A REPORT ON THE STABILITY OF QUASI-STATIC PATHS FOR FINITE DIMENSIONAL ELASTIC-PLASTIC SYSTEMS WITH HARDENING

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Abstract. In this paper we prove the stability of quasi-static paths of finite dimensional mechanical systems that have an elastic-plastic behavior with linear hardening. The concept of stability of quasi-static paths used here is essentially a continuity property relatively to the size of the initial perturbations (as in Lyapunov stability) and to the smallness of the rate of application of the external forces (which plays here the role of the small parameter in singular perturbation problems). The discussion of stability is preceded by the presentation of mathematical formulations (plus existence and uniqueness results) for those dynamic and quasi-static problems, in a form that is convenient for the subsequent discussion of stability.

Key words: differential inclusions, plasticity, hardening, existence, stability
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1. Introduction

The Newton law, *force equals mass times acceleration*, determines the governing equation for the dynamic evolution of mechanical systems. A classical approximation for the equations that govern the slow evolution of mechanical systems is to neglect inertia effects and take the balance equations as static equilibrium equations, i.e. *force equals zero*. The slow evolutions made up of the successive equilibrium configurations are called *quasi-static evolutions*.

Martins et al. [7] have established the relation that exists between dynamic and quasi-static evolutions and theory of singular perturbations. For this purpose a change of variables is performed that consists of replacing the physical time t by a (slow) load parameter λ, whose rate of change with respect to time, $\varepsilon = d\lambda/dt$, is eventually decreased to zero. This leads to a system of dynamic (ordinary or partial) differential equations that defines (in finite or infinite dimensions) a singular perturbation problem, i.e. a problem governed by a system of equations where, in some equations, the highest order derivative with respect to λ appears multiplied by the small parameter ε.

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An issue that is relevant in the study of quasi-static trajectories is their "stability". The concept of Lyapunov stability has been used for long to study the stability of dynamic trajectories of mechanical systems, namely the zero acceleration trajectories of the equilibrium configurations under constant applied loads. But the application of that concept to quasi-static paths with slowly varying loads faces the difficulty that such paths are not, in general, true dynamic solutions [7].

In order to overcome the limitations of some criteria or procedures used earlier in the literature to study the stability of quasi-static paths (cf. [4], [7], and the references therein), a mathematical definition of stability of quasi-static paths was recently proposed by Martins et al. ([4], [3]). That proposal (i) takes inertia into account, (ii) recognizes the distinction between quasi-static and dynamic governing equations and time scales, (iii) considers a finite interval of the load parameter, which, for vanishing load rates, corresponds to infinitely large intervals of physical time, and (iv) selects the quasi-static paths close to which the dynamic evolutions will remain when: (a) the dynamic evolutions initiate sufficiently close to the quasi-static path, and (b) the load is applied sufficiently slowly.

After the study of some finite dimensional smooth cases in [3], the present paper applies the same definition to a class of problems that has a not very severe non-smoothness: the finite dimensional elastic-plastic problems with linear hardening.

The structure of the article is the following. In Section 2, the mathematical formulations for dynamic and quasi-static elastic-plastic systems with hardening are presented, and in Section 3, existence and uniqueness results are recalled, which use the theory of *m-accretive* operators (see [1], [2], [5], [6], [8]). In Section 4.2, *a priori* estimates are obtained which enable us to prove the stability of the quasi-static path in the sense of the definition proposed in ([4], [3]).

2. Governing equations

We consider a finite dimensional elastic-plastic system with linear kinematic hardening and we assume geometrical linearity. The governing dynamic equations can be non-dimensionalized by using the non-dimensional time (τ) and load paramater (λ, $\lambda = \lambda_1 + \varepsilon\tau$), yielding

$$\varepsilon^2 \boldsymbol{M}\boldsymbol{u}'' = \boldsymbol{f}_{\text{ext}}(\lambda) + \boldsymbol{f}_{\text{int}}(\boldsymbol{u}, \boldsymbol{r}), \tag{1}$$

where $\boldsymbol{M}$ is the non-dimensional (constant, symmetric, positive definite) mass matrix, and $\boldsymbol{u}$, $\boldsymbol{f}_{\text{ext}}$ and $\boldsymbol{f}_{\text{int}}$ are the non-dimensional vectors of generalized displacements, external forces and internal forces, respectively; the latter are related to the forces σ_i, $i = 1, \ldots, n$, that act on each of the elastic-plastic elements of the system and are grouped in the vector $\boldsymbol{\sigma}$

$$\boldsymbol{f}_{\text{int}} = \boldsymbol{f}_{\text{int}}(\boldsymbol{u}, \boldsymbol{r}) = -\boldsymbol{L}^T \boldsymbol{\sigma}(\boldsymbol{u}, \boldsymbol{r}). \tag{2}$$

We assume that the $(n \times N)$ matrix $\boldsymbol{L}$ has linearly independent rows. The forces r_i, $i = 1, \ldots, n$, in the plastic elements are grouped in the vector $\boldsymbol{r}$. The non-dimensional elongations $e_i, i = 1, \ldots, n$, of the system elastic-plastic elements are grouped in the vector $\boldsymbol{e}$, and can be related to the non-dimensional generalized displacements $\boldsymbol{u}$ by means of the constant matrix $\boldsymbol{L}$.

In view of the presence of the elastic-plastic elements, it can be decomposed into elastic, $\boldsymbol{e}^e$, and plastic, $\boldsymbol{e}^p$, parts:

$$\boldsymbol{e} = \boldsymbol{L}\boldsymbol{u} = \boldsymbol{e}^e + \boldsymbol{e}^p. \tag{3}$$

The forces $\boldsymbol{\sigma}$ are related to the elastic parts of the non-dimensional elongations by means of Hooke's law,

$$\boldsymbol{\sigma} = \boldsymbol{r} + \boldsymbol{H}\boldsymbol{e}^p = \boldsymbol{E}\boldsymbol{e}^e = \boldsymbol{E}(\boldsymbol{L}\boldsymbol{u} - \boldsymbol{e}^p), \tag{4}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are diagonal positive definite $(n \times n)$ matrices. Therefore using (4) in (2), we get

$$\boldsymbol{f}_{\text{int}} = -\boldsymbol{L}^T\boldsymbol{\sigma}(\boldsymbol{u},\boldsymbol{r}) = -\boldsymbol{L}^T\boldsymbol{D}\big(\boldsymbol{L}\boldsymbol{u} + \boldsymbol{H}^{-1}\boldsymbol{r}\big), \tag{5}$$

where $\boldsymbol{D} = \big(\boldsymbol{E}^{-1} + \boldsymbol{H}^{-1}\big)^{-1}$. Carrying (5) into (1), we obtain

$$\varepsilon^2\boldsymbol{M}\boldsymbol{u}'' + \boldsymbol{L}^T\boldsymbol{D}\big(\boldsymbol{L}\boldsymbol{u} + \boldsymbol{H}^{-1}\boldsymbol{r}\big) = \boldsymbol{f}_{\text{ext}}. \tag{6}$$

The behavior of the plastic elements is characterized by the non-dimensional inequalities and flow rule:

$$|r_i| \leq 1,\ \frac{de_i^p}{d\lambda}\begin{cases}\geq 0 \text{ if } r_i = +1,\\ = 0 \text{ if } -1 < r_i < +1,\\ \leq 0 \text{ if } r_i = -1,\end{cases} \quad \forall i = 1,\dots,n. \tag{7}$$

The governing dynamic equations (6), together with the conditions (7) that characterize the behavior of the plastic elements, can be put in the form of a singular perturbation system of first order differential equations and inclusions. For that purpose, let $\mathcal{C}$ denote the following closed convex set in $\mathbb{R}^n$

$$\mathcal{C} = \{\boldsymbol{r} \in \mathbb{R}^n : |r_i| \leq 1, \forall i = 1,\dots,n\}, \tag{8}$$

and let $\mathbf{sign^{-1}}(\boldsymbol{r})$ be the normal cone to $\mathcal{C}$ at $\boldsymbol{r} \in \mathbb{R}^n$ defined by

if $\boldsymbol{r} \notin \mathcal{C}$ then $\mathbf{sign^{-1}}(\boldsymbol{r}) = \emptyset$,

if $\boldsymbol{r} \in \mathcal{C}$ then $\mathbf{sign^{-1}}(\boldsymbol{r}) = \{\boldsymbol{x} \in \mathbb{R}^n : x_i \geq 0,\ \text{if } r_i = +1; x_i = 0,$
$\text{if } -1 < r_i < +1; x_i \leq 0,\ \text{if } r_i = -1, \forall i = 1,\dots,n\}.$

Then we observe that (7) can be written in the differential inclusion form:

$$(\boldsymbol{e}^p)' \in \mathbf{sign^{-1}}(\boldsymbol{r}). \tag{9}$$

Relations (4) lead to

$$\widetilde{\boldsymbol{D}}(\boldsymbol{e}^p)' = \boldsymbol{E}\boldsymbol{L}\boldsymbol{u}' - \boldsymbol{r}' \text{ where } \widetilde{\boldsymbol{D}} = \boldsymbol{E} + \boldsymbol{H}. \tag{10}$$

Substituting (10) in (9), we get

$$\boldsymbol{E}\boldsymbol{L}\boldsymbol{u}' - \boldsymbol{r}' \in \widetilde{\boldsymbol{D}}\mathbf{sign^{-1}}(\boldsymbol{r}). \tag{11}$$

From (6) and (11) we finally obtain the governing dynamic system

$$\begin{cases}\varepsilon\boldsymbol{u}' - \boldsymbol{v} = 0,\\ \varepsilon\boldsymbol{M}\boldsymbol{v}' + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{u} + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r} = \boldsymbol{f}_{\text{ext}},\\ \boldsymbol{E}\boldsymbol{L}\boldsymbol{u}' - \boldsymbol{r}' \in \widetilde{\boldsymbol{D}}\mathbf{sign^{-1}}(\boldsymbol{r}),\end{cases} \tag{12}$$

which must be satisfied together with some initial conditions

$$\big(\boldsymbol{u}(\lambda_1), \boldsymbol{v}(\lambda_1), \boldsymbol{r}(\lambda_1)\big) = (\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{r}_1) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}. \tag{13}$$

The corresponding quasi-static system is then (let $\varepsilon = 0$ in (12))

$$\begin{cases} \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\bar{\boldsymbol{u}} + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\bar{\boldsymbol{r}} = \boldsymbol{f}_{\text{ext}}, \\ \boldsymbol{E}\boldsymbol{L}\bar{\boldsymbol{u}}' - \bar{\boldsymbol{r}}' \in \widetilde{\boldsymbol{D}}\mathbf{sign}^{-\mathbf{1}}(\bar{\boldsymbol{r}}), \end{cases} \tag{14}$$

with initial conditions

$$\bar{\boldsymbol{r}}(\lambda_1) = \bar{\boldsymbol{r}}_1 \in \mathcal{C}. \tag{15}$$

Note that, consistently with the above, the quasi-static displacement rate with respect to the physical time vanishes ($\bar{\boldsymbol{v}} \equiv 0$). Besides, if X is a space of scalar functions, the bold-face notation $\boldsymbol{X}_d$ will denote the space X^d and $\boldsymbol{N}^T$ will denote the transpose of a matrix $\boldsymbol{N}$.

3. Existence and uniqueness of solution for the dynamic and the quasi-static systems

We observe that the dynamic and the quasi-static systems introduced in Section 2 can be rewritten in a form that may be studied with the theory of *m-accretive* operators. The definition and some properties of *m-accretive* operators are recalled in Section 3.1. Existence and uniqueness results for the systems of Section 2 are presented in Section 3.2.

3.1. **Reminder about m-accretive operators.** We recall now the definition of *m-accretive* operators which is contained in many text books, see, e.g., [1] or [8].

Definition 3.1. *A mapping* $\boldsymbol{A} : \mathcal{D}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^p : \boldsymbol{A}\boldsymbol{x} \neq \emptyset\} \subset \mathbb{R}^p \to \mathbb{R}^p$ *is called* ***m-accretive*** *operator, if it is monotone,*

$$(\boldsymbol{A}\boldsymbol{x}_1 - \boldsymbol{A}\boldsymbol{x}_2) \cdot (\boldsymbol{x}_1 - \boldsymbol{x}_2) \geq 0, \ \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{D}(\boldsymbol{A}),$$

and if it is maximal in the set of monotone operators, i.e. for all $[\boldsymbol{x}, \boldsymbol{y}] \in \mathbb{R}^p \times \mathbb{R}^p$ *such that*

$$(\boldsymbol{y} - \boldsymbol{A}\boldsymbol{\zeta}) \cdot (\boldsymbol{x} - \boldsymbol{\zeta}) \geq 0, \ \forall \boldsymbol{\zeta} \in \mathcal{D}(\boldsymbol{A}) \ \textit{then} \ \boldsymbol{y} \in \boldsymbol{A}\boldsymbol{x}.$$

If $\boldsymbol{\varphi} = \sum_{i=1}^{p} \varphi(x_i)$ is a convex proper and lower semi-continuous function from $\mathbb{R}^p$ to $(-\infty, +\infty]$, we can define its sub-differential $\boldsymbol{\partial\varphi} : \mathcal{D}(\boldsymbol{\partial\varphi}) = \{\boldsymbol{x} \in \mathbb{R}^p : \boldsymbol{\partial\varphi}(\boldsymbol{x}) \neq \emptyset\} \to \mathbb{R}^p \times \mathbb{R}^p$ by

$$\boldsymbol{y} \in \boldsymbol{\partial\varphi}(\boldsymbol{x}) \Leftrightarrow \forall \boldsymbol{h} \in \mathbb{R}^p, \ \boldsymbol{\varphi}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{\varphi}(\boldsymbol{x}) \geq \boldsymbol{y} \cdot \boldsymbol{h}.$$

Notice that $\boldsymbol{\partial\varphi}$ is an *m-accretive* operator. On the other hand, the *m-accretive* operator $\partial\varphi(x) = \text{sign}^{-1}(x)$ is a sub-differential of a convex proper and lower semi-continuous function defined by

$$\forall x \in \mathbb{R}, \ \varphi(x) = \begin{cases} 0 & \text{if}\, x \in [-1, 1], \\ +\infty & \text{if}\, x \notin [-1, 1]. \end{cases} \tag{16}$$

For more details, the reader can see the example 2.3.4, p.25 of [1].

3.2. **Existence and uniqueness of solution.** Recall that existence and uniqueness of solution to the differential inclusion problem

$$\boldsymbol{x}' + \mathbb{A}\boldsymbol{x} \ni \boldsymbol{g} \text{ a.e. on } (\lambda_1, \lambda_2), \tag{17a}$$

$$\boldsymbol{x}(\lambda_1) = \boldsymbol{x}_1, \tag{17b}$$

follows from the following Proposition (cf. [1], [5] and Appendix):

Proposition 3.2. *Assume that* $\mathbb{A} : \mathcal{D}(\mathbb{A}) \subset \mathbb{R}^p \to \mathbb{R}^p$ *is an m-accretive operator,* $\boldsymbol{g}$ *belongs to* $\boldsymbol{W}_p^{1,\infty}(\lambda_1, \lambda_2)$ *and* $x_1 \in \mathcal{D}(\mathbb{A})$. *Then there exists a unique solution* $\boldsymbol{x}$ *of* (17) *belonging to* $\boldsymbol{W}_p^{1,\infty}(\lambda_1, \lambda_2)$.

By applying Proposition 3.2, we prove existence and uniqueness of solution for the dynamic system (12)-(13) and for the corresponding quasi-static system (14)-(15). We introduce the following notations:

$$\widetilde{\boldsymbol{u}} = \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{1/2}\boldsymbol{u},\ \widetilde{\boldsymbol{v}} = \boldsymbol{M}^{1/2}\boldsymbol{v},\ \widetilde{\boldsymbol{r}} = \widetilde{\boldsymbol{D}}^{-1/2}\boldsymbol{r}. \tag{18}$$

Carrying (18) into (12) and denoting $\boldsymbol{x} = (\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{r}})$, we get (17a) where $\mathbb{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{x}$ with

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ \widetilde{\boldsymbol{D}}^{1/2}\mathbf{sign}^{-1}\big(\widetilde{\boldsymbol{D}}^{1/2}\widetilde{\boldsymbol{r}}\big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \widetilde{\boldsymbol{D}}^{1/2}\boldsymbol{\partial\varphi}\big(\widetilde{\boldsymbol{D}}^{1/2}\widetilde{\boldsymbol{r}}\big) \end{pmatrix},$$

and

$$\boldsymbol{B}\boldsymbol{x} = \frac{1}{\varepsilon}\begin{pmatrix} -\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{1/2}\boldsymbol{M}^{-1/2}\widetilde{\boldsymbol{v}} \\ \boldsymbol{M}^{-1/2}\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{1/2}\widetilde{\boldsymbol{u}} + \boldsymbol{M}^{-1/2}\boldsymbol{L}^T\boldsymbol{E}\widetilde{\boldsymbol{D}}^{-1/2}\widetilde{\boldsymbol{r}} \\ -\widetilde{\boldsymbol{D}}^{-1/2}\boldsymbol{E}\boldsymbol{L}\boldsymbol{M}^{-1/2}\widetilde{\boldsymbol{v}} \end{pmatrix},$$

and

$$\boldsymbol{g} = \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ \boldsymbol{M}^{-1/2}\boldsymbol{f}_{\text{ext}} \\ 0 \end{pmatrix}.$$

$\boldsymbol{A}$ is *m-accretive* since $\boldsymbol{E}$ and $\boldsymbol{H}$ are diagonal positive definite matrices and $\boldsymbol{\varphi}(\,\cdot\,)$ is a convex proper and lower semi-continuous function. Moreover $\boldsymbol{B}$ is a monotone and Lipschitzian operator. Then $\mathbb{A}$ is a *m-accretive* operator (cf. [1]) and Proposition 3.2 yields the following Corollary:

Corollary 3.3. *Assume that* $\boldsymbol{f}_{\text{ext}}$ *belongs to* $\boldsymbol{W}_N^{1,\infty}(\lambda_1, \lambda_2)$ *and that* (13) *holds. Then there exists a unique solution* $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r})$ *of* (12)–(13) *belonging to* $\big(\boldsymbol{W}_N^{1,\infty}(\lambda_1, \lambda_2)\big)^2 \times \boldsymbol{W}_n^{1,\infty}(\lambda_1, \lambda_2)$.

On the other hand, we deduce from the identity in (14) that

$$\bar{\boldsymbol{u}} = \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{-1}\boldsymbol{f}_{\text{ext}} - \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{-1}\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\bar{\boldsymbol{r}}. \tag{19}$$

Carrying (19) into (14) and using the injectivity of $\boldsymbol{L}^T$, we get

$$\bar{\boldsymbol{r}}' + \boldsymbol{H}\mathbf{sign}^{-1}(\bar{\boldsymbol{r}}) \ni \boldsymbol{D}\boldsymbol{L}\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{-1}\boldsymbol{f}'_{\text{ext}}. \tag{20}$$

As for the dynamic system, the sub-differential $\boldsymbol{\partial\varphi}(\bar{\boldsymbol{r}}) = \boldsymbol{H}\mathbf{sign}^{-1}(\bar{\boldsymbol{r}})$ is an *m-accretive* operator since $\boldsymbol{H}$ is a diagonal positive definite matrix and $\boldsymbol{\varphi}(\bar{\boldsymbol{r}})$ is a proper convex and lower semi-continuous function. Denoting $\boldsymbol{x} = \bar{\boldsymbol{r}}$,

$\mathbb{A} = \boldsymbol{H}\mathbf{sign^{-1}}$, $\boldsymbol{g} = \boldsymbol{D}\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{-1}\boldsymbol{f}'_{\text{ext}}$ with $p = n$ in (17), then we apply Proposition 3.2 and we obtain the following Corollary:

Corollary 3.4. *Assume that* $\boldsymbol{f}_{\text{ext}}$ *belongs to* $\boldsymbol{W}_N^{1,\infty}(\lambda_1, \lambda_2)$ *and* $\bar{\boldsymbol{r}}_1 \in \mathcal{C}$. *Then there exists a unique solution* $\bar{\boldsymbol{r}}$ *of* (20) *belonging to* $\boldsymbol{W}_n^{1,\infty}(\lambda_1, \lambda_2)$, *with initial conditions* $\bar{\boldsymbol{r}}_1 \in \mathcal{C}$.

Remark 3.5. *According to Corollary 3.4 and identity* (19), $\bar{\boldsymbol{u}}$ *belongs to* $\boldsymbol{W}_N^{1,\infty}(\lambda_1, \lambda_2)$.

4. Stability of quasi-static paths of elastic-plastic systems

In Section 4.1 we adapt the definition of stability of a quasi-static path ([4],[3]) to the present elastic-plastic problem, and in Section 4.2 we prove *a priori* estimates that show that, in order to guarantee that those two solutions remain close to each other in some finite interval of load, it suffices that the dynamic solution of (12) is initially close to the quasi-static solution of (14) and the loading rate ε is sufficiently small.

4.1. Definition of stability of a quasi-static path.

The mathematical definition of stability of a quasi-static path at an equilibrium point is presented in the context of the governing dynamic system (12)-(13) and the quasi-static system (14)-(15).

Definition 4.1. *The quasi-static path* $(\bar{\boldsymbol{u}}(\lambda), \bar{\boldsymbol{r}}(\lambda))$ *is said to be stable at* λ_1 *if there exists* $0 < \Delta\lambda \leq \lambda_2 - \lambda_1$, *such that, for all* $\delta > 0$ *there exists* $\bar{\rho}(\delta) > 0$ *and* $\bar{\varepsilon}(\delta) > 0$ *such that for all initial conditions* $\boldsymbol{u}_1$, $\boldsymbol{v}_1$, $\boldsymbol{r}_1$ *and* $\bar{\boldsymbol{r}}_1$ $(\boldsymbol{r}_1 \in \mathcal{C}, \bar{\boldsymbol{r}}_1 \in \mathcal{C})$ *and all* $\varepsilon > 0$ *such that*

$$|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2 \leq \bar{\rho}(\delta) \text{ and } \varepsilon \leq \bar{\varepsilon}(\delta),$$

the solution $(\boldsymbol{u}(\lambda), \boldsymbol{v}(\lambda), \boldsymbol{r}(\lambda))$ *of the dynamic system* (12)-(13) *satisfies*

$$|\boldsymbol{v}(\lambda)|^2 + |\boldsymbol{u}(\lambda) - \bar{\boldsymbol{u}}(\lambda)|^2 + |\boldsymbol{r}(\lambda) - \bar{\boldsymbol{r}}(\lambda)|^2 \leq \delta,\ \forall \lambda \in [\lambda_1, \lambda_1 + \Delta\lambda].$$

For more details, the reader is referred to [4].

4.2. A priori estimates and stability.

Let us introduce the regularized problem:

$$\begin{cases} \varepsilon^2 \boldsymbol{M}\boldsymbol{u}''_\mu + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}_\mu + \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}_\mu = \boldsymbol{f}_{\text{ext}}, \\ \boldsymbol{E}\boldsymbol{L}\boldsymbol{u}'_\mu - \boldsymbol{r}'_\mu = \dfrac{\widetilde{\boldsymbol{D}}}{\mu}\left(\boldsymbol{r}_\mu - \mathbf{proj}_{\mathcal{C}}\boldsymbol{r}_\mu\right), \end{cases} \tag{21}$$

with initial conditions

$$\big(\boldsymbol{u}_\mu(\lambda_1), \boldsymbol{v}_\mu(\lambda_1), \boldsymbol{r}_\mu(\lambda_1)\big) = (\boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{r}_1) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{C}. \tag{22}$$

Here $\mathbf{proj}_{\mathcal{C}}$ denotes the projection on the convex $\mathcal{C}$, i.e.

$$\mathbf{proj}_{\mathcal{C}} = \begin{pmatrix} \mathrm{proj}_{\mathcal{C}_1} \\ \vdots \\ \mathrm{proj}_{\mathcal{C}_n} \end{pmatrix} \text{ where } \mathcal{C}_i = \{r_i \in \mathbb{R} : |r_i| \leq 1\},\ \forall i = 1, \ldots, n,$$

and $\boldsymbol{v}_\mu = \varepsilon \boldsymbol{u}'_\mu$. We introduce the following notations:

$$\widetilde{\boldsymbol{u}}_\mu = \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\big)^{1/2}\boldsymbol{u}_\mu,\ \widetilde{\boldsymbol{v}}_\mu = \boldsymbol{M}^{1/2}\boldsymbol{v}_\mu,\ \widetilde{\boldsymbol{r}} = \widetilde{\boldsymbol{D}}^{-1/2}\boldsymbol{r}_\mu. \tag{23}$$

Carrying (48) into (21) and denoting $\boldsymbol{x}_\mu = (\widetilde{\boldsymbol{u}}_\mu, \widetilde{\boldsymbol{v}}_\mu, \widetilde{\boldsymbol{r}}_\mu)$, we get

$$\boldsymbol{x}'_\mu + \boldsymbol{A}_\mu \boldsymbol{x}_\mu + \boldsymbol{B}\boldsymbol{x}_\mu = \boldsymbol{g}, \tag{24}$$

where

$$\boldsymbol{A}_\mu \boldsymbol{x}_\mu = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\widetilde{\boldsymbol{D}}^{1/2}}{\mu}\Big(\widetilde{\boldsymbol{D}}^{1/2}\widetilde{\boldsymbol{r}}_\mu + \mathbf{proj}_{\mathcal{C}}\big(\widetilde{\boldsymbol{D}}^{1/2}\widetilde{\boldsymbol{r}}_\mu\big)\Big) \end{pmatrix}.$$

Since $\boldsymbol{A}_\mu$ is Lipschitzian with constant $1/\mu$ and *m-accretive* operator (see Proposition 2.6. p.28 of [1]) and $\boldsymbol{B}$ is monotone and Lipschitzian operator, then $\boldsymbol{A}_\mu + \boldsymbol{B}$ is an *m-accretive* operator (cf. [1]) and Proposition 3.2 yields the following Corollary:

Corollary 4.1. *Assume that $\boldsymbol{f}_{\text{ext}}$ belongs to $\boldsymbol{W}_N^{1,\infty}(\lambda_1,\lambda_2)$ and that (22) holds. Then there exists a unique solution $(\boldsymbol{u}_\mu, \boldsymbol{v}_\mu, \boldsymbol{r}_\mu)$ of (21)-(22) belonging to $\big(\boldsymbol{W}_N^{1,\infty}(\lambda_1,\lambda_2)\big)^2 \times \boldsymbol{W}_n^{1,\infty}(\lambda_1,\lambda_2)$. Moreover, as μ tends to zero, $(\boldsymbol{u}_\mu, \boldsymbol{v}_\mu, \boldsymbol{r}_\mu)$ converges strongly to the solution of (12)-(13).*

Lemma 4.2. *Assume that (13) holds and $\boldsymbol{f}_{ext}$ belongs to $\boldsymbol{W}_N^{2,\infty}(\lambda_1,\lambda_2)$. Then there exists a positive constant $c(\lambda_1,\lambda_2)$ that depends on the interval of λ and such that*

$$\begin{aligned} |\varepsilon \boldsymbol{v}'(\lambda)|^2 \le c(\lambda_1,\lambda_2)\Big(&|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2 \\ &+ \varepsilon^2 |\boldsymbol{f}'_{ext}(\lambda_1)|^2 + \varepsilon^2 \|\boldsymbol{f}'_{ext}\|^2_{L^\infty(\lambda_1,\lambda_2)} + \varepsilon^2 \|\boldsymbol{f}''_{ext}\|^2_{L^2(\lambda_1,\lambda_2)}\Big). \end{aligned} \tag{25}$$

Proof. This estimate results from the application of Gronwall's lemma to energy estimates that are obtained by differentiating the governing system (21) with respect to λ and multiplying the result by $\varepsilon^2 \boldsymbol{u}''_\mu$. Integrating the resulting expression over (λ_1,λ), we get

$$\begin{aligned} &\int_{\lambda_1}^{\lambda} \varepsilon^4 (\boldsymbol{M}\boldsymbol{u}'''_\mu)\cdot \boldsymbol{u}''_\mu \, d\xi + \int_{\lambda_1}^{\lambda} \varepsilon^2 \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}'_\mu\big)\cdot \boldsymbol{u}''_\mu \, d\xi \\ &\quad + \int_{\lambda_1}^{\lambda} \varepsilon^2 \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}'_\mu\big)\cdot \boldsymbol{u}''_\mu \, d\xi = \int_{\lambda_1}^{\lambda} \varepsilon^2 \boldsymbol{f}'_{\text{ext}}\cdot \boldsymbol{u}''_\mu \, d\xi. \end{aligned} \tag{26}$$

We shall pass to the limit in (26) when μ tends to zero. Denoting $\boldsymbol{v}_\mu = \varepsilon \boldsymbol{u}'_\mu$ and integrating (26) by parts (except the third integral on the left hand side) and using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} &\Big[\varepsilon^2(\boldsymbol{M}\boldsymbol{v}'_\mu)\cdot \boldsymbol{v}'_\mu + \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{v}_\mu\big)\cdot \boldsymbol{v}_\mu\Big]_{\lambda_1}^{\lambda} + 2\int_{\lambda_1}^{\lambda} \varepsilon\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}'_\mu\big)\cdot \boldsymbol{v}'_\mu \, d\xi \\ &\le 2\varepsilon(|(\boldsymbol{f}'_{\text{ext}}\cdot \boldsymbol{v}_\mu)(\lambda)| + |\boldsymbol{f}'_{\text{ext}}(\lambda_1)\cdot \boldsymbol{v}_1|) + \int_{\lambda_1}^{\lambda} |\varepsilon \boldsymbol{f}''_{\text{ext}}|^2 \, d\xi + \int_{\lambda_1}^{\lambda} |\boldsymbol{v}_\mu|^2 d\xi. \end{aligned} \tag{27}$$

On one hand, we subtract the first equation in (21) at λ_1 to the first one in (14) at λ_1. From (22) and since $\boldsymbol{M}$ and $\boldsymbol{D}$ are respectively symmetric positive definite and diagonal positive definite matrices, then there exists a positive constant c_1 such that

$$\begin{aligned} &\varepsilon^2\big(\boldsymbol{M}\boldsymbol{v}'_\mu(\lambda_1)\big)\cdot \boldsymbol{v}'_\mu(\lambda_1) + \big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{v}_\mu(\lambda_1)\big)\cdot \boldsymbol{v}_\mu(\lambda_1) \\ &\le c_1\big(|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2\big). \end{aligned} \tag{28}$$

On the other hand, the second identity in (21) leads to

$$\begin{aligned}\int_{\lambda_1}^{\lambda} &\varepsilon\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}'_\mu\big)\cdot\boldsymbol{v}'_\mu\,d\xi \\ &= \int_{\lambda_1}^{\lambda}\varepsilon^2\big(\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}'_\mu\big)\cdot\boldsymbol{r}''_\mu\,d\xi + \int_{\lambda_1}^{\lambda}\frac{\varepsilon^2}{\mu}\,\boldsymbol{r}'_\mu\cdot\big(\boldsymbol{r}_\mu-\mathbf{proj}_{\mathcal{C}}\boldsymbol{r}_\mu\big)'\,d\xi,\end{aligned} \tag{29}$$

which implies immediately, since the second integral on the right hand side in (29) is nonnegative and $\widetilde{\boldsymbol{D}}^{-1}$ is a diagonal positive definite matrix, that

$$\int_{\lambda_1}^{\lambda}\varepsilon\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}'_\mu\big)\cdot\boldsymbol{v}'_\mu\,d\xi \geq \frac{\varepsilon^2}{2}\Big[\big(\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}'_\mu\big)\cdot\boldsymbol{r}'_\mu\Big]_{\lambda_1}^{\lambda}. \tag{30}$$

Let us remark that the initial conditions (22) and the second identity in (21) imply that there exists a positive constant c_2 such that

$$\varepsilon^2\big(\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}'_\mu(\lambda_1)\big)\cdot\boldsymbol{r}'_\mu(\lambda_1)\leq c_2|\boldsymbol{v}_1|^2. \tag{31}$$

Carrying (31) into (30), we obtain

$$\int_{\lambda_1}^{\lambda}\varepsilon\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}'_\mu\big)\cdot\boldsymbol{v}'_\mu\,d\xi \geq \frac{\varepsilon^2}{2}\,\big(\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}'_\mu(\lambda)\big)\cdot\boldsymbol{r}'_\mu(\lambda)-c_2|\boldsymbol{v}_1|^2. \tag{32}$$

Introducing (28) and (32) in (27), and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\Big(\varepsilon^2(\boldsymbol{M}\boldsymbol{v}'_\mu)\cdot\boldsymbol{v}'_\mu+\big(\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{v}_\mu\big)\cdot\boldsymbol{v}_\mu+\varepsilon^2\big(\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{r}'_\mu\big)\cdot\boldsymbol{r}'_\mu\Big)(\lambda)\\ &\leq c(\varepsilon)+2\varepsilon|\boldsymbol{f}'_{\mathrm{ext}}(\lambda)\cdot\boldsymbol{v}_\mu(\lambda)|+\varepsilon^2\int_{\lambda_1}^{\lambda}|\boldsymbol{f}''_{\mathrm{ext}}|^2\,d\xi+\int_{\lambda_1}^{\lambda}|\boldsymbol{v}_\mu|^2\,d\xi,\end{aligned} \tag{33}$$

where

$$c(\varepsilon)=(1+c_1+c_2)|\boldsymbol{v}_1|^2+c_1\big(|\boldsymbol{u}_1-\bar{\boldsymbol{u}}(\lambda_1)|^2+|\boldsymbol{r}_1-\bar{\boldsymbol{r}}_1|^2\big)+\varepsilon^2|\boldsymbol{f}'_{\mathrm{ext}}(\lambda_1)|^2.$$

Since $\boldsymbol{M}$, $\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}$ and $\widetilde{\boldsymbol{D}}$ are positive definite matrices, we deduce from (33) that there exist strictly positive numbers γ_i, $i=1,\,2,\,3$, such that

$$\begin{aligned}&\gamma_1|\varepsilon\boldsymbol{v}'_\mu(\lambda)|^2+\gamma_2|\boldsymbol{v}_\mu(\lambda)|^2+\gamma_3|\varepsilon\boldsymbol{r}'_\mu(\lambda)|^2\\ &\leq c(\varepsilon)+2|\varepsilon\boldsymbol{f}'_{\mathrm{ext}}(\lambda)\cdot\boldsymbol{v}_\mu(\lambda)|+\varepsilon^2\int_{\lambda_1}^{\lambda}|\boldsymbol{f}''_{\mathrm{ext}}|^2\,d\xi+\int_{\lambda_1}^{\lambda}|\boldsymbol{v}_\mu|^2\,d\xi.\end{aligned} \tag{34}$$

We estimate the product $|\varepsilon\boldsymbol{f}'_{\mathrm{ext}}(\lambda)\cdot\boldsymbol{v}_\mu(\lambda)|$ by $\varepsilon^2|\boldsymbol{f}'_{\mathrm{ext}}(\lambda)|^2/\gamma_2+\gamma_2|\boldsymbol{v}_\mu(\lambda)|^2/4$, and then the inequality (34) leads to

$$\frac{\gamma_2}{2}\,|\boldsymbol{v}_\mu(\lambda)|^2\leq g(\varepsilon)+\int_{\lambda_1}^{\lambda}|\boldsymbol{v}_\mu|^2\,d\xi,$$

where

$$g(\varepsilon)=c(\varepsilon)+\frac{2\varepsilon^2}{\gamma_2}\,\|\boldsymbol{f}'_{\mathrm{ext}}\|^2_{L^\infty(\lambda_1,\lambda_2)}+\varepsilon^2\|\boldsymbol{f}''_{\mathrm{ext}}\|^2_{L^2(\lambda_1,\lambda_2)}.$$

By classical Gronwall's lemma, it is clear that

$$|\boldsymbol{v}_\mu(\lambda)|^2\leq\frac{2g(\varepsilon)}{\gamma_2}\,\exp\left(\frac{2(\lambda_2-\lambda_1)}{\gamma_2}\right). \tag{35}$$

Therefore the last term on the right hand side of (34) can be estimated by using Hölder's inequality and (35):

$$
\begin{aligned}
&\gamma_1|\varepsilon \boldsymbol{v}'_\mu(\lambda)|^2 + \frac{\gamma_2}{2}\,|\boldsymbol{v}_\mu(\lambda)|^2 + \gamma_3|\varepsilon \boldsymbol{r}'_\mu(\lambda)|^2 \\
&\leq g(\varepsilon)\left(1 + \frac{2(\lambda_2-\lambda_1)}{\gamma_2}\,\exp\left(\frac{2(\lambda_2-\lambda_1)}{\gamma_2}\right)\right).
\end{aligned} \tag{36}
$$

Differentiating the first identity in the system (21) and integrating the result over (η_1, η_2), η_1 and η_2 belonging to (λ_1, λ_2), we get, since $\boldsymbol{v}_\mu = \varepsilon \boldsymbol{u}'_\mu$,

$$
\varepsilon \boldsymbol{M}\boldsymbol{v}'_\mu(\eta_2) - \varepsilon \boldsymbol{M}\boldsymbol{v}'_\mu(\eta_1) = \int_{\eta_1}^{\eta_2} \left(\boldsymbol{f}'_{\text{ext}} - \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}\boldsymbol{u}'_\mu - \boldsymbol{L}^T\boldsymbol{D}\boldsymbol{H}^{-1}\boldsymbol{r}'_\mu\right) d\xi,
$$

which implies, thanks to (36) and Cauchy-Schwarz's inequality, that for fixed $\varepsilon > 0$ there exists a positive constant c_3 such that for every positive μ,

$$
|\varepsilon \boldsymbol{v}'_\mu(\eta_2) - \varepsilon \boldsymbol{v}'_\mu(\eta_1)| \leq c_3|\eta_2 - \eta_1|. \tag{37}
$$

As a consequence, (36) and (37) show that the sequence $\varepsilon \boldsymbol{v}'_\mu$ is equicontinuous and bounded in $\boldsymbol{C}^0_N(\lambda_1, \lambda_2)$. Therefore, thanks to Ascoli's theorem, there exists $\boldsymbol{z}$ belonging to $\boldsymbol{C}^0_N(\lambda_1, \lambda_2)$ and a subsequence, still denoted by $\varepsilon \boldsymbol{v}'_\mu$, such that

$$
\varepsilon \boldsymbol{v}'_\mu \to \boldsymbol{z} \text{ in } \boldsymbol{C}^0_N(\lambda_1, \lambda_2) \text{ as } \mu \text{ tends to } 0. \tag{38}
$$

By uniqueness of the limit, $\boldsymbol{z} = \varepsilon \boldsymbol{v}'$ and thanks to (36), we obtain the result in the Lemma. □

Remark 4.3. *The inclusions in* (12) *and* (14) *can be written in slightly different but equivalent forms: for all* $\boldsymbol{r}^*$ *belonging to* $\mathcal{C}$ *and for all* λ *belonging to* $[\lambda_1, \lambda_2]$, *we have*

$$
\int_{\lambda_1}^{\lambda} \left(\widetilde{\boldsymbol{D}}^{-1}(\boldsymbol{E}\boldsymbol{L}\boldsymbol{u}' - \boldsymbol{r}')\right)\cdot(\boldsymbol{r} - \boldsymbol{r}^*)\, d\xi \geq 0, \tag{39}
$$

and

$$
\int_{\lambda_1}^{\lambda} \left(\widetilde{\boldsymbol{D}}^{-1}(\boldsymbol{E}\boldsymbol{L}\bar{\boldsymbol{u}}' - \bar{\boldsymbol{r}}')\right)\cdot(\bar{\boldsymbol{r}} - \boldsymbol{r}^*)\, d\xi \geq 0. \tag{40}
$$

Proposition 4.4. *(Stability). Assume that* (13) *and* (15) *hold and that* $\boldsymbol{f}_{ext}$ *belongs to* $\boldsymbol{W}^{2,\infty}_N(\lambda_1, \lambda_2)$. *Then there exist* $\gamma_i > 0$, $i = 1, 2$, *such that*

$$
\begin{aligned}
&|\boldsymbol{v}(\lambda)|^2 + |\boldsymbol{u}(\lambda) - \bar{\boldsymbol{u}}(\lambda)|^2 + |\boldsymbol{r}(\lambda) - \bar{\boldsymbol{r}}(\lambda)|^2 \\
&\leq \gamma_1\left(|\boldsymbol{v}_1|^2 + |\boldsymbol{u}_1 - \bar{\boldsymbol{u}}(\lambda_1)|^2 + |\boldsymbol{r}_1 - \bar{\boldsymbol{r}}_1|^2\right) + \varepsilon\gamma_2.
\end{aligned} \tag{41}
$$

Proof. We subtract the equality in the quasi-static system (14) to (6), then we multiply the resulting expression by $(\boldsymbol{u}' - \bar{\boldsymbol{u}}')$ and we integrate over (λ_1, λ), λ belonging to (λ_1, λ_2). On the other hand, we choose $\boldsymbol{r}^* = \bar{\boldsymbol{r}}$ in (39) and $\boldsymbol{r}^* = \boldsymbol{r}$ in (40), and we add (39) to (40). We get the following system:

$$
(42) \quad \begin{cases} \displaystyle\int_{\lambda_1}^{\lambda} \varepsilon^2(\boldsymbol{M}\boldsymbol{u}'') \cdot \boldsymbol{u}'\, d\xi + \int_{\lambda_1}^{\lambda} \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{L}(\boldsymbol{u}-\bar{\boldsymbol{u}})\big) \cdot (\boldsymbol{u}'-\bar{\boldsymbol{u}}')\, d\xi \\ \displaystyle + \int_{\lambda_1}^{\lambda} \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{H}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})\big) \cdot (\boldsymbol{u}'-\bar{\boldsymbol{u}}')\, d\xi = \int_{\lambda_1}^{\lambda} \varepsilon^2(\boldsymbol{M}\boldsymbol{u}'') \cdot \bar{\boldsymbol{u}}'\, d\xi, \\ \displaystyle\int_{\lambda_1}^{\lambda} \Big(\widetilde{\boldsymbol{D}}^{-1}\big(\boldsymbol{E}\boldsymbol{L}(\boldsymbol{u}'-\bar{\boldsymbol{u}}') - (\boldsymbol{r}'-\bar{\boldsymbol{r}}')\big)\Big) \cdot (\boldsymbol{r}-\bar{\boldsymbol{r}})\, d\xi \geq 0. \end{cases}
$$

Since $\boldsymbol{M}$ is a symmetric positive definite matrix, $\boldsymbol{D}$ is a diagonal positive definite matrix, and $\boldsymbol{v} = \varepsilon \boldsymbol{u}'$, we have

$$
(43) \quad \begin{aligned} &\int_{\lambda_1}^{\lambda} \varepsilon^2(\boldsymbol{M}\boldsymbol{u}'') \cdot \boldsymbol{u}'\, d\lambda + \int_{\lambda_1}^{\lambda} \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{L}(\boldsymbol{u}-\bar{\boldsymbol{u}})\big) \cdot (\boldsymbol{u}'-\bar{\boldsymbol{u}}')\, d\xi \\ &= \frac{1}{2}\Big[(\boldsymbol{M}\boldsymbol{v}) \cdot \boldsymbol{v} + \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{L}(\boldsymbol{u}-\bar{\boldsymbol{u}})\big) \cdot (\boldsymbol{u}-\bar{\boldsymbol{u}})\Big]_{\lambda_1}^{\lambda}. \end{aligned}
$$

Notice that $\boldsymbol{D}\boldsymbol{H}^{-1} = \widetilde{\boldsymbol{D}}^{-1}\boldsymbol{E}$, so that the inequality in the system (42) leads to

$$
\int_{\lambda_1}^{\lambda} \big(\widetilde{\boldsymbol{D}}^{-1}(\boldsymbol{r}'-\bar{\boldsymbol{r}}')\big) \cdot (\boldsymbol{r}-\bar{\boldsymbol{r}})\, d\xi \leq \int_{\lambda_1}^{\lambda} \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{H}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})\big) \cdot (\boldsymbol{u}'-\bar{\boldsymbol{u}}')\, d\xi,
$$

which immediately implies

$$
(44) \quad \frac{1}{2}\Big[\big(\widetilde{\boldsymbol{D}}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})\big) \cdot (\boldsymbol{r}-\bar{\boldsymbol{r}})\Big]_{\lambda_1}^{\lambda} \leq \int_{\lambda_1}^{\lambda} \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{H}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})\big) \cdot (\boldsymbol{u}'-\bar{\boldsymbol{u}}')\, d\xi.
$$

Define now

$$
(45) \quad \begin{aligned} h(\xi) = &\big((\boldsymbol{M}\boldsymbol{v}) \cdot \boldsymbol{v} + \big(\boldsymbol{L}^T \boldsymbol{D}\boldsymbol{L}(\boldsymbol{u}-\bar{\boldsymbol{u}})\big) \cdot (\boldsymbol{u}-\bar{\boldsymbol{u}})\big)(\xi) \\ &+ \Big(\big(\widetilde{\boldsymbol{D}}^{-1}(\boldsymbol{r}-\bar{\boldsymbol{r}})\big) \cdot (\boldsymbol{r}-\bar{\boldsymbol{r}})\Big)(\xi). \end{aligned}
$$

Carrying (43) and (44) into the identity in (42) and using the Cauchy-Schwarz inequality, the notation (45) and denoting $\|\boldsymbol{M}\|_\infty = \max_i \sum_j |M_{ij}|$, we obtain the following inequality

$$
(46) \quad h(\lambda) \leq h(\lambda_1) + \|\boldsymbol{M}\|_\infty \left(\int_{\lambda_1}^{\lambda} |\varepsilon \boldsymbol{v}'|^2\, d\xi\right)^{1/2} \left(\int_{\lambda_1}^{\lambda} |\bar{\boldsymbol{u}}'|^2\, d\xi\right)^{1/2}.
$$

Observing that $\boldsymbol{M}$, $\boldsymbol{L}^T\boldsymbol{D}\boldsymbol{L}$ and $\widetilde{\boldsymbol{D}}^{-1}$ are symmetric positive definite matrices, we conclude from (46) that there exists $\alpha > 0$ such that

$$
\begin{aligned} &|\boldsymbol{v}(\lambda)|^2 + |\boldsymbol{u}(\lambda)-\bar{\boldsymbol{u}}(\lambda)|^2 + |\boldsymbol{r}(\lambda)-\bar{\boldsymbol{r}}(\lambda)|^2 \\ &\leq \alpha h(\lambda_1) + \alpha \left(\int_{\lambda_1}^{\lambda} |\varepsilon \boldsymbol{v}'|^2\, d\xi\right)^{1/2} \left(\int_{\lambda_1}^{\lambda} |\bar{\boldsymbol{u}}'|^2\, d\xi\right)^{1/2}. \end{aligned}
$$

The conclusion follows then from Lemma 4.2. $\square$

Appendix

We prove existence and uniqueness of solution to the differential inclusion problem (17) with $\boldsymbol{g}$ depending on λ and on $\boldsymbol{x}$ and not only on λ, $\lambda \in (\lambda_1, \lambda_2)$.

Proposition A.1. *Let $\boldsymbol{g}(\,\cdot\,,\boldsymbol{x})$ be a function from $[\lambda_1,\lambda_2]\times\mathbb{R}^p$ to $\mathbb{R}^p$ such that, for some $\omega > 0$, the following assumptions are satisfied*

$$\forall\lambda \in [\lambda_1,\lambda_2],\ \forall(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^{2p},\ |\boldsymbol{g}(\,\cdot\,,\boldsymbol{x}) - \boldsymbol{g}(\,\cdot\,,\boldsymbol{y})| \le \omega|\boldsymbol{x}-\boldsymbol{y}|, \tag{47a}$$

$$\forall\boldsymbol{x} \in \mathbb{R}^p,\ \boldsymbol{g}(\,\cdot\,,\boldsymbol{x}) \in \boldsymbol{L}_p^\infty(\lambda_1,\lambda_2). \tag{47b}$$

Assume that $\mathbb{A} : \mathbb{R}^p \to \mathbb{R}^p$ is an m-accretive operator. Then there exists a unique solution $\boldsymbol{x}$ of (17) belonging to $\boldsymbol{W}_p^{1,\infty}(\lambda_1,\lambda_2)$.

Proof. Let $\boldsymbol{J}_\mu$ and $\mathbb{A}_\mu$ be the resolvent and the Yosida regularization of the *m-accretive* operator $\mathbb{A}$, respectively,

$$\boldsymbol{J}_\mu = (\mathbf{1} + \mu\mathbb{A})^{-1} \quad \text{and} \quad \mathbb{A}_\mu = \frac{1}{\mu}(\mathbf{1} - \boldsymbol{J}_\mu). \tag{48}$$

The proof has two steps: first, using the Carathéodory's theorem, we prove that for all $\mu > 0$, there exists a unique solution $\boldsymbol{x}_\mu \in \boldsymbol{W}_p^{1,\infty}(\lambda_1,\lambda_2)$ of the regularized problem

$$\boldsymbol{x}_\mu' + \mathbb{A}_\mu\boldsymbol{x}_\mu = \boldsymbol{g}(\,\cdot\,,\boldsymbol{x}_\mu) \text{ a.e. on } (\lambda_1,\lambda_2), \tag{49a}$$

$$\boldsymbol{x}_\mu(\lambda_1) = \boldsymbol{x}_1; \tag{49b}$$

next, passing to the limit, in the regularized problem, as $\mu \to 0$, we prove the existence of a solution to (17). Uniqueness is obtained thanks to a classical Gronwall lemma.

Let $\mu > 0$ be fixed. Define

$$\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_\mu) = -\mathbb{A}_\mu\boldsymbol{x}_\mu + \boldsymbol{g}(\,\cdot\,,\boldsymbol{x}_\mu). \tag{50}$$

Recall that $\mathbb{A}_\mu$ is Lipschitzian with constant $1/\mu$ and is an *m-accretive* operator (see Proposition 2.6, p.28 of [1]). Therefore (50) and (47) yield

$$\forall\lambda \in [\lambda_1,\lambda_2],\ \forall(\boldsymbol{x}_\mu,\boldsymbol{y}_\mu) \in \mathbb{R}^{2p}, \tag{51a}$$
$$|\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_\mu) - \boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{y}_\mu)| \le \big(\omega + 1/\mu\big)|\boldsymbol{x}_\mu - \boldsymbol{y}_\mu|,$$

$$\forall\boldsymbol{x}_\mu \in \mathbb{R}^p,\ \boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_\mu) \in \boldsymbol{L}_p^\infty(\lambda_1,\lambda_2). \tag{51b}$$

We conclude from (49a) that

$$|\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_\mu)| \le |\boldsymbol{h}_\mu(\,\cdot\,,0)| + \big(\omega + 1/\mu\big)|\boldsymbol{x}_\mu|, \forall\lambda \in [\lambda_1,\lambda_2], \forall x_\mu \in \mathbb{R}^p. \tag{52}$$

Thanks to the Carathéodory theorem, (51) and (52) imply that for all $\lambda \in (\lambda_1,\lambda_2)$, there exists a solution $\boldsymbol{x}_\mu \in \boldsymbol{W}_p^{1,\infty}(\lambda_1,\lambda)$ to (49). By a standard reasoning, we are able to extend the previous local solution to an interval $[\lambda_1,\lambda_2]$. Let $\boldsymbol{x}_\mu$ be a solution of (49) on $[\lambda_1,\lambda]$. We integrate (17) over (λ_1,λ), and we obtain

$$\boldsymbol{x}_\mu(\lambda) - \boldsymbol{x}_1 = \int_{\lambda_1}^{\lambda} \boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_\mu)\, d\xi. \tag{53}$$

Using (51), we show that

$$\begin{aligned}|\boldsymbol{x}_\mu(\lambda)-\boldsymbol{x}_1| \leq &(\lambda_2-\lambda_1)\|\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_1)\|_{\boldsymbol{L}_p^\infty(\lambda_1,\lambda_2)}\\ &+\big(\omega+1/\mu\big)\int_{\lambda_1}^{\lambda_2}|\boldsymbol{x}^\mu-\boldsymbol{x}_1|\,d\xi.\end{aligned}$$

A classical Gronwall lemma leads then to

$$|\boldsymbol{x}_\mu(\lambda)| \leq |\boldsymbol{x}_1| + (\lambda_2-\lambda_1)\|\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_1)\|_{\boldsymbol{L}_p^\infty(\lambda_1,\lambda_2)}\exp\big((\lambda_2-\lambda_1)(\omega+1/\mu)\big).$$

Since the previous estimate is uniform in λ, we conclude that $\boldsymbol{x}_\mu$ is a global solution of (49).

We show now that $\boldsymbol{x}_\mu$ is unique. Let $\boldsymbol{x}_\mu$ and $\boldsymbol{y}_\mu$ be two solutions of (49) belonging to $\boldsymbol{W}_p^{1,\infty}(\lambda_1,\lambda_2)$. We subtract (49a) applied to $\boldsymbol{y}_\mu$ and (49a) applied to $\boldsymbol{x}_\mu$, we multiply the resulting expression by $\boldsymbol{x}_\mu-\boldsymbol{y}_\mu$ and we integrate over (λ_1,λ). Since $\mathbb{A}_\mu$ is *m-accretive*, we get for all $\lambda\in[\lambda_1,\lambda_2]$,

$$|\boldsymbol{x}_\mu(\lambda)-\boldsymbol{y}_\mu(\lambda)|^2 \leq 2\int_{\lambda_1}^{\lambda}|(\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{x}_\mu)-\boldsymbol{h}_\mu(\,\cdot\,,\boldsymbol{y}_\mu))(\boldsymbol{x}_\mu-\boldsymbol{y}_\mu)|\,d\xi. \tag{54}$$

Introducing (51a) in (54), we see that, for all $\lambda\in[\lambda_1,\lambda_2]$,

$$|\boldsymbol{x}_\mu(\lambda)-\boldsymbol{y}_\mu(\lambda)|^2 \leq 2\omega\int_{\lambda_1}^{\lambda}|\boldsymbol{x}_\mu-\boldsymbol{y}_\mu|^2\,d\xi$$

which implies, according to Gronwall's lemma, that $\boldsymbol{x}_\mu=\boldsymbol{y}_\mu$.

Let $\mathbb{A}^0\boldsymbol{x}$ be the element of $\mathbb{A}\boldsymbol{x}$ having the minimal norm. The proposition 2.6, p.28 of [1], yields

$$|\mathbb{A}_\mu\boldsymbol{x}_\mu| \leq |\mathbb{A}^0\boldsymbol{x}_\mu| \leq p. \tag{55}$$

Applying (53) to the global solution $\boldsymbol{x}_\mu$ of the regularized problem (49), using (55) and Gronwall's lemma, it is possible to show that there exists a positive constant c_1 such that

$$|\boldsymbol{x}_\mu(\lambda)| \leq c_1,\ \forall\mu>0,\ \forall\lambda\in[\lambda_1,\lambda_2]. \tag{56}$$

We apply once again (53) to the global solution $\boldsymbol{x}_\mu$ of (49); we subtract (53) applied to η_1 and (53) applied to η_2, we get, for all $\mu>0$ and for all η_1,η_2 belonging to $[\lambda_1,\lambda_2]$,

$$|\boldsymbol{x}_\mu(\eta_1)-\boldsymbol{x}_\mu(\eta_2)| \leq \int_{\eta_1}^{\eta_2}\big(|\mathbb{A}_\mu\boldsymbol{x}_\mu|+|\boldsymbol{g}(\,\cdot\,,\boldsymbol{x}_\mu)|\big)\,d\xi.$$

Therefore (47), (56) imply that there exists a positive constant c_2 such that

$$|\boldsymbol{x}_\mu(\eta_1)-\boldsymbol{x}_\mu(\eta_2)| \leq c_2|\eta_1-\eta_2|,\ \forall\mu>0,\ \forall\eta_1,\eta_2\in[\lambda_1,\lambda_2]. \tag{57}$$

As a consequence, (56) and (57) show that the sequence $\boldsymbol{x}_\mu$ is equicontinuous and bounded in $\boldsymbol{C}_p^0(\lambda_1,\lambda_2)$. Therefore, according to Ascoli's theorem, there exists $\boldsymbol{z}$ belonging to $\boldsymbol{C}_p^0(\lambda_1,\lambda_2)$ and a subsequence, still denoted by $\boldsymbol{x}_\mu$, such that

$$\boldsymbol{x}_\mu \to \boldsymbol{z} \text{ in } \boldsymbol{C}_p^0(\lambda_1,\lambda_2). \tag{58}$$

Moreover we deduce from (49a) that for all $\mu>0$,

$$|\boldsymbol{x}_\mu'| \leq |\mathbb{A}_\mu\boldsymbol{x}_\mu| + |\boldsymbol{g}(\,\cdot\,,\boldsymbol{x}_\mu)|. \tag{59}$$

Substituting (47a), (56) in (59), we immediately see that there exists a positive constant c_3 such that

$$\|\boldsymbol{x}'_\mu\|_{\boldsymbol{L}^\infty_p(\lambda_1,\lambda_2)} \leq c_3,\ \forall \mu > 0.$$

Then there exists $\bar{\boldsymbol{z}}$ belonging to $\boldsymbol{L}^\infty_p(\lambda_1, \lambda_2)$ and a subsequence, still denoted by $\boldsymbol{x}'_\mu$, such that

$$\boldsymbol{x}'_\mu \rightharpoonup \bar{\boldsymbol{z}} \text{ weakly } \star \text{ in } \boldsymbol{L}^\infty_p(\lambda_1, \lambda_2). \tag{60}$$

By uniqueness of the limit, $\boldsymbol{z}$ belongs to $\boldsymbol{W}^{1,\infty}_p(\lambda_1, \lambda_2)$ and

$$\boldsymbol{x}'_\mu \rightharpoonup \boldsymbol{z}' \text{ weakly } \star \text{ in } \boldsymbol{L}^\infty_p(\lambda_1, \lambda_2). \tag{61}$$

With the help of (48), (49a) can be written now

$$\forall \mu > 0,\ \frac{1}{\mu}\,(\mathbf{1} - \boldsymbol{J}_\mu)\boldsymbol{x}_\mu = \boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu \text{ a.e. on } (\lambda_1, \lambda_2)$$

where $\mathbf{1}$ is the identity matrix. It is easy to see that

$$\forall \mu > 0,\ -\mu\big(\boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu\big) + \boldsymbol{x}_\mu = \boldsymbol{J}_\mu \text{ a.e. on } (\lambda_1, \lambda_2)$$

which implies, using the definition of the resolvent $\boldsymbol{J}_\mu$, that for all $\mu > 0$,

$$\mathbb{A}\big(\boldsymbol{x}_\mu - \mu(\boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu)\big) \ni \boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu \text{ a.e. on } (\lambda_1, \lambda_2). \tag{62}$$

Observe that (61) implies, thanks to (47a), (58), that

$$\boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}' \rightharpoonup \boldsymbol{g}(\,\cdot\,, \boldsymbol{z}) - \boldsymbol{z}' \text{ weakly in } \boldsymbol{L}^2_p(\lambda_1, \lambda_2), \tag{63}$$

and thus there exists a positive constant c_4 such that

$$\|\boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'\|_{\boldsymbol{L}^2_p(\lambda_1,\lambda_2)} \leq c_4,\ \forall \mu > 0. \tag{64}$$

It follows from (58) and (64) that

$$\boldsymbol{x}_\mu - \mu\big(\boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu\big) \to \boldsymbol{z} \text{ in } \boldsymbol{L}^2_p(\lambda_1, \lambda_2). \tag{65}$$

Let $\boldsymbol{\mathcal{A}}$ be a *m-accretive* operator on $\boldsymbol{L}^2_p(\lambda_1, \lambda_2)$ (see examples 2.1.3, p.21, and 2.3.3, p.25, of [1]) such that for all $\boldsymbol{x}_\mu, \boldsymbol{y}_\mu$ belonging to $\boldsymbol{L}^2_p(\lambda_1, \lambda_2)$, we get

$$\boldsymbol{x}_\mu \in \boldsymbol{\mathcal{A}}\boldsymbol{y}_\mu \Leftrightarrow \boldsymbol{x}_\mu \in \mathbb{A}\boldsymbol{y}_\mu \text{ a.e. on } (\lambda_1, \lambda_2). \tag{66}$$

Thanks to (66), (62) is equivalent to

$$\boldsymbol{\mathcal{A}}\big(\boldsymbol{x}_\mu - \mu(\boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu)\big) \ni \boldsymbol{g}(\,\cdot\,, \boldsymbol{x}_\mu) - \boldsymbol{x}'_\mu,\ \forall \mu > 0. \tag{67}$$

In view of Proposition 2.5, p.27 of [1], and (63), (65), (67), we may conclude that

$$\boldsymbol{\mathcal{A}}\boldsymbol{z} \ni \boldsymbol{g}(\,\cdot\,, \boldsymbol{z}) - \boldsymbol{z}',$$

in other words, we get

$$\boldsymbol{z}' + \mathbb{A}\boldsymbol{z} \ni \boldsymbol{g}(\,\cdot\,, \boldsymbol{z}) \text{ a.e. on } (\lambda_1, \lambda_2).$$

We also see, by (49b) and (58), that

$$\boldsymbol{z}(\lambda_1) = \boldsymbol{x}_1.$$

$\square$

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