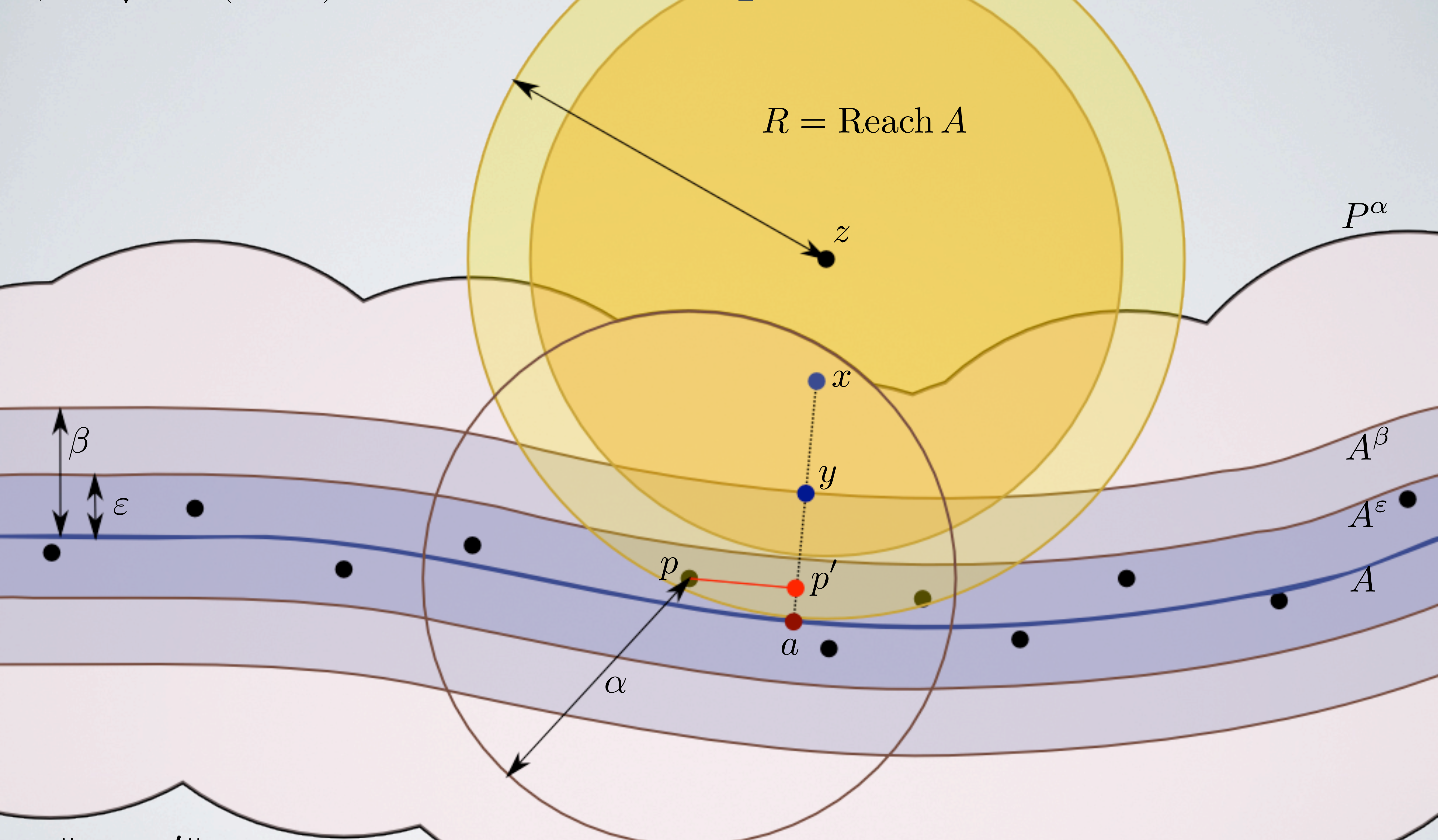


prove that $\|a - p'\| \leq \beta \implies y$ lies between x and p'

$R = \text{Reach } A$

$\beta = \sqrt{R - (R-\varepsilon)^2 - \alpha^2}$

Short proof

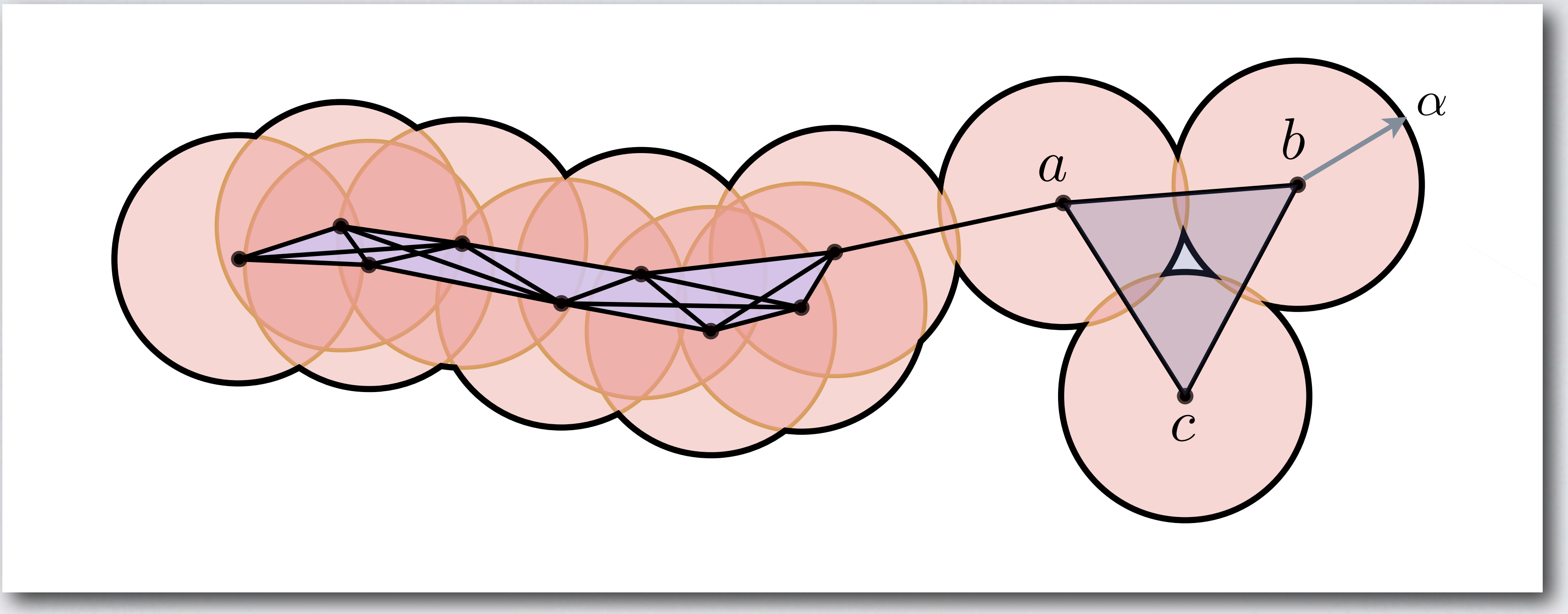


$\|p - p'\| \leq \alpha$

$\|z - p\| \geq R - \varepsilon$

$\Rightarrow \|a - p'\| = R - \sqrt{\|z-p\|^2 - \|p-p'\|^2} \leq \beta$

Rips complexes



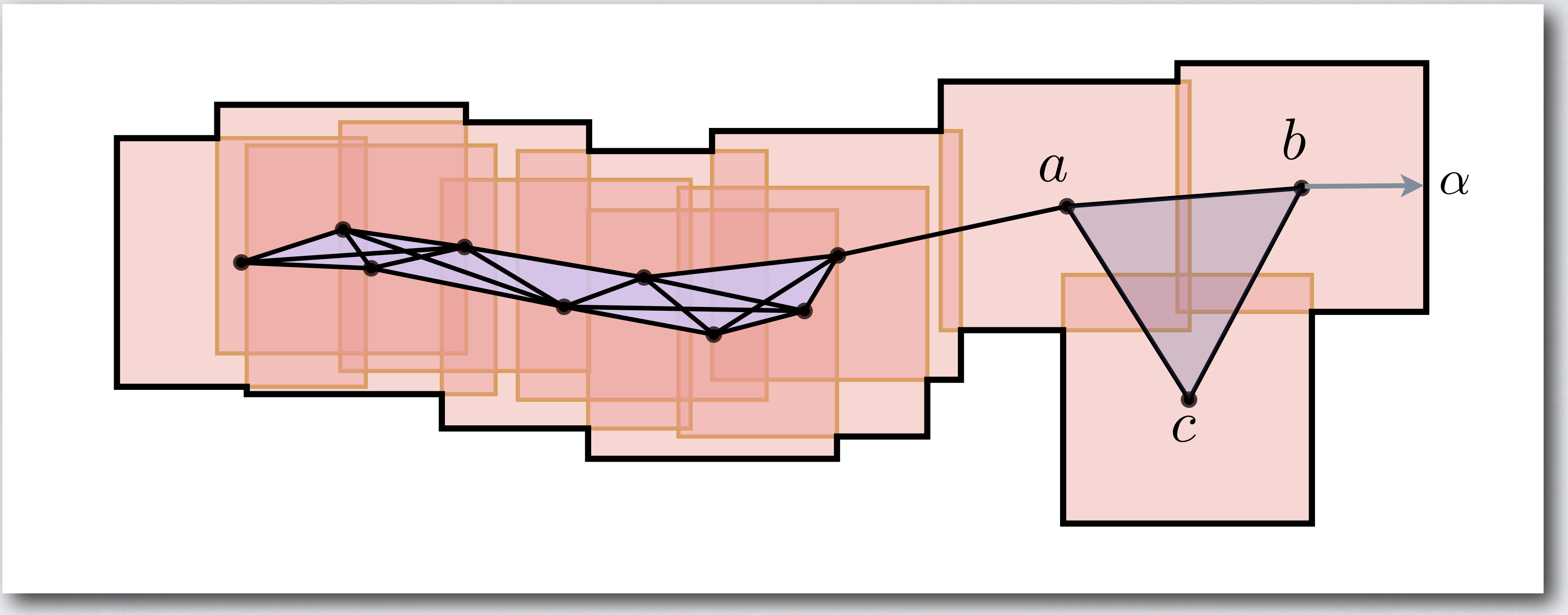
$$\text{Rips}(P, \alpha) = \{\sigma \subset P \mid \text{Diameter}(\sigma) \leq 2\alpha\}$$

$$\text{Rips}(P, \alpha) \supset \text{Cech}(P, \alpha)$$

- proximity graph G_α connects every pair of points within 2α
- $\text{Rips}(P, \alpha) = \text{Flag}\, G_\alpha$ [Flag G = largest complex whose 1-skeleton is G]
- compressed form of storage through the 1-skeleton
- easy to compute

Rips complexes with L_∞

When distances are measured using L_∞



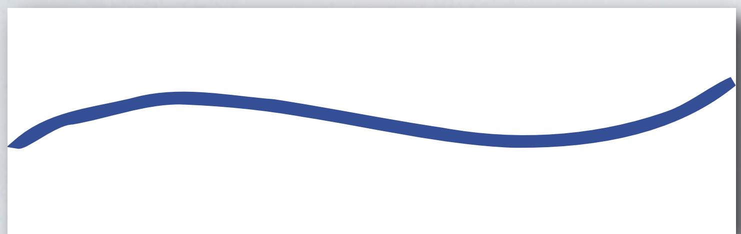
$$\text{Rips}(P,\alpha) = \{\sigma \subset P \mid \text{Diameter}(\sigma) \leq 2\alpha\}$$

$$\text{Rips}(P,\alpha) = \text{Cech}(P,\alpha)$$

- proximity graph G_α connects every pair of points within 2α
- $\text{Rips}(P,\alpha) = \text{Flag}\, G_\alpha$ [Flag G = largest complex whose 1-skeleton is G]
- compressed form of storage through the 1-skeleton
- easy to compute

Rips complexes with L_∞

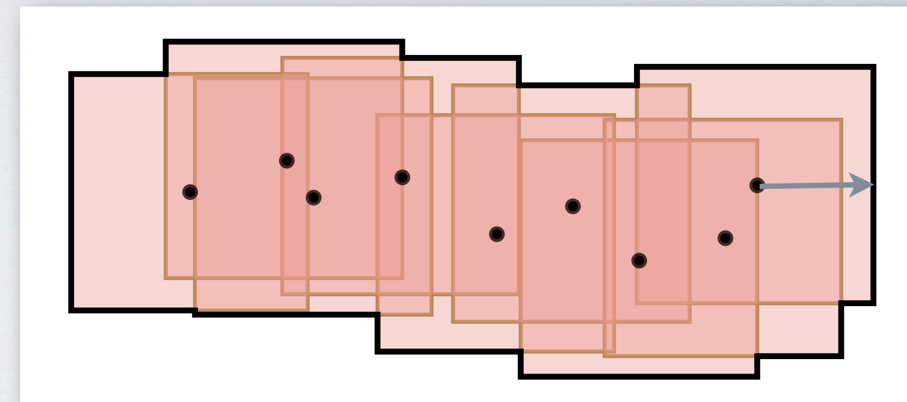
A



$\simeq$

?

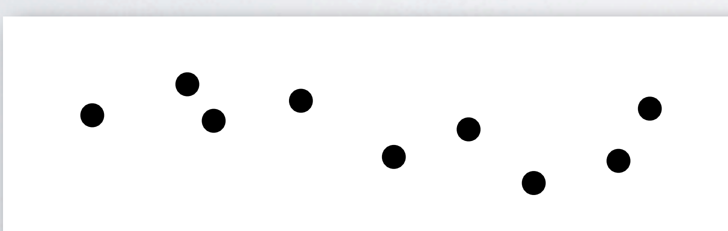
$P + \alpha\, B^{\infty}(0,1)$



$\simeq$

Nerve Lemma.

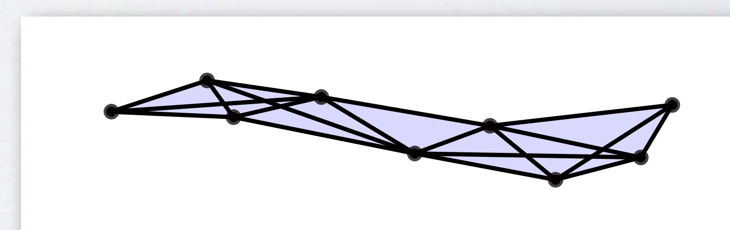
P



Input

RECONSTRUCTION

$\mathrm{Rips}(P,\alpha) = \mathrm{Cech}(P,\alpha)$

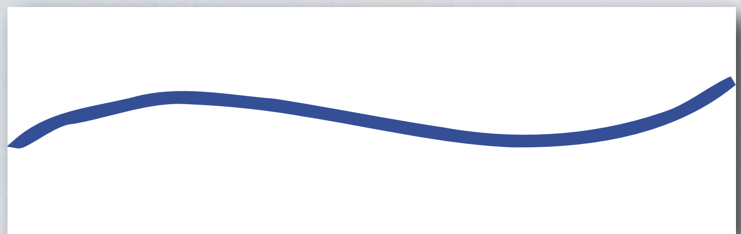


Output

easy to compute

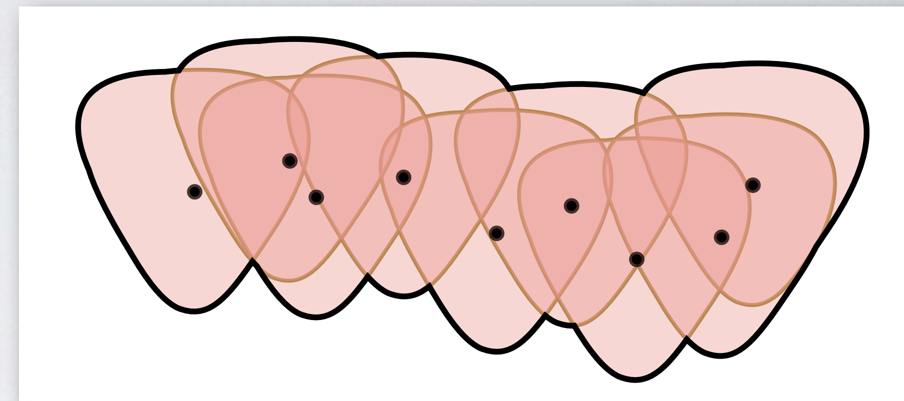
Minkowski sum

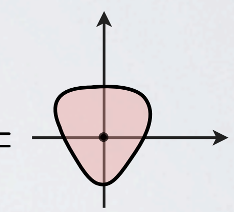
A



$\simeq$?

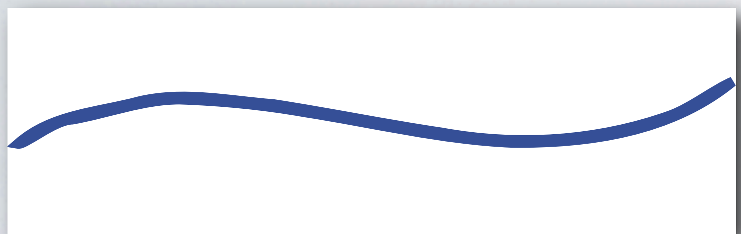
$P + \alpha C$



where $C =$  compact convex set

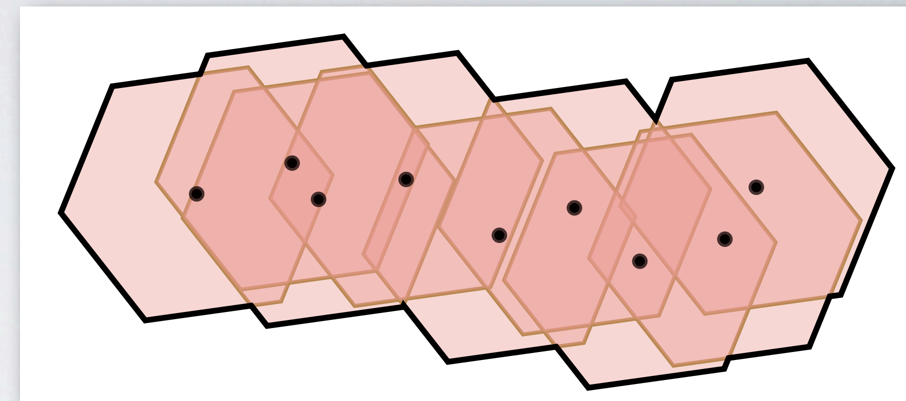
Minkowski sum

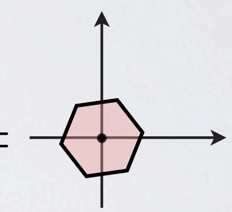
A



$\simeq$?

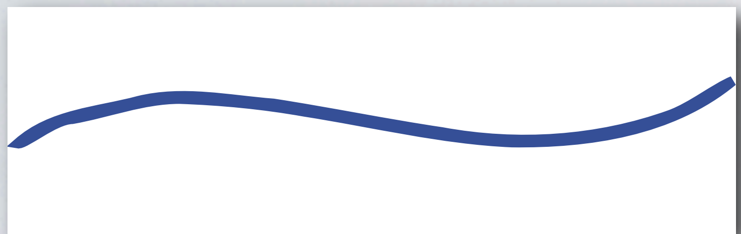
$P + \alpha C$



where $C =$  compact convex set

Minkowski sum

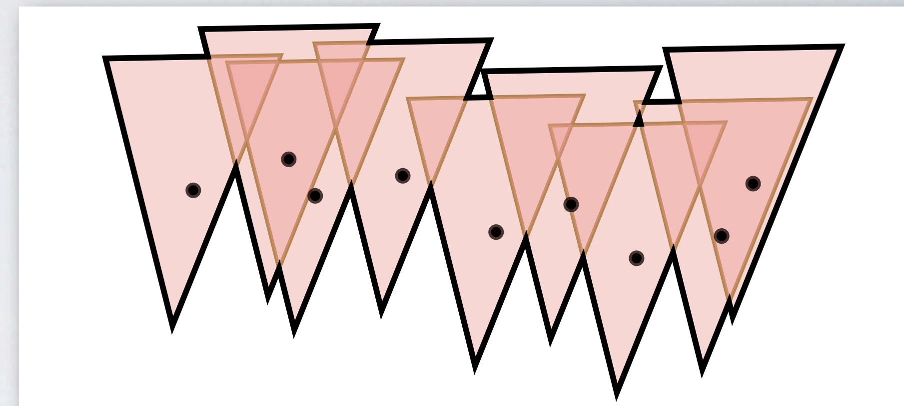
A

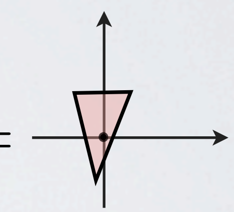


$\simeq$

?

$P + \alpha C$



where $C =$  compact convex set

Minkowski sum

A $\xleftrightarrow{\text{inclusion homotopy equivalence}}$ $P + \alpha C$

if

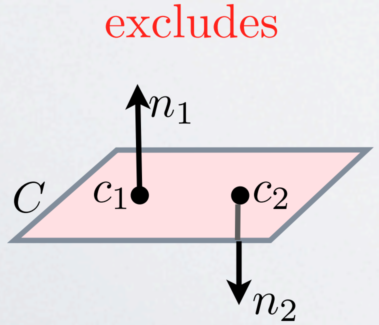
$P \subset A^{\varepsilon}$ and $A \subset P + \varepsilon C$ and $\frac{\varepsilon}{\text{Reach } A}$ small enough and $\frac{\alpha}{\varepsilon} = \frac{4}{1-\xi}$

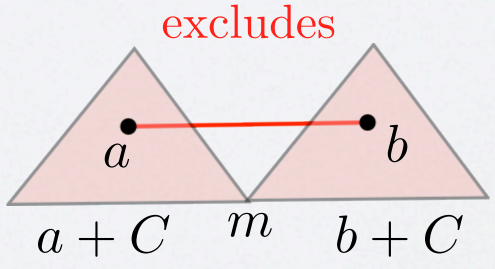
where C compact convex set that satisfies:

(i) $B(0,1) \subset C \subset \delta B(0,1)$ for some $\delta \geq 1$; ("distortion" to unit ball)

(ii) C is $(\theta, \varkappa)$-round for $\theta = \arccos(-\frac{1}{d})$ and $\varkappa > 0$; ("curvature")

(iii) C is ξ-eccentric for $\xi < 1$. ("distance" between $\bigcap_{q \in Q}(q + C)$ and $\text{Hull}(Q)$)





Minkowski sum

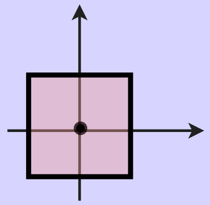
A — inclusion homotopy equivalence — $P + \alpha C$

if

$P \subset A^\varepsilon$ and $A \subset P + \varepsilon C$ and $\frac{\varepsilon}{\text{Reach } A}$ small enough and $\frac{\alpha}{\varepsilon} = \frac{4}{1-\xi}$

❶ d-balls satisfy (i) (ii) and (iii) for $\delta = 1$, $\varkappa = 1$ and $\xi = 0$.

❷ d-cubes satisfy (i) (ii) and (iii) for $\delta = \sqrt{d}$



$$\varkappa = \begin{cases} \frac{1}{2\sqrt{2}}\left(\cos\frac{\pi}{4} + \cos\frac{\pi}{12}\right) & \text{if } d = 2, \\ \frac{1}{\sqrt{6}} & \text{if } d = 3, \\ \frac{1}{(d-2)\sqrt{d}} & \text{if } d \geq 4, \end{cases}$$

$\xi = 1 - \frac{2}{d}$

Minkowski sum

A	inclusion homotopy equivalence ⟷ if	$P + \alpha C$

$P \subset A^{\varepsilon}$ and $A \subset P + \varepsilon C$ and $\frac{\varepsilon}{\text{Reach } A} < \lambda$ and $\frac{\alpha}{\varepsilon} = \eta$

Admissible values of ε and α are solutions of a system of equations that depends upon $(\delta, \varkappa, \xi)$.

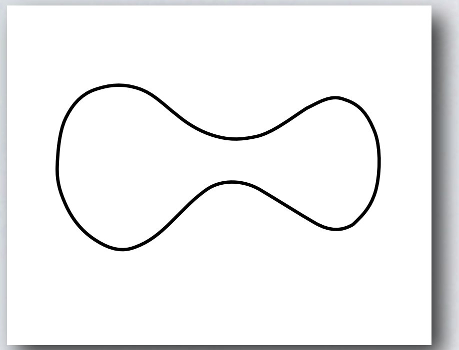
C	d	λ	η
d-ball with [NSW04]	$\forall d$	$3 - \sqrt{8} \approx 0.17$	$2 + \sqrt{2} \approx 3.41$
d-ball with this proof	$\forall d$	0.077	3.96
d-cube [Rips(P, α) with ℓ_∞]	2	0.04	4.04
	3	0.01	6.14
	4	0.004	8.18
	5	0.002	10.2
	10	0.0002	20.23
	100	0.0000002	200.23

What now?

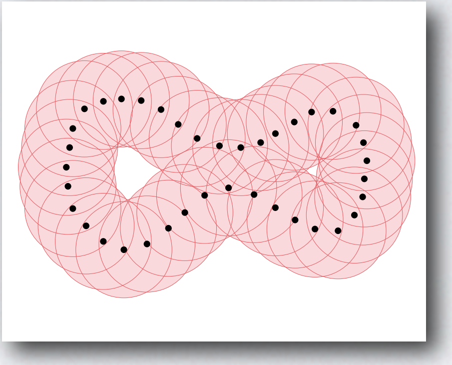
- The largest ratio $\frac{\varepsilon}{\text{Reach } A}$ that we get for $\text{Rips}(P, \alpha)$ with ℓ_∞:
  - Decreases quickly with d
  - Is it tight?
- ℓ_∞ ➠ ℓ_2

Rips complexes with L_2

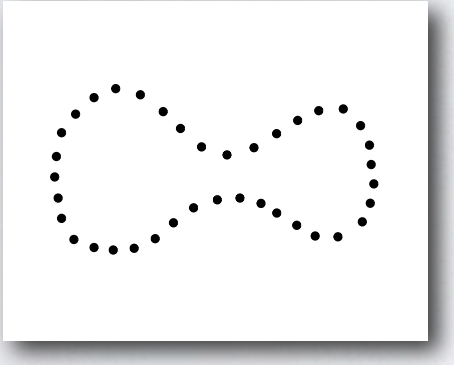
A



P^α

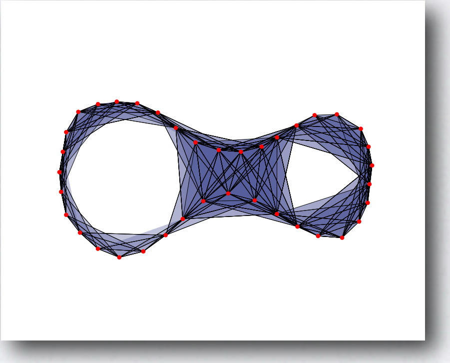


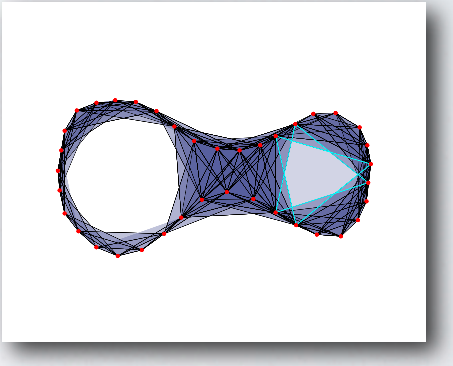
P



Input

$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$



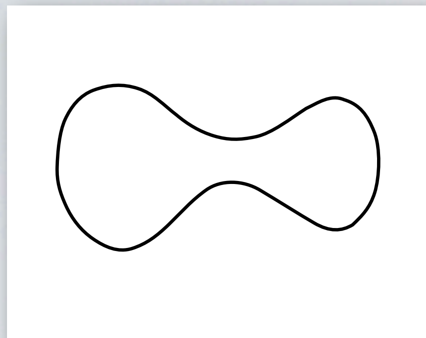


Output

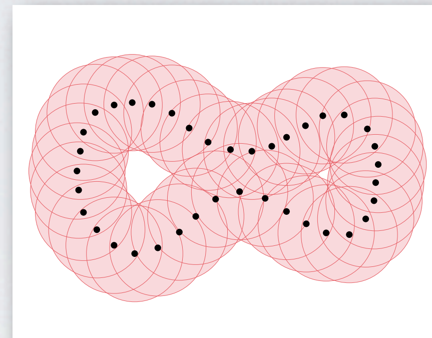
easy to compute

Rips complexes with L_2

A

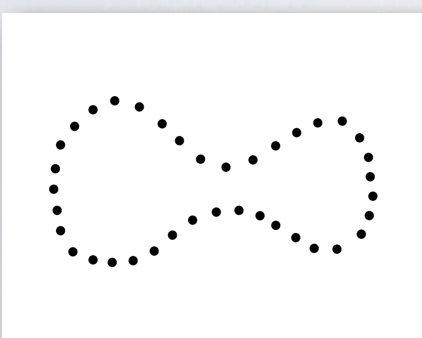


P^α



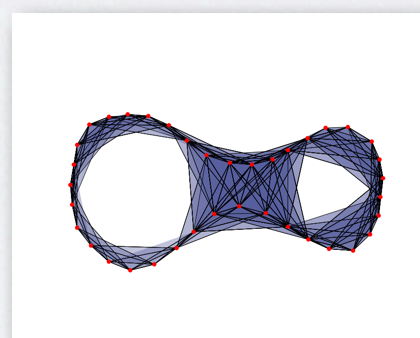
$\simeq$ **Nerve Lemma**

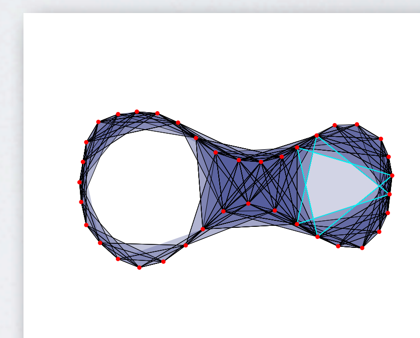
P



Input

$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$

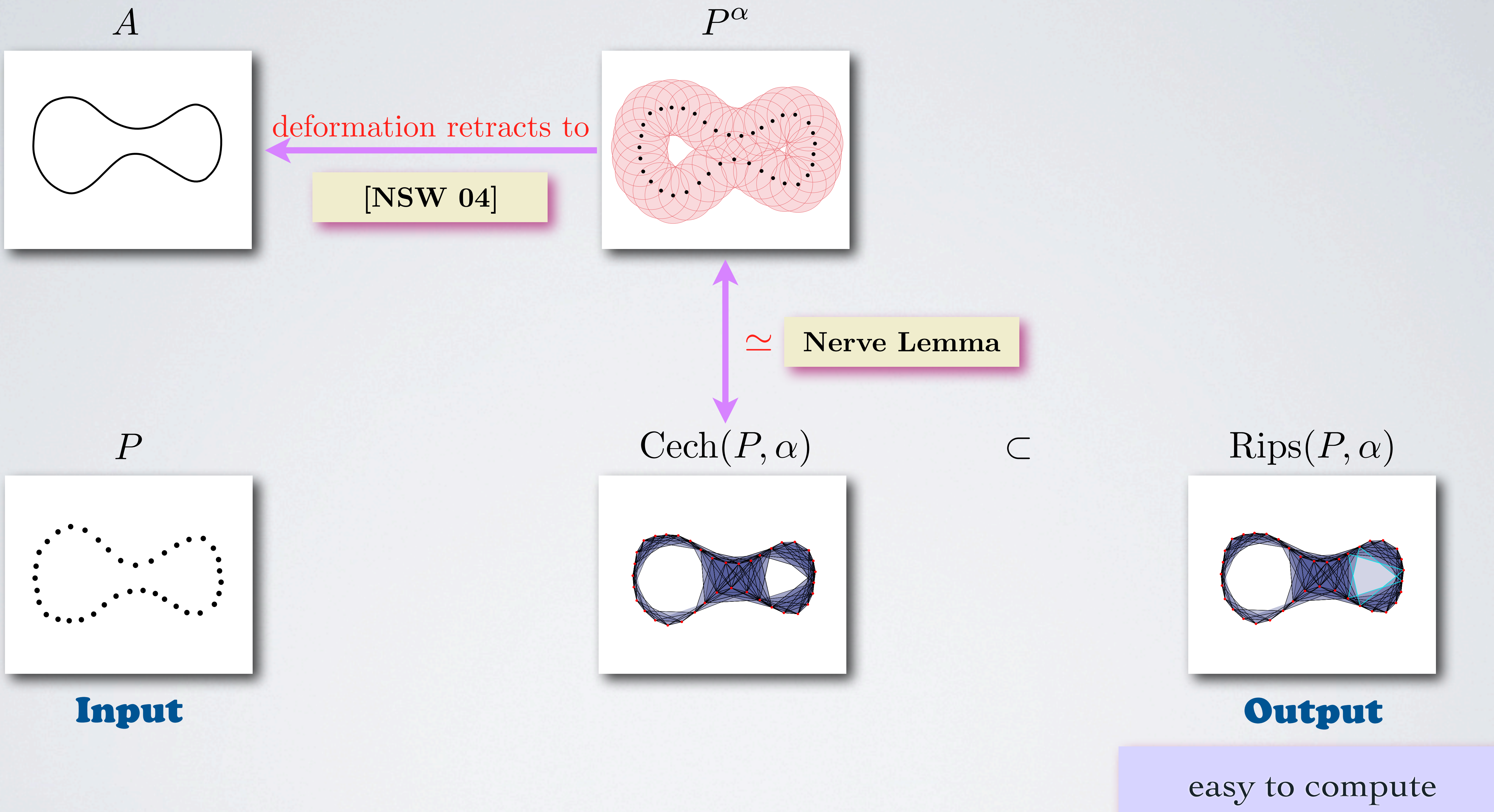




Output

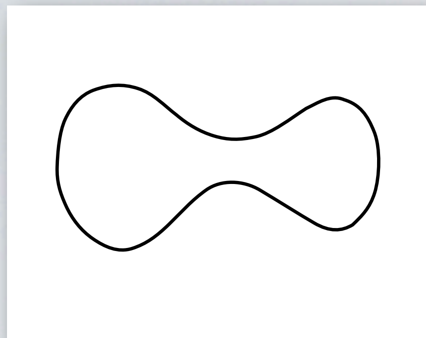
easy to compute

Rips complexes with L_2



Rips complexes with L$_2$

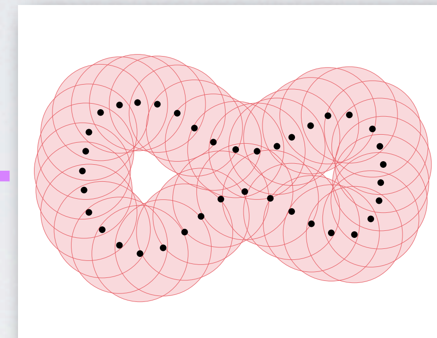
A



deformation retracts to

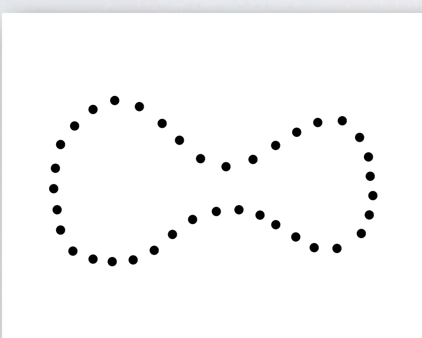
[NSW 04]

P^α



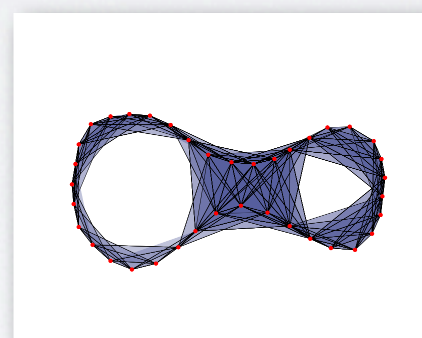
$\simeq$ **Nerve Lemma**

P



Input

$\text{Cech}(P, \alpha)$

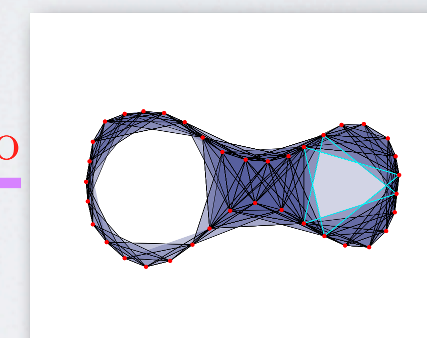


$\subset$

deformation retracts to

?

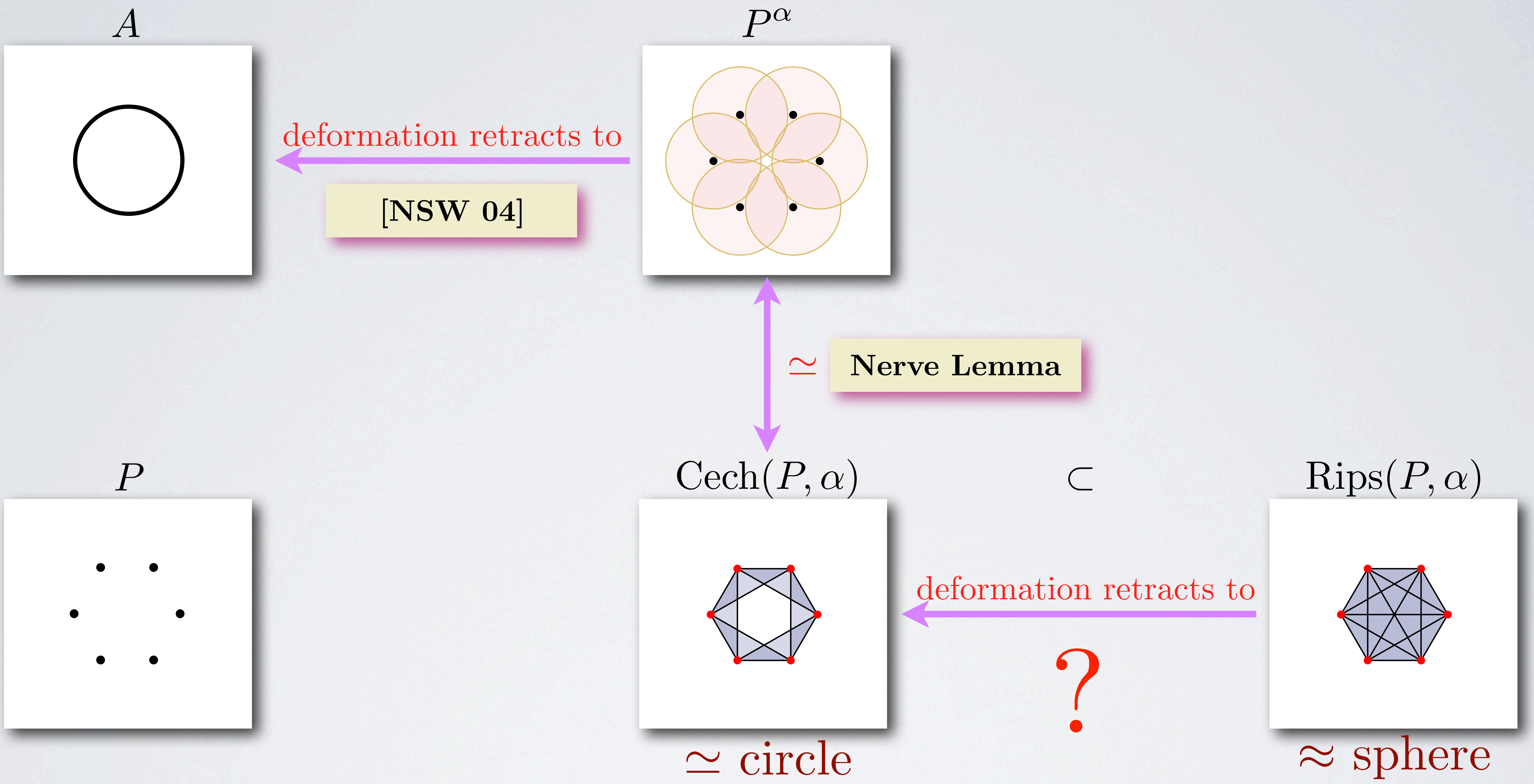
$\text{Rips}(P, \alpha)$



Output

easy to compute

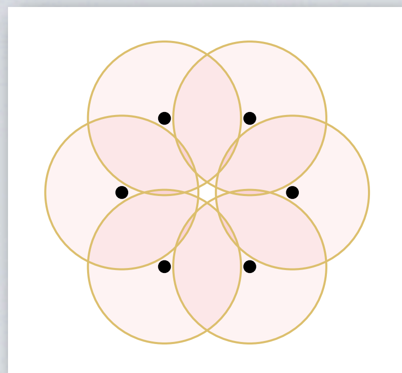
Rips complexes with L_2



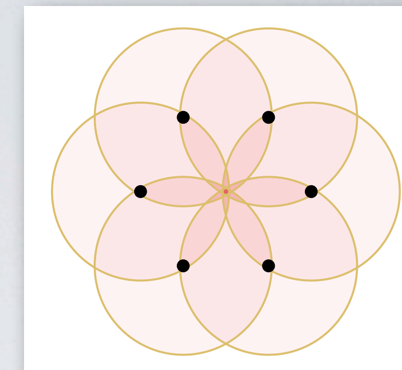
Rips and Cech complexes generally don’t share the same topology, but ...

Roadmap

P^{α}

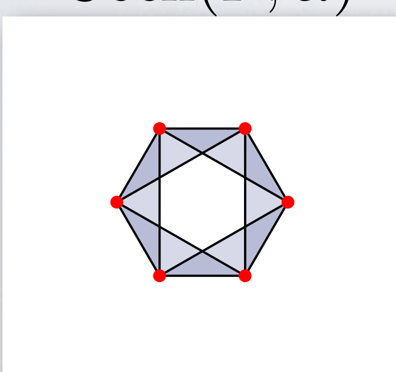


$P^{\vartheta_d \alpha}$

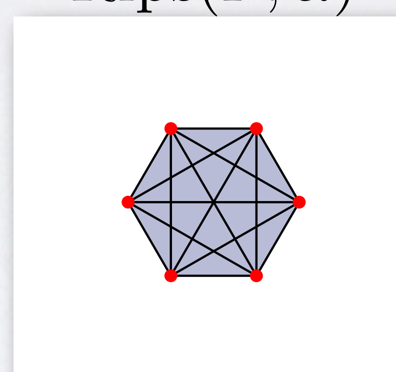


$\simeq$ $\simeq$

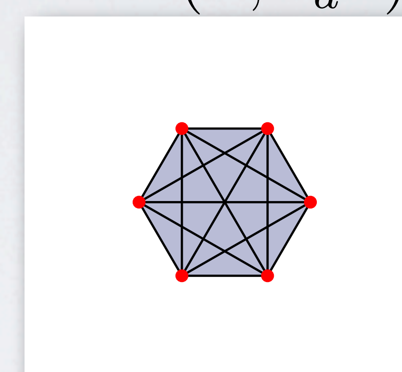
$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$ $\subset$ $\mathrm{Cech}(P, \vartheta_d \alpha)$



$\simeq$ circle



$\approx$ sphere

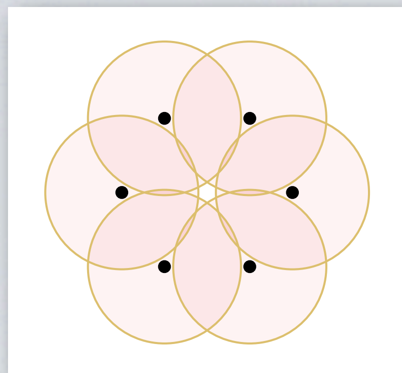


$\approx$ 5-ball

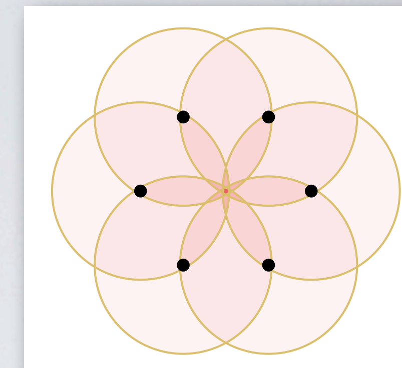
for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Roadmap

P^{α}



$P^{\vartheta_d \alpha}$

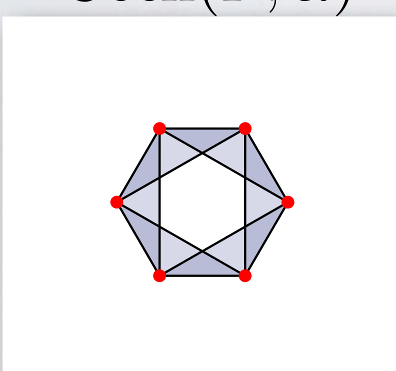


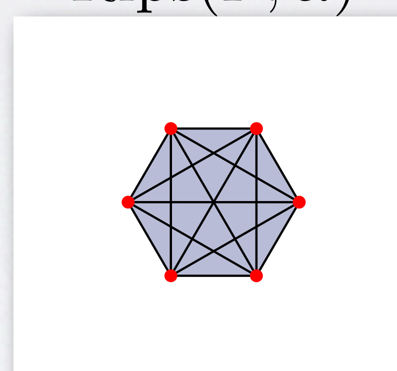
❶ Find a condition under which the topology of $\{\, \mathrm{Cech}(P, t)\, \}_{\alpha \leq t \leq \vartheta_d \alpha}$ is "stable"

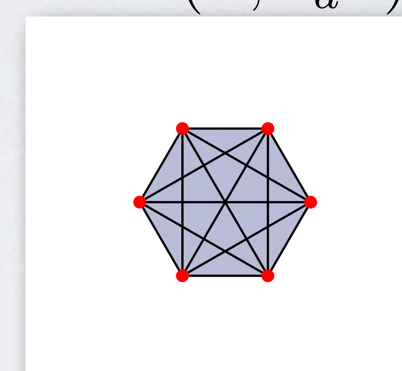
$\simeq$ $\simeq$

sequence of collapses ?

$\mathrm{Cech}(P, \alpha)$ $\subset$ $\mathrm{Rips}(P, \alpha)$ $\subset$ $\mathrm{Cech}(P, \vartheta_d \alpha)$



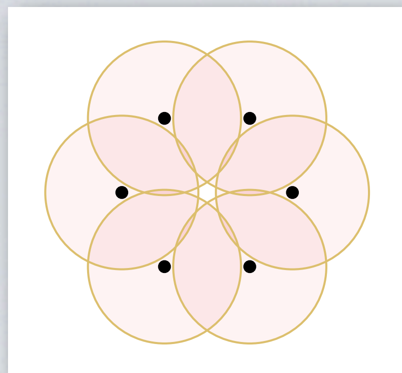




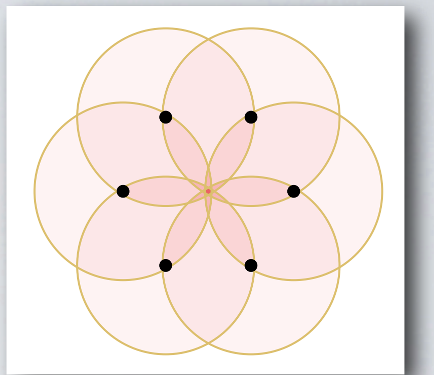
for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Roadmap

P^{α}



$P^{\vartheta_d \alpha}$

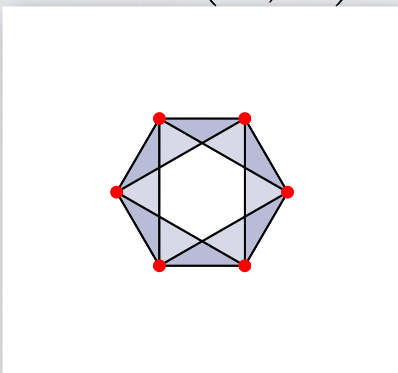


❶ Find a condition under which the topology of $\{\,\mathrm{Cech}(P,t)\,\}_{\alpha \leq t \leq \vartheta_d \alpha}$ is "stable"

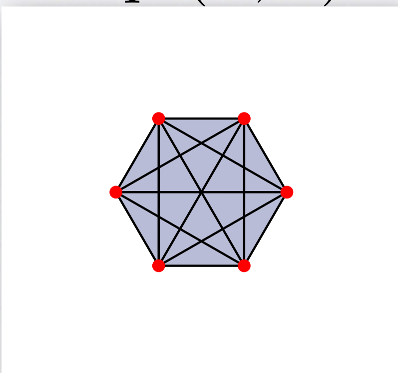
$\simeq$ $\simeq$

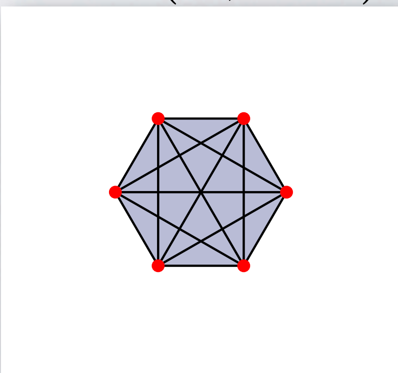
sequence of collapses ?

$\mathrm{Cech}(P,\alpha)$ $\subset$ $\mathrm{Rips}(P,\alpha)$ $\subset$ $\mathrm{Cech}(P,\vartheta_d \alpha)$



sequence of collapses ?



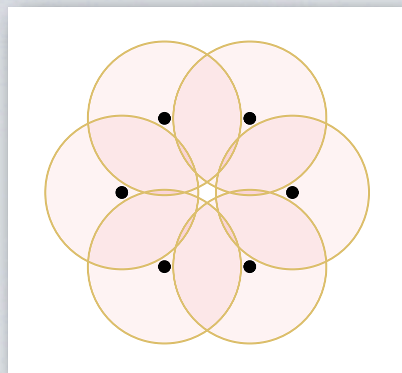


for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

❷ Deduce a condition under which the topology of $\{\,\mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)\,\}_{\alpha \leq t \leq \vartheta_d \alpha}$ is "stable"

Roadmap

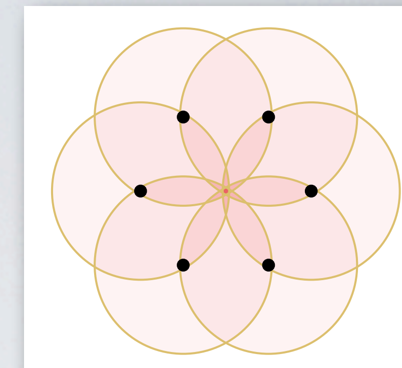
P^{α}



deformation retracts to

?

$P^{\vartheta_d \alpha}$



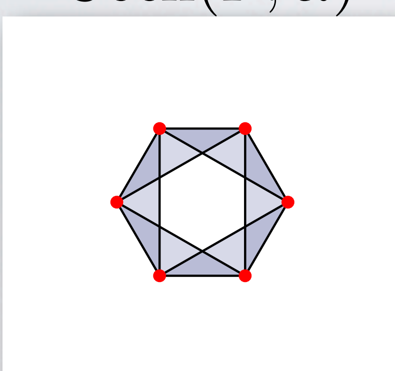
$\simeq$

Find a condition under which the topology of

$\{\, \text{Cech}(P, t) \,\}_{\alpha \leq t \leq \vartheta_d \alpha}$ is "stable"

$\simeq$

$\text{Cech}(P, \alpha)$

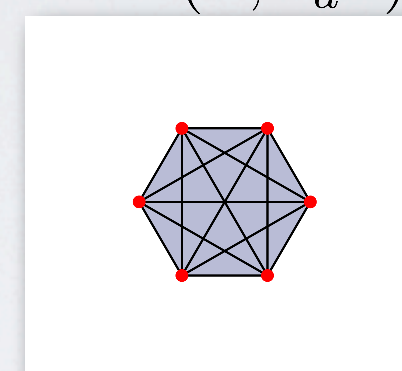


$\simeq$ circle

sequence of collapses

?

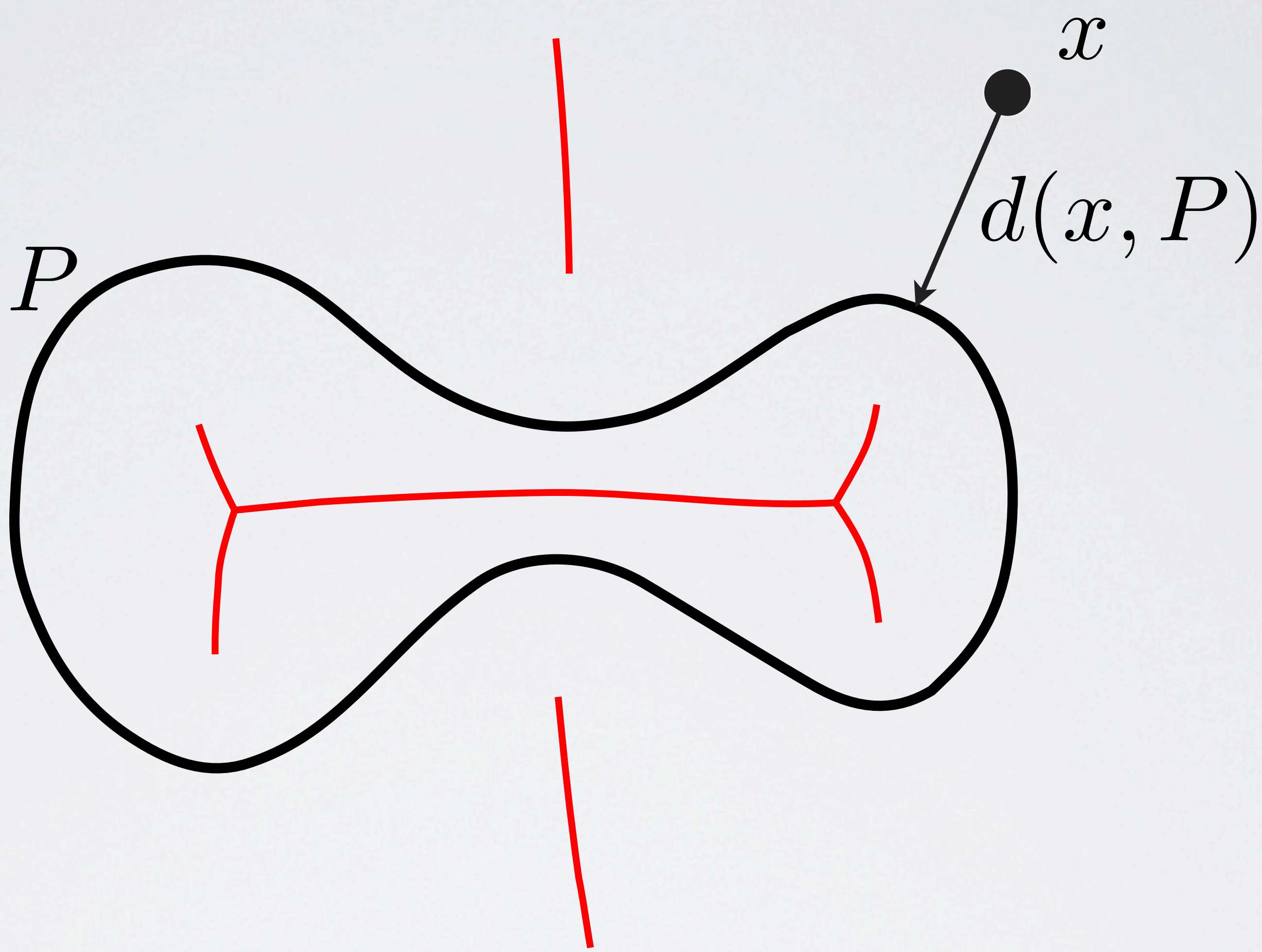
$\text{Cech}(P, \vartheta_d \alpha)$



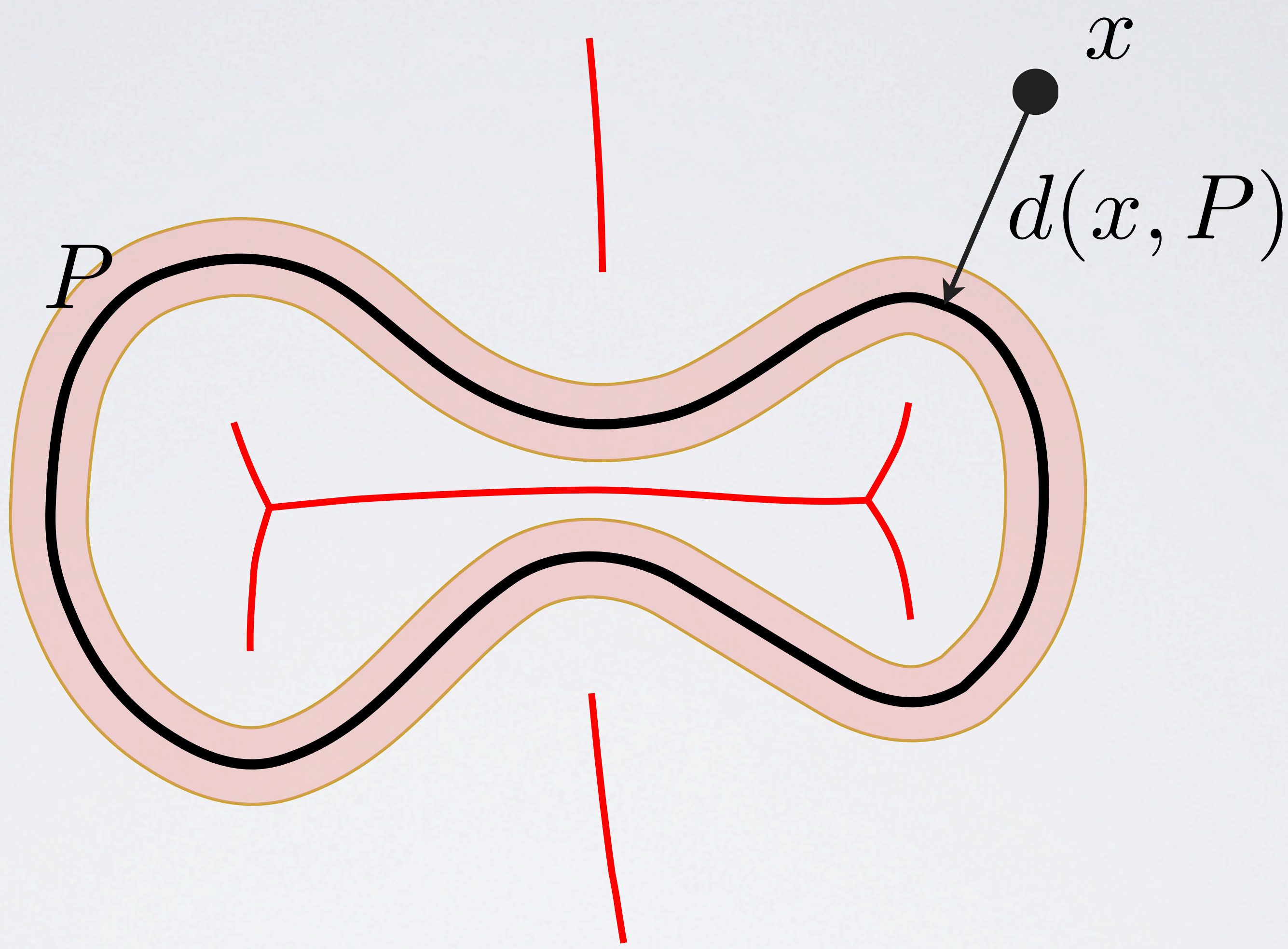
$\approx$ 5-ball

for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Distance function

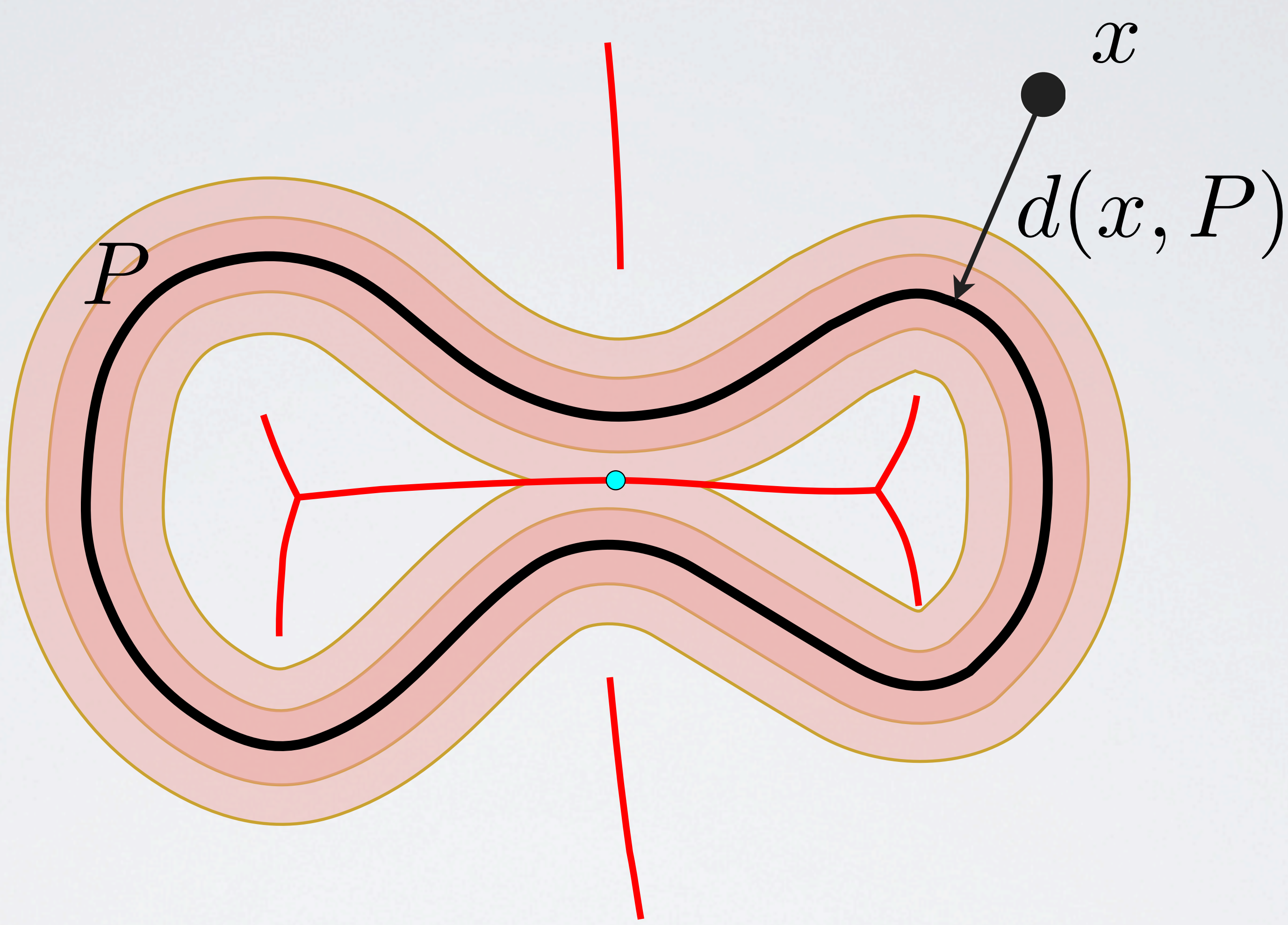


Distance function



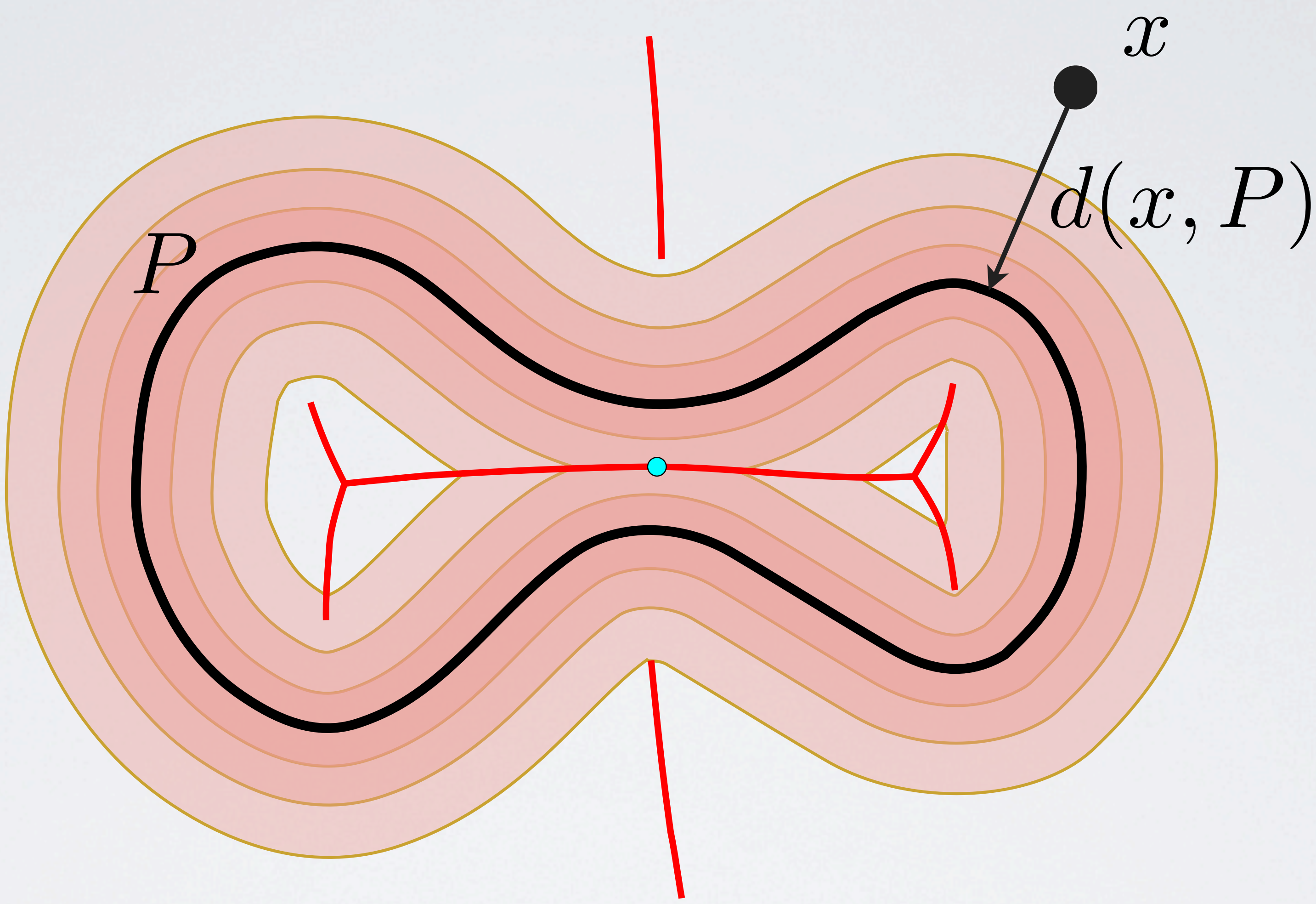
- Sublevel sets of $d(\cdot, P)$ are offsets of P.

Distance function



- Sublevel sets of $d(\cdot, P)$ are offsets of P.
- Topology of sublevel sets changes at critical values t_0.

Distance function



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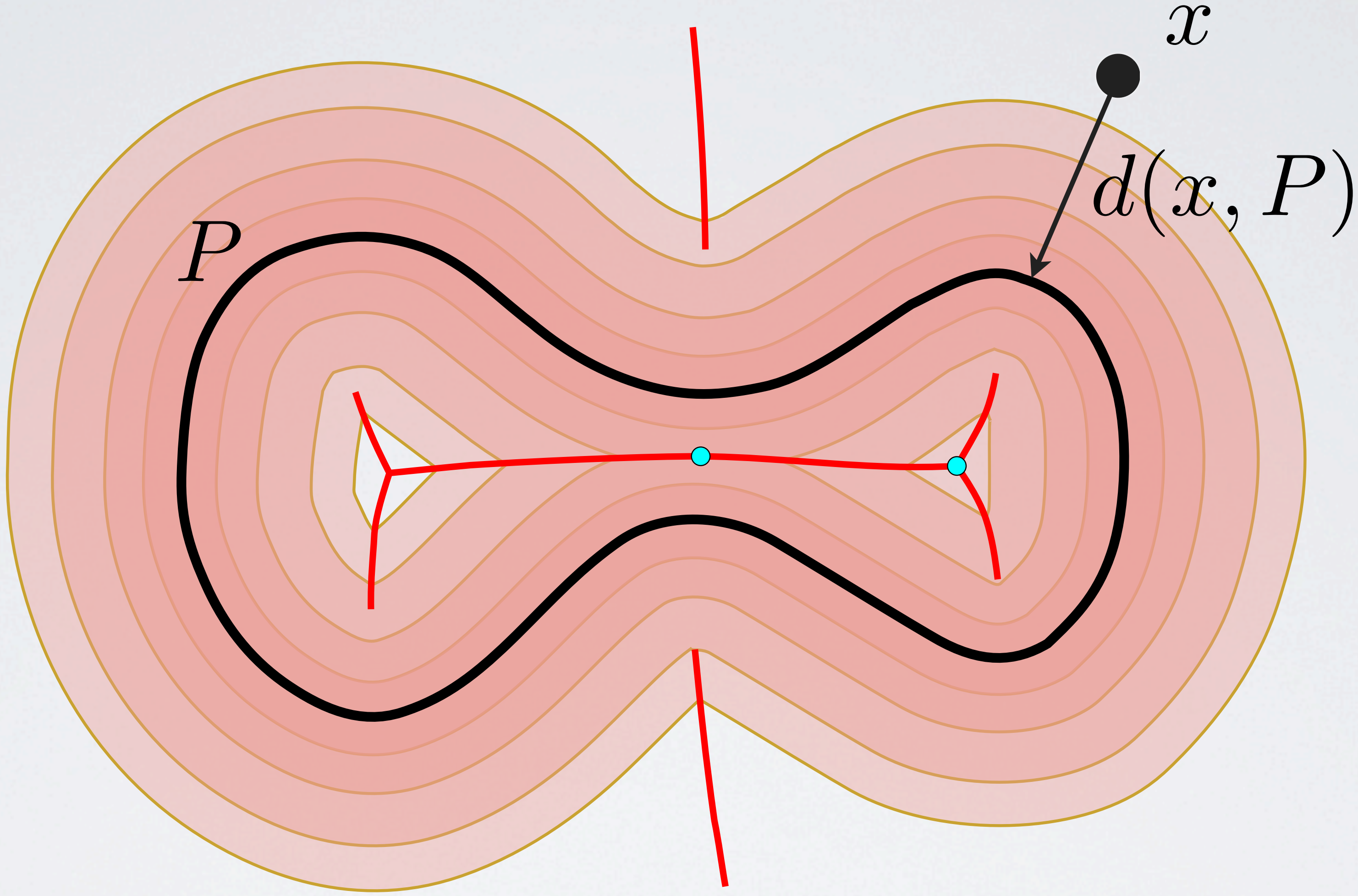

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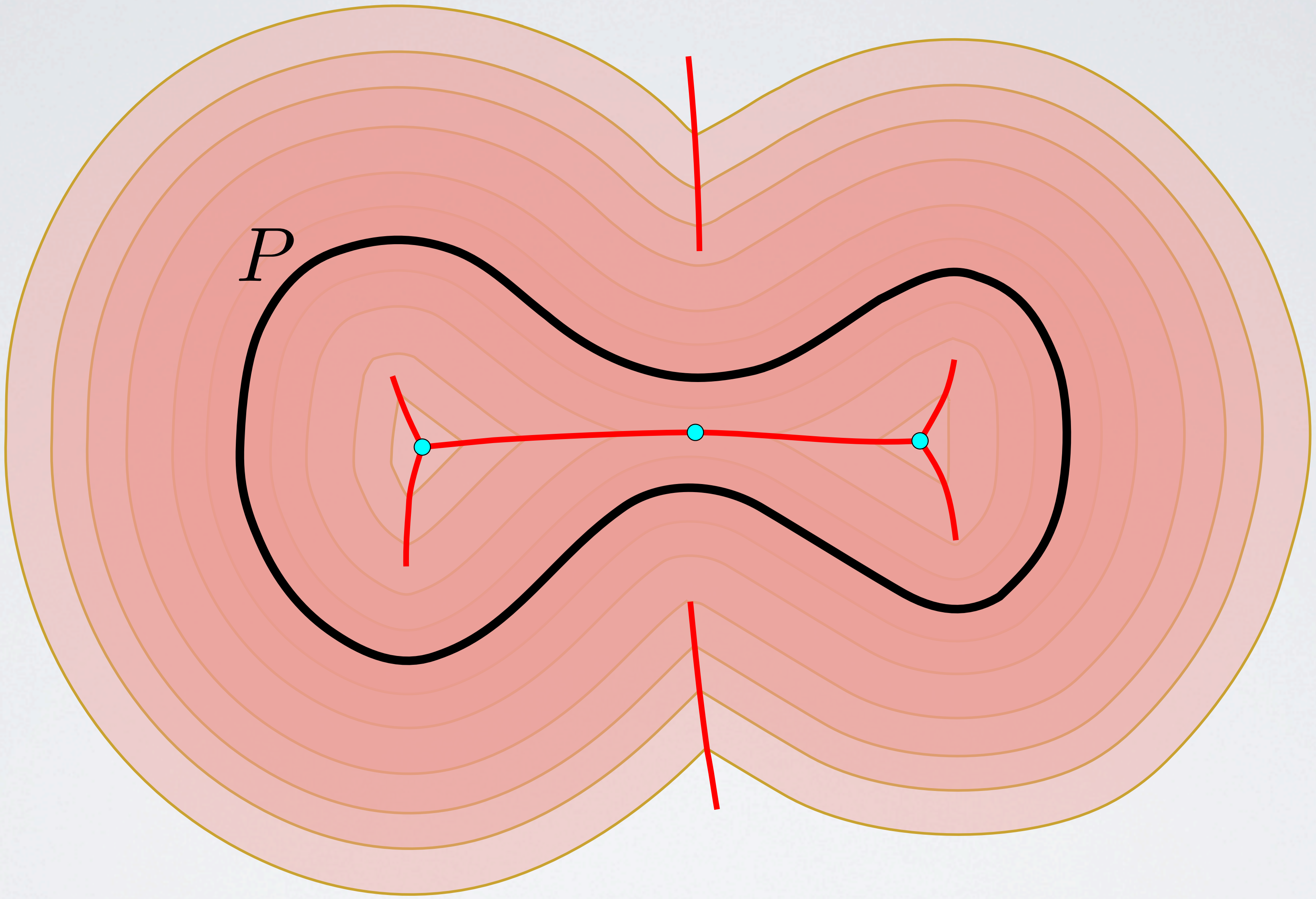
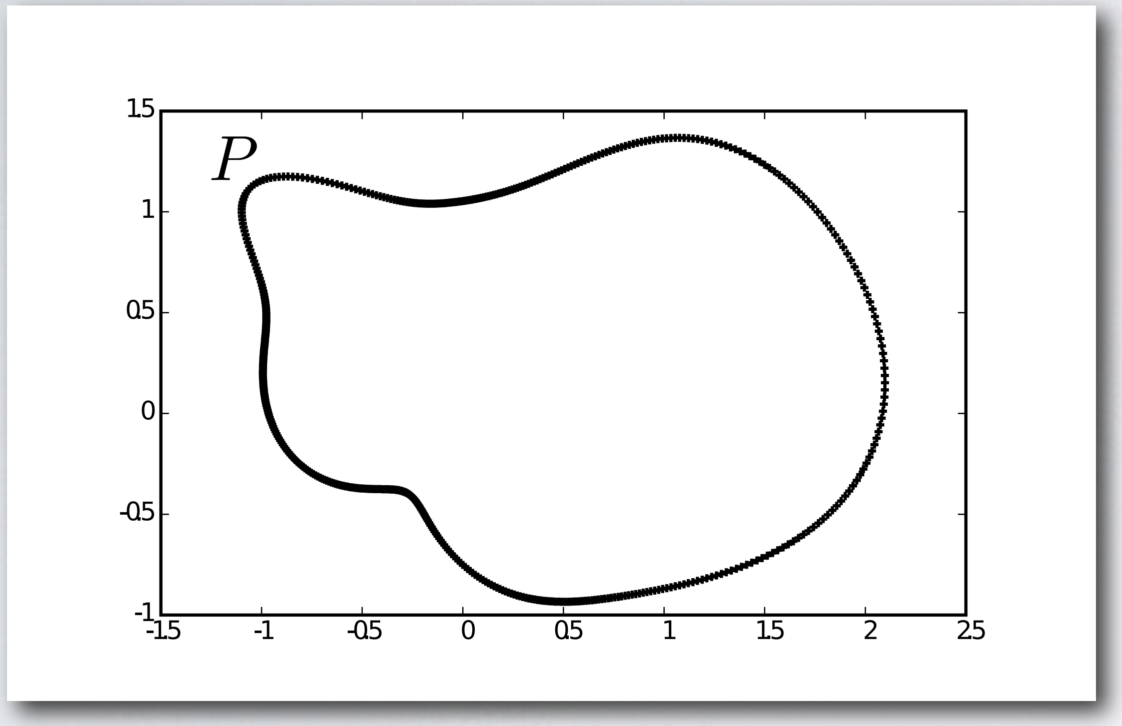
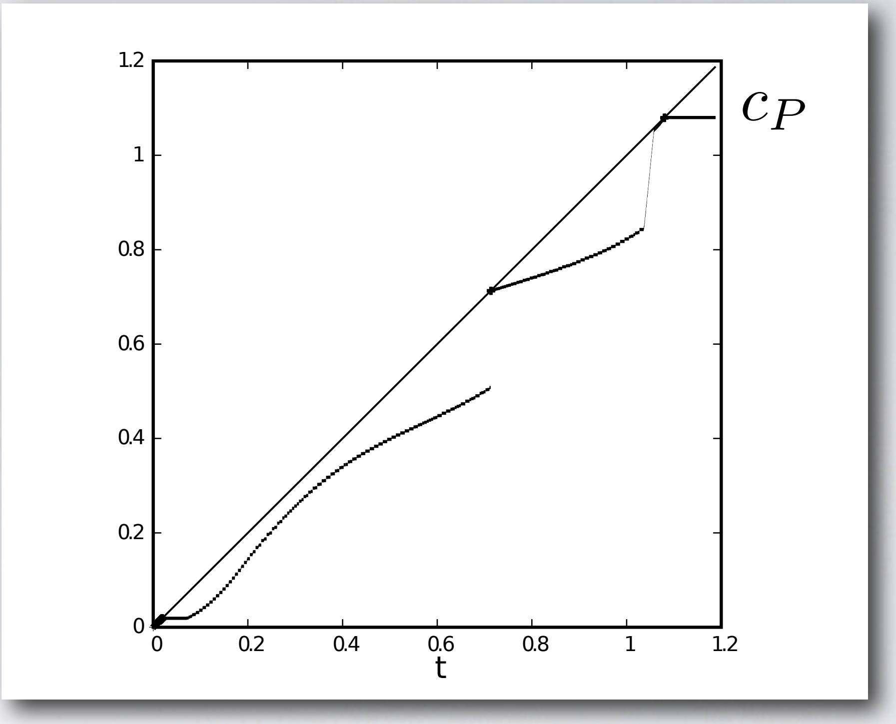


- Sublevel sets of $d(\cdot, P)$ are offsets of P.
- Topology of sublevel sets changes at critical values t_0.
- t_0 critical value $\iff$ $c_P(t_0) = t_0$

Convexity defects





$$\text{Centers}(P,t) = \bigcup_{\substack{\emptyset \neq \sigma \subset P \\ \text{Rad}(\sigma) \leq t}} \{\text{Center}(\sigma)\}.$$

$$c_P(t) = d_H(\text{Centers}(P,t) \mid P)$$

- For a compact set P: P convex $\iff c_P = 0$
- c_P non decreasing
- $c_P(t) \leq t$
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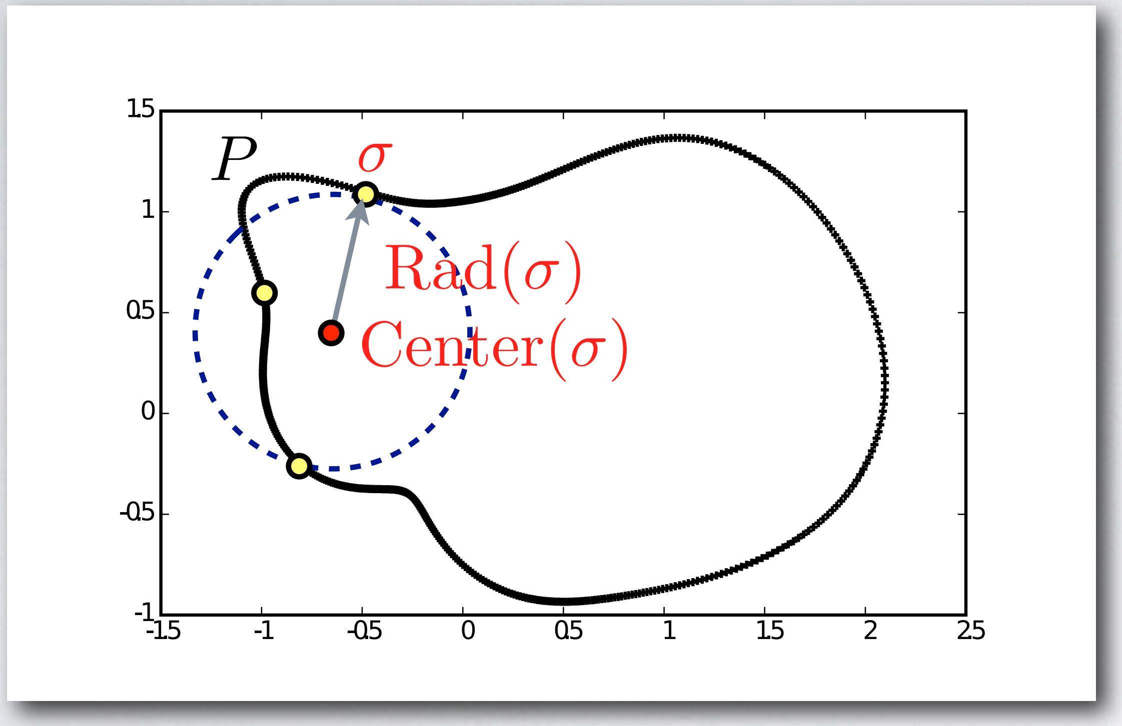
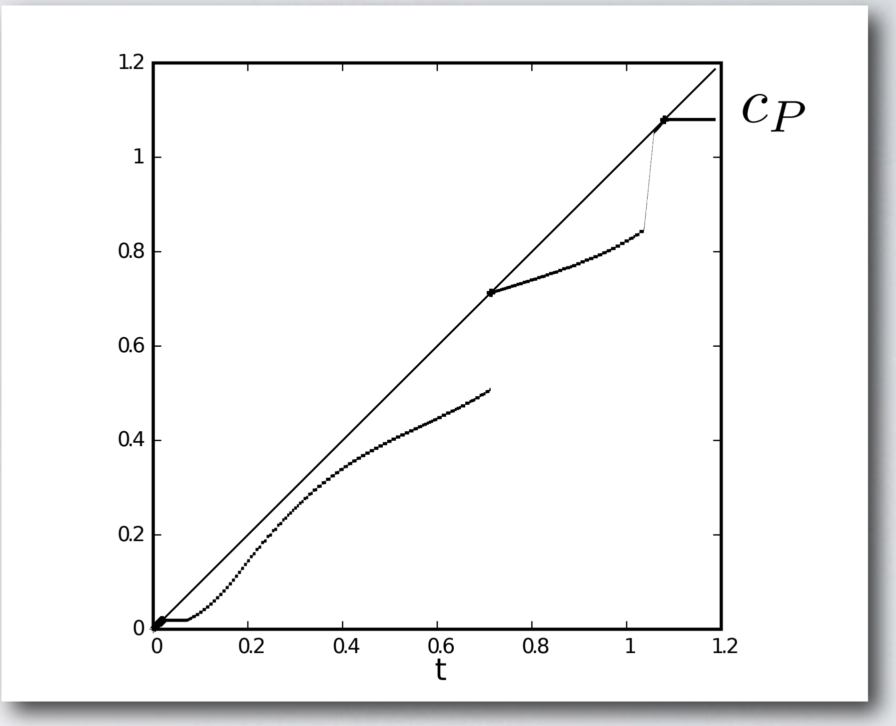
Convexity defects





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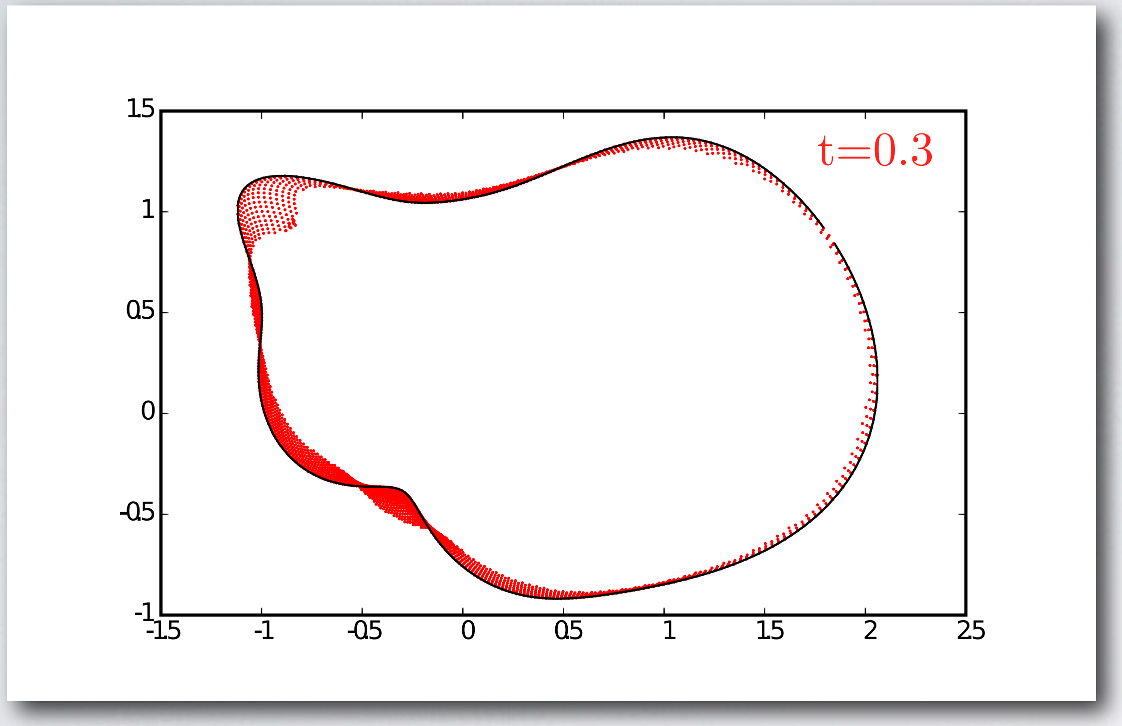

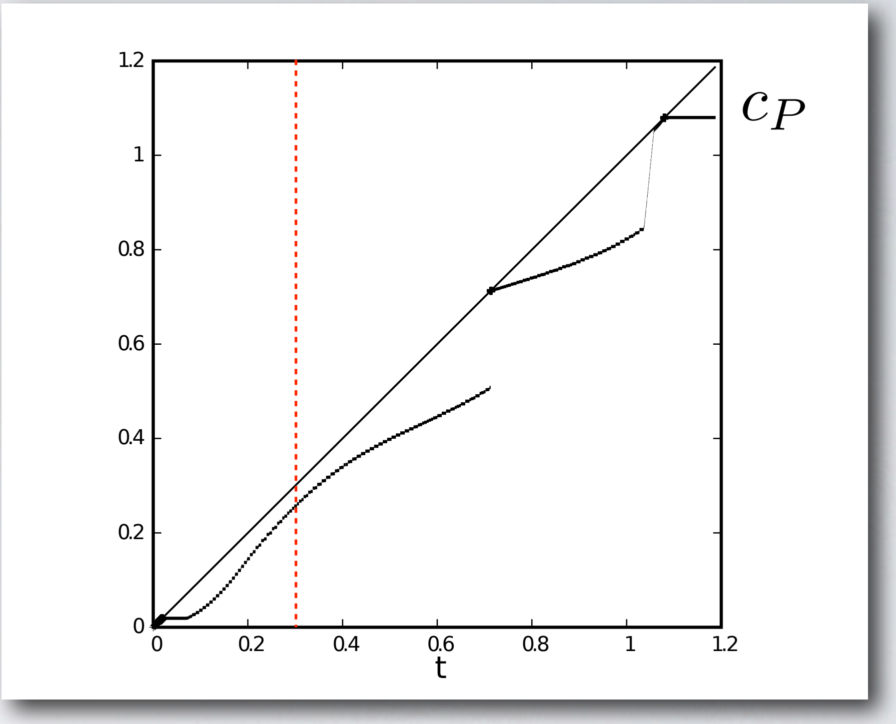

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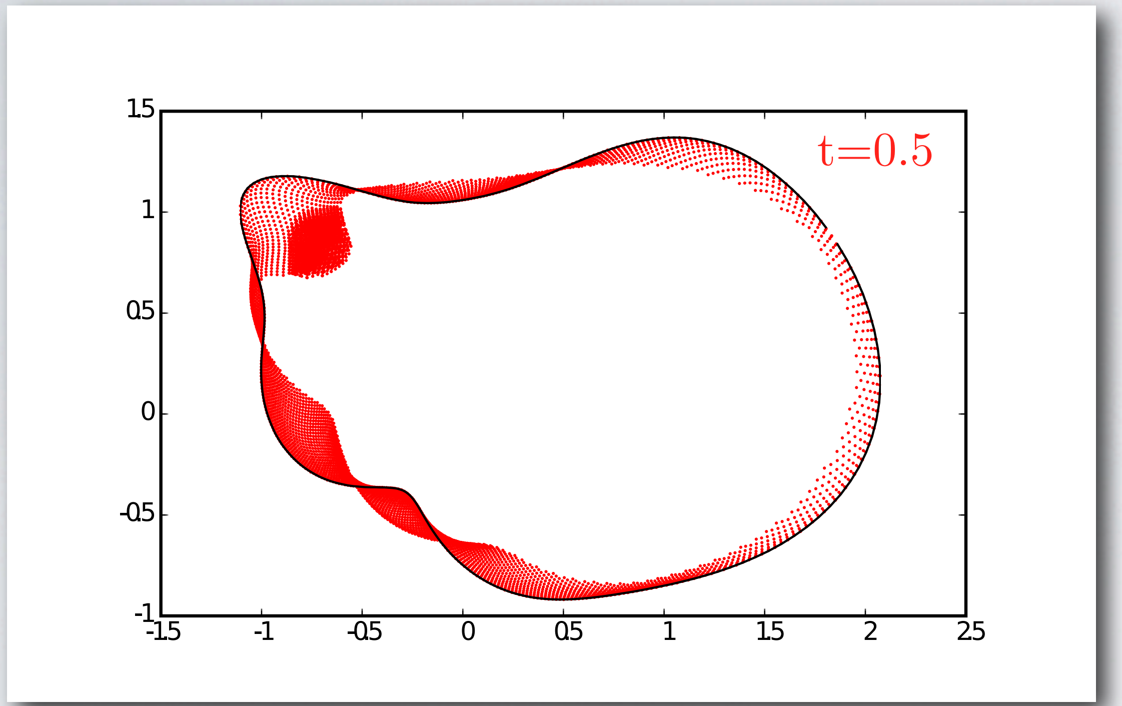

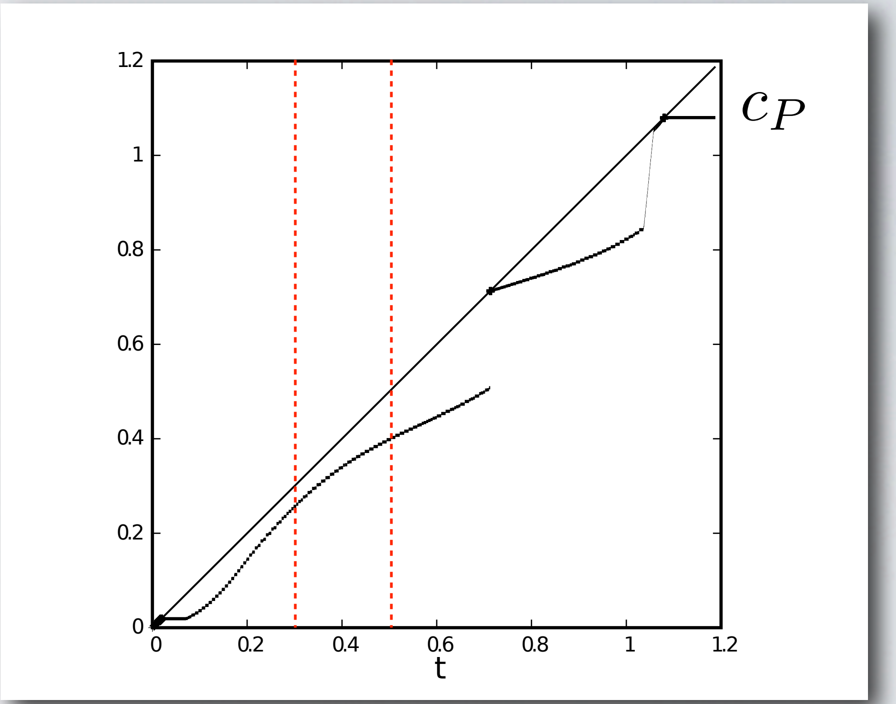

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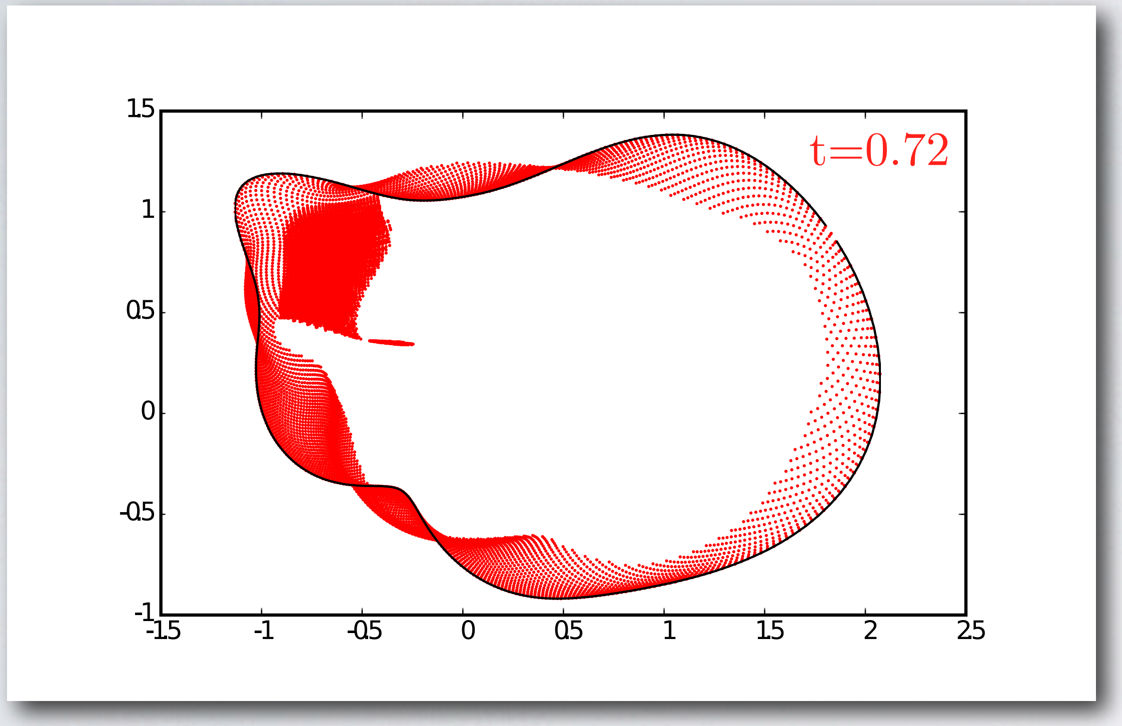
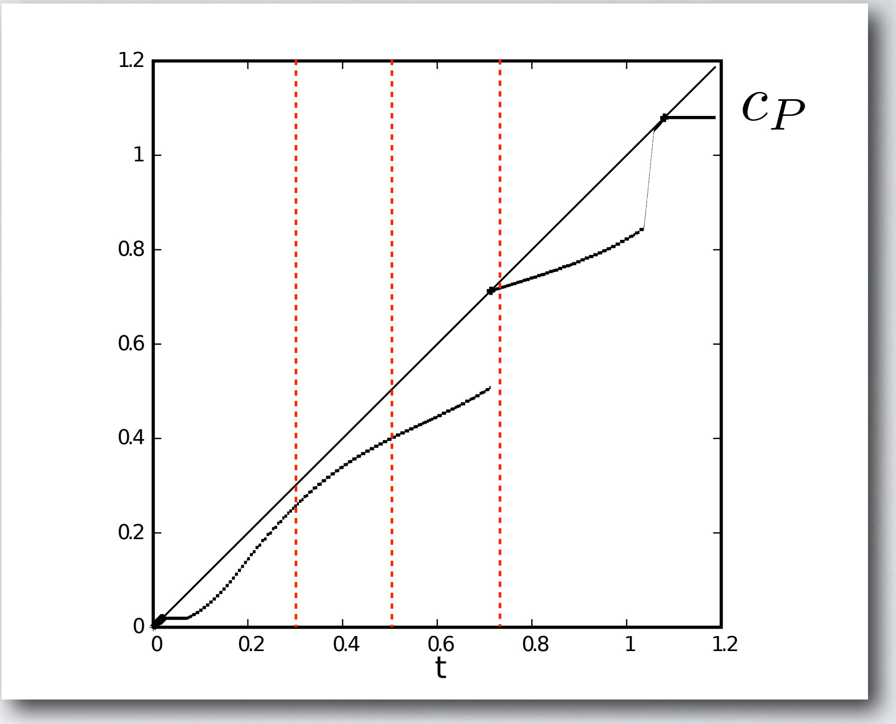
Convexity defects





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- For a compact set P: P convex $\iff c_P = 0$
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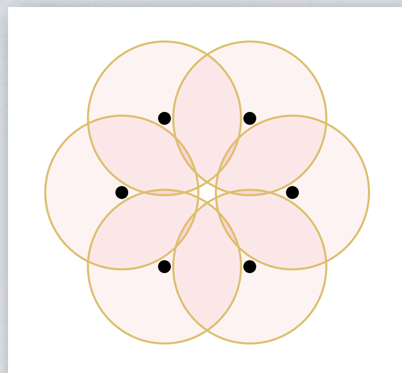
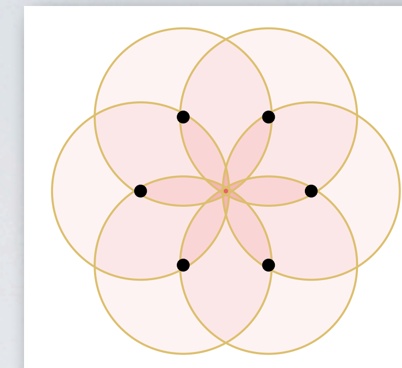
Roadmap

P^{α}



$P^{\vartheta_d \alpha}$



deformation retracts to

$c_P(t) < t, \quad \forall t \in [\alpha, \vartheta_d \alpha]$

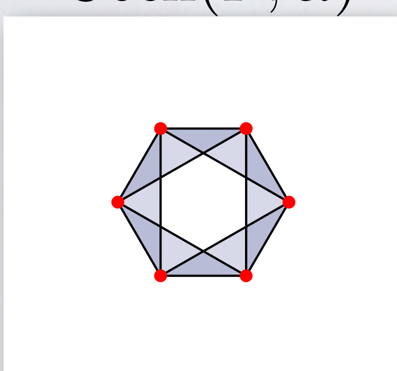
sequence of collapses

$\{\, \mathrm{Cech}(P, t) \,\}_{\alpha \le t \le \vartheta_d \alpha}$

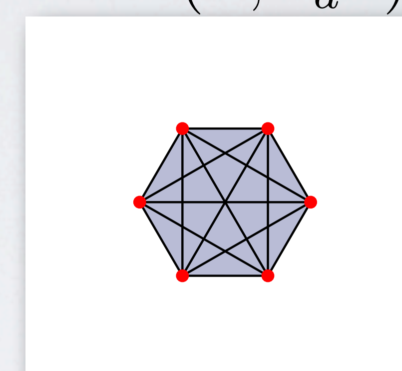
$\simeq$

$\simeq$

$\mathrm{Cech}(P, \alpha)$



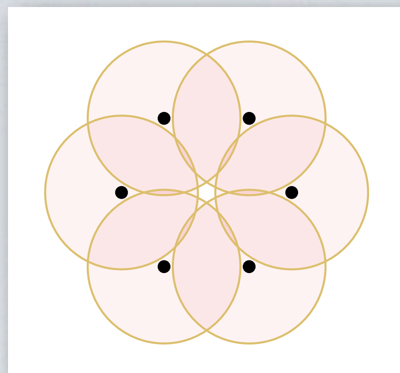
$\mathrm{Cech}(P, \vartheta_d \alpha)$



for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

Roadmap

P^{α}



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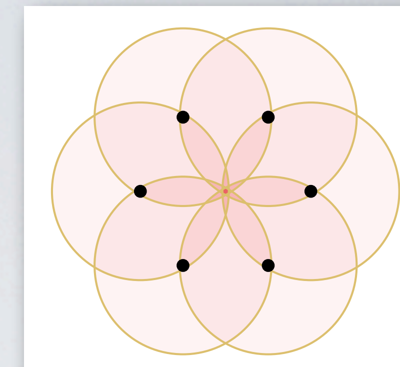

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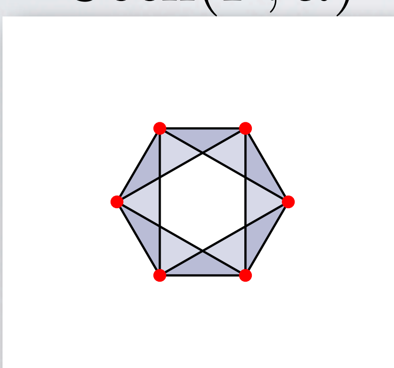
sequence of collapses

$\{\,\mathrm{Cech}(P,t)\,\}_{\alpha \le t \le \vartheta_d \alpha}$

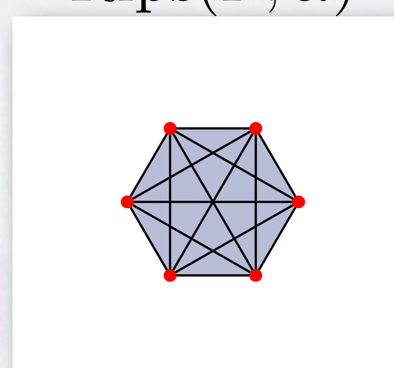
$\simeq$

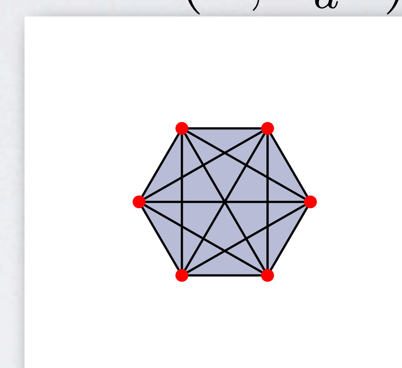
$\mathrm{Cech}(P,\alpha)$ $\subset$ $\mathrm{Rips}(P,\alpha)$ $\subset$ $\mathrm{Cech}(P,\vartheta_d \alpha)$



sequence of collapses

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$



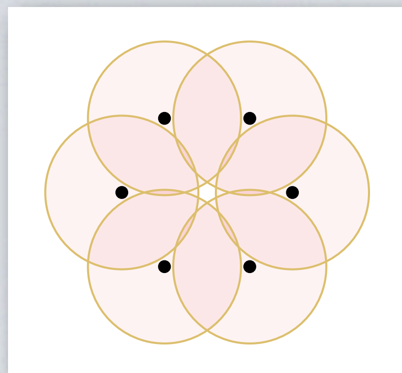


for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

$\{\,\mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)\,\}_{\alpha \le t \le \vartheta_d \alpha}$

Roadmap

P^{α}



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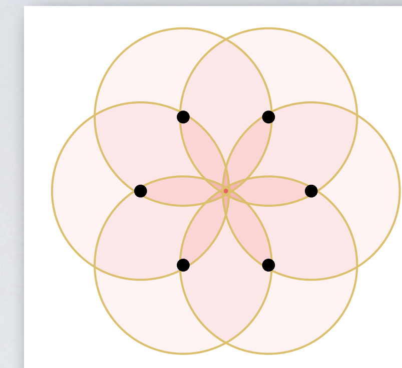

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$c_P(t) < t, \quad \forall t \in [\alpha, \vartheta_d \alpha]$

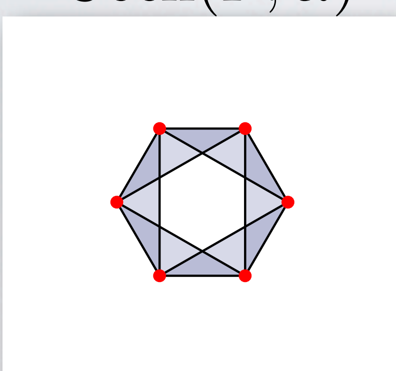
sequence of collapses

$\{\, \mathrm{Cech}(P,t)\, \}_{\alpha \le t \le \vartheta_d \alpha}$

$\simeq$

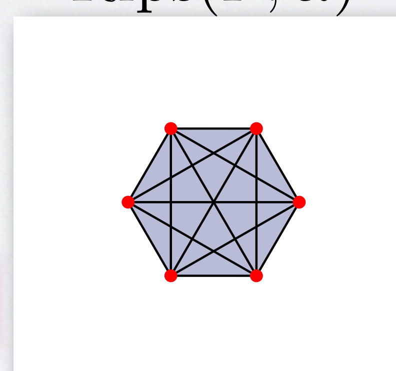
$\simeq$

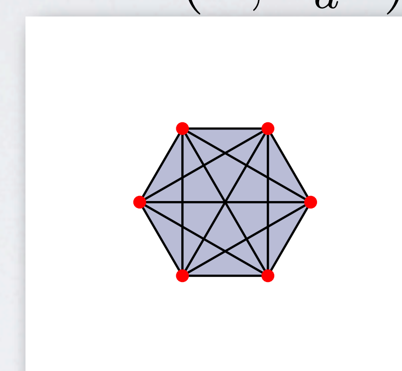
$\mathrm{Cech}(P,\alpha)$ $\subset$ $\mathrm{Rips}(P,\alpha)$ $\subset$ $\mathrm{Cech}(P,\vartheta_d \alpha)$



sequence of collapses

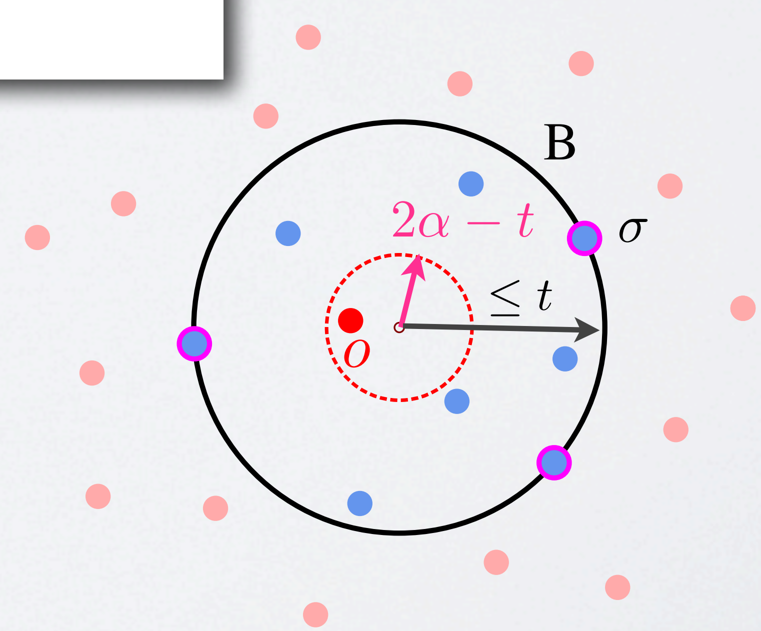
$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$





for $\vartheta_d = \sqrt{\frac{2d}{d+1}}$

$\{\, \mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)\, \}_{\alpha \le t \le \vartheta_d \alpha}$

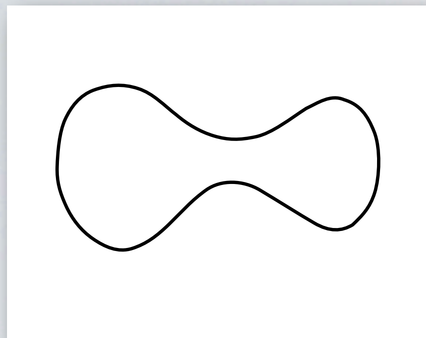


$\forall t \in [\alpha, \vartheta_d \alpha], \forall \sigma \in \mathrm{Cech}(P,t)$:

the link of $\sigma \in \mathrm{Cech}(P,t) \cap \mathrm{Rips}(P,\alpha)$ is a cone.

Rips complexes with L_2

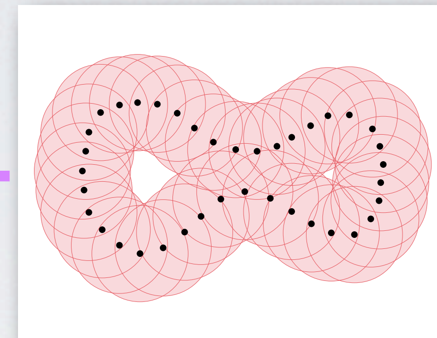
A



deformation retracts to

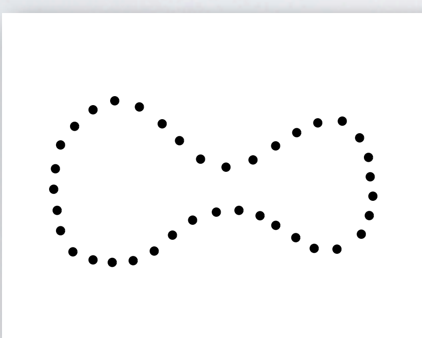
[NSW 04]

P^α

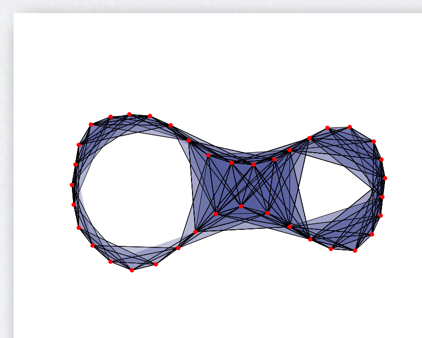


$\simeq$ **Nerve Lemma**

P

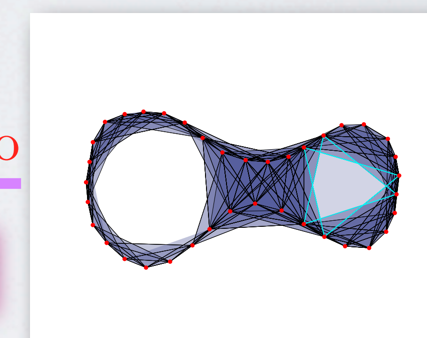


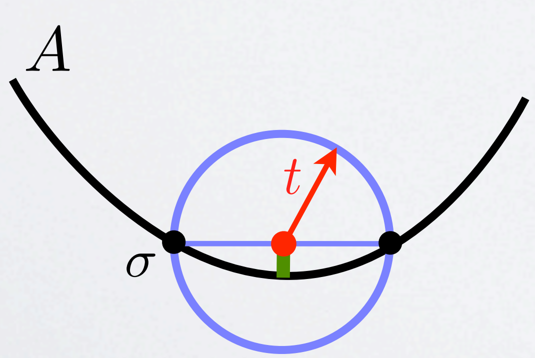
$\text{Cech}(P,\alpha)$ $\subset$ $\text{Rips}(P,\alpha)$



deformation retracts to

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$





if $d_H(A,P) \leq \varepsilon$, then for $t < \text{Reach}(A) - \varepsilon$

$$c_P(t) \leq \text{Reach}(A) - \sqrt{\text{Reach}(A)^2 - (t+\varepsilon)^2} + 2\varepsilon$$

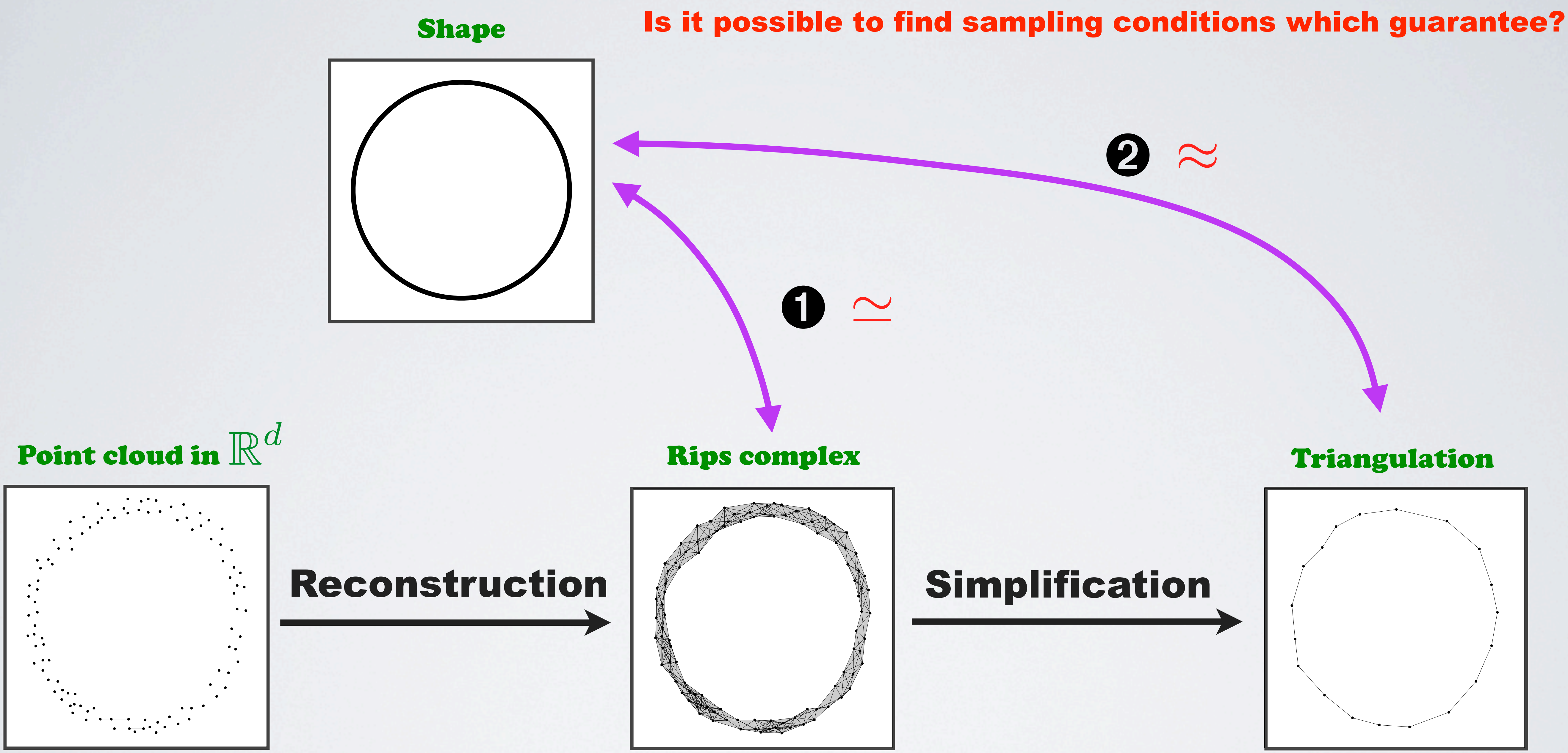
Shapes with a positive reach

A $\xleftrightarrow{\simeq}$ $\text{Rips}(P, \alpha)$ with ℓ_2

if $d_H(A, P) \leq \varepsilon$ and $\frac{\varepsilon}{\text{Reach } A} < \lambda$ and $\frac{\alpha}{\varepsilon} = \eta$

Reconstruction	d	λ	η
P^α with [NSW04]	$\forall d$	$3 - \sqrt{8} \approx 0.17$	$2 + \sqrt{2} \approx 3.41$
Rips (P, α)	2	0.063	5.00
	3	0.055	5.46
	4	0.050	5.76
	5	0.047	5.97
	10	0.041	6.50
	100	0.035	7.22
	$+\infty$	$\frac{2\sqrt{2-\sqrt{2}}-\sqrt{2}}{2+\sqrt{2}} \approx 0.0340$	7.22

Overview

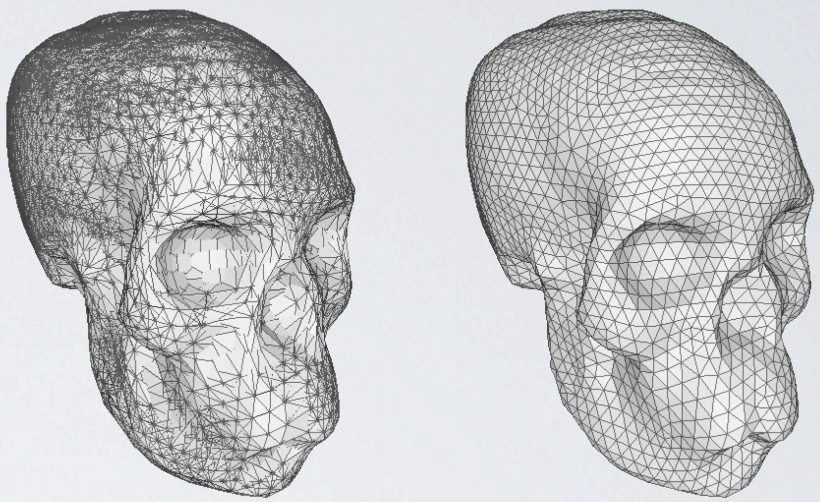


Can be high-dimensional!

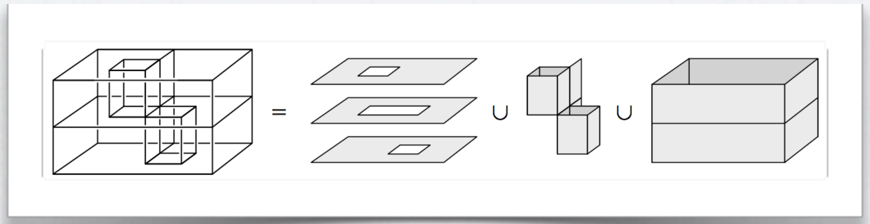
Does simplification exist?

How to get an object with the right dimension?

- Different strategies:
  - Edge contractions;
  - Vertex and edge collapses …
  - Seems to work well in practice



- And yet, not all obvious that the Rips complex whose vertices sample a shape contains a subcomplex homeomorphic to that shape.
  - A triangulated Bing's house is contractible but not collapsible



  - Geometry has to play a key role.

Ongoing work [A & Lieutier SoCG 2013]

Shape A

$\approx$

Triangulation of A

$\mathrm{Rips}(P, \alpha)$

Ongoing work [A & Lieutier SoCG 2013]

Shape A

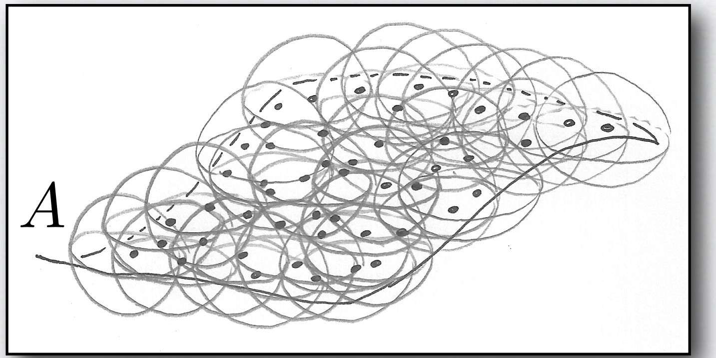
$\approx$

Triangulation of A

$\mathrm{Cech}(P, \alpha)$ ← sequence of collapses — $\mathrm{Rips}(P, \alpha)$

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$

$\mathrm{Nerve}\{B(p, \alpha) \mid p \in P\}$



Ongoing work [A & Lieutier SoCG 2013]

Shape A

$\approx$

Triangulation of A

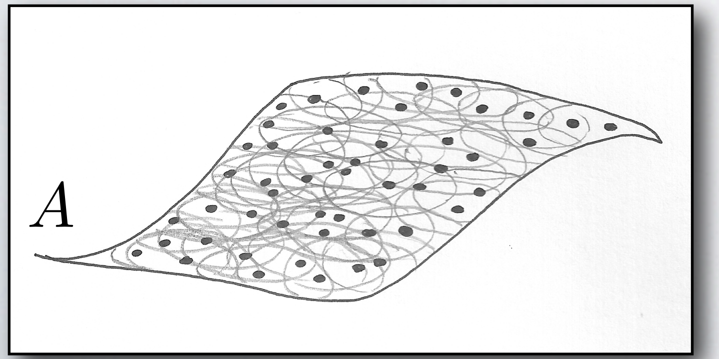
$\mathrm{Cech}_A(P,\alpha)$ ← sequence of collapses ← $\mathrm{Cech}(P,\alpha)$ ← sequence of collapses ← $\mathrm{Rips}(P,\alpha)$

$d_H(A,P) \leq \varepsilon < (3-\sqrt{8})\,\mathrm{Reach}\,A$

$\alpha = (2+\sqrt{2})\varepsilon$

$c_P(\vartheta_d\alpha) < 2\alpha - \vartheta_d\alpha$

$\mathrm{Nerve}\{A \cap B(p,\alpha) \mid p \in P\}$



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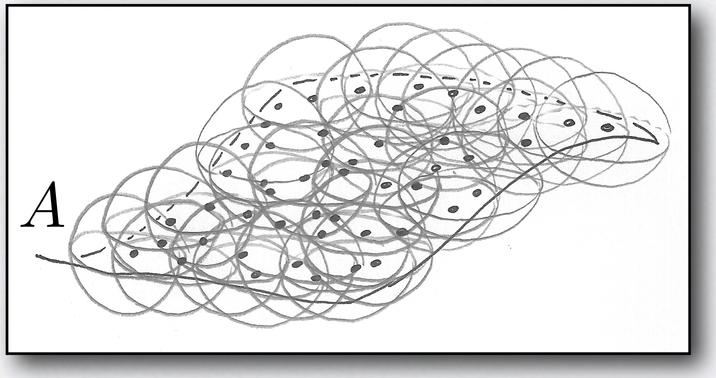


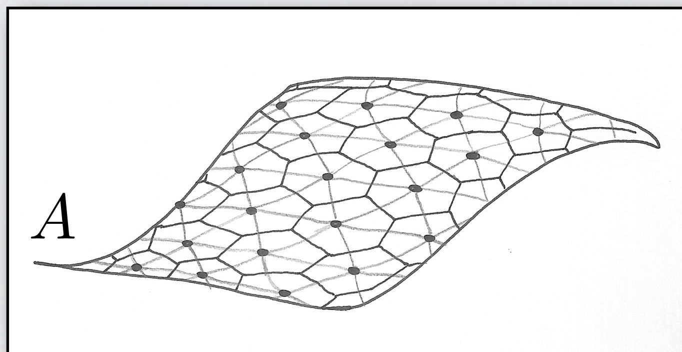
Ongoing work [A & Lieutier SoCG 2013]

Shape A

$\approx$

$\exists$? α-Nice Triangulation of A

Nerve$\{A \cap \mathrm{Hull}_\alpha(\mathrm{Cell}(v)) \mid v \in V\}$



with $A = \bigcup_{v \in V} \mathrm{Cell(v)}$ and

$\mathrm{Cell}(v) \subset B(p, \alpha)$ for some $p \in P$

sequence of collapses

$\alpha < \mathrm{Reach}\, A$

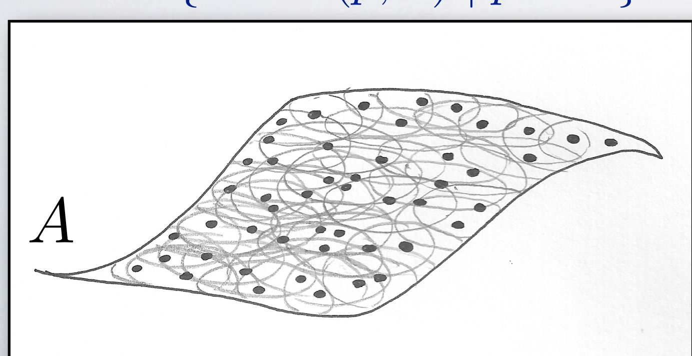
$\mathrm{Cech}_A(P, \alpha)$ ← sequence of collapses ← $\mathrm{Cech}(P, \alpha)$ ← sequence of collapses ← $\mathrm{Rips}(P, \alpha)$

$d_H(A, P) \leq \varepsilon < (3 - \sqrt{8})\, \mathrm{Reach}\, A$

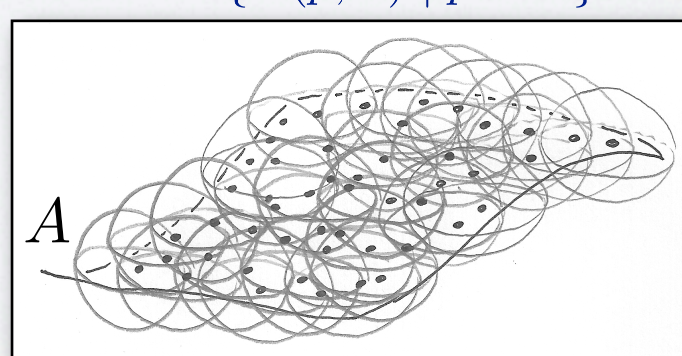
$\alpha = (2 + \sqrt{2})\varepsilon$

$c_P(\vartheta_d \alpha) < 2\alpha - \vartheta_d \alpha$

Nerve$\{A \cap B(p, \alpha) \mid p \in P\}$

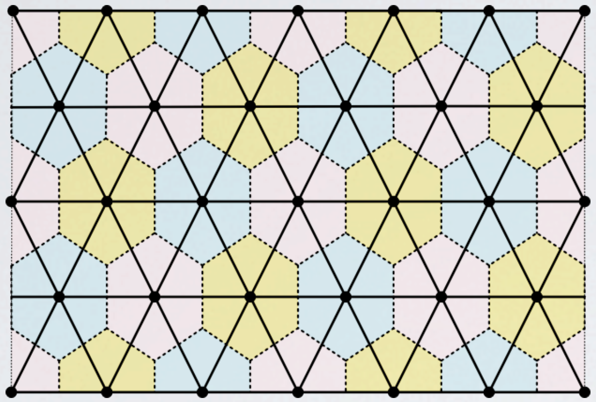


Nerve$\{B(p, \alpha) \mid p \in P\}$

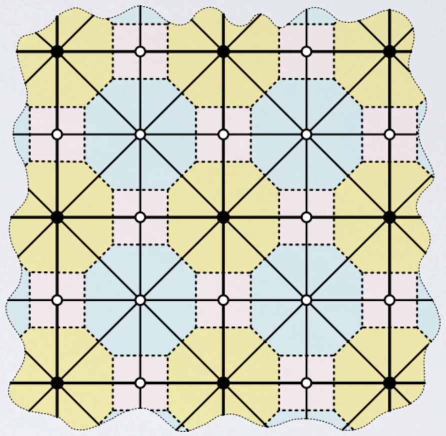


Future work

Shapes with α-nice triangulations?



Flat torus $\mathbb{T}^2$ in $\mathbb{R}^4$

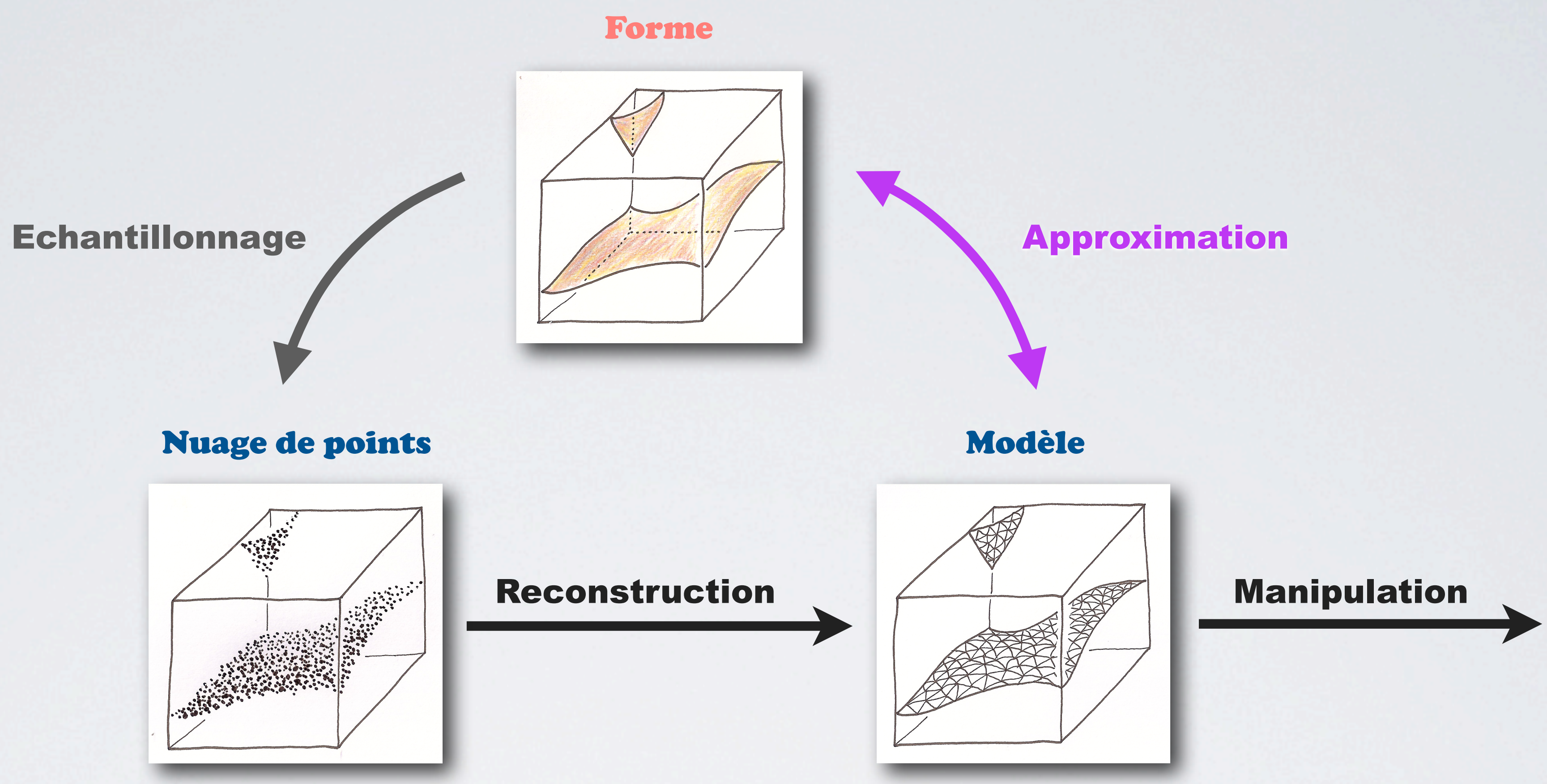


$\mathbb{R}^m$

How to turn all this into a practical algorithm?

- In general
  - Collapsibility of 3-complexes is NP-hard [Martin Tancer 2012]
  - Geometry has to play a key role.
- For Rips complexes
  - whose vertices sample a convex set, a 0-manifold or a 1-manifold
  - How to go beyond?

Wrap-up



Géométrie élémentaire | **Topologie** | **Algorithmique**

- Fonction de distance, théorie de Morse, points critiques, gradient, axe médian, reach, ...
- Homéomorphisme, type d'homotopie, se rétracte par déformation, ...
- Complexes simpliciaux abstraits, Nerves, Flag complexes, collapse, ...
- Triangulation de Delaunay, Cech complex, Rips complex, ...
- Structure de données, complexité, preuves de NP-complétude, ...

References

[AL10] D. Attali and A. Lieutier. **Reconstructing shapes with guarantees by unions of convex sets**. In *Proc. 26th Ann. Sympos. Comput. Geom.*, pages 344–353, Snowbird, Utah, June 13-16 2010.

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