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ON THE FORMULATION OF DYNAMIC PROBLEMS WITH FRICTION AND PERSISTENT CONTACT

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Abstract. *In this communication we re-formulate the finite dimensional dynamic friction problem with unilateral contact as a standard L^1 differential inclusion when the data is chosen such that contact persistently holds. Existence of solution to the problem is re-established by using standard theoretical results for differential inclusions and for algebraic inclusions. The most relevant results in the literature on the same problem are due to Jean and Pratt [0].*

1 Introduction

The work presented in this communication is part of a long-term research effort dedicated to understanding instability phenomena in frictional contact problems. Various results involving the normal compliance model were presented, in e.g. Martins, Oden and Simões [0]. Other results using the perfect unilateral contact conditions, either in the finite dimensional or the continuum cases, can be found in Martins, Pinto da Costa and Simões [0] and the references therein. The relation between dynamic and quasi-static evolutions in frictional contact systems and some sort of "dynamic instability" of the latter were addressed in Martins, Simões, Gastaldi and Marques [0]; later this was identified in a related context (Loret, Simões, Martins [0]) as a singular perturbation problem.

Our purpose in this communication is to set the stage for the application to finite dimensional problems with perfect unilateral contact and Coulomb friction of some mathematical results in the literature on stability or singular perturbation issues in some non-smooth evolution problems. We do this in a framework that is less non-smooth than the one needed to deal with collisions or "frictional catastrophes" (Monteiro Marques [0]).

This communication summarizes recent results (Martins, Marques, Petrov [0]) that show that, with appropriate conditions on the data, the finite dimensional dynamic problem with unilateral but persistent contact can be formulated as a standard L^1 differential inclusion, to which standard theoretical results (Aubin and Celina [0]) can be applied to establish existence of solution.

The re-formulation and study of this problem in the framework of the L^1 differential inclusions has the purpose of preparing the application to frictional contact problems of some mathematical results on instability and singular perturbation issues that have been recently developed for some classes of non-smooth evolution problems.

2 Governing dynamic equations

We start by considering a three-dimensional holonomic and scleronomic finite dimensional mechanical system whose configuration at each time $t \geq 0$ is described by the values $X_i(t)$, $1 \leq i \leq N$, of the independent generalized coordinates; the corresponding column vector of the values at time t of those generalized coordinates is denoted by $\boldsymbol{X}(t) \in \mathbb{R}^N$. A finite number of particles of that mechanical system is subjected to unilateral contact constraints with fixed curved obstacles. The set $\mathcal{P}_C \subset \mathbb{N}$ groups the labels of the particles (p) of those *Contact* candidate particles (see Fig. 1).

Each point in the plane of the system is identified by the column vector $\boldsymbol{x}$ of the components x_k, $k = 1, 2, 3$, of its position vector in some fixed orthonormal reference frame $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. For each contact candidate particle p, the corresponding obstacle is identified by the set of vectors $\boldsymbol{x} \in \mathbb{R}^3$ such that

$$\phi^p(\boldsymbol{x}) = 0, \tag{2.1}$$

where the function $\phi^p : \mathbb{R}^3 \to \mathbb{R}$ is twice continuously differentiable, and $\partial\phi^p/\partial\boldsymbol{x} \neq 0$ at the points on or sufficiently close to the obstacle. On each point of these obstacles, unit normal and tangent vectors are defined such that

$$\boldsymbol{n}^p(\boldsymbol{x}) = \frac{\partial\phi^p/\partial\boldsymbol{x}}{\partial\phi^p/\partial\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{t}_1^p(\boldsymbol{x}) \times \boldsymbol{t}_2^p(\boldsymbol{x}). \tag{2.2}$$

In view of the assumptions above, the orthogonal basis $(\boldsymbol{t}_1^p(\boldsymbol{x}), \boldsymbol{t}_2^p(\boldsymbol{x}), \boldsymbol{n}^p(\boldsymbol{x}))$ may be extended to all points of the space that are sufficiently close to the obstacle p.

The position of each particle $p \in \mathcal{P}_C$ at each time $t \geq 0$ is identified by the column vector $\boldsymbol{x}^p(t) = \boldsymbol{x}^p(\boldsymbol{X}(t)) \in \mathbb{R}^3$, and the column vector of the normal and tangential components of the particle velocity is given by

$$\boldsymbol{v}^p(t) = \left\{ \begin{array}{c} v_{t1}^p(t) \\ v_{t2}^p(t) \\ v_n^p(t) \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{v}_t^p(t) \\ v_n^p(t) \end{array} \right\} = \left[\begin{array}{c} \boldsymbol{G}_t^p(\boldsymbol{X}(t)) \\ \boldsymbol{G}_n^p(\boldsymbol{X}(t)) \end{array} \right] \dot{\boldsymbol{X}}(t) = \boldsymbol{G}^p(\boldsymbol{X}(t))\dot{\boldsymbol{X}}(t) \tag{2.3}$$

where the $(2 \times N)$ and $(1 \times N)$ row matrices $\boldsymbol{G}_t^p(\boldsymbol{X})$ and $\boldsymbol{G}_n^p(\boldsymbol{X})$, respectively, have the components

$$G_{tki}^p(\boldsymbol{X}) = \boldsymbol{t}_k^p\big(\boldsymbol{x}^p(\boldsymbol{X})\big) \cdot \frac{\partial \boldsymbol{x}^p}{\partial X_i}(\boldsymbol{X}), \quad G_{ni}^p(\boldsymbol{X}) = \boldsymbol{n}^p\big(\boldsymbol{x}^p(\boldsymbol{X})\big) \cdot \frac{\partial \boldsymbol{x}^p}{\partial X_i}(\boldsymbol{X}), \quad k = 1, 2, \quad i = 1, \ldots, N. \tag{2.4}$$

As usual, the notation $(\dot{\ })$ denotes the time derivative $d(\cdot)/dt$. The velocities () of the contact particles p are grouped in a single column vector $\boldsymbol{v}(t)$ of dimension $3n_C$ ($n_C = \#\mathcal{P}_C$) and, accordingly, the $(3n_C \times N)$ matrix $\boldsymbol{G}(\boldsymbol{X})$ is constructed such that

$$\boldsymbol{v}(t) = \boldsymbol{G}(\boldsymbol{X}(t))\dot{\boldsymbol{X}}(t). \tag{2.5}$$

We denote by

$$\boldsymbol{r}^p(t) = \left\{ \begin{array}{c} \boldsymbol{r}_t^p(t) \\ r_n^p(t) \end{array} \right\} = \left\{ \begin{array}{c} r_{t1}^p(t) \\ r_{t2}^p(t) \\ r_n^p(t) \end{array} \right\} \tag{2.6}$$

the column vector of the tangential and normal components of the reaction force that acts at some time $t \geq 0$ on the contact particle p. The column vector (of dimension $3n_C$) that groups all the reaction vectors $\boldsymbol{r}^p(t)$ is denoted by $\boldsymbol{r}(t)$. For some contact reaction $\boldsymbol{r}(t) \in \mathbb{R}^{3n_C}$ at some configuration $\boldsymbol{X}(t)$ of the system, the vector of generalized reactions $\boldsymbol{R}(t) \in \mathbb{R}^N$ is given by

$$\boldsymbol{R}(t) = \boldsymbol{G}^T(\boldsymbol{X}(t))\boldsymbol{r}(t). \tag{2.7}$$

In the following it will be always assumed that

the lines of the $(3n_C \times N)$ *matrix* $\boldsymbol{G}(\boldsymbol{X})$ *are linearly independent.* (2.8)

The classical *unilateral contact conditions*

$$\phi^p\big(\boldsymbol{x}^p(t)\big) \leq 0, \quad r_n^p(t) \leq 0, \quad \phi^p\big(\boldsymbol{x}^p(t)\big) r_n^p(t) = 0, \tag{2.9}$$

and the *friction law of Coulomb*

$$\boldsymbol{r}_t^p(t) \in \mu r_n^p(t)\boldsymbol{d}(\boldsymbol{v}_t^p(t)) \tag{2.10}$$

are satisfied at all contact particles $p \in \mathcal{P}_C$; $\mu \geq 0$ is the coefficient of friction, $\boldsymbol{d}(\cdot)$ denotes the multi-valued application such that, for each $\boldsymbol{y} \in \mathbb{R}^2$,

$$\boldsymbol{d}(\boldsymbol{y}) = \begin{cases} \boldsymbol{y}/\boldsymbol{y}, & \text{if} \quad \boldsymbol{y} \neq \boldsymbol{0}, \\ \left\{ \left\{ \begin{array}{c} z_1 \\ z_2 \end{array} \right\} : z_1^2 + z_2^2 \leq 1 \right\}, & \text{if} \quad \boldsymbol{y} = \boldsymbol{0}. \end{cases} \tag{2.11}$$

The mechanical system is assumed to be acted by external time dependent forces, and by internal elastic or viscous forces such that $Q_i = Q_i(\boldsymbol{X}, \dot{\boldsymbol{X}}, t)$, $i = 1, \ldots, N$, are the corresponding generalized forces. We denote by $T(\boldsymbol{X}, \dot{\boldsymbol{X}}) = (1/2)\dot{\boldsymbol{X}} \cdot \boldsymbol{M}(\boldsymbol{X})\dot{\boldsymbol{X}}$ the kinetic energy of the system, where $\boldsymbol{M}(\boldsymbol{X})$ is the symetric, positive definite mass matrix.

Along portions of the system trajectory where $\boldsymbol{X}(t)$ is twice continuously differentiable and $\boldsymbol{r}(t)$ is continuous, the motion of the system is governed by the N *Lagrange equations*

$$\boldsymbol{M}(\boldsymbol{X}(t))\ddot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}(t), \dot{\boldsymbol{X}}(t), t) + \boldsymbol{G}^T(\boldsymbol{X}(t))\boldsymbol{r}(t), \tag{2.12}$$

where

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t) &= \boldsymbol{Q}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t) - \boldsymbol{D}(\boldsymbol{X}, \dot{\boldsymbol{X}}), \\ D_i(\boldsymbol{X}, \dot{\boldsymbol{X}}) &= \sum_{j=1}^{N}\sum_{k=1}^{N} \left[\frac{\partial M_{ij}}{\partial X_k}(\boldsymbol{X}) - \frac{1}{2}\frac{\partial M_{jk}}{\partial X_i}(\boldsymbol{X}) \right] \dot{X}_j \dot{X}_k, \quad i = 1, \ldots, N. \end{aligned} \tag{2.13}$$

Note that the vector $\boldsymbol{D}(\boldsymbol{X}, \dot{\boldsymbol{X}})$ groups inertia terms quadratically dependent on the generalized velocities.

3 THE CASE OF PERSISTENT CONTACT

In the case of persistent contact we wish to solve the dynamic equations (3.12) together with the Coulomb friction law (3.10) and the persistent contact conditions

$$\Phi^p(\boldsymbol{X}(t)) = \phi^p(\boldsymbol{x}^p(\boldsymbol{X}(t))) = 0,\ r_n^p \leq 0, \text{ for all } t \geq 0 \text{ and all } p \in \mathcal{P}_C. \tag{3.14}$$

Then, observing that, for each contact candidate particle, an equality of the type

$$\frac{d}{dt}\left(\phi^p(\boldsymbol{x}^p(\boldsymbol{X}(t)))\right) = \left|\frac{\partial \phi^p}{\partial \boldsymbol{x}}(\boldsymbol{x}^p(t))\right| \boldsymbol{v}_n^p(t) = \left|\frac{\partial \phi^p}{\partial \boldsymbol{x}}(\boldsymbol{x}^p(t))\right| \boldsymbol{G}_n^p(\boldsymbol{X}(t))\dot{\boldsymbol{X}}(t). \tag{3.15}$$

holds, we conclude that, for persistent contact,

$$\boldsymbol{G}_n(\boldsymbol{X}(t))\dot{\boldsymbol{X}}(t) = \boldsymbol{0}. \tag{3.16}$$

The set of admissible reactions for some admissible configuration $\boldsymbol{X}$ and velocity $\boldsymbol{V}$ is

$$\mathcal{K}_{\boldsymbol{r}}(\boldsymbol{X},\boldsymbol{V}) = \big\{\boldsymbol{r}\in\mathbb{R}^{3n_C} : r_n^p \leq 0,\, \boldsymbol{r}_t^p \in \mu r_n^p \boldsymbol{d}(\boldsymbol{G}_t^p(\boldsymbol{X})\boldsymbol{V}),\ \text{ for all } p\in\mathcal{P}_C\big\}. \tag{3.17}$$

Let $\boldsymbol{X}_0$, $\boldsymbol{V}_0$ be the given vectors of $\mathbb{R}^N$ that represent the initial configuration and velocity of the system, which satisfy $\Phi^p(\boldsymbol{X}_0)=0$ for all $p\in\mathcal{P}_C$ and $\boldsymbol{G}_n(\boldsymbol{X}_0)\boldsymbol{V}_0=\boldsymbol{0}$, and let $\widetilde{\boldsymbol{a}}$ be the vector of dimension $3n_C$ that contains the contributions to the contact accelerations that depend quadratically on the generalized velocities,

$$\widetilde{\boldsymbol{a}} = \widetilde{\boldsymbol{a}}(\boldsymbol{X},\boldsymbol{V}) = \left(\frac{\partial \boldsymbol{G}(\boldsymbol{X})}{\partial \boldsymbol{X}}\boldsymbol{V}\right)\boldsymbol{V}\in\mathbb{R}^{3n_C}. \tag{3.18}$$

Then the dynamic problem that we wish to solve has the following differential inclusion fomulation:

$$\begin{cases}\text{Find } (\boldsymbol{X}(t),\boldsymbol{V}(t))\in\mathbb{R}^{2N} \text{ such that for almost every } t\in[0,T],\\ \dfrac{d}{dt}\left\{\begin{array}{c}\boldsymbol{X}\\ \boldsymbol{V}\end{array}\right\}\in\mathcal{F}(\boldsymbol{X},\boldsymbol{V},t),\\ \text{and } \boldsymbol{X}(0)=\boldsymbol{X}_0,\ \boldsymbol{V}(0)=\boldsymbol{V}_0,\end{cases} \tag{3.19}$$

where

$$\mathcal{F}(\boldsymbol{X},\boldsymbol{V},t) = \left\{\begin{array}{c}\boldsymbol{V}\\ \boldsymbol{M}^{-1}\big[\boldsymbol{F}(\boldsymbol{X},\boldsymbol{V},t)+\boldsymbol{G}^T\mathcal{K}_{\boldsymbol{r}}(\boldsymbol{X},\boldsymbol{V})\big]\end{array}\right\}\cap\mathcal{T}(\boldsymbol{X},\boldsymbol{V}) \tag{3.20}$$

$$\mathcal{T}(\boldsymbol{X},\boldsymbol{V}) = \left\{\left\{\begin{array}{c}\boldsymbol{Y}\\ \boldsymbol{A}\end{array}\right\}\in\mathbb{R}^{2N} : \boldsymbol{G}_n(\boldsymbol{X})\boldsymbol{Y}=\boldsymbol{0},\, \boldsymbol{G}_n(\boldsymbol{X})\boldsymbol{A}+\widetilde{\boldsymbol{a}}_n(\boldsymbol{X},\boldsymbol{V})=\boldsymbol{0}\right\}. \tag{3.21}$$

In the present note, we shall use the theory of differential inclusions to establish existence of a solution to (3.19). Sufficient conditions for existence of solution are (cf. Aubin and Cellina [0]: Theorem 4, page 98) that $\mathcal{F}$ is upper semi-continuous, closed, convex,

$$\mathcal{F}(\boldsymbol{X},\boldsymbol{V},t)\neq\varnothing \text{ on } \mathbb{R}^{2N}\times[0,T], \tag{3.22}$$

and $(\boldsymbol{X},\boldsymbol{V},t)\to m(\mathcal{F}(\boldsymbol{X},\boldsymbol{V},t))$ is locally bounded, where m is the minimal norm defined as follows:

$$m(\mathcal{F}(\boldsymbol{X},\boldsymbol{V},t)) = \inf\big\{\|(\boldsymbol{Y},\boldsymbol{A})\| : (\boldsymbol{Y},\boldsymbol{A})\in\mathcal{F}(\boldsymbol{X},\boldsymbol{V},t)\big\}.$$

The non-emptiness condition (3.22) reduces, at each $(\boldsymbol{X},\boldsymbol{V},t)\in\mathbb{R}^{2N}\times[0,T]$, to finding solutions to the auxiliary problem:

Find $\boldsymbol{A}\in\mathbb{R}^N$ and $\boldsymbol{r}\in\mathbb{R}^{3n_C}$ such that [cf. (3.19), (3.20), (3.21)],

$$\boldsymbol{A} = \boldsymbol{M}^{-1}[\boldsymbol{F}+\boldsymbol{G}^T\boldsymbol{r}], \tag{3.23}$$

$$\boldsymbol{r}\in\mathcal{K}_{\boldsymbol{r}}, \tag{3.24}$$

$$\boldsymbol{a}_n = \boldsymbol{G}_n\boldsymbol{A}+\widetilde{\boldsymbol{a}}_n = \boldsymbol{0}, \tag{3.25}$$

where, for simplicity, the dependence on $\boldsymbol{X}$, $\boldsymbol{V}$, and t has been omitted.

In order to solve the auxiliary problem (3.23)-(3.25), we substitute (3.23) in (3.25) and we decompose $\boldsymbol{G}^T\boldsymbol{r}$ into its normal and tangential contributions

$$\begin{aligned} \mathbf{0} = \boldsymbol{a}_n &= \boldsymbol{G}_n\boldsymbol{M}^{-1}(\boldsymbol{F} + \boldsymbol{G}_t^T\boldsymbol{r}_t + \boldsymbol{G}_n^T\boldsymbol{r}_n) + \widetilde{\boldsymbol{a}}_n \\ &= (\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{F} + \widetilde{\boldsymbol{a}}_n) + (\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{G}_t^T)\boldsymbol{r}_t + (\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{G}_n^T)\boldsymbol{r}_n. \end{aligned} \tag{3.26}$$

Premultiplying then by $(\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{G}_n^T)^{-1}$ and using the friction law (3.10) we get

$$-(\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{G}_n^T)^{-1}(\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{F} + \widetilde{\boldsymbol{a}}_n) \in \big[\boldsymbol{I} + \mu(\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{G}_n^T)^{-1}(\boldsymbol{G}_n\boldsymbol{M}^{-1}\boldsymbol{G}_t^T)\mathbb{D}\big]\boldsymbol{r}_n \tag{3.27}$$

where $\mathbb{D}$ is the $(2n_C \times n_C)$ block-diagonal multivalued matrix, whose p diagonal blocks are the (2×1) multivalued vector applications defined by (3.11):

$$\mathbb{D} = \mathbb{D}(\boldsymbol{X}, \boldsymbol{V}) = \big(\mathrm{diag}\ \boldsymbol{d}(\boldsymbol{G}_t^p(\boldsymbol{X})\boldsymbol{V}),\ p \in \mathcal{P}_C\big). \tag{3.28}$$

The inclusion (3.27) becomes thus

$$\boldsymbol{\rho}_n \in [\boldsymbol{I} + \mu\boldsymbol{\mathcal{M}}\mathbb{D}]\boldsymbol{r}_n, \tag{3.29}$$

where the $(n_C \times 1)$ vector $\boldsymbol{\rho}_n$ is defined by

$$\begin{aligned} \boldsymbol{\rho}_n &= \boldsymbol{\rho}_n(\boldsymbol{X}, \boldsymbol{V}, t) \\ &= -\big[\boldsymbol{G}_n(\boldsymbol{X})\boldsymbol{M}^{-1}(\boldsymbol{X})\boldsymbol{G}_n^T(\boldsymbol{X})\big]^{-1}\big[\boldsymbol{G}_n(\boldsymbol{X})\boldsymbol{M}^{-1}(\boldsymbol{X})\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{V}, t) + \widetilde{\boldsymbol{a}}_n(\boldsymbol{X}, \boldsymbol{V})\big] \end{aligned} \tag{3.30}$$

and the $(n_C \times 2n_C)$ coupling matrix $\boldsymbol{\mathcal{M}}$ is

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}(\boldsymbol{X}) = \big[\boldsymbol{G}_n(\boldsymbol{X})\boldsymbol{M}^{-1}(\boldsymbol{X})\boldsymbol{G}_n^T(\boldsymbol{X})\big]^{-1}\big[\boldsymbol{G}_n(\boldsymbol{X})\boldsymbol{M}^{-1}(\boldsymbol{X})\boldsymbol{G}_t(\boldsymbol{X})\big]. \tag{3.31}$$

If the coefficient of friction μ or the coupling matrix $\boldsymbol{\mathcal{M}}$ vanish, it is clear that the algebraic inclusion (3.29) can always be solved in $\mathbb{R}^N \times [0, T]$, with actually $\boldsymbol{r}_n = \boldsymbol{\rho}_n$. When $\mu > 0$ and the coupling matrix $\boldsymbol{\mathcal{M}}$ does not vanish, it can be shown by continuity that the algebraic inclusion (3.29) can also be solved for $\boldsymbol{r}_n$ with

$$\boldsymbol{r}_n = [\boldsymbol{I} + \mu\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{D}}]^{-1}\boldsymbol{\rho}_n, \tag{3.32}$$

$\boldsymbol{\mathcal{D}} = \boldsymbol{\mathcal{D}}(\boldsymbol{X}, \boldsymbol{V})$ being one of the matrices in the set of matrices $\mathbb{D}(\boldsymbol{X}, \boldsymbol{V})$ (see (3.28)). For that purpose we start by assuming that there exists $M > 0$ such that

$$\forall \boldsymbol{X} \in \mathbb{R}^N,\ \forall i = 1, \ldots, n_C \text{ and } \forall j = 1, \ldots, 2n_C,\ |\mathcal{M}_{ij}(\boldsymbol{X})| \leq M. \tag{3.33}$$

A sufficient condition for invertibility of every $[\boldsymbol{I} + \mu\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{D}}]$ is that $[\boldsymbol{I} + \mu\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{D}}]$ is positive definite for all $\boldsymbol{\mathcal{D}}$ belonging to $\mathbb{D}(\boldsymbol{X}, \boldsymbol{V})$ which means

$$\boldsymbol{s} \cdot [\boldsymbol{I} + \mu\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{D}}]\boldsymbol{s} > 0,\ \forall \boldsymbol{s} \in \mathbb{R}^{n_C},\ \boldsymbol{s} \neq \mathbf{0}. \tag{3.34}$$

Thanks to (3.33) and Hölder's inequality, and since $|\boldsymbol{d}| \leq 1$, we may establish the inequality

$$\boldsymbol{s} \cdot [\boldsymbol{I} + \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}]\boldsymbol{s} \geq \left(1 - \mu M n_C^2\right) \boldsymbol{s} \cdot \boldsymbol{s}, \forall \boldsymbol{s} \in \mathbb{R}^{n_C}, \boldsymbol{s} \neq \boldsymbol{0}, \tag{3.35}$$

so that (3.34) is satisfied if the right hand side of (3.35) is strictly positive, i.e. if

$$\mu < \frac{1}{M n_C^2}. \tag{3.36}$$

Assume now that $\rho_n^p(\boldsymbol{X}_0, \boldsymbol{V}_0, 0)$ is strictly negative and that ρ_n^p is continuous with respect to $\boldsymbol{X}$, $\boldsymbol{V}$ and t for all $p \in \mathcal{P}_C$. Then in a neighbourhood $\mathcal{N}(\boldsymbol{X}_0) \times \mathcal{N}(\boldsymbol{V}_0) \times [0, T]$ of the initial conditions $(\boldsymbol{X}_0, \boldsymbol{V}_0, 0)$, there exists a strictly positive real number $\bar{\rho}$ such that

$$\underline{\rho} \leq -\rho_n^p(\boldsymbol{X}, \boldsymbol{V}, t) \leq \bar{\rho}, \forall p \in \mathcal{P}_C \text{ and } \forall (\boldsymbol{X}, \boldsymbol{V}, t) \in \mathcal{N}(\boldsymbol{X}_0) \times \mathcal{N}(\boldsymbol{V}_0) \times [0, T] \tag{3.37}$$

where $\underline{\rho}$ is a strictly positive real number. We infer from (3.32) and from the identity:

$$[\boldsymbol{I} + \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}]^{-1} = \boldsymbol{I} - [\boldsymbol{I} + \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}]^{-1} \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}, \forall p \in \mathcal{P}_C$$

that

$$r_n^p = \rho_n^p - \sum_{q,k} \left([\boldsymbol{I} + \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}]^{-1} \right)_{pk} (\mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}})_{kq} \rho_n^q, \forall p \in \mathcal{P}_C. \tag{3.38}$$

Using the inequalities (3.37) and the identity (3.38), we get

$$-r_n^p \geq \underline{\rho} - \bar{\rho} \max_{\boldsymbol{d}} (\kappa) \tag{3.39}$$

where the maximum is taken with respect to all $\mathbb{D}$ given by (3.28) and (3.11) and

$$\kappa = \max_p \sum_{q,k} \left| \left([\boldsymbol{I} + \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}]^{-1} \right)_{pk} (\mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}})_{kq} \right|. \tag{3.40}$$

For any matrix $\boldsymbol{N}$, we denote

$$\|\boldsymbol{N}\|_\infty = \max_i \sum_j |\boldsymbol{N}_{ij}|.$$

Therefore with this notation, it is easy to deduce the following inequality:

$$\kappa \leq \|\mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}\|_\infty \|[\boldsymbol{I} + \mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}]^{-1}\|_\infty. \tag{3.41}$$

Since $\|\boldsymbol{I}\|_\infty = 1$, we obtain the estimate

$$\kappa \leq \|\mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}\|_\infty \left(1 - \|\mu \boldsymbol{\mathcal{M}} \boldsymbol{\mathcal{D}}\|_\infty\right)^{-1}. \tag{3.42}$$

Introducing (3.42) in (3.39) it is possible to deduce, using Hölder's inequality, (3.33) and the fact that $|\boldsymbol{d}| \leq 1$, that

$$-r_n^p \geq \underline{\rho} - (\mu M n_C)(1 - \mu M n_C)^{-1} \bar{\rho}. \tag{3.43}$$

Therefore we choose

$$\mu < \frac{1}{\big(1 + \bar{\rho}/\underline{\rho}\big) M n_C} \tag{3.44}$$

which garantees that

$$r_n^p < 0, \, \forall p \in \mathcal{P}_C. \tag{3.45}$$

To complete the solution of the auxiliary problem (3.23)-(3.25) it suffices to let

$$\boldsymbol{r}_t = \mu \boldsymbol{\mathcal{D}} \boldsymbol{r}_n,$$

where $\boldsymbol{\mathcal{D}}$ is the matrix in (3.32), and compute $\boldsymbol{A}$ by (3.23).

Next we remark that $m(\mathcal{F}(\boldsymbol{X}, \boldsymbol{V}, t))$ is locally bounded iff $m(\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{V}, t) + \boldsymbol{G}^T(\boldsymbol{X}) \mathcal{K}_{\boldsymbol{r}}(\boldsymbol{X}, \boldsymbol{V}))$ is locally bounded. Therefore it is sufficient to prove that $\mathcal{K}_{\boldsymbol{r}}(\boldsymbol{X}, \boldsymbol{V})$ is locally bounded, which holds due to (3.17), (3.32), and (3.37). Finally, since it is quite routine to prove that $\mathcal{F}$ is upper semi-continuous, closed and convex, we leave the verification to the reader.

Remark 3.1 *Jean and Pratt in [0] have obtained conditions that are quite similar to* (3.36) *and* (3.44)*. In their work some constant in the interval* $(0,1)$ *plays the role of* $1/n_C$ *in* (3.36) *and* (3.44)*, but the dependency of such constant on* n_C *is not made explicit.*

The application of Aubin-Cellina's result yields only a local solution. However, a priori estimates may be available as in the following lemma:

Lemma 3.2 *Assume that* $\boldsymbol{Q}(\boldsymbol{X}, \boldsymbol{V}, t)$ *is Lipschitzian with respect to* $\boldsymbol{X}$*,* $\boldsymbol{V}$ *and continuous with respect to* t*,* $\boldsymbol{G}(\boldsymbol{X})$ *is Lipschitzian with respect to* $\boldsymbol{X}$*. Then* $T(\boldsymbol{X}, \dot{\boldsymbol{X}})$ *is bounded and* $\dot{\boldsymbol{X}}$ *is bounded in* $L^2(0,T)$*.*

Proof. This estimate is simply an application of Gronwall's lemma to the energy estimate. Let us enter into details. We multiply (3.12) by $\dot{\boldsymbol{X}}$ and we integrate over $(0,\tau)$, $\tau \in [0,T]$ then we get

$$\int_0^\tau \frac{dT}{dt} (\boldsymbol{X}, \dot{\boldsymbol{X}}) \, dt = \int_0^\tau \dot{\boldsymbol{X}} \cdot \boldsymbol{Q}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t) \, dt + \int_0^\tau \dot{\boldsymbol{X}} \cdot \big(\boldsymbol{G}^T(\boldsymbol{X}) \boldsymbol{r}\big) \, dt. \tag{3.46}$$

We observe that

$$\Big| \int_0^\tau \dot{\boldsymbol{X}} \cdot \boldsymbol{Q}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t) \, dt \Big| \leq \frac{1}{2} \, \|\boldsymbol{Q}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t)\|_{L^2(0,\tau)}^2 + \frac{1}{2} \, \|\dot{\boldsymbol{X}}\|_{L^2(0,\tau)}^2. \tag{3.47}$$

On the other hand, we have

$$\int_0^\tau \dot{\boldsymbol{X}} \cdot \big(\boldsymbol{G}^T(\boldsymbol{X})\boldsymbol{r}\big)\,dt = \int_0^\tau \dot{\boldsymbol{X}} \cdot \big(\boldsymbol{G}_n^T(\boldsymbol{X})\boldsymbol{r}_n\big)\,dt + \int_0^\tau \dot{\boldsymbol{X}} \cdot \big(\boldsymbol{G}_t^T(\boldsymbol{X})\boldsymbol{r}_t\big). \tag{3.48}$$

Since the first integral on the right hand side of (3.48) vanishes, the friction law of Coulomb gives the following inequality:

$$\int_0^\tau \dot{\boldsymbol{X}} \cdot \big(\boldsymbol{G}^T(\boldsymbol{X})\boldsymbol{r}\big)\,dt \leq 0. \tag{3.49}$$

Carrying (3.47) and (3.49) into (3.46) and since there exists $c_1 > 0$ for all $\boldsymbol{V}$ such that

$$c_1 \int_0^\tau T(\boldsymbol{X}, \boldsymbol{V})\,dt \geq \|\boldsymbol{V}\|^2_{L^2(0,\tau)}, \tag{3.50}$$

we obtain the following inequality:

$$T(\boldsymbol{X}(\tau), \dot{\boldsymbol{X}}(\tau)) \leq T(\boldsymbol{X}_0, \boldsymbol{V}_0) + \frac{1}{2}\|\boldsymbol{Q}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t)\|^2_{L^2(0,\tau)} + \frac{c_1}{2}\int_0^\tau T(\boldsymbol{X}, \dot{\boldsymbol{X}})\,dt.$$

A classical Gronwall's lemma enables us to deduce that $T(\boldsymbol{X}, \dot{\boldsymbol{X}})$ is bounded and, by (3.50), $\dot{\boldsymbol{X}}$ is bounded in $L^2(0, T)$. $\square$

Then we are able to extend the previous local solution to a maximal interval $[0, T^*]$ where T^* is only limited by loss of contact and we deduce the following proposition:

Proposition 3.3 *(Existence) Let the assumptions of Lemma 3.2 hold on* $[0, T^*]$*, assume that* $\rho_n^p(\boldsymbol{X}_0, \boldsymbol{V}_0, 0)$ *is strictly negative and* ρ_n^p *is continuous with respect to* $\boldsymbol{X}$*,* $\boldsymbol{V}$ *and* t *for all* $p \in \mathcal{P}_C$*,* (3.33) *and* (3.37) *are satisfied and*

$$\mu < \frac{1}{Mn_C} \inf\left(\frac{1}{n_C}, \frac{1}{(1 + \bar{\rho}/\underline{\rho})}\right). \tag{3.51}$$

Then there exists an absolutely continuous $(\boldsymbol{X}, \boldsymbol{V})$ *solution to* (3.19) *defined on* $[0, T^*]$.

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