Big Data & Greedy Trees

Gérard Biau & Luc Devroye

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Outline

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- Big Data requires massively parallel softwares running on tens, hundreds, or even thousands of servers.

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- Greedy methods have an autonomy that makes them ideally suited for distributive or parallel computation.
- In the short term, parallelism will take hold in massive datasets and complex systems.

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- They may find use in a world with new computational models in which parallel or distributed computation is feasible and even the norm.

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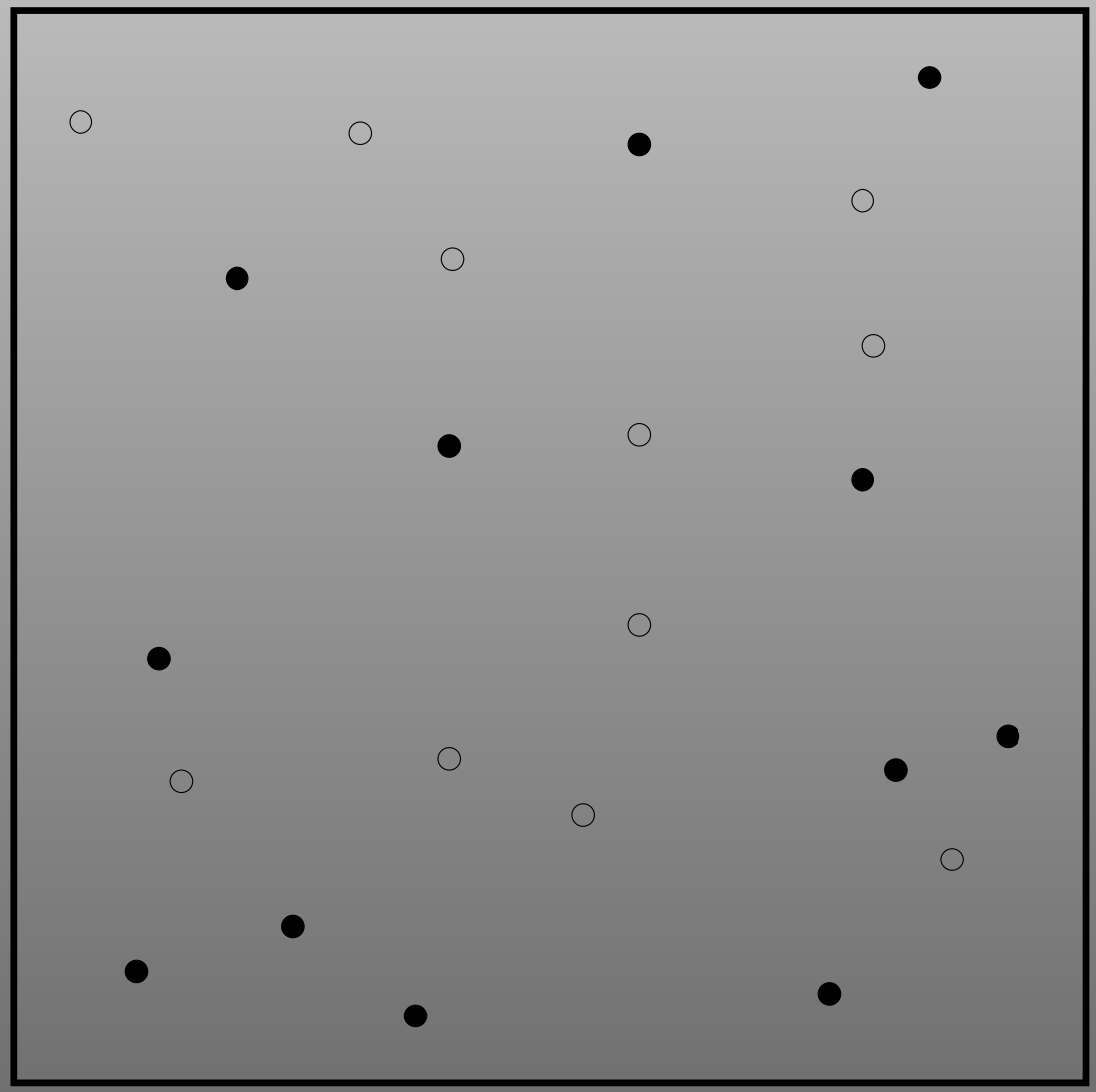
Classification



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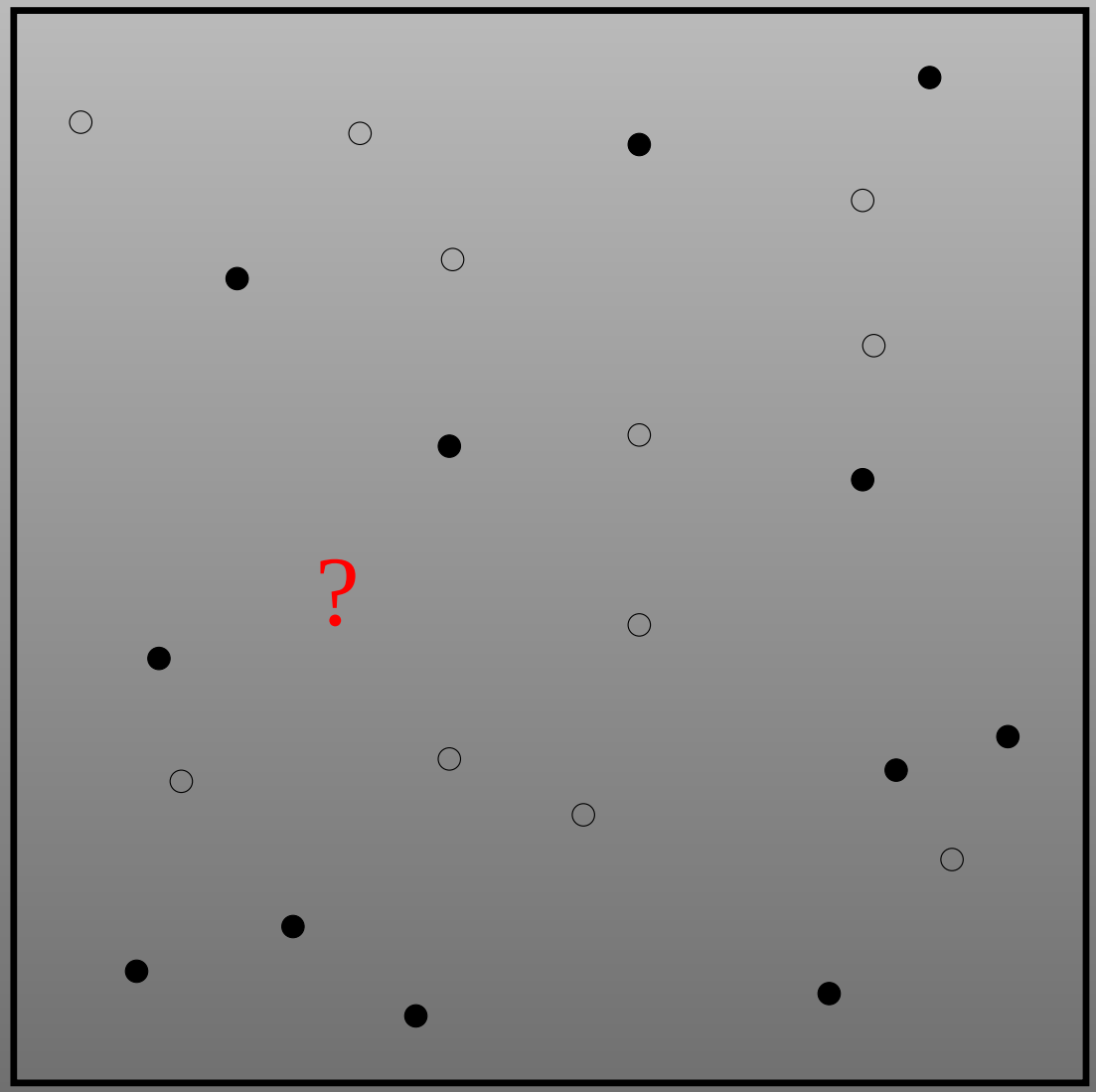

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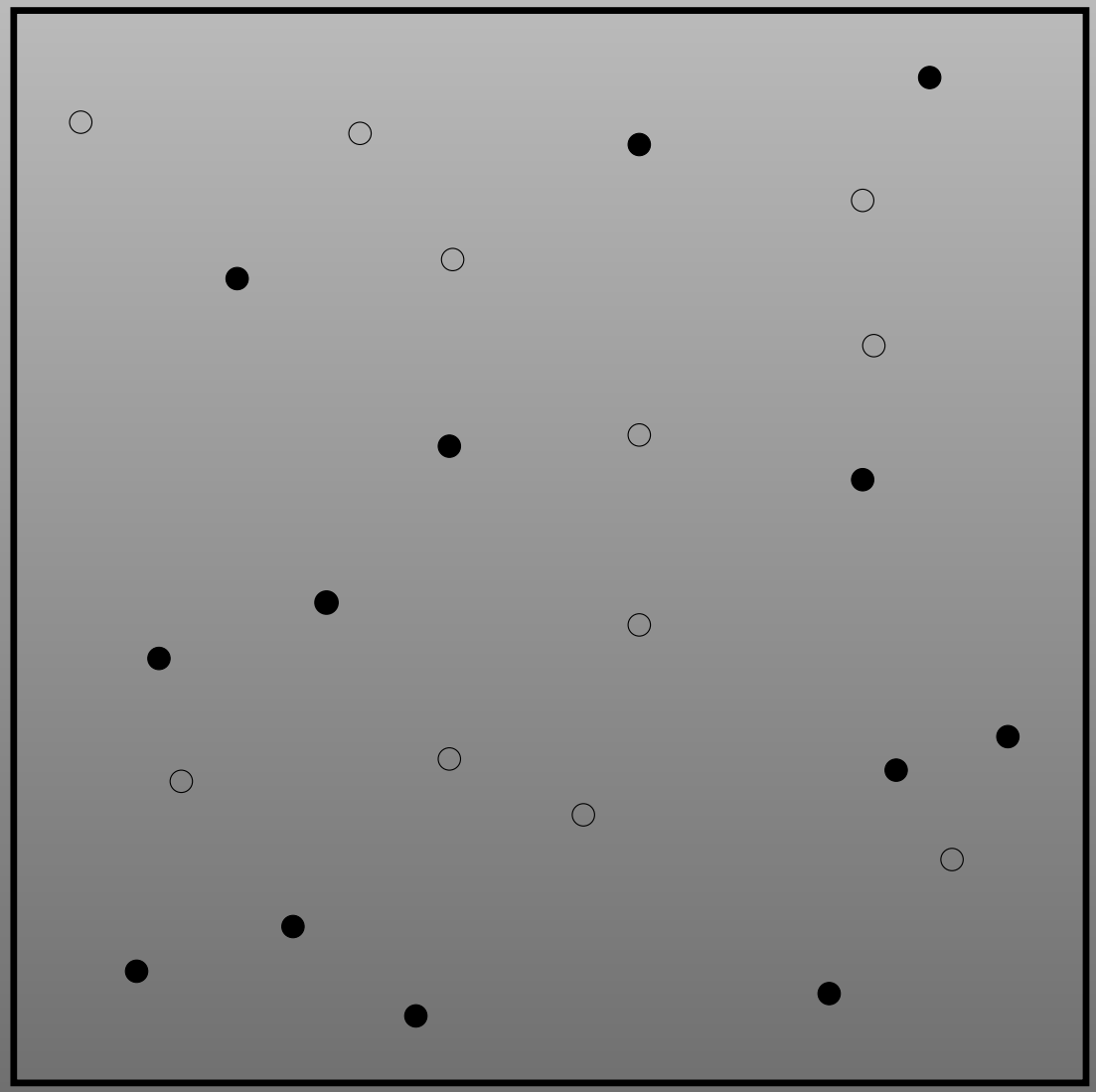

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- The Bayes classifier

$$g^\star(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbb{P}\{Y = 1 | \mathbf{X} = \mathbf{x}\} > 1/2 \\ 0 & \text{otherwise} \end{cases}$$

has the smallest probability of error, that is

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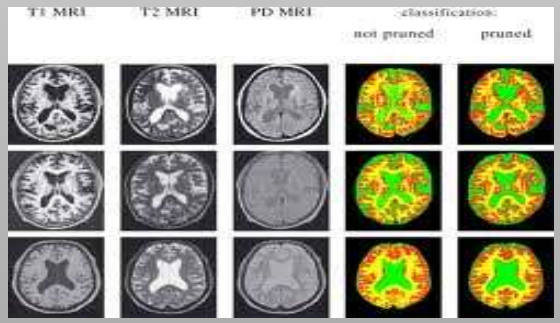
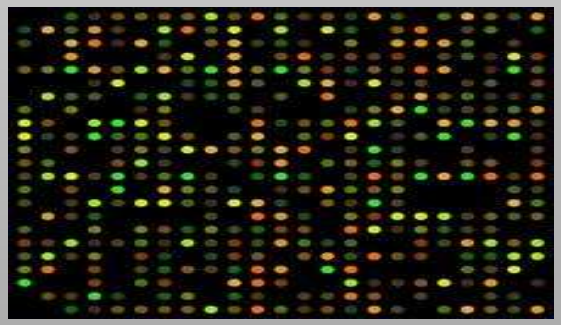
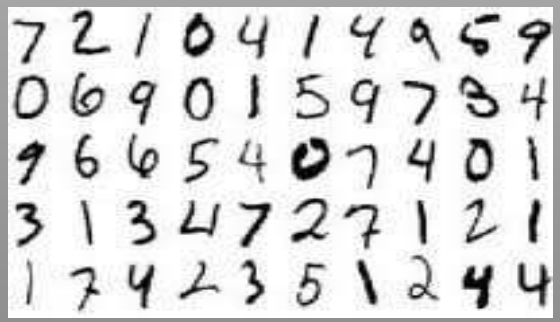
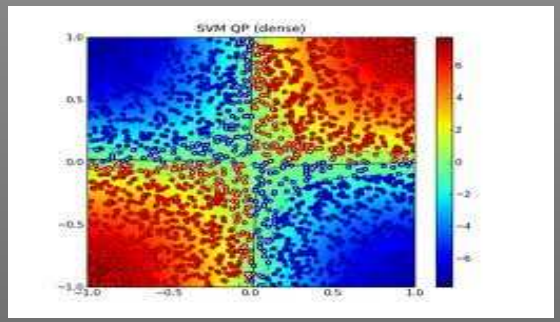
- It is universally consistent if it is consistent for all possible distributions of $(\mathbf{X}, Y)$.


T1 MRI
T2 MRI
PD MRI
classification:
not pruned
pruned






SVM QP (dense)

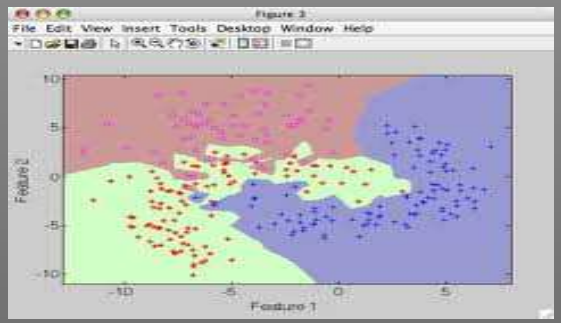


Figure 3
File Edit View Insert Tools Desktop Window Help
Feature 2
Feature 1

Tree classifiers

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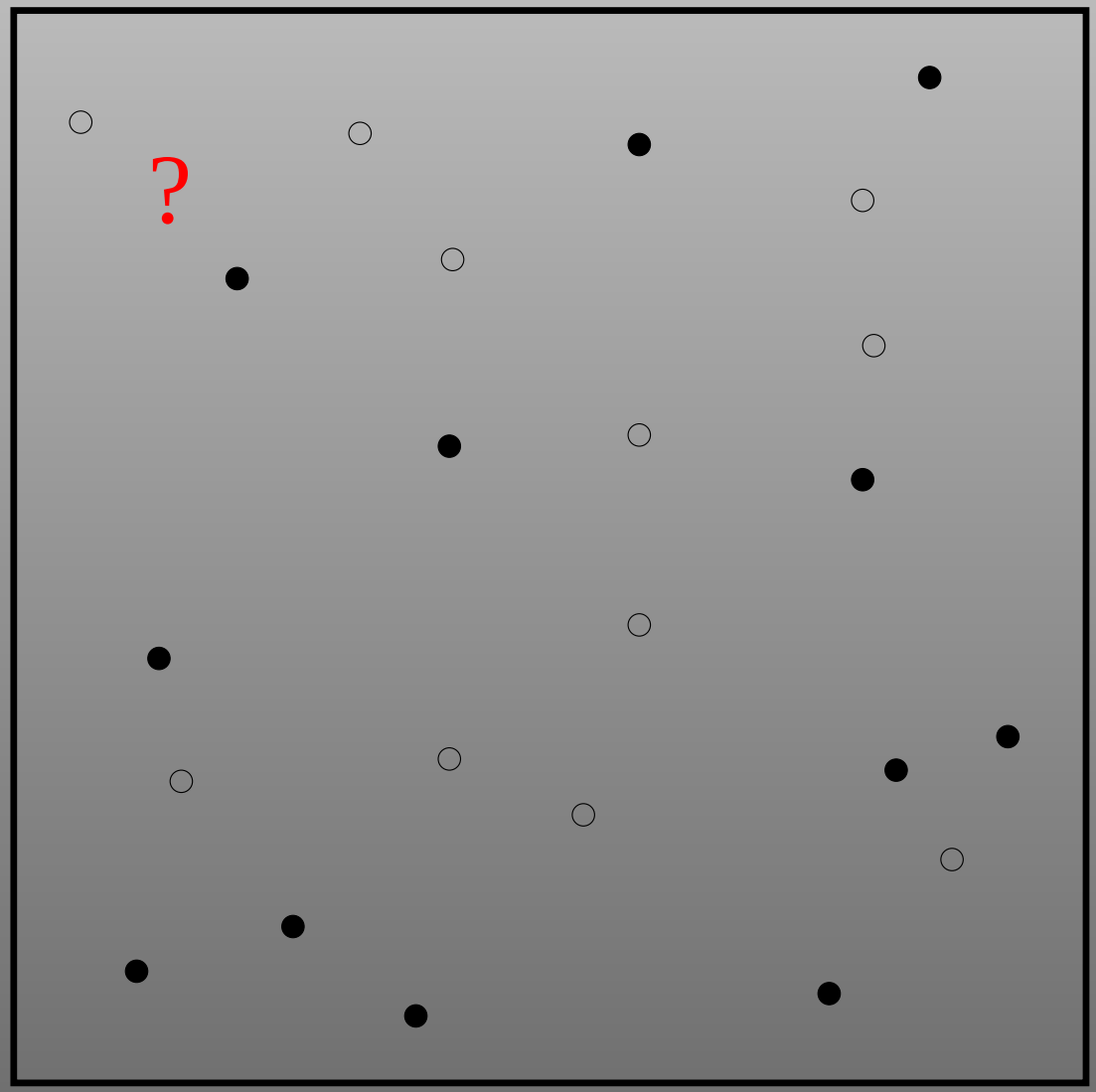
  - All procedures that partition space can be viewed as special cases of partitions generated by trees.
  - Simple neural networks that use voting methods can also be regarded as trees.
  - Tree classifiers are conceptually simple, and explain the data very well.

Trees



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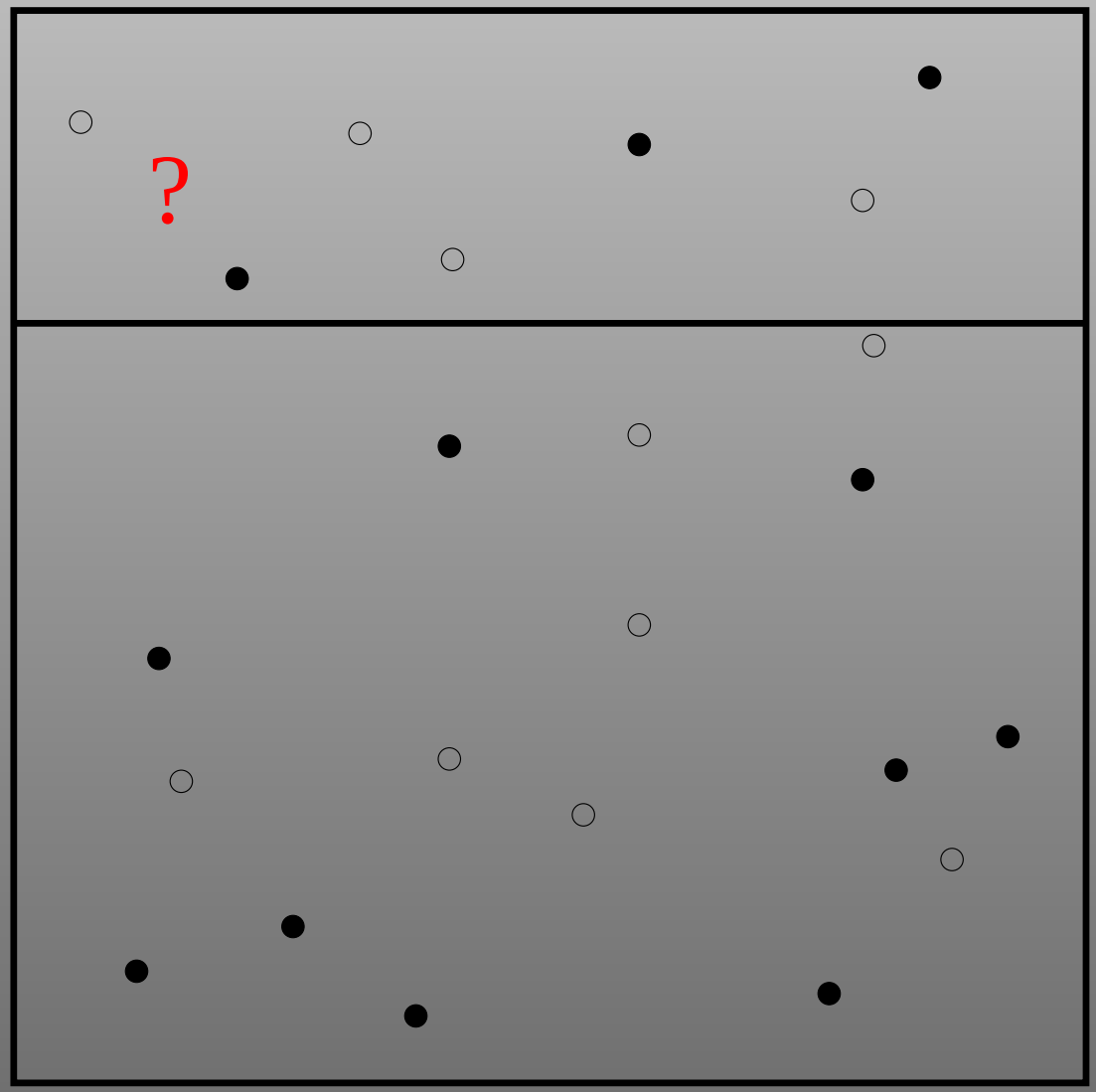

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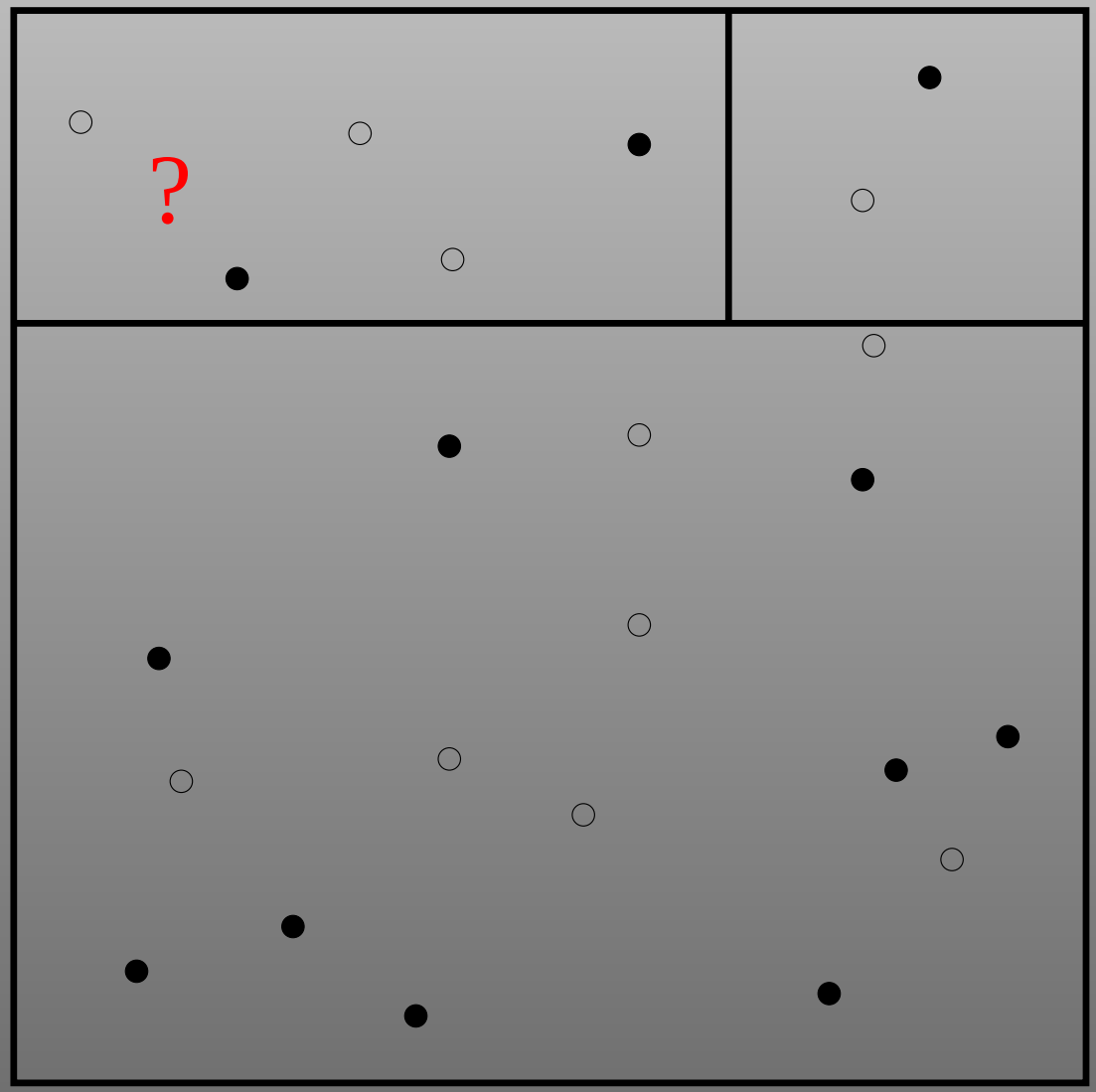

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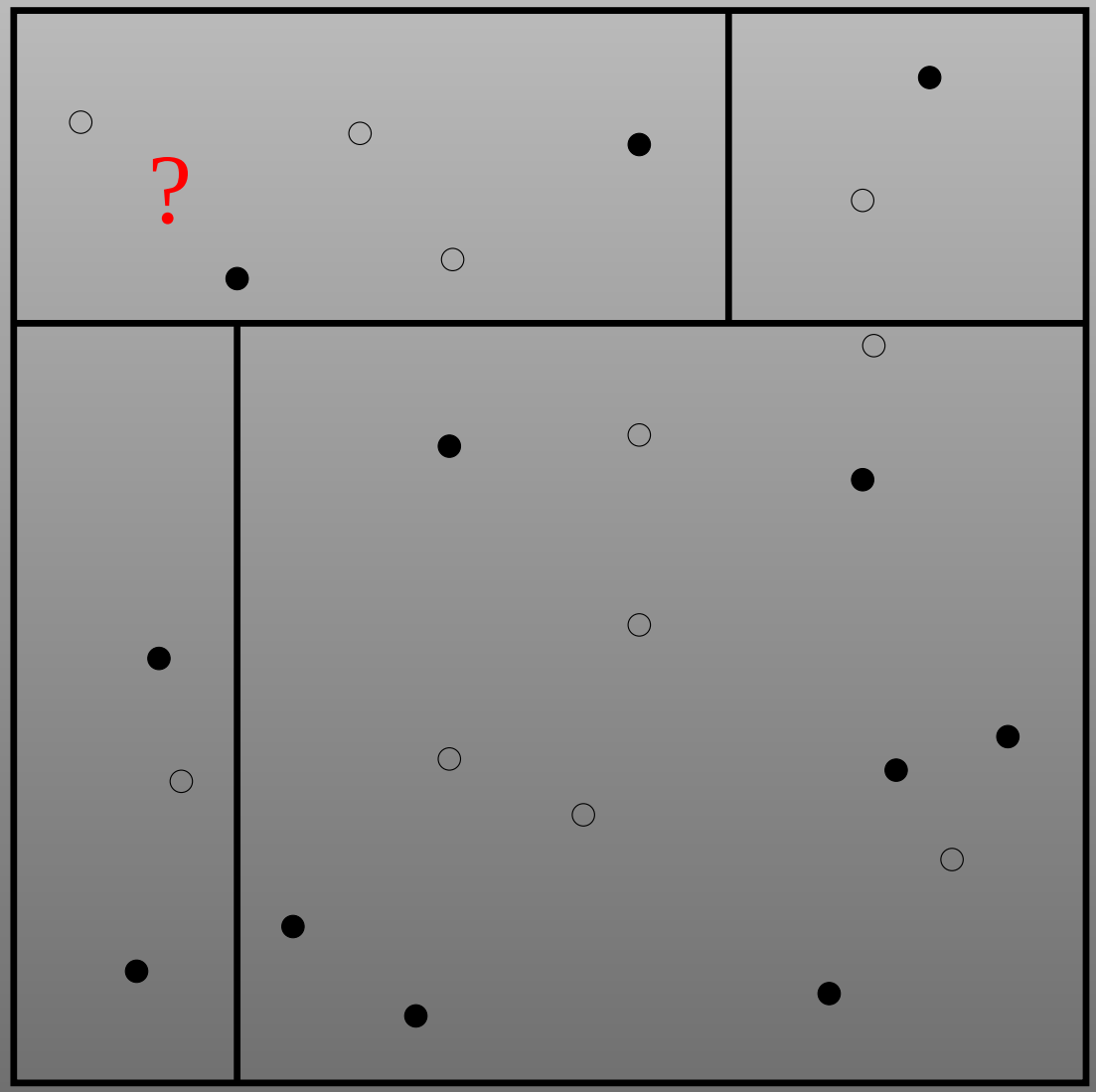

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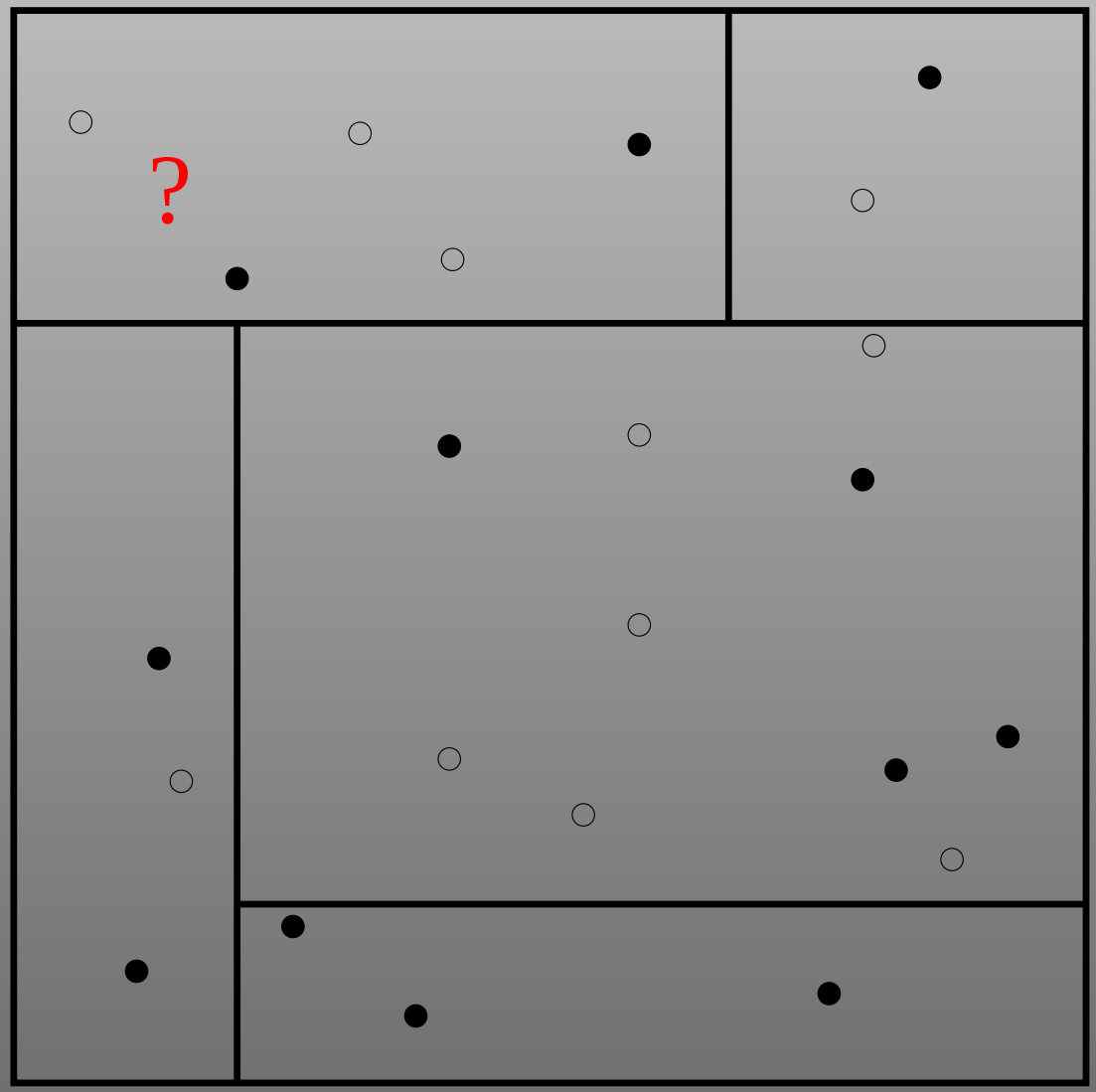

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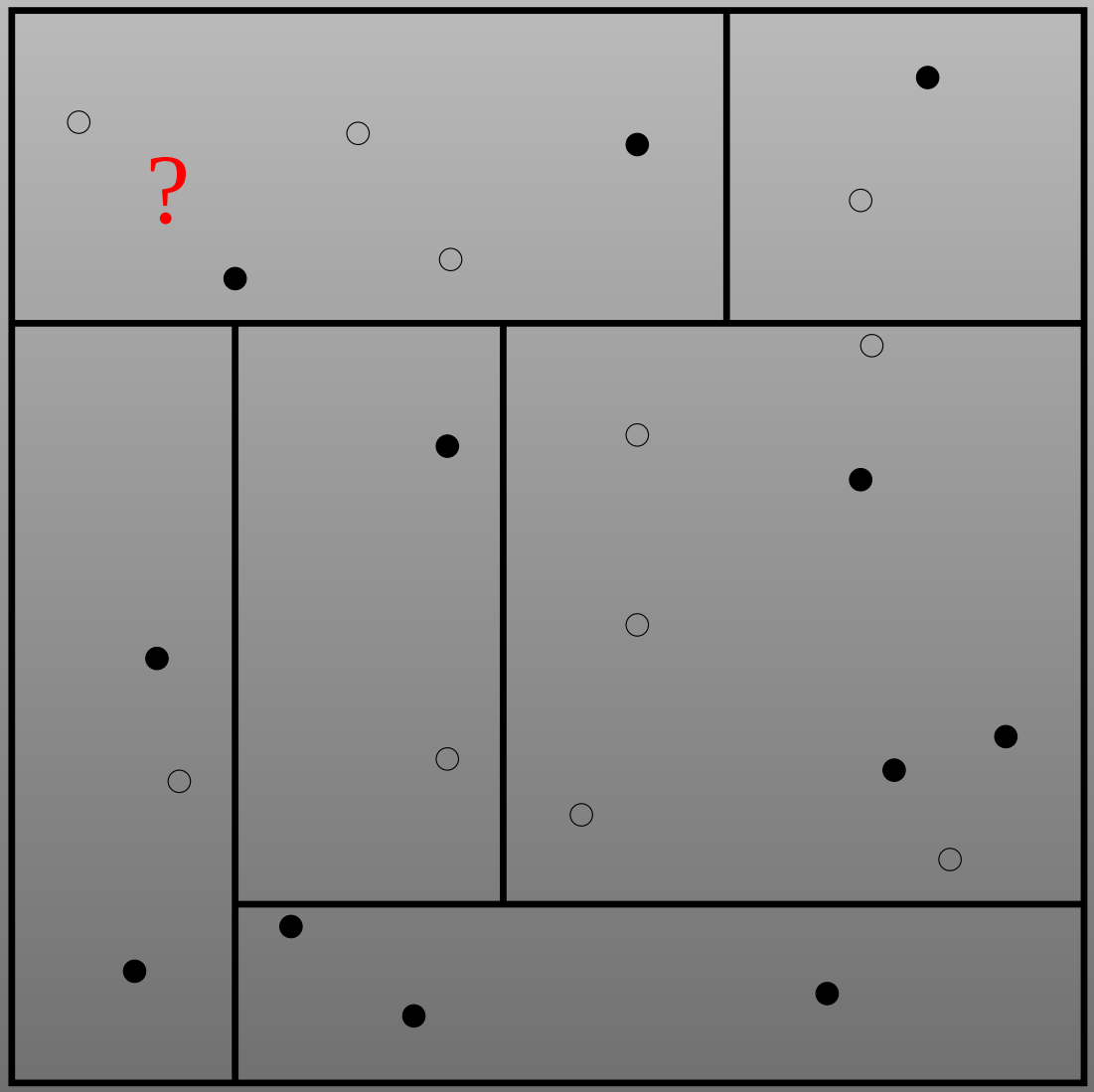

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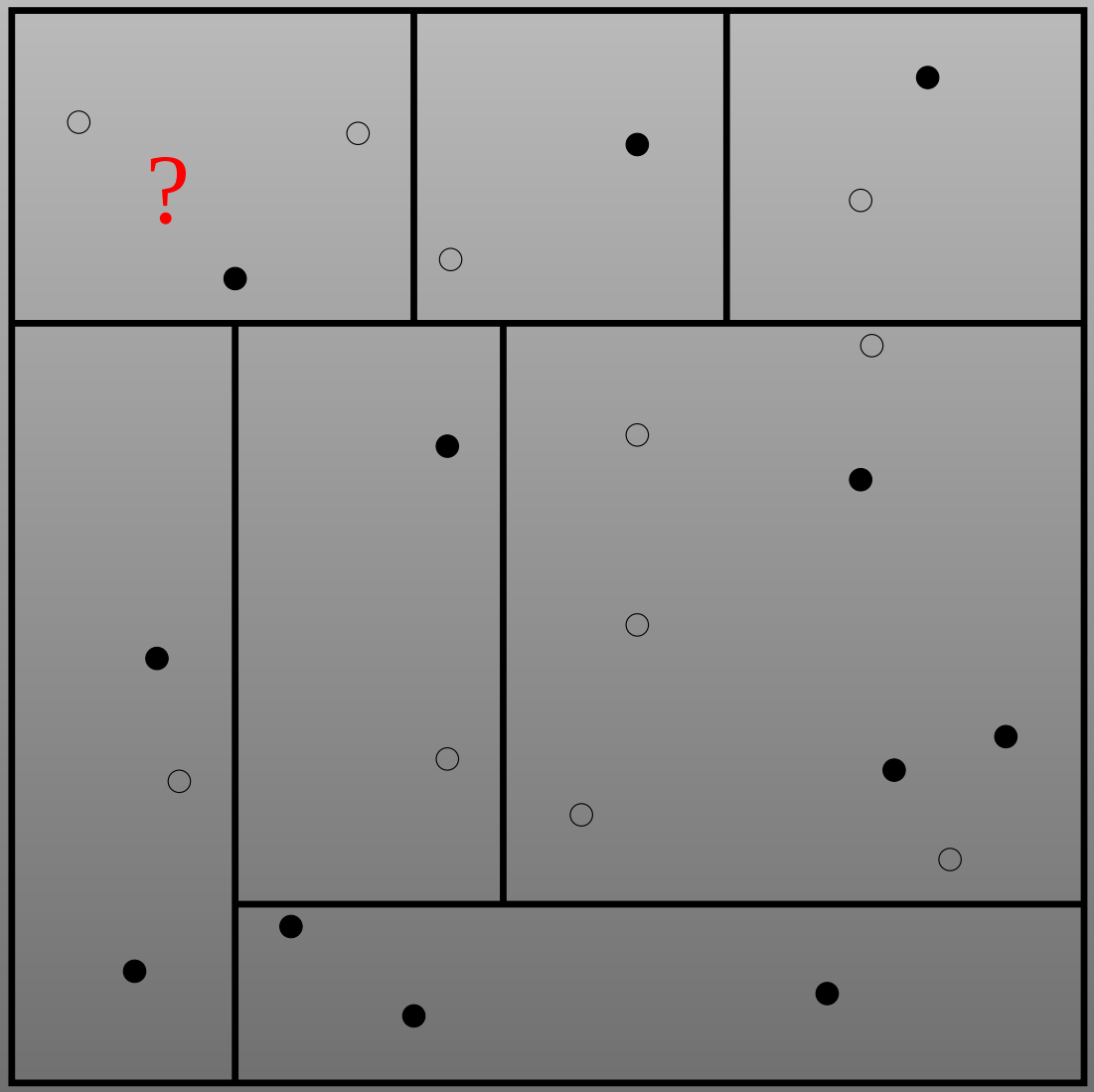

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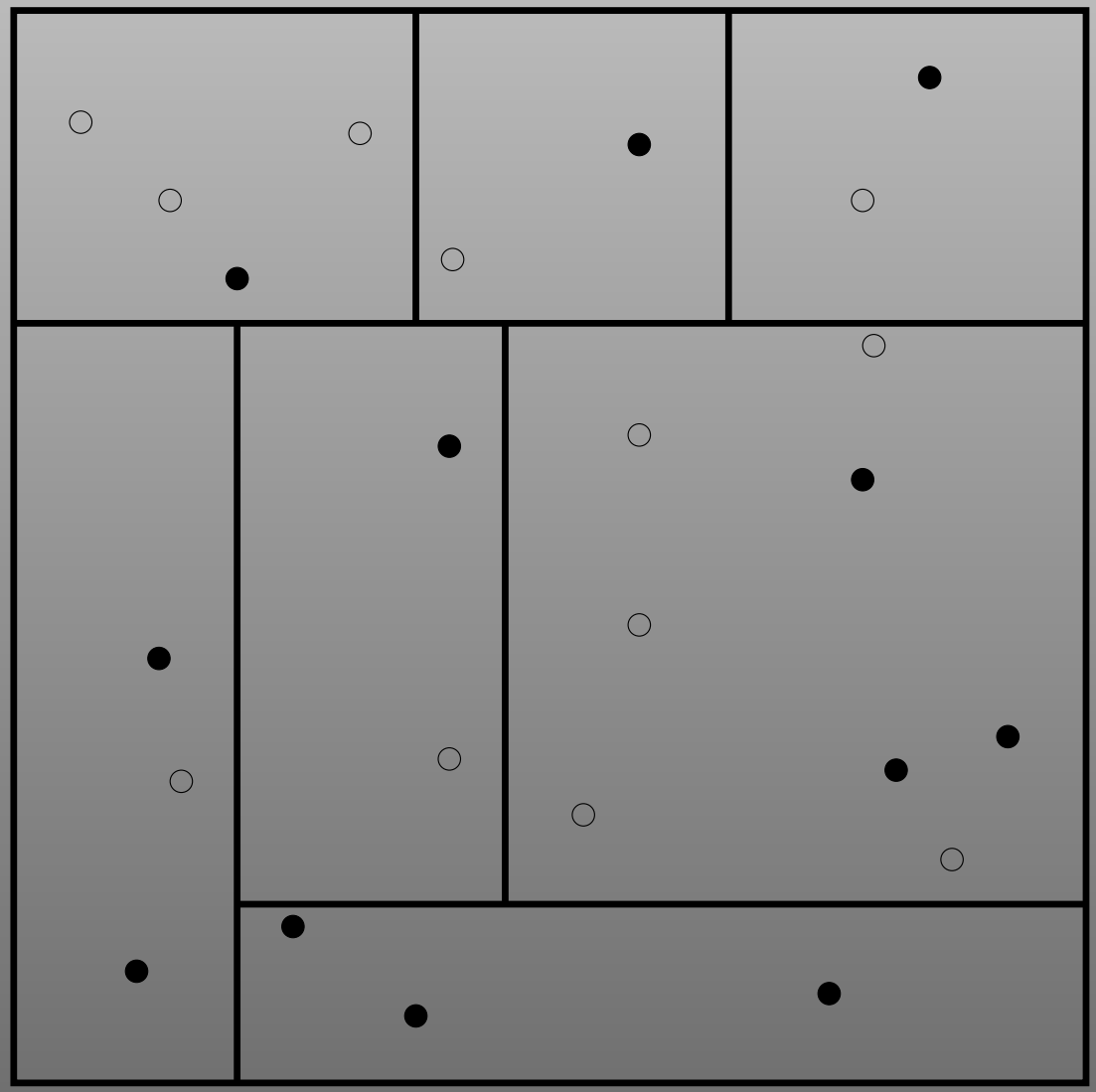

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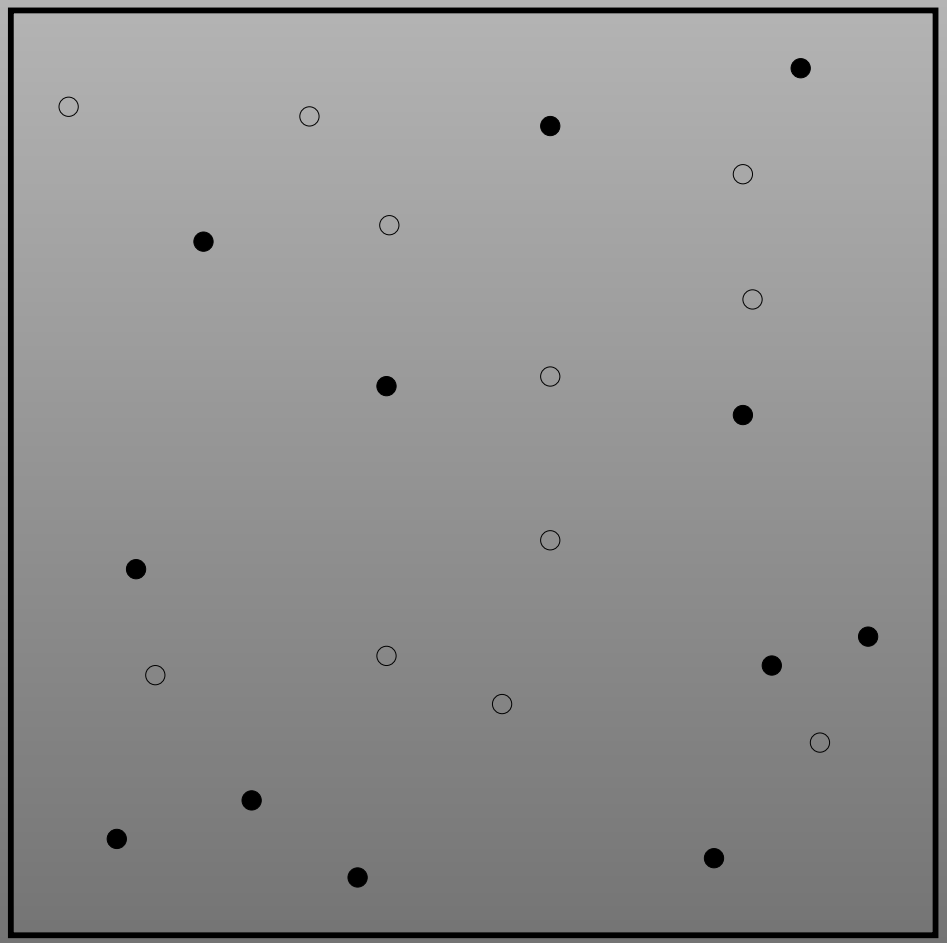

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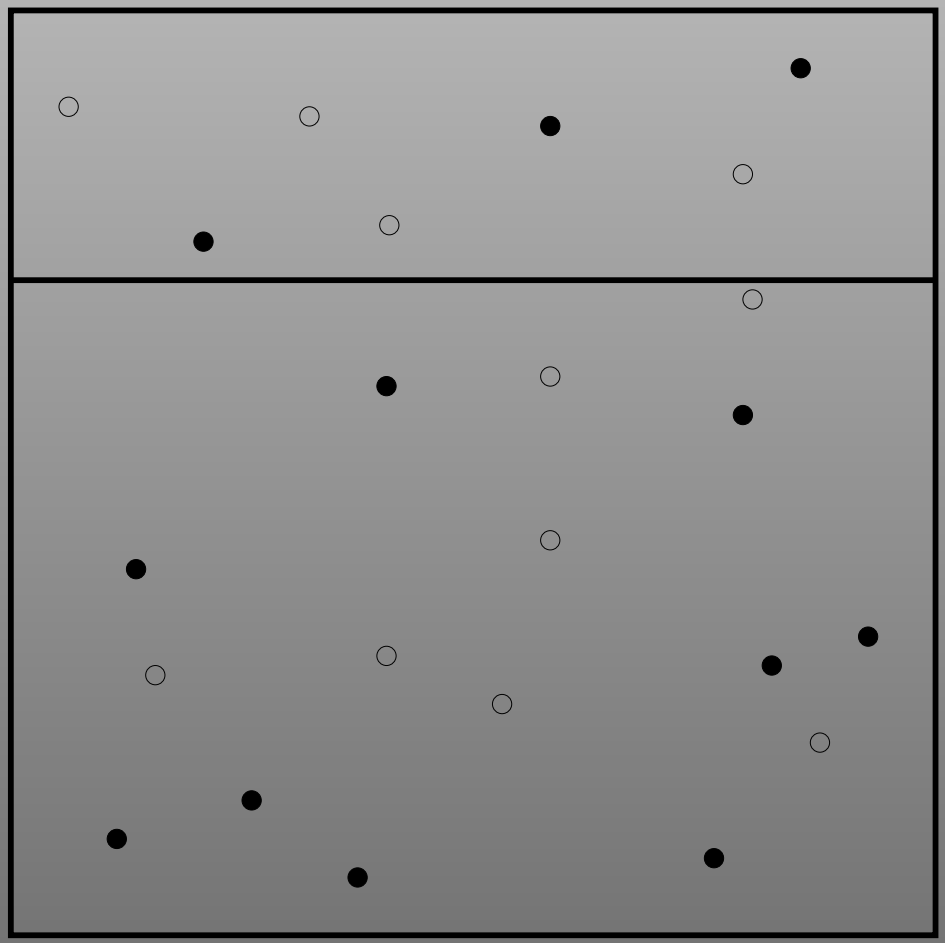

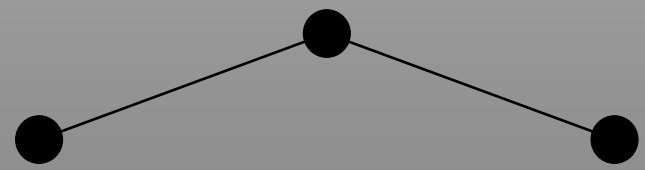

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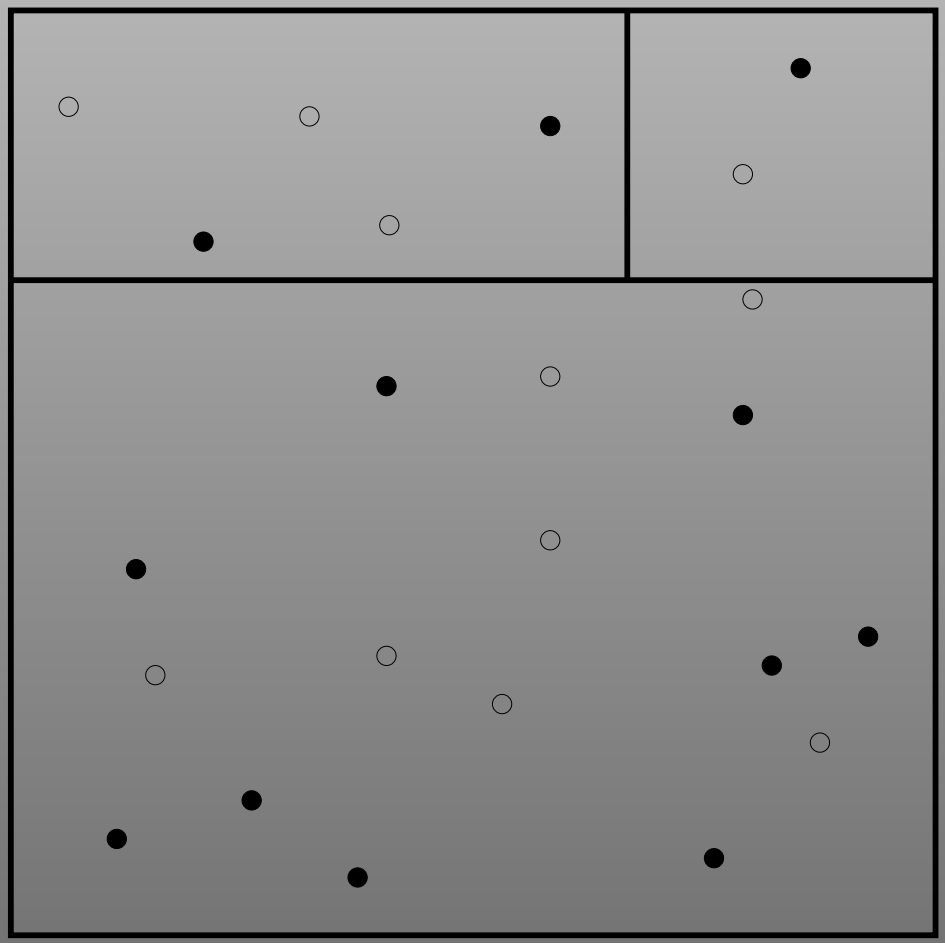

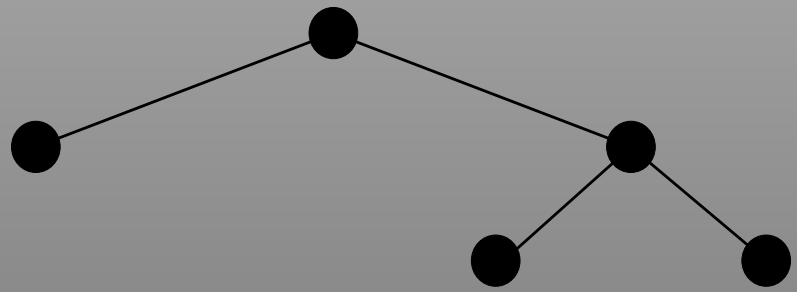

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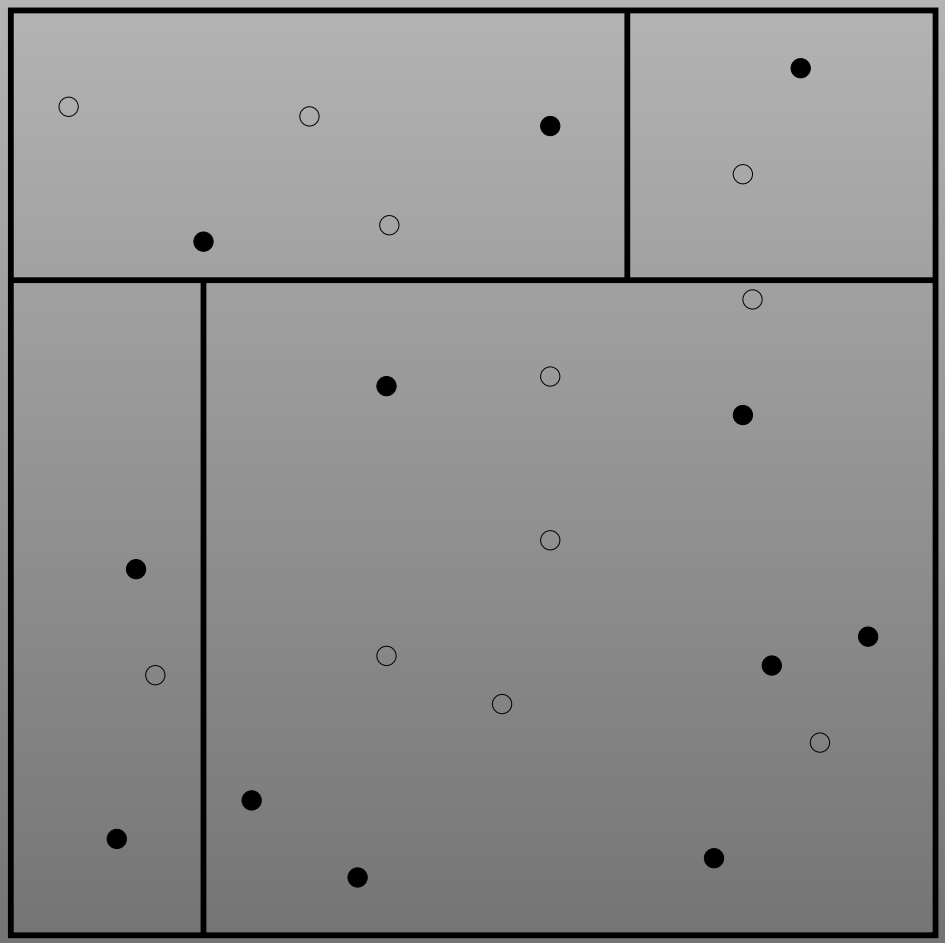

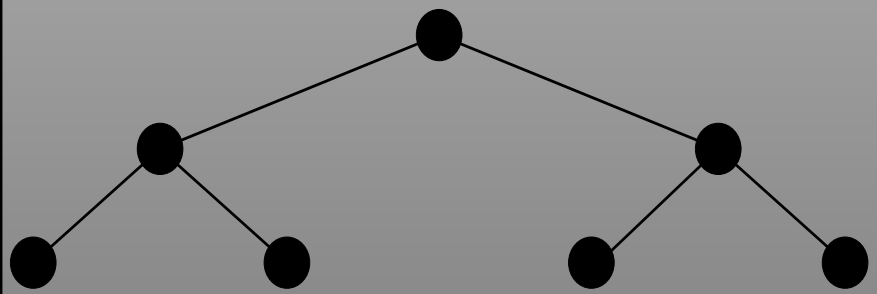

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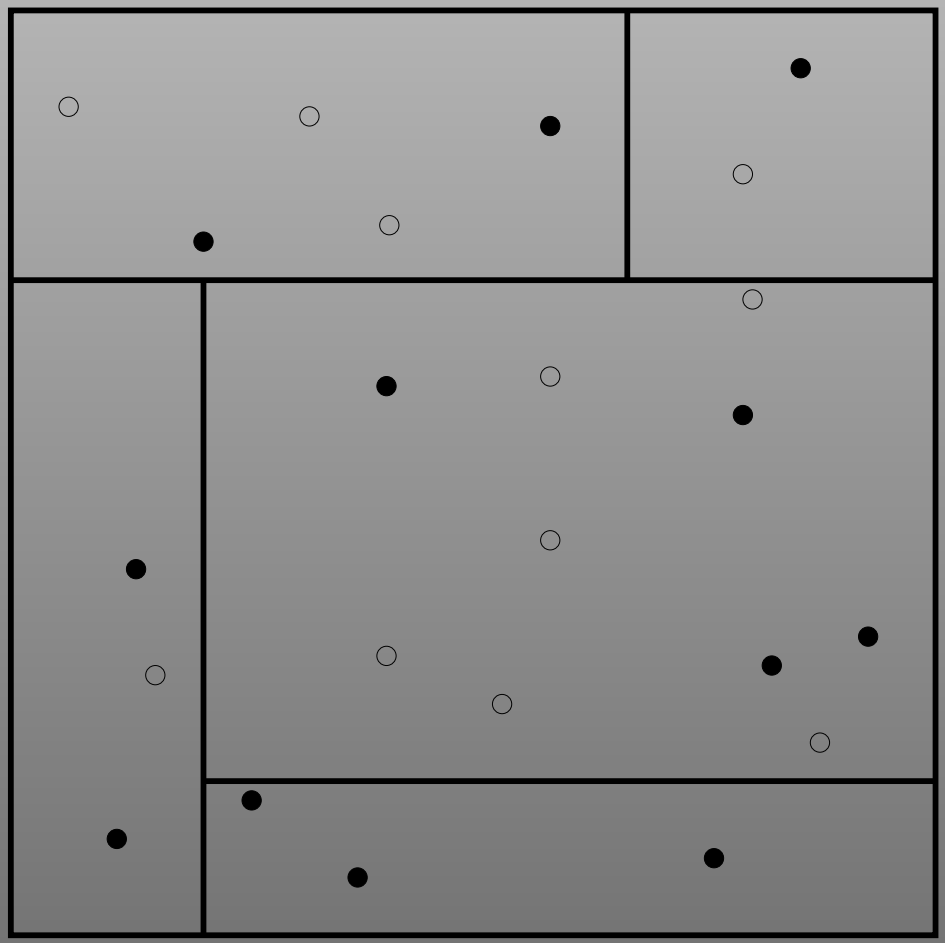

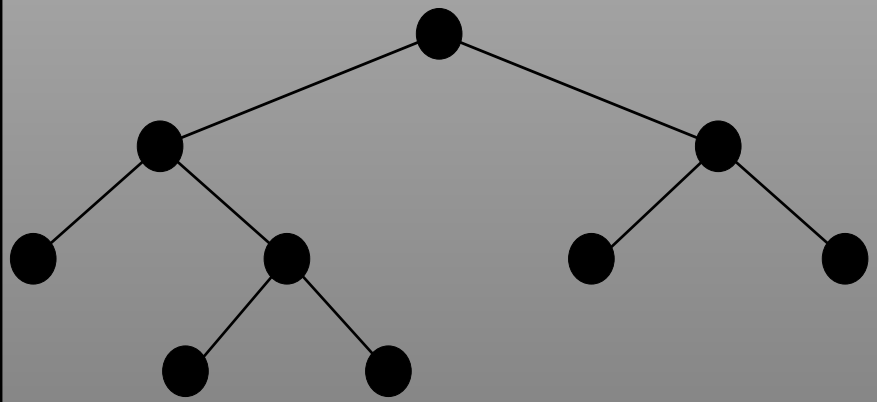

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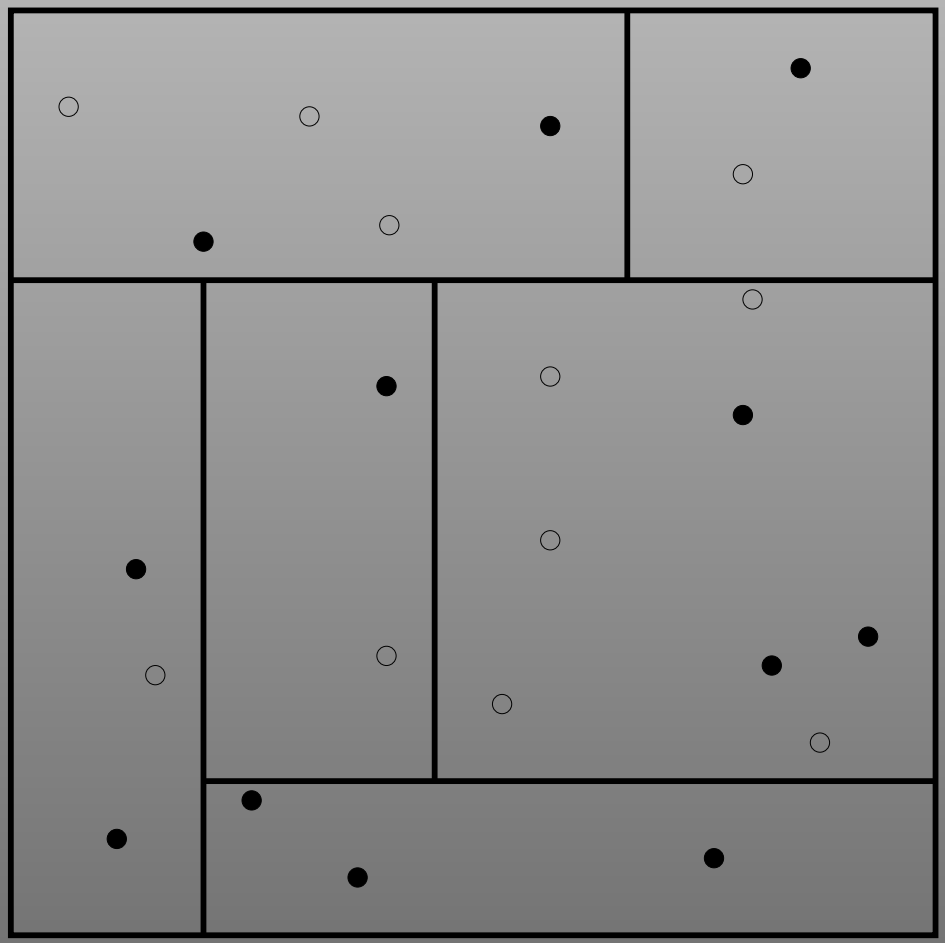

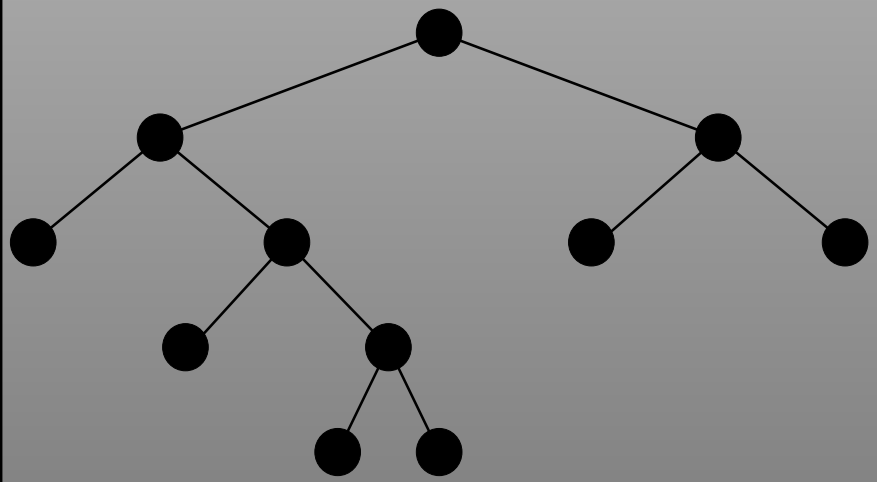

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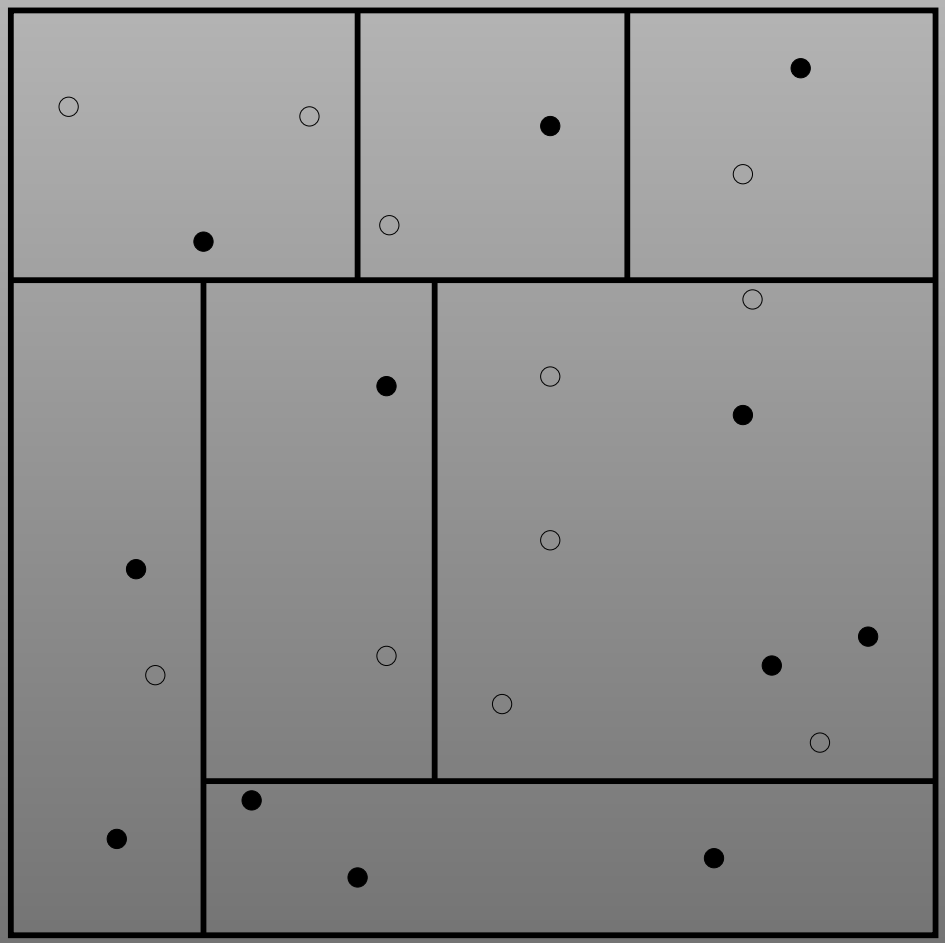

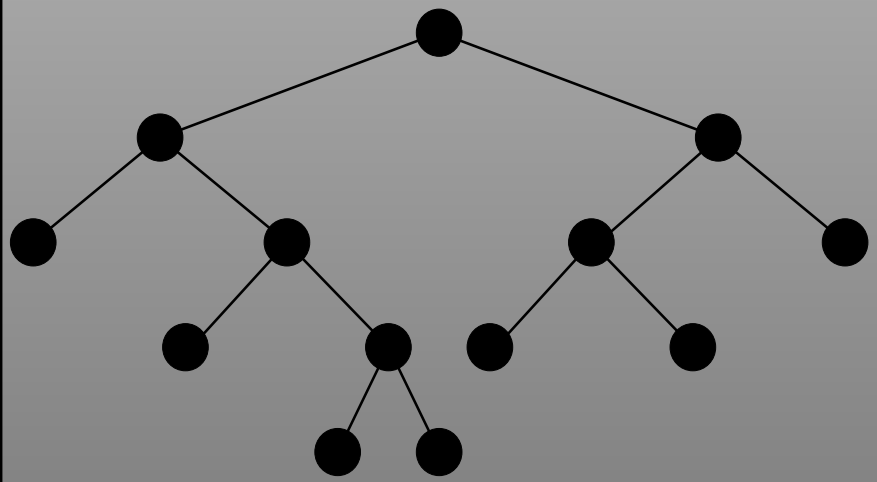

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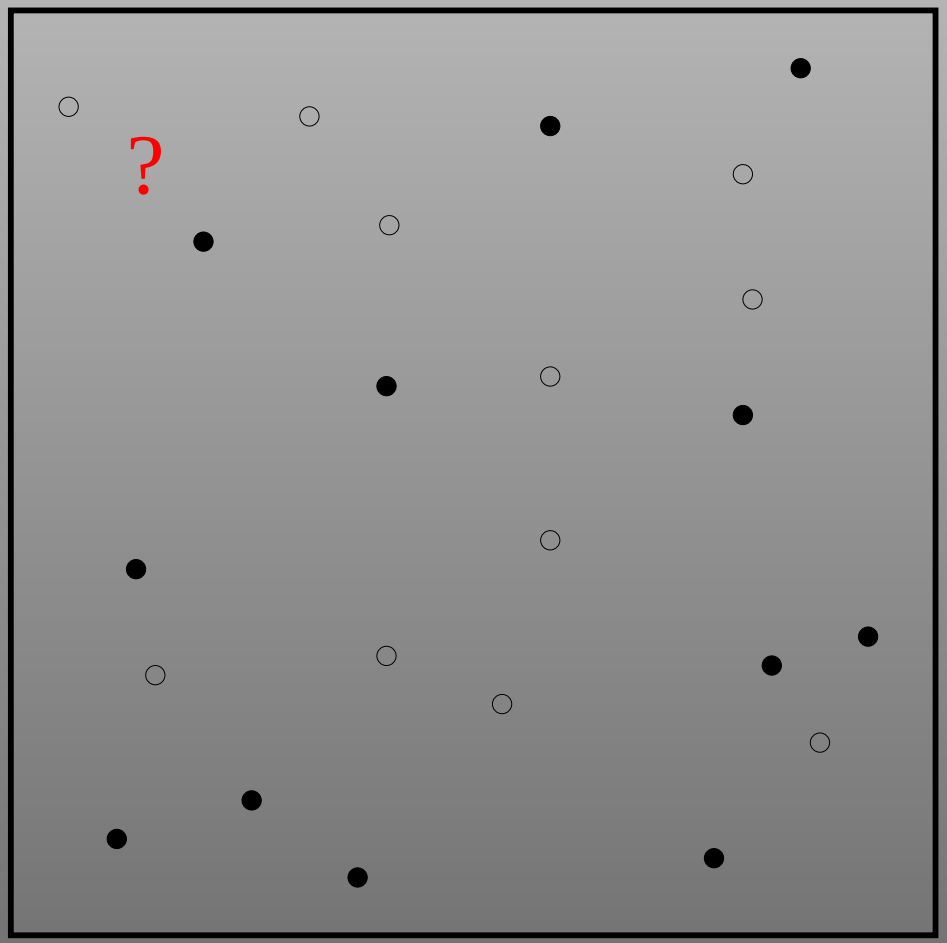

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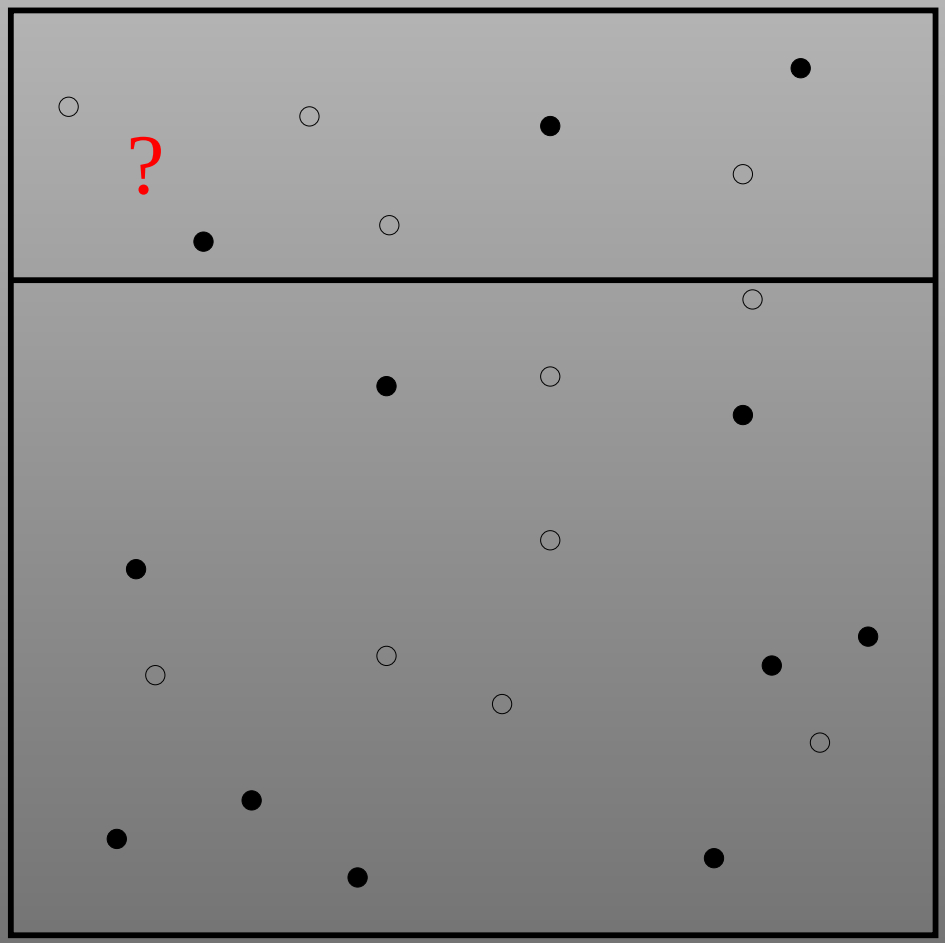

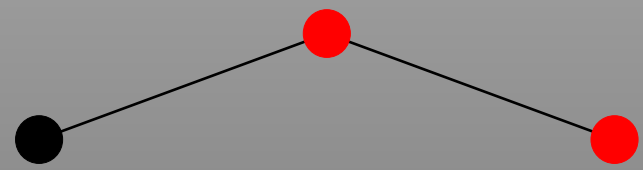

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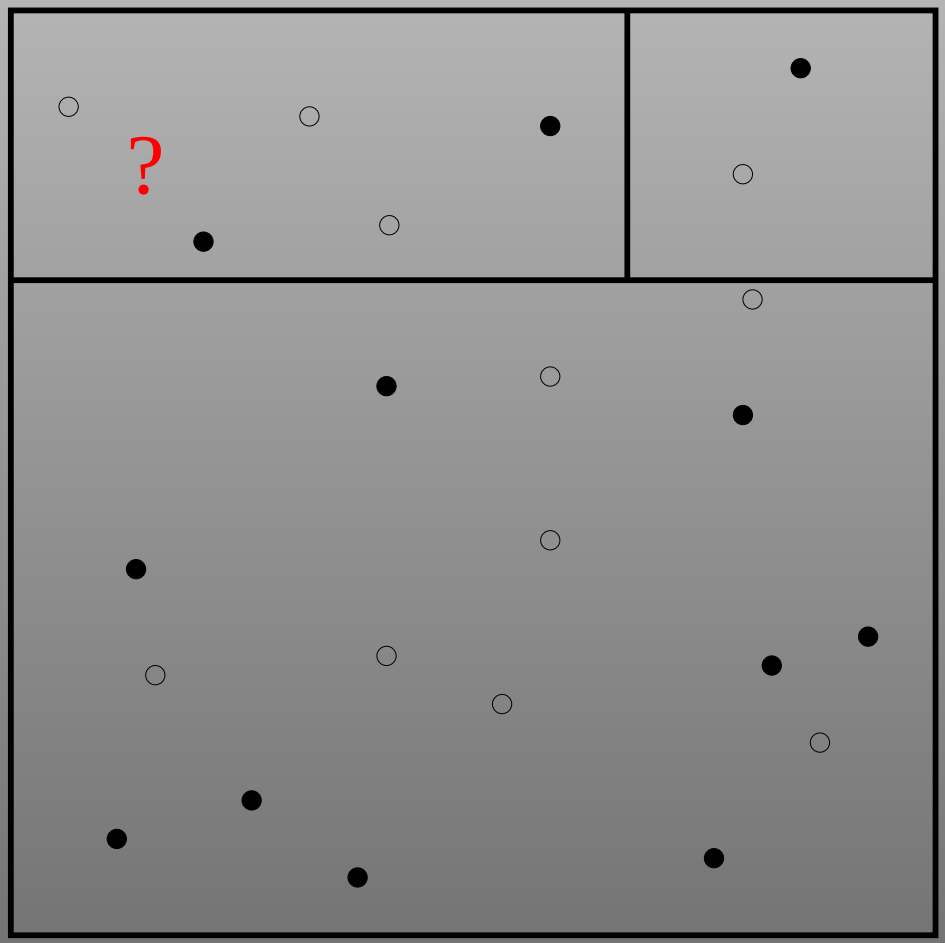

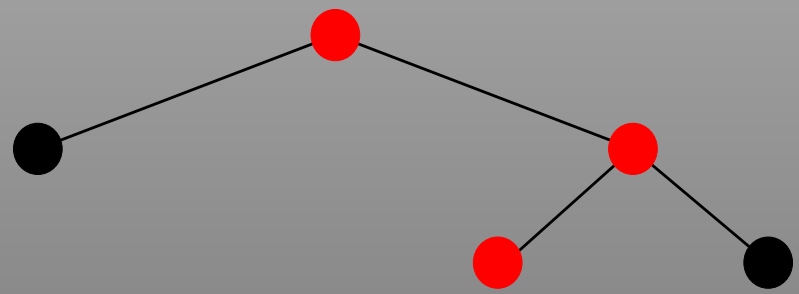

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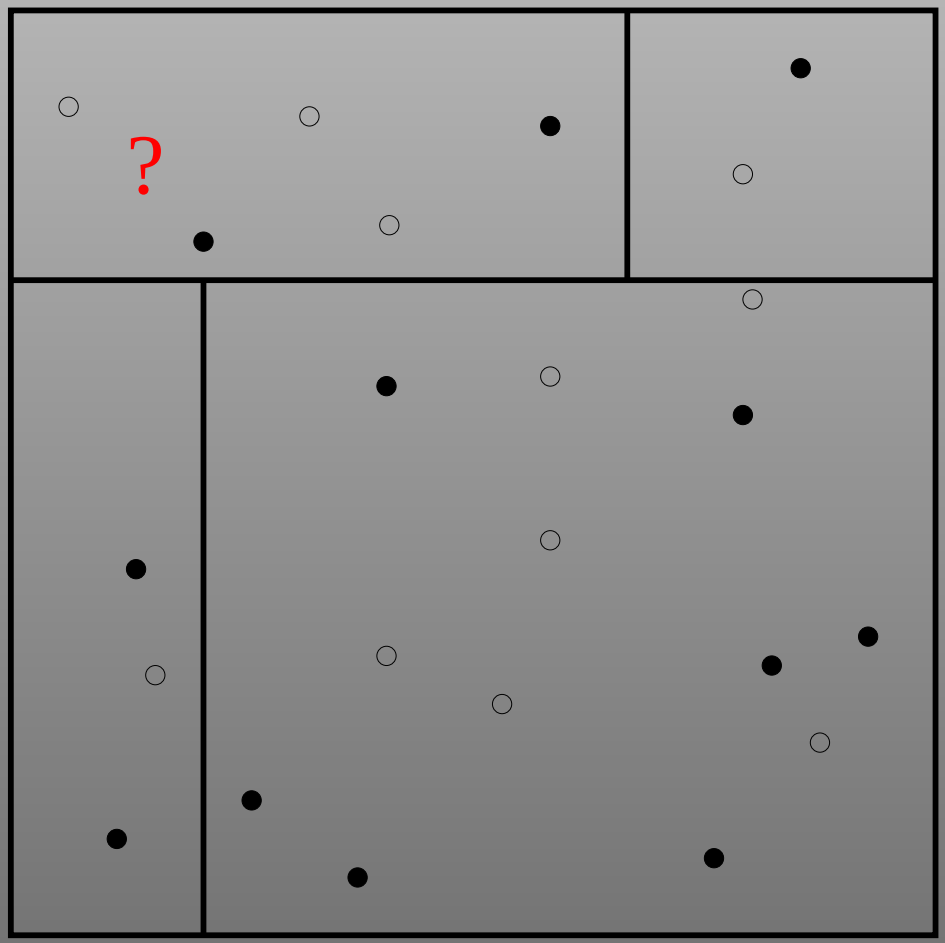

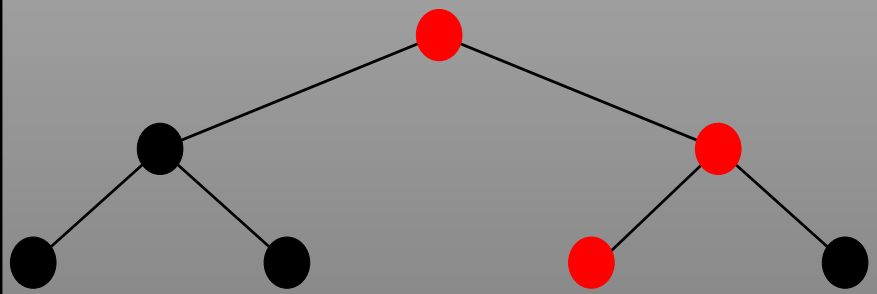

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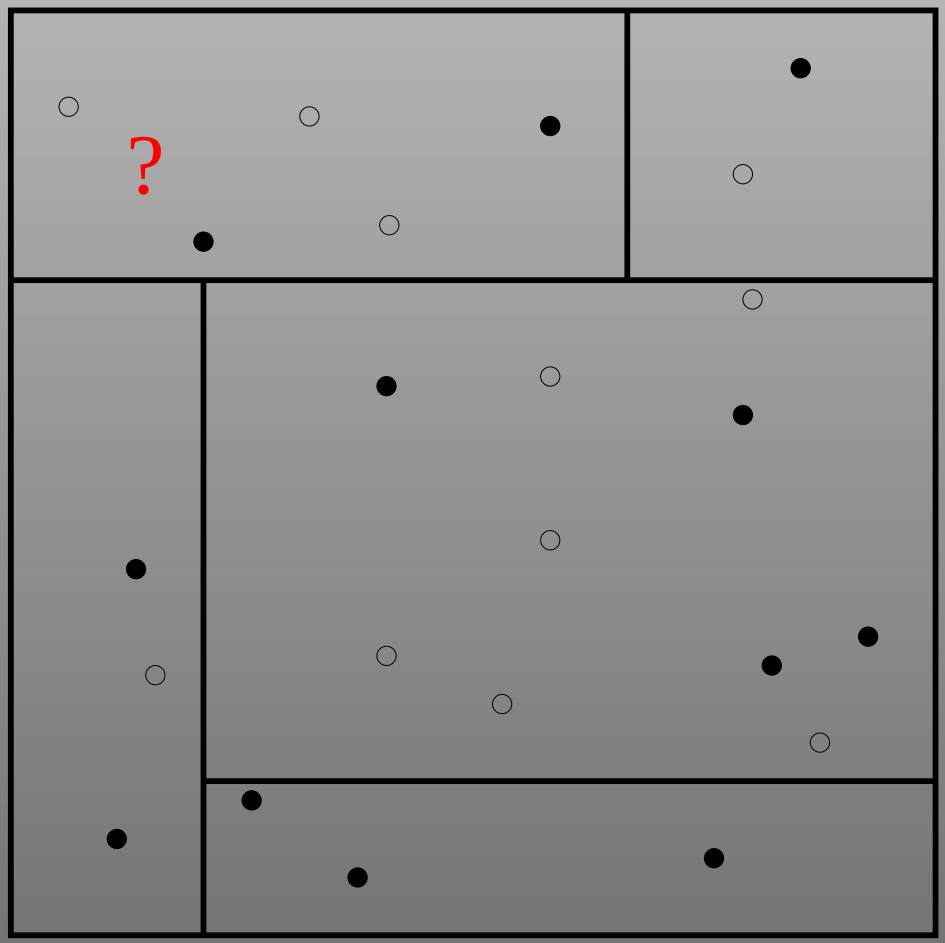

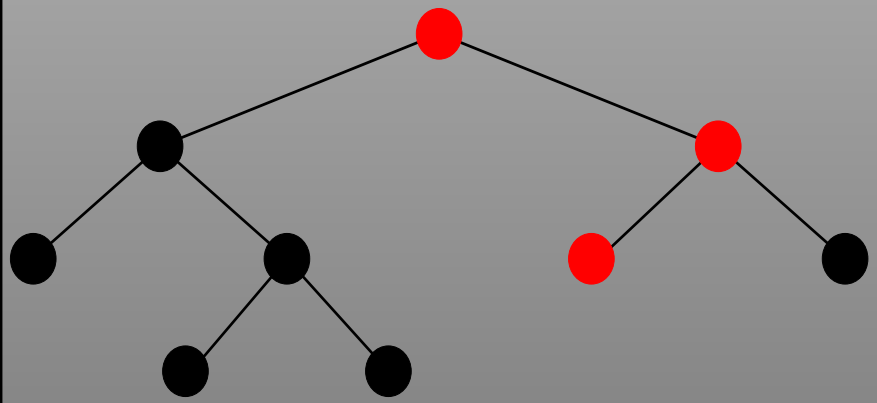

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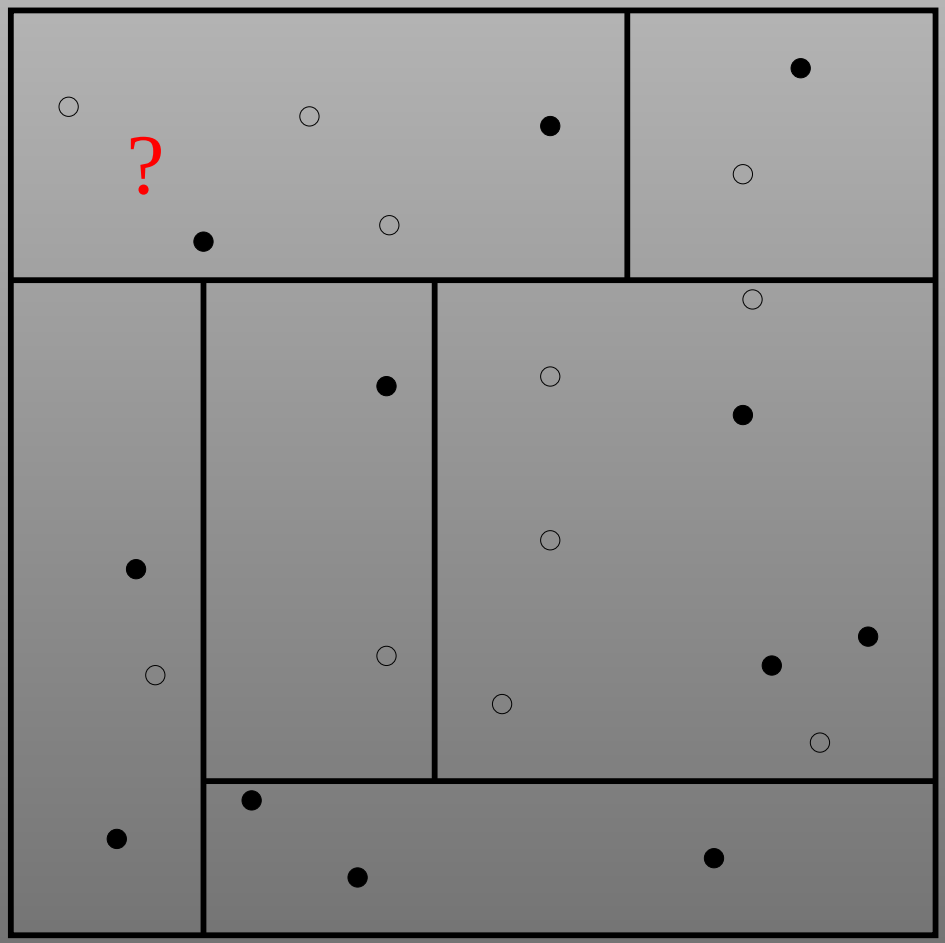

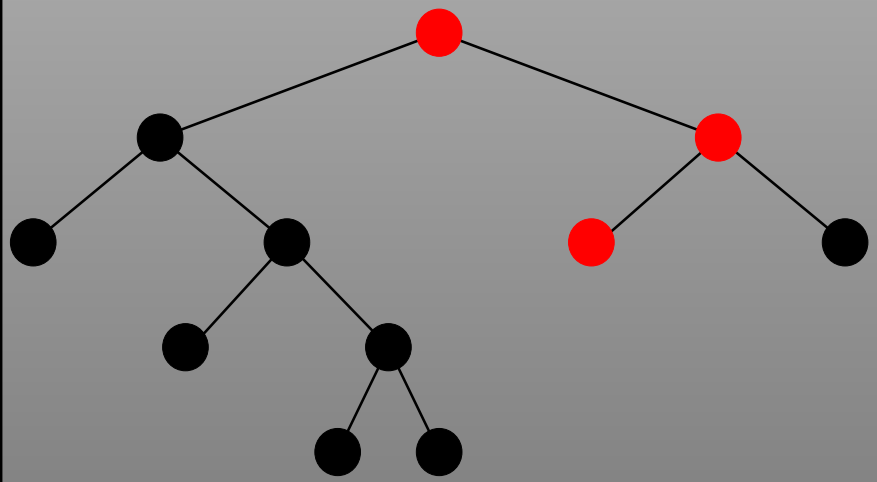

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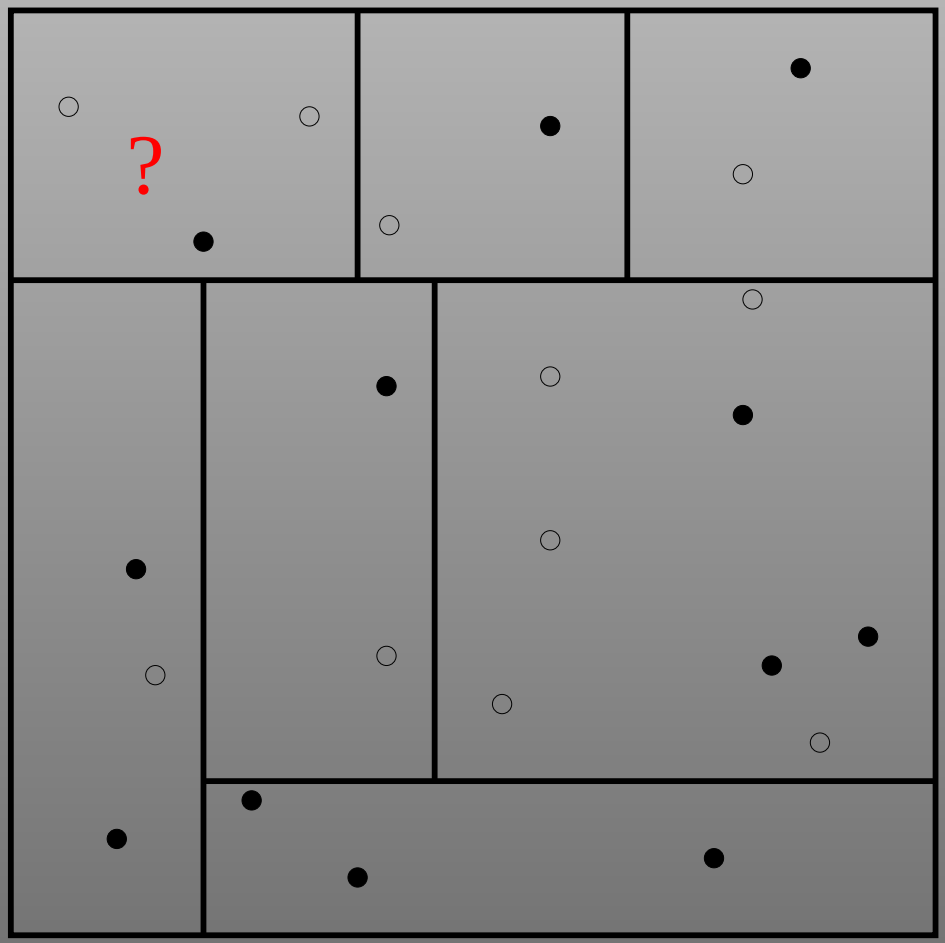

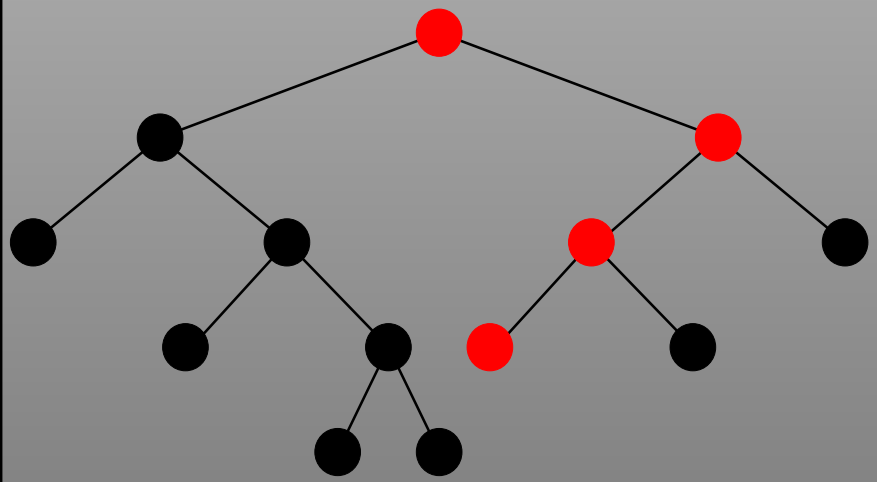

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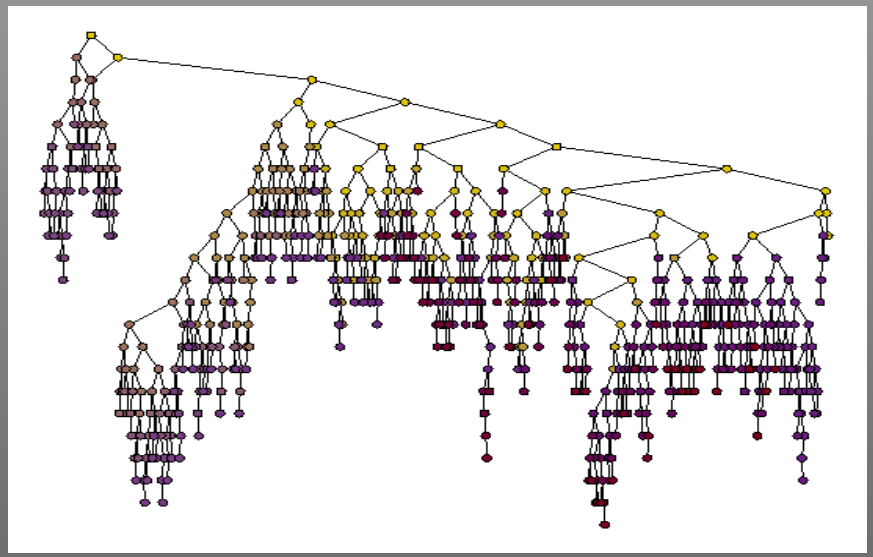
- Optimizations could face a huge combinatorial and computational hurdle.
- The greedy paradigm addresses these concerns.



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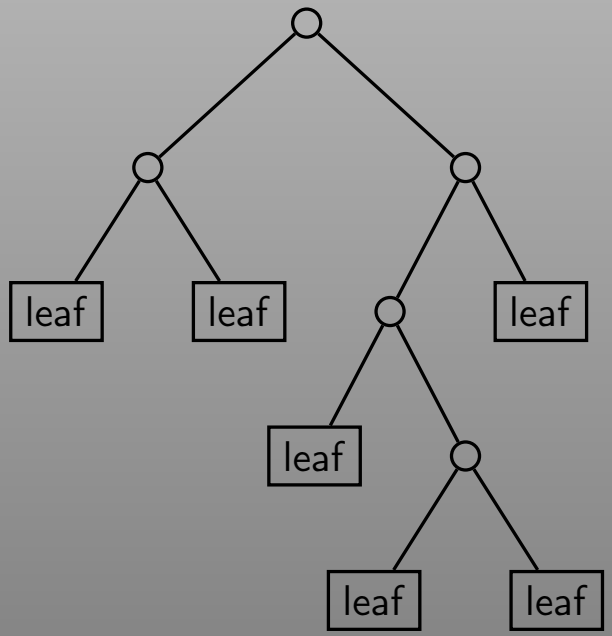

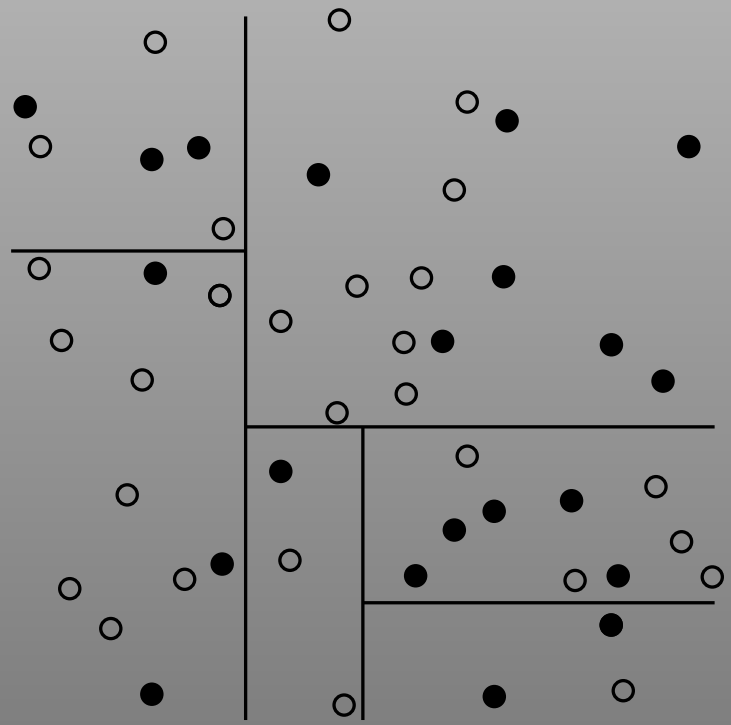

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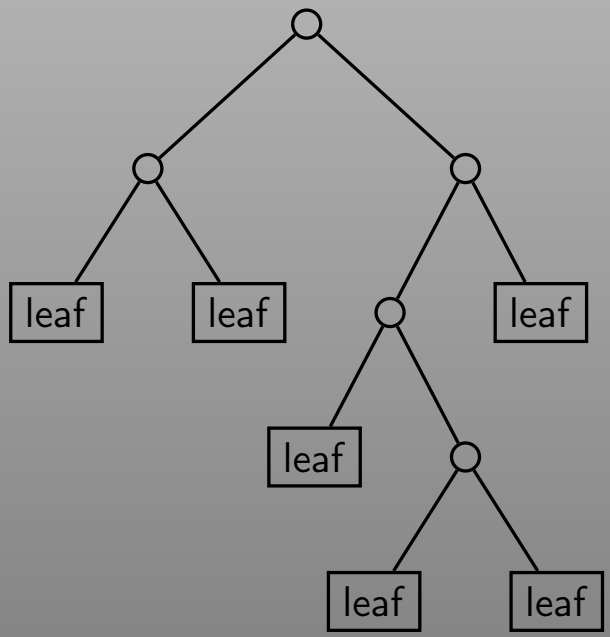

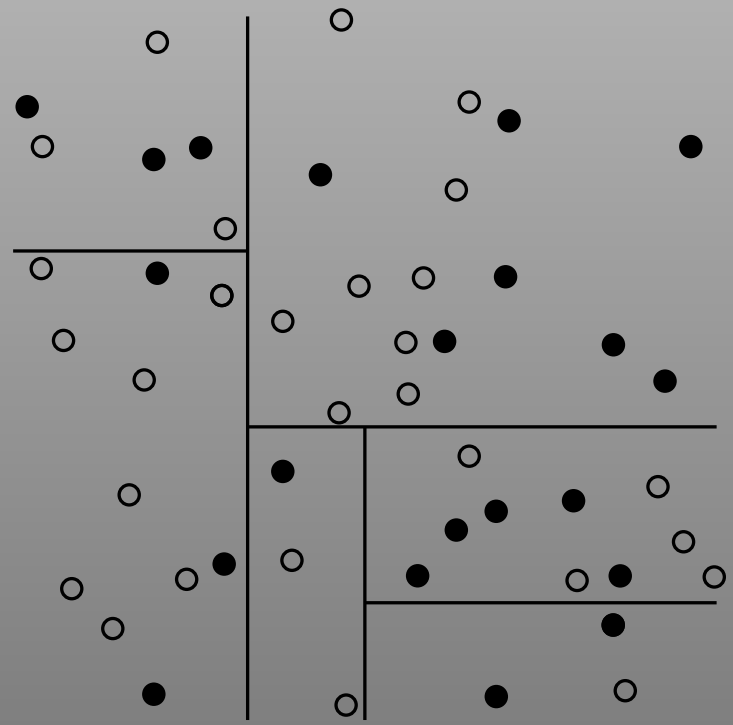

- The tree classifier takes the simple form

$$g_n(\mathbf{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n \mathbf{1}_{[\mathbf{X}_i \in A, Y_i = 1]} > \sum_{i=1}^n \mathbf{1}_{[\mathbf{X}_i \in A, Y_i = 0]}, \quad \mathbf{x} \in A \\ 0 & \text{otherwise.} \end{cases}$$

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- Thus, there are virtually infinitely many possible strategies to build classification trees.
- Despite this great diversity, all tree species end up with two fundamental questions at each node:

① Should the node be split?

② In the affirmative, what are its children?

The greedy spirit

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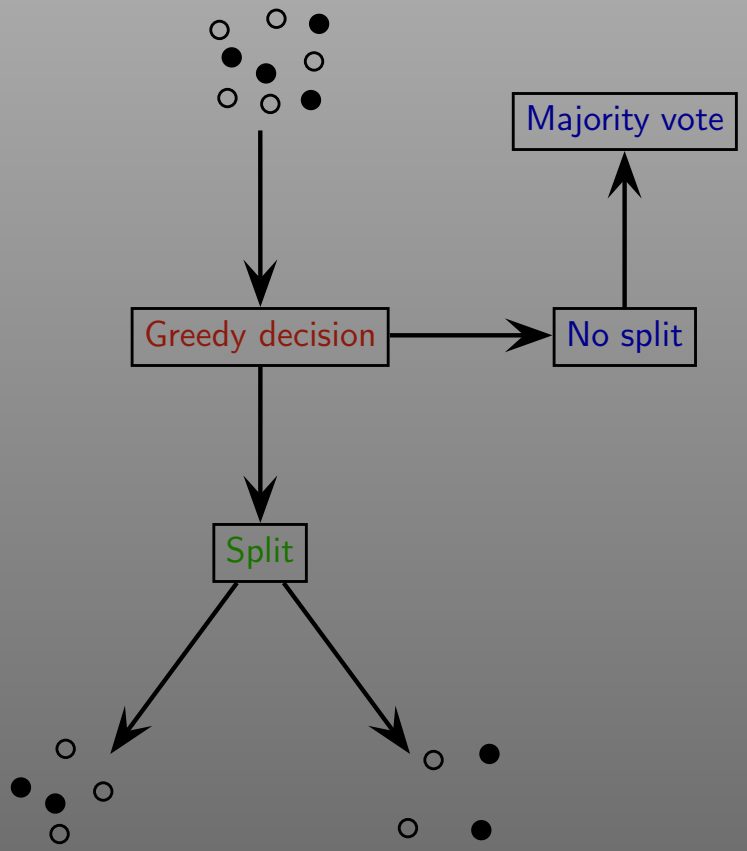
- A greedy tree should be able to answer questions ① and ② using local information only.



Outline

A model

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- Let $\{(\mathbf{X}_i, Y_i) : i \in I\}$ be the subset of observations falling in a given node.
- A split that uses a set $C \in \mathcal{C}$ results in two sets of indices:

$$I' = \{i \in I : \mathbf{X}_i \in C\} \quad \text{and} \quad I'' = \{i \in I : \mathbf{X}_i \notin C\}.$$

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- In addition, there is a second mapping θ, but this time with a boolean output.
- It is a stopping rule and models the greedy decision ①.

Greedy procedure

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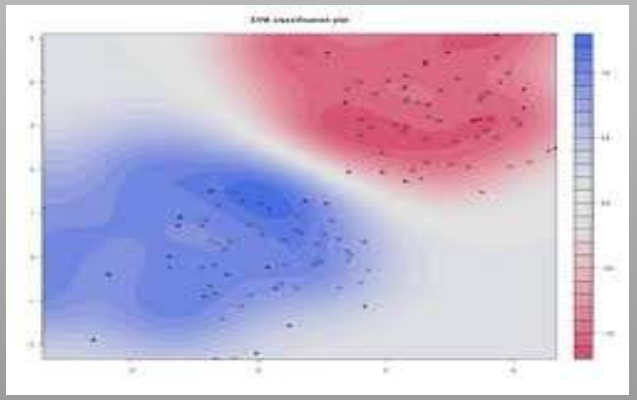
- Final classification proceeds by a majority vote.

An important remark

Is any classifier a greedy tree?



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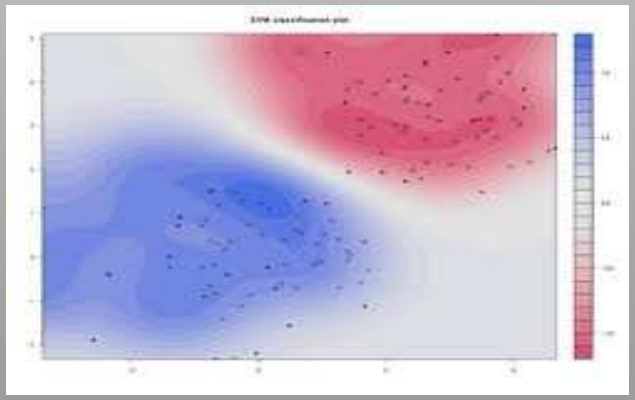

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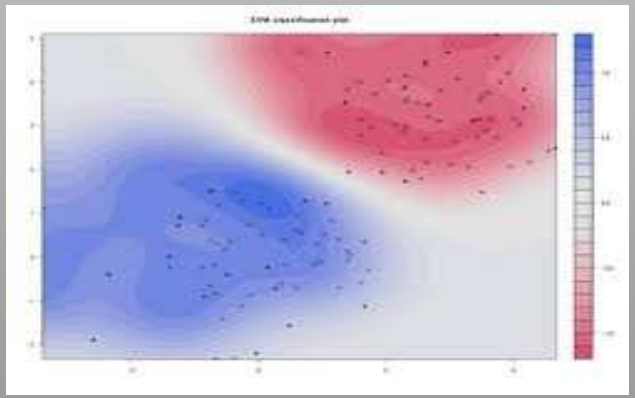

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- The selected dimension is then split at the median.
- The novelty is in the choice of the decision function θ.
- This function ignores the data altogether and uses a randomized decision that is based on the size of the input.

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Theorem

Let β be a real number in $(0, 1)$. Define

$$p(n) = \begin{cases} 1 & \text{if } n < 3 \\ 1/\log^{\beta} n & \text{if } n \geq 3. \end{cases}$$

Then

$$\lim_{n \to \infty} \mathbb{E}L(g_n) = L^{\star} \quad \text{as } n \to \infty.$$

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There is sufficient randomness at hand to do the job.

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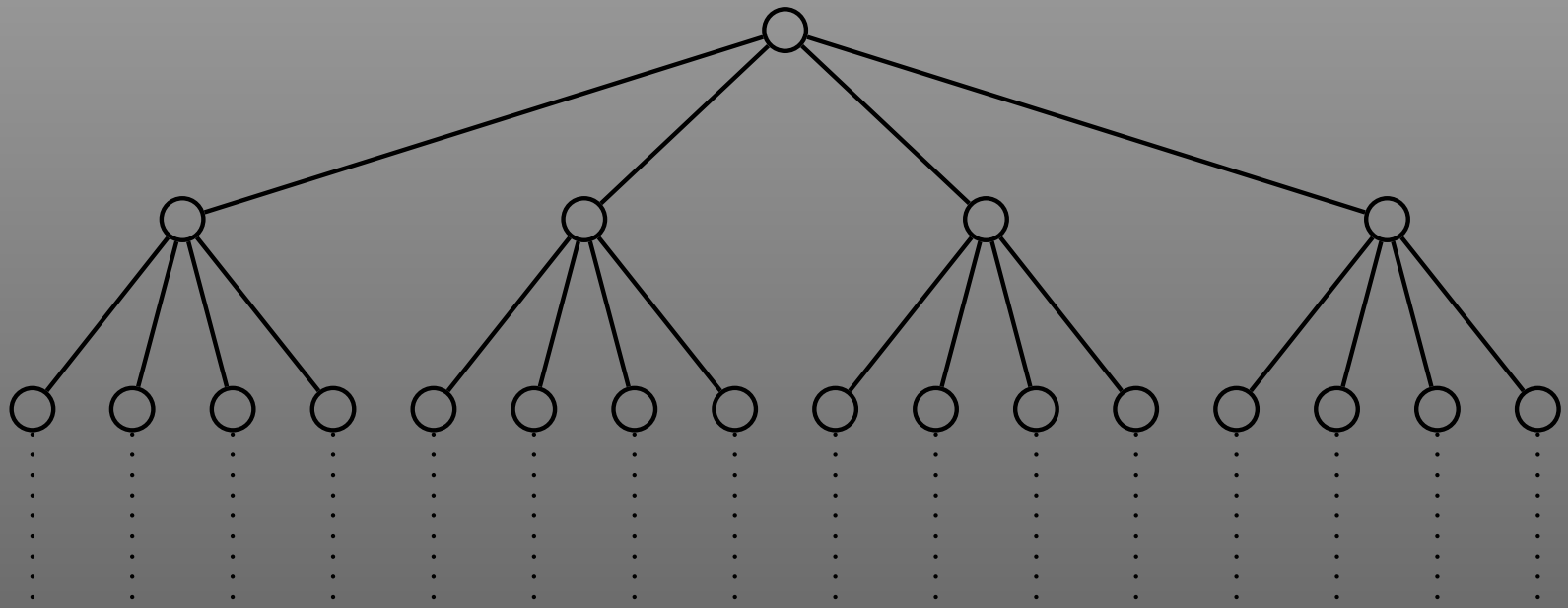
- Repeating this for k levels of nodes leads to 2^{dk} leaf regions.



The stopping rule θ

- The quality of the classifier at node A is assessed by

$$\hat{L}_n(A) = \frac{1}{N(A)} \min \left(\sum_{i=1}^{n} \mathbf{1}_{[\mathbf{X}_i \in A, Y_i = 1]}, \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{X}_i \in A, Y_i = 0]} \right).$$

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This is greedy.

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Theorem

Take $1 - d\alpha - 2\beta > 0$. Then

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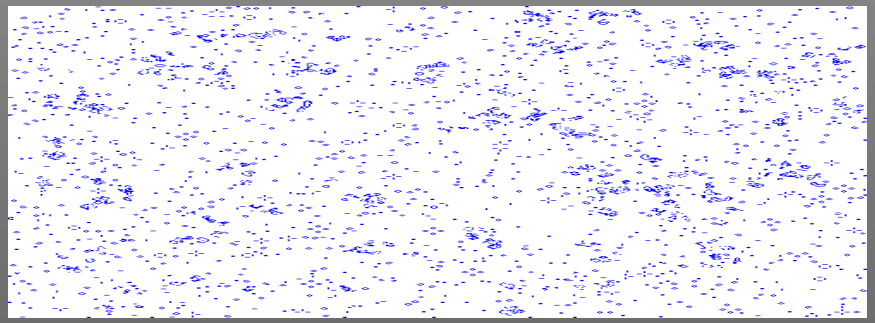
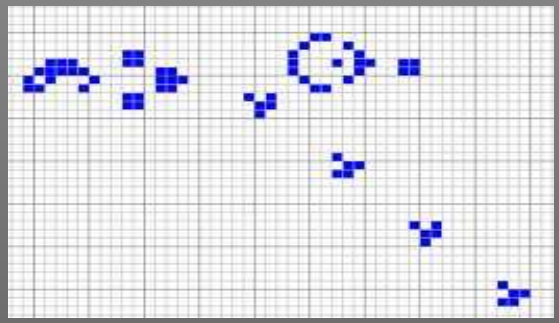
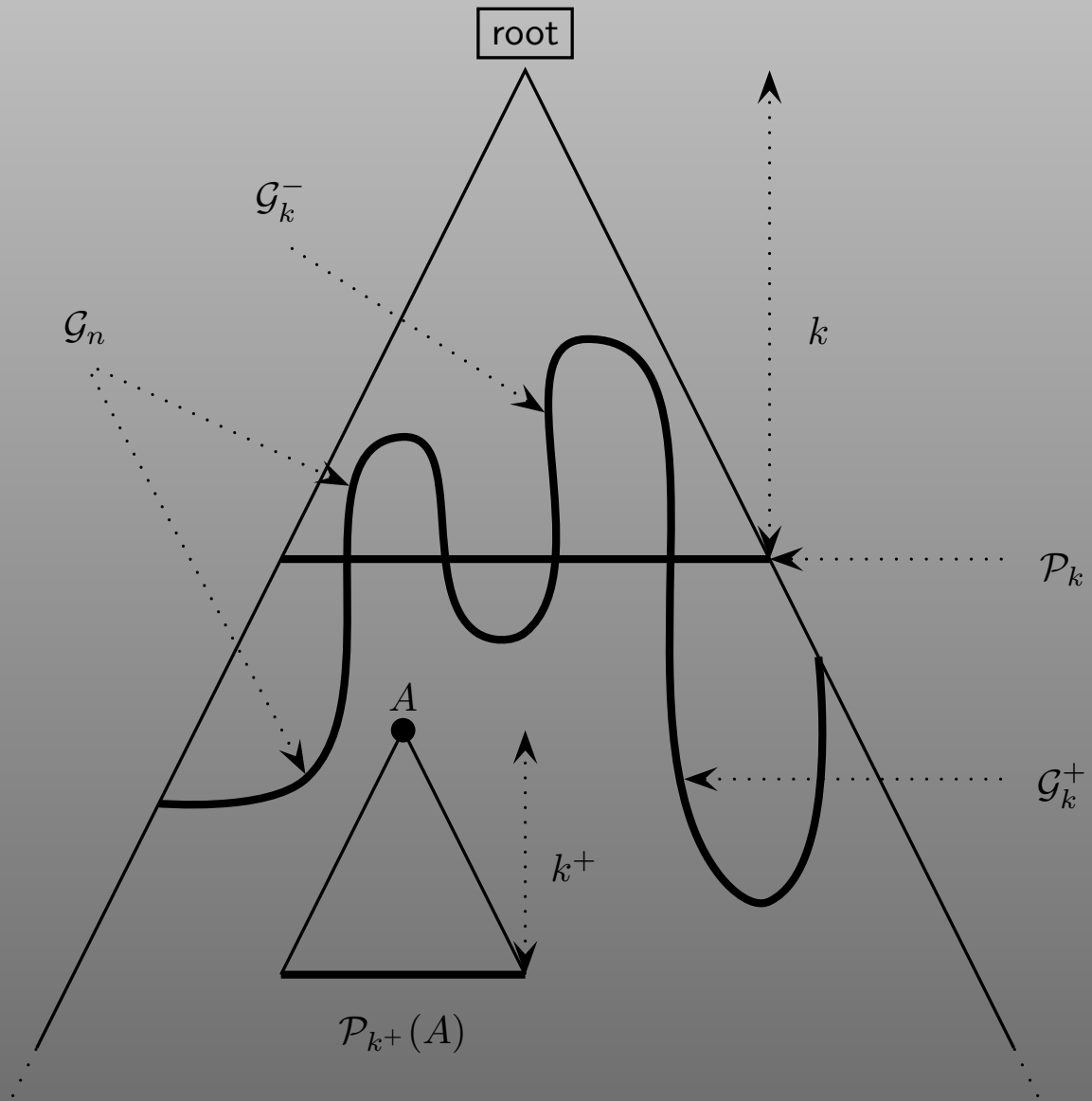




Proof


root
$\mathcal{G}_k^-$
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Proof

Let $\psi(n,k) = L_k^\star - L^\star$. Set

$$k_n^\star = \min\left\{\ell \geq 0 : \psi(n,\ell) < \sqrt{\left(\frac{2^{d\ell}}{n}\right)^{1-d\alpha}}\right\}.$$

Then

$$\frac{2^{dk_n^\star}}{n} \to 0 \quad \text{as } n \to \infty.$$