Mathematical result on the stability of quasi-static paths of elastic-plastic systems with hardening

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In this paper, existence and uniqueness results for a class of dynamic and quasi-static problems with elastic-plastic systems are recalled, and a stability result is obtained for the quasi-static paths of those systems. The studied elastic-plastic systems are continuum 1D (bar) systems that have linear hardening, and the concept of stability of quasi-static paths used here takes into account the existence of fast (dynamic) and slow (quasi-static) times scales in the system. That concept is essentially a continuity property relatively to the size of the initial perturbations (as in Lyapunov stability) and relatively to the smallness of the rate of application of the forces (which plays here the role of the small parameter in singular perturbation problems).

1. Introduction

Martins and co-workers studied in [1,2] the relation that exists between, on one hand, dynamic and quasi-static problems and, on the other hand, the theory of singular perturbations. More precisely, they performed a change of variables in the governing system of dynamic equations that consists of replacing the (fast) physical time t by a (slow) loading parameter λ whose rate of change with respect to time, $\varepsilon = d\lambda/dt$, is eventually decreased to zero. In this manner they obtained a system of equations or inclusions defining a singular perturbation problem, i.e. a problem where the highest order derivative with respect to the loading parameter appears multiplied by the time rate of change (ε) of that parameter.

The variational formulations for elastic perfect plastic and elastic-visco-plastic systems were established by Duvaut and Lions [3]. Hardening effects were introduced in the formulations by Johnson [4,5], who proved existence of a strong solution and, under some assumptions, a regularity result for the velocity field.

In the present paper, the definition of stability of quasi-static paths given in [2] is adapted to the present continuum case. And we establish that, similarly to the finite dimensional elastic-plastic systems with hardening discussed in [6], the dynamic evolutions remain close to a quasi-static path when they start sufficiently close to that quasi-static path and the load is applied sufficiently slowly.

The structure of the article is the following. In Section 2, the mathematical formulations for dynamic and quasi-static elastic-plastic systems with hardening are presented, and in Section 3, existence and uniqueness results are recalled, which use the theory of *m-accretive* operators [3,7–10]. The proof of stability of a quasi-static path is presented in Section 4.

2. Governing equations

We consider an elastic-plastic bar with linear kinematic hardening that has the length L along the x axis. Geometrical linearity is assumed. The governing dynamic equation can be non-dimensionalized by using the non-dimensional time (τ) and load parameter (λ, $\lambda = \lambda_1 + \varepsilon\tau$), yielding

$$\varepsilon^2 u'' - \sigma_x(u, r) = f(x, \lambda), \tag{1}$$

where u, r, f are the non-dimensional axial displacement, stress in the plastic element, and applied force per unit length along the bar, respectively; σ is the stress in the elastic-plastic element, which depends on u and r; and the subscript x denotes a derivative with respect to x. The extension e is the derivative in space of the non-dimensional generalized displacement u, and it can be decomposed into elastic, e^e, and plastic, e^p, parts:

$$e = u_x = e^e + e^p. \tag{2}$$

The stress σ is related to the elastic part of the extension by means of Hooke's law,

$$\sigma = r + He^p = Ee^e = E(u_x - e^p),\ E > 0,\ H > 0. \tag{3}$$

Therefore (3) leads to

$$\sigma(u,r) = Du_x + DH^{-1}r \text{ where } D = (E^{-1} + H^{-1})^{-1}. \tag{4}$$

Carrying (4) into (1), we obtain

$$\varepsilon^2 u'' - Du_{xx} - DH^{-1}r_x = f. \tag{5}$$

The behavior of the plastic element is characterized by the non-dimensional inequality and flow rule:

$$|r| \leq 1,\ (e^p)' \begin{cases} \geq 0 \text{ if } r = +1, \\ = 0 \text{ if } -1 < r < +1, \\ \leq 0 \text{ if } r = -1. \end{cases} \tag{6}$$

The governing dynamic equations (5), together with the conditions (6) can be put in the form of a singular perturbation system of first order differential equation and inclusion. For that purpose, let $\mathcal{C}$ denote the following closed convex set in $L^2(0,L)$

$$\mathcal{C} = \{r \in L^2(0,L) : |r| \leq 1\}, \tag{7}$$

and let $\text{sign}^{-1}(r)$ be the normal cone to $\mathcal{C}$ at $r \in L^2(0,L)$. Then we observe that (6) can be written in the differential inclusion form:

$$(e^p)' \in \text{sign}^{-1}(r). \tag{8}$$

Relations (3) lead to

$$(e^p)' = \tilde{D}^{-1}(Eu'_x - r') \text{ where } \tilde{D} = E + H. \tag{9}$$

Substituting (9) in (8), we get

$$Eu'_x - r' \in \tilde{D}\text{sign}^{-1}(r). \tag{10}$$

We now introduce the following spaces

$$\mathcal{H} = L^2(0,L),\ \mathcal{V} = H^1(0,L),\ \mathcal{V}_0 = H^1_0(0,L),$$
$$\mathcal{W} = \{(u,r) \in \mathcal{V}_0 \times \mathcal{C} : \sigma = D(u_x + H^{-1}r) \in \mathcal{V}\}.$$

We will denote the norm in $\mathcal{H}$ (resp. $\mathcal{V}$) by $|\cdot|$ (resp. $||\cdot||$) and the scalar product in $\mathcal{H}$ by $(\cdot\,,\cdot)$. From (5) and (10) we finally obtain the governing dynamic system

$$\begin{cases} \varepsilon u' - v = 0, \\ \varepsilon v' - Du_{xx} - DH^{-1}r_x = f, \\ Eu'_x - r' \in \tilde{D}\text{sign}^{-1}(r), \end{cases} \tag{11}$$

together with the Dirichlet boundary conditions

$$u = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2), \tag{12}$$

and the initial conditions

$$\big(v(\lambda_1), u(\lambda_1), r(\lambda_1)\big) = (v_1, u_1, r_1) \in \mathcal{V}_0 \times \mathcal{W}. \tag{13}$$

The corresponding quasi-static system is then (let $\varepsilon = 0$ in (11))

$$\begin{cases} -D\bar{u}_{xx} - DH^{-1}\bar{r}_x = f, \\ E\bar{u}'_x - \bar{r}' \in \tilde{D}\text{sign}^{-1}(\bar{r}), \end{cases} \tag{14}$$

with the Dirichlet boundary conditions

$$\bar{u} = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2), \tag{15}$$

and the initial conditions

$$\big(\bar{u}(\lambda_1), \bar{r}(\lambda_1)\big) = (\bar{u}_1, \bar{r}_1) \in \mathcal{W}. \tag{16}$$

Note that, consistently with the above, the quasi-static displacement rate with respect to the physical time vanishes ($\bar{v} \equiv 0$).

Note that the dynamic system (11)-(13) can be written in the equivalent variational form:

$$\begin{cases} \text{Find } (u, r) \in \mathcal{W} \text{ such that } \forall (u^*, r^*) \in \mathcal{W}, \\ (\varepsilon^2 u'', u^*) + \dfrac{1}{2}(u_x, u_x^*) + \dfrac{1}{2}(r, u_x^*) = (f, u^*), \\ (r', r - r^*) - (u'_x, r - r^*) \le 0, \end{cases} \tag{17}$$

with the initial conditions (13). The corresponding variational formulation of the quasi-static problem (14)-(16) is:

$$\begin{cases} \text{Find } (\bar{u}, \bar{r}) \in \mathcal{W} \text{ such that } \forall (\bar{u}^*, \bar{r}^*) \in \mathcal{W}, \\ \dfrac{1}{2}(\bar{u}_x, \bar{u}_x^*) + \dfrac{1}{2}(\bar{r}, \bar{u}_x^*) = (f, \bar{u}^*), \\ (\bar{r}', \bar{r} - \bar{r}^*) - (\bar{u}'_x, \bar{r} - \bar{r}^*) \le 0, \end{cases} \tag{18}$$

with the initial conditions (16).

Finally note that if X is a space of scalar functions, the bold-face notation $\boldsymbol{X}_d$ will denote the space X^d.

3. Existence and uniqueness of solution for the dynamic and the quasi-static systems

We observe that the dynamic and the quasi-static systems introduced in Section 2 can be rewritten in a form that may be studied with the theory of *m-accretive* operators. Recall that existence and uniqueness of solution to the differential inclusion problem

$$\boldsymbol{x}' + \mathbb{A}\boldsymbol{x} \ni \boldsymbol{g} \text{ a.e. on } (\lambda_1, \lambda_2), \tag{19a}$$

$$\boldsymbol{x}(\lambda_1) = \boldsymbol{x}_1, \tag{19b}$$

can be obtained from the following Proposition:

Proposition 3.1. *Assume that $\mathbb{A}$ is an m-accretive operator in the Hilbert space $\mathcal{Y}$, $\boldsymbol{g}$ belongs to $\boldsymbol{W}_p^{1,\infty}(\lambda_1, \lambda_2; \mathcal{Y})$ and $\boldsymbol{x}_1 \in \mathcal{D}(\mathbb{A})$. Then there exists a unique solution $\boldsymbol{x}$ of* (19) *belonging to $\boldsymbol{W}_p^{1,\infty}(\lambda_1, \lambda_2; \mathcal{Y})$.*

The reader can find a detailed proof of this Proposition in [5] or in [4]. By applying Proposition 3.1, we prove existence and uniqueness of solution for the dynamic system (11)–(13) and for the corresponding quasi-static system (14)–(16). Differentiating with respect to x the first equation in the system (11), performing a change of unknown function by using $e = u_x$ and denoting $\boldsymbol{x} = (D^{1/2}e, v, \tilde{D}^{-1/2}r)$, we get the inclusion (19a) with

$$\mathbb{A} = \frac{1}{\varepsilon}\begin{pmatrix} 0 & -D^{1/2}\partial/\partial x & 0 \\ -D^{1/2}\partial/\partial x & 0 & -E\tilde{D}^{-1/2}\partial/\partial x \\ 0 & -E\tilde{D}^{-1/2}\partial/\partial x & \varepsilon\tilde{D}^{1/2}\text{sign}^{-1}(\tilde{D}^{1/2}\,\cdot\,) \end{pmatrix},$$

$$\boldsymbol{g} = \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}.$$

Let us define $\mathcal{Z} = \{(e, r) \in \boldsymbol{\mathcal{H}}_2 : De + DH^{-1}r \in \mathcal{V}\}$. First we check by direct estimate that $\mathbb{A}$ is a monotone operator (see [7,11]). Second, if $(g_1, f, g_2) \in \boldsymbol{\mathcal{H}}_3$ and $(v, e, r) \in \mathcal{V} \times \mathcal{Z}$, there exists $h(r) \in \text{sign}^{-1}(r) \in \mathcal{H}$ for which the resolvent equation $(\mathbf{1} + \mathbb{A})(D^{1/2}e, v, \tilde{D}^{-1/2}r)^T \ni (g_1, f/\varepsilon, g_2)^T$ is equivalent to solving the system

$$\begin{cases} \varepsilon D^{1/2}e - D^{1/2}v_x = \varepsilon g_1, \\ \varepsilon v - \dfrac{1}{2}D^{1/2}e_x - E\tilde{D}^{-1}r_x = f, \\ \varepsilon\tilde{D}^{-1/2}r - E\tilde{D}^{-1/2}v_x + \varepsilon h(r) = \varepsilon g_2. \end{cases} \tag{20}$$

This is equivalent to solve for $v \in \mathcal{V}$ the following equation:

$$v - D^{1/2}\frac{\partial}{\partial x}\Big(\frac{1}{\varepsilon^2}D^{1/2}v_x + \frac{1}{2\varepsilon}g_1\Big)$$
$$- E\tilde{D}^{-1}\frac{\partial}{\partial x}\Big(\big(1 + D^{-1/2}h(\,\cdot\,)\big)^{-1}\Big(\frac{1}{\varepsilon^2}\tilde{D}^{-1/2}v_x + \frac{1}{\varepsilon}g_2\Big)\Big) = \frac{1}{\varepsilon}f \text{ in } \mathcal{V}'.$$

The form is coercive, and existence of a solution follows. The components of $(e, r) \in \mathcal{Z}$ are obtained directely from the first and third terms in (20) respectively. Hence, we conclude that $\mathbb{A}$ is *m-accretive*. For more details, see [8,9]. Observing that with $p = 3$ and $\mathcal{Y} = \mathcal{V} \times \mathcal{Z}$, Proposition 3.1 yields the following Corollary:

Corollary 3.1. *Assume that f belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and that* (13) *holds. Then there exists a unique solution (v, e, r) of* (11)–(13) *such that (v, e, r) and (v', e', r') belong respectively to $L^\infty(\lambda_1, \lambda_2; \mathcal{V}) \times \boldsymbol{L}_2^\infty(\lambda_1, \lambda_2; \mathcal{H})$ and $\boldsymbol{L}_3^\infty(\lambda_1, \lambda_2; \mathcal{H})$ and $(De + DH^{-1}r)$ belongs to $L^\infty(\lambda_1, \lambda_2; \mathcal{V})$.*

Remark 3.1. According to Corollary 3.1 and since $e = u_x$, $u = 0$ on $\{0, L\}$, u and u' belong respectively to $L^\infty(\lambda_1, \lambda_2; \mathcal{V}_0)$ and $L^\infty(\lambda_1, \lambda_2; \mathcal{V}_0)$.

In what concerns the quasi-static problem, we differentiate the first identity of (14) with respect to λ to get

$$-D\bar{u}'_{xx} = DH^{-1}\bar{r}'_x + f', \tag{21}$$

with Dirichlet boundary conditions. We deduce that (21) has a unique solution $\bar{u}'$. On the other hand, $\bar{u}'_x + H^{-1}\bar{r}'$ depends linearly and continuously on f', i.e.

$$\bar{u}'_x + DH^{-1}\bar{r}' = \mathcal{B}f'.$$

Inserting this in the inclusion in (14) leads to

$$\bar{r}' + H\text{sign}^{-1}(\bar{r}) \ni D\mathcal{B}f'. \tag{22}$$

The sub-differential $\partial\varphi(\bar{r}) = \text{sign}^{-1}(\bar{r})$ is an *m-accretive* operator since $\varphi(\bar{r})$ is a proper convex and lower semi-continuous function. For $\bar{r}$, $\mathbb{A}\boldsymbol{x} = H\text{sign}^{-1}(\bar{r})$, $\boldsymbol{g} = D\mathcal{B}f'$ with $p = 1$ in (19) and $\mathcal{Y} = \mathcal{V}$, we apply Proposition 3.1 and we obtain the following Corollary:

Corollary 3.2. *Assume that f belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and* (16) *holds. Then there exists a unique solution $(\bar{u}, \bar{r})$ of* (14)–(16) *such that $(\bar{u}, \bar{r})$ and $(\bar{u}', \bar{r}')$ belong respectively to $L^\infty(\lambda_1, \lambda_2; \mathcal{V}_0) \times L^\infty(\lambda_1, \lambda_2; \mathcal{H})$ and $L^\infty(\lambda_1, \lambda_2; \mathcal{V}_0) \times L^\infty(\lambda_1, \lambda_2, \mathcal{H})$ and $(D\bar{u}_x + DH^{-1}\bar{r})$ belongs to $L^\infty(\lambda_1, \lambda_2; \mathcal{V})$.*

4. Stability of quasi-static paths of elastic-plastic systems

In Section 4.1, we adapt the definition of stability of a quasi-static path [2,6,12] to the present elastic-plastic problem with hardening, which can be seen as the limit of a sequence of elastic-visco-plastic problems. In Section 4.2, we introduce such elastic-visco-plastic problems and we recall existence and uniqueness results for them. In Section 4.3, a priori estimates on the elastic-visco-plastic system are obtained which, in Section 4.4, lead to the proof that the dynamic and the quasi-static solutions remain close to each other if they start sufficiently close, and the loading rate ε is sufficiently small.

From now on we assume, without loss of generality, that $E = H = 1$.

4.1. *Definition of stability of a quasi-static path*

The mathematical definition of stability of a quasi-static path at an equilibrium point is presented in the context of the governing dynamic system (11)–(13) and the quasi-static system (14)–(16).

Definition 4.1. The quasi-static path $(\bar{u}(\lambda), \bar{r}(\lambda))$ is said to be stable at λ_1 if there exists $0 < \Delta\lambda \leq \lambda_2 - \lambda_1$, such that, for all $\delta > 0$ there exists $\bar{\rho}(\delta) > 0$ and $\bar{\varepsilon}(\delta) > 0$ such that for all initial conditions u_1, v_1, r_1 and $\bar{u}_1$, $\bar{r}_1$ and all $\varepsilon > 0$ such that

$$|v_1|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 \leq \bar{\rho}(\delta) \text{ and } \varepsilon \leq \bar{\varepsilon}(\delta),$$

the solution $(u(\lambda), v(\lambda), r(\lambda))$ of the dynamic system (11)–(13) satisfies

$$|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \delta,$$

for all $\lambda \in [\lambda_1, \lambda_1 + \Delta\lambda]$.

For more details, the reader is referred to [2].

4.2. *Existence and uniqueness of solution for the elastic-visco-plastic systems*

We introduce here the elastic-visco-plastic systems:

$$\begin{cases} \varepsilon^2 u''_\mu - \dfrac{1}{2} u_{\mu xx} - \dfrac{1}{2} r_{\mu x} = f, \\ u'_{\mu x} - r'_\mu = 2\mathcal{J}_\mu(r_\mu), \end{cases} \quad \text{where} \quad \mathcal{J}_\mu(r_\mu) = \frac{1}{\mu}\left(r_\mu - \text{proj}_{\mathcal{C}} r_\mu\right), \tag{23}$$

with the Dirichlet boundary conditions

$$u_\mu = 0 \text{ on } \{0, L\} \times (\lambda_1, \lambda_2), \tag{24}$$

and the initial conditions

$$\big(v_\mu(\lambda_1), u_\mu(\lambda_1), r_\mu(\lambda_1)\big) = (v_1, u_1, r_1) \in \mathcal{V}_0 \times \mathcal{W}. \tag{25}$$

Here $\mathrm{proj}_{\mathcal{C}}$ denotes the projection on the convex $\mathcal{C}$. The variational formulation of the problem (23)–(25) is the following:

$$\begin{cases} \text{Find } (u_\mu, r_\mu) \in \mathcal{W} \text{ such that } \forall (u^*, r^*) \in \mathcal{W}, \\ (\varepsilon^2 u''_\mu, u^*) + \dfrac{1}{2}(u_{\mu x}, u^*_x) + \dfrac{1}{2}(r_\mu, u^*_x) = (f, u^*), \\ (r'_\mu, r^*) - (u'_{\mu x}, r^*) + 2(\mathcal{J}_\mu(r_\mu), r^*) = 0, \end{cases} \tag{26}$$

with the initial conditions (25). Note that this elastic-visco-plastic problem is an Yosida regularization of the original elastic-plastic problem. For a similar approximation in the corresponding finite-dimensional system see [6]. Let us define $v_\mu = \varepsilon u'_\mu$.

Proposition 4.1. *Assume that f belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and that (25) holds. Then there exists a unique solution (u_μ, v_μ, r_μ) of (23)–(25) such that (u_μ, v_μ, r_μ) and (u'_μ, v'_μ, r'_μ) belong respectively to $\boldsymbol{L}^\infty_2(\lambda_1, \lambda_2; \mathcal{V}_0) \times L^\infty(\lambda_1, \lambda_2; \mathcal{H})$ and $\boldsymbol{L}^\infty_3(\lambda_1, \lambda_2; \mathcal{H})$ and $(u_{\mu xx} + r_{\mu x})$ belongs to $L^\infty(\lambda_1, \lambda_2; \mathcal{H})$. Moreover, as μ tends to zero, $(u_\mu, u_{\mu x} + r_\mu)$ converges strongly to its limit.*

Idea of the proof. We regularize (26) in the space variable. Then *a priori* estimates and the Galerkin method (cf. [13]) lead us to the desired result. The reader can find a detailed proof in the Appendix of [11] or in [3]. Observe that this Proposition can be also proved using the theory of *m-accretive* operators. □

4.3. *A priori estimates*

Lemma 4.1. *Assume that (25) holds and f belongs to $W^{1,\infty}(\lambda_1, \lambda_2; \mathcal{H})$. Then independently of $\mu > 0$, for all λ belonging to (λ_1, λ_2), $v_\mu(\lambda)$, $u_{\mu x}(\lambda)$ and $r_\mu(\lambda)$ are bounded in $\mathcal{H}$.*

Proof. This estimate results from the application of Gronwall's lemma to energy estimates. Choosing $u^* = 2u'_\mu$ and $r^* = r_\mu$ in (26), and adding both identities, we obtain

$$2(\varepsilon^2 u''_\mu, u'_\mu) + (u_{\mu x}, u'_{\mu x}) + (r'_\mu, r_\mu) + 2(\mathcal{J}_\mu(r_\mu), r_\mu) = 2(f, u'_\mu). \tag{27}$$

Observing that $(\mathcal{J}_\mu(r_\mu), r_\mu)$ is non negative we conclude from (27) that

$$\frac{d}{d\xi}\big(2|\varepsilon u_\mu'|^2 + |u_{\mu x}|^2 + |r_\mu|^2\big) \leq 4(f, u_\mu'). \tag{28}$$

We integrate (28) over (λ_1, λ), $\lambda \in [\lambda_1, \lambda_2]$, and since $v_\mu = \varepsilon u_\mu'$, we get

$$\big[2|v_\mu|^2 + |u_{\mu x}|^2 + |r_\mu|^2\big]_{\lambda_1}^{\lambda} \leq 4\int_{\lambda_1}^{\lambda} (f, u_\mu')\, d\xi. \tag{29}$$

Integrating by parts in time the right hand side of (29), we obtain

$$\big[2|v_\mu|^2 + |u_{\mu x}|^2 + |r_\mu|^2\big]_{\lambda_1}^{\lambda} \leq 4\big[(f, u_\mu)\big]_{\lambda_1}^{\lambda} - 4\int_{\lambda_1}^{\lambda} (f', u_\mu)\, d\xi.$$

We estimate the product (z, y) by $|z|^2/2\gamma_i + \gamma_i|y|^2/2$, and, choosing different values for γ_i, $i = 1, 2, 3$, in different terms, we have

$$2|v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda)|^2 + |r_\mu(\lambda)|^2 \leq c_1 + \frac{1}{\gamma_1}\,|u_\mu(\lambda)|^2 + \frac{1}{\gamma_3}\int_{\lambda_1}^{\lambda} |u_\mu|^2\, d\xi, \tag{30}$$

where

$$\begin{aligned} c_1 =& 2|v_1|^2 + |u_{1x}|^2 + |r_1|^2 + \gamma_2|u_1|^2 + \frac{1}{\gamma_2}\,|f(\lambda_1)|^2 \\ &+ \gamma_1||f||^2_{L^\infty(\lambda_1,\lambda_2;\mathcal{H})} + \gamma_3||f'||^2_{L^2(\lambda_1,\lambda_2;\mathcal{H})}. \end{aligned}$$

On the other hand, the Poincaré inequality (see [14,15]) shows that there exists a strictly positive constant c such that

$$|u_\mu(\xi)|^2 \leq c|u_{\mu x}(\xi)|^2, \ \forall \xi \in (\lambda_1, \lambda_2). \tag{31}$$

Using (31) in (30) and choosing $\gamma_1 = \gamma_3 = 2c$ and $\gamma_2 = 1$ in (30), we may infer that

$$2|v_\mu(\lambda)|^2 + \frac{1}{2}\,|u_{\mu x}(\lambda)|^2 + |r_\mu(\lambda)|^2 \leq c_1 + \frac{1}{2}\int_{\lambda_1}^{\lambda} |u_{\mu x}|^2 d\xi. \tag{32}$$

By classical Gronwall's lemma, we get

$$|u_{\mu x}(\lambda)|^2 \leq 2c_1 \exp(\lambda_2 - \lambda_1). \tag{33}$$

As the last term on the right hand side of (32) is now easily estimated, we finally obtain

$$2|v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda)|^2 + |r_\mu(\lambda)|^2 \leq c_1\big(1 + (1 + (\lambda_2 - \lambda_1))\exp(\lambda_2 - \lambda_1)\big),$$

from which the desired result follows. □

Lemma 4.2. *Assume that* (25) *holds and f belongs to $L^2(\lambda_1,\lambda_2;\mathcal{H})$. Then for all λ belonging to (λ_1,λ_2),*

$$v_\mu(\lambda) \to v(\lambda) \text{ strongly in } \mathcal{H},$$
$$u_{\mu x}(\lambda) \to u_x(\lambda) \text{ strongly in } \mathcal{H},$$
$$r_\mu(\lambda) \to r(\lambda) \text{ strongly in } \mathcal{H},$$

as μ tends to 0.

Proof. These convergence properties are obtained by energy estimating the difference between the elastic-visco-plastic system and the elastic-plastic system with hardening. Choosing $u^* = u'_\mu - u'$ and $u^* = u' - u'_\mu$ respectively the first identities in (23) and (17), and adding both identities, we get

$$(\varepsilon^2 u''_\mu - \varepsilon^2 u'', u'_\mu - u') + \frac{1}{2}(u_{\mu x} - u_x, u'_{\mu x} - u'_x) + \frac{1}{2}(r_\mu - r, u'_{\mu x} - u'_x) = 0. \quad (34)$$

Observing that the second identity in the system (23) implies that

$$(r_\mu - r, u'_{\mu x} - u'_x) = (r'_\mu - r', r_\mu - r) + 2(\mathcal{J}_\mu(r_\mu), r_\mu - r) + (r' - u'_x, r_\mu - r). \quad (35)$$

Carrying (35) into (34) and integrating over (λ_1,λ), $\lambda \in [\lambda_1,\lambda_2]$, and using the initial conditions (25) and (12) lead to the following identity

$$\begin{aligned} &|\varepsilon(u'_\mu(\lambda) - u'(\lambda))|^2 + \frac{1}{2}|u_{\mu x}(\lambda) - u_x(\lambda)|^2 + \frac{1}{2}|r_\mu(\lambda) - r(\lambda)|^2 \\ &+ 2\int_{\lambda_1}^{\lambda} (\mathcal{J}_\mu(r_\mu), r_\mu - r)\, d\xi + \int_{\lambda_1}^{\lambda} (r' - u'_x, r_\mu - r)\, d\xi = 0. \end{aligned} \quad (36)$$

Since $(\mathcal{J}_\mu(r_\mu), r_\mu - r)$ is non negative, $v_\mu = \varepsilon u'_\mu$ and $v = \varepsilon u'$, then we may deduce from (36) that

$$|v_\mu(\lambda) - v(\lambda)|^2 + \frac{1}{2}|u_{\mu x}(\lambda) - u_x(\lambda)|^2 + \frac{1}{2}|r_\mu(\lambda) - r(\lambda)|^2 \le \int_{\lambda_1}^{\lambda} (r' - u'_x, r - r_\mu)\, d\xi.$$

The conclusion follows from Lemma 4.1. □

Lemma 4.3. *Assume that* (25) *holds and f belongs to $W^{2,\infty}(\lambda_1,\lambda_2;\mathcal{H})$. Then there exists a subsequence, still denoted by v'_{μ_n}, such that*

$$v'_{\mu_n} \rightharpoonup v'_\mu \text{ weakly } * \text{ in } L^\infty(\lambda_1,\lambda_2;\mathcal{H}). \quad (37)$$

Moreover there exists a positive constant $c(\lambda_1,\lambda_2)$ that depends on the interval of λ and such that

$$\begin{aligned} &|\varepsilon v'_{\mu_n}(\lambda)|^2 \le c(\lambda_1,\lambda_2)\big(||v_1||^2 + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2 \\ &+ \varepsilon^2|f'(\lambda_1)|^2 + \varepsilon^2||f'||^2_{L^\infty(\lambda_1,\lambda_2;\mathcal{H})} + \varepsilon^2||f''||^2_{L^2(\lambda_1,\lambda_2;\mathcal{H})}\big). \end{aligned} \quad (38)$$

Proof. This estimate results from the energy estimate, Gronwall's lemma and the proof can be completed by a classical Galerkin method. Let $\{w_j\}_{j=1}^{\infty}$ be a complete orthonormal sequence in $\mathcal{H}$ whose elements belong to $H^2(0,L)$. Let $u_{\mu_n} = \sum_{i=1}^n g_{in}(\lambda)w_i(x)$ and $r_{\mu_n} = \sum_{i=1}^n h_{in}(\lambda)w_i(x)$ satisfying the following variational formulation

$$\begin{cases} \text{For all } u^* = \sum_{i=1}^n g^*_{in}(\lambda)w_i(x) \text{ and } r^* = \sum_{i=1}^n h^*_{in}(\lambda)w_i(x), \\ (\varepsilon^2 u''_{\mu_n}, u^*) + \dfrac{1}{2}(u_{\mu_n x}, u^*_x) + \dfrac{1}{2}(r_{\mu_n}, u^*_x) = (f, u^*), \\ (r'_{\mu_n}, r^*) - (u'_{\mu_n x}, r^*) + 2(\mathcal{J}_\mu(r_{\mu_n}), r^*) = 0, \end{cases} \tag{39}$$

with $\lim_{n\to\infty}\sum_{i=1}^n g'_{in}(\lambda_1)w_i(x) = v_1$, $\lim_{n\to\infty}\sum_{i=1}^n g_{in}(\lambda_1)w_i(x) = u_1$ and $\lim_{n\to\infty}\sum_{i=1}^n h_{in}(\lambda_1)w_i(x) = r_1$. We drop now the subscript n. Differentiating the governing system (39) with respect to λ, taking $u^* = 2\varepsilon^2 u''_\mu$ and $r^* = \varepsilon^2 r'_\mu$ and finally adding both identities, we get

$$\begin{aligned} &2(\varepsilon^2 u'''_\mu, \varepsilon^2 u''_\mu) + (u'_{\mu x}, \varepsilon^2 u''_{\mu x}) + (r''_\mu, \varepsilon^2 r'_\mu) \\ &+ 2\big((\mathcal{J}_\mu(r_\mu))', \varepsilon^2 r'_\mu\big) = 2(f', \varepsilon^2 u''_\mu). \end{aligned} \tag{40}$$

The monotonicity of $r_\mu \mapsto \mathcal{J}_\mu(r_\mu)$ leads to

$$\begin{aligned} &\big((\mathcal{J}_\mu(r_\mu(\xi)))', r'_\mu(\xi)\big) \\ &= \lim_{\Delta\xi\to 0}\frac{1}{(\Delta\xi)^2}\big(\mathcal{J}_\mu(r_\mu(\xi+\Delta\xi)) - \mathcal{J}_\mu(r_\mu(\xi)), r_\mu(\xi+\Delta\xi) - r_\mu(\xi)\big) \geq 0. \end{aligned}$$

Then we deduce from (40) that

$$\frac{d}{d\xi}\big(2|\varepsilon^2 u''_\mu|^2 + |\varepsilon u'_{\mu x}|^2 + |\varepsilon r'_\mu|^2\big) \leq 4(f', \varepsilon^2 u''_\mu). \tag{41}$$

We integrate (41) over (λ_1, λ), $\lambda \in [\lambda_1, \lambda_2]$, and since $v_\mu = \varepsilon u'_\mu$, we get

$$\big[2|\varepsilon v'_\mu|^2 + |v_{\mu x}|^2 + |\varepsilon r'_\mu|^2\big]_{\lambda_1}^{\lambda} \leq 4\int_{\lambda_1}^{\lambda}(\varepsilon f', v'_\mu)\,d\xi. \tag{42}$$

On one hand, we subtract the first equation in (23) at λ_1 to the first one in (14) at λ_1. From (25), we deduce that

$$|\varepsilon v'(\lambda_1)|^2 \leq |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2. \tag{43}$$

Moreover the initial condition $r_\mu(\lambda_1) = r_1 \in \mathcal{C}$ implies that $\mathcal{J}_\mu(r_1) = 0$ and then the second identity in (23) leads to the following identity

$$|\varepsilon r'_\mu(\lambda_1)|^2 = |v_{1x}|^2. \tag{44}$$

On the other hand, we integrate by parts the right hand side of (42), and we estimate the product (z, y) by $|z|^2/2\gamma_i + \gamma_i|y|^2/2$, and, choosing different values for γ_i, $i = 1, 2, 3$, we get

$$\begin{aligned} 4\int_{\lambda_1}^{\lambda} (\varepsilon f', v'_\mu)\, d\xi &\le \frac{\varepsilon^2\gamma_1}{2}||f'||^2_{L^\infty(\lambda_1,\lambda_2;\mathcal{H})} + \frac{1}{2\gamma_1}\,|v_\mu(\lambda)|^2 \\ &+ \frac{1}{2\gamma_2}\,|v_1|^2 + \frac{\varepsilon^2\gamma_2}{2}|f'(\lambda_1)|^2 + \frac{\varepsilon^2\gamma_3}{2}||f''||^2_{L^2(\lambda_1,\lambda_2;\mathcal{H})} + \frac{1}{2\gamma_3}\int_{\lambda_1}^{\lambda} |v_\mu|^2\, d\xi. \end{aligned} \tag{45}$$

Since $v = \varepsilon u'$ then the Dirichlet boundary conditions and the Poincaré inequality show that there exists a strictly positive constant c such that

$$|v_\mu(\xi)|^2 \le c|v_{\mu x}(\xi)|^2, \ \forall\xi \in (\lambda_1, \lambda_2). \tag{46}$$

Carrying (46) into (45), choosing $\gamma_1 = \gamma_3 = c$ and $\gamma_2 = 1$, we have

$$\begin{aligned} 4\int_{\lambda_1}^{\lambda} (\varepsilon f', v'_\mu)\, d\xi &\le \frac{c\varepsilon^2}{2}||f'||^2_{L^\infty(\lambda_1,\lambda_2;\mathcal{H})} + \frac{1}{2}\,|v_{\mu x}(\lambda)|^2 \\ &+ \frac{1}{2}|v_1|^2 + \frac{\varepsilon^2}{2}|f'(\lambda_1)|^2 + \frac{c\varepsilon^2}{2}||f''||^2_{L^2(\lambda_1,\lambda_2;\mathcal{H})} + \frac{1}{2}\int_{\lambda_1}^{\lambda} |v_{\mu x}|^2\, d\xi. \end{aligned} \tag{47}$$

Introducing (43), (44) and (47) in (42), we obtain

$$2|\varepsilon v'_\mu(\lambda)|^2 + \frac{1}{2}\,|v_{\mu x}(\lambda)|^2 + |\varepsilon r'_\mu(\lambda)|^2 \le g(\varepsilon) + \frac{1}{2}\int_{\lambda_1}^{\lambda} |v_{\mu x}|^2\, d\xi, \tag{48}$$

where

$$\begin{aligned} g(\lambda_1, \varepsilon) &= \frac{1}{2}|v_1|^2 + 2|v_{1x}| + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2 \\ &+ \frac{\varepsilon^2}{2}\big(|f'(\lambda_1)|^2 + c||f'||^2_{L^\infty(\lambda_1,\lambda_2;\mathcal{H})} + c||f''||^2_{L^2(\lambda_1,\lambda_2;\mathcal{H})}\big). \end{aligned}$$

By classical Gronwall's lemma, it is clear that

$$|v_{\mu x}(\lambda)|^2 \le 2g(\lambda_1, \varepsilon)\exp(\lambda_2 - \lambda_1). \tag{49}$$

Therefore the last term on the right hand side of (48) is now easily estimated. We finally obtain

$$2|\varepsilon v'_\mu(\lambda)|^2 + \frac{1}{2}\,|v_{\mu x}(\lambda)|^2 + |\varepsilon r'_\mu(\lambda)|^2 \le g(\lambda_1, \varepsilon)\big(1 + (\lambda_2 - \lambda_1)\exp(\lambda_2 - \lambda_1)\big),$$

which proves the Lemma. □

Proposition 4.2. *Assume that f belongs to $W^{2,\infty}(\lambda_1, \lambda_2; \mathcal{H})$ and that* (13) *and* (16) *hold. Then there exist $\gamma_i > 0$, $i = 1, 2$, such that*

$$\begin{aligned} &|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \le \gamma_1\big(||v_1||^2 \\ &+ |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2\big) + \varepsilon\gamma_2. \end{aligned} \tag{50}$$

Proof. This result follows from an energy estimate of the difference between the dynamic elastic-visco-plastic system and the quasi-static elastic-plastic system. Choosing $u^* = u'_\mu - \bar{u}'$ and $r^* = (r_\mu + \bar{r})/2$ in (26), $\bar{u}^* = \bar{u}' - u'_\mu$ and $\bar{r}^* = (\bar{r} + r)/2$ in (18), and adding the resulting expressions, we obtain the following inequality:

$$\begin{aligned} &(\varepsilon^2 u''_\mu, u'_\mu) + \frac{1}{2}(u_{\mu x} - \bar{u}_x, u'_{\mu x} - \bar{u}'_x) + \frac{1}{2}(r'_\mu - \bar{r}', r_\mu - \bar{r}) \\ &+ \frac{1}{2}(\bar{r}' - \bar{u}'_x, r_\mu - r) + (\mathcal{J}_\mu(r_\mu), r_\mu - \bar{r}) \le (\varepsilon^2 u''_\mu, \bar{u}'). \end{aligned} \tag{51}$$

Since $\bar{r} \in \mathcal{C}$ then $\mathcal{J}_\mu(\bar{r}) = 0$, and due to the monotonicity of $\mathcal{J}_\mu$, we get

$$(\mathcal{J}_\mu(r_\mu), r_\mu - \bar{r}) = (\mathcal{J}_\mu(r_\mu) - \mathcal{J}_\mu(\bar{r}), r_\mu - \bar{r}) \ge 0. \tag{52}$$

Using (52) in (51) and since $v_\mu = \varepsilon u'_\mu$, we infer that

$$\frac{1}{2}\frac{d}{d\xi}\left(2|v_\mu|^2 + |u_{\mu x} - \bar{u}_x|^2 + |r_\mu - \bar{r}|^2\right) + (\bar{r}' - \bar{u}'_x, r_\mu - r) \le 2(\varepsilon v'_\mu, \bar{u}'). \tag{53}$$

We integrate (53) over (λ_1, λ), $\lambda \in [\lambda_1, \lambda_2]$ and we obtain

$$\begin{aligned} &|v_\mu(\lambda)|^2 + \frac{1}{2}|u_{\mu x}(\lambda) - \bar{u}_x(\lambda)|^2 + \frac{1}{2}|r_\mu(\lambda) - \bar{r}(\lambda)|^2 \\ &+ \int_{\lambda_1}^{\lambda} (\bar{r}' - \bar{u}'_x, r_\mu - r)\, d\xi \le c(\lambda_1) + 2\int_{\lambda_1}^{\lambda} (\varepsilon v'_\mu, \bar{u}')\, d\xi, \end{aligned} \tag{54}$$

where

$$c(\lambda_1) = |v_1|^2 + \frac{1}{2}|u_{1x} - \bar{u}_{1x}|^2 + \frac{1}{2}|r_1 - \bar{r}_1|^2.$$

Let us observe that

$$\begin{aligned} &|v(\lambda)|^2 + \frac{1}{2}|u_x(\lambda) - \bar{u}_x(\lambda)|^2 + \frac{1}{2}|r(\lambda) - \bar{r}(\lambda)|^2 - g_\mu(\lambda) \\ &\le 2|v_\mu(\lambda)|^2 + |u_{\mu x}(\lambda) - \bar{u}_x(\lambda)|^2 + |r_\mu(\lambda) - \bar{r}(\lambda)|^2, \end{aligned} \tag{55}$$

where

$$g_\mu(\lambda) = 2|v_\mu(\lambda) - v(\lambda)|^2 + |u_{\mu x}(\lambda) - u_x(\lambda)|^2 + |r_\mu(\lambda) - r(\lambda)|^2.$$

Carrying (55) into (54) and using Cauchy-Schwarz's inequality we have

$$\begin{aligned} &|v(\lambda)|^2 + \frac{1}{2}|u_x(\lambda) - \bar{u}_x(\lambda)|^2 + \frac{1}{2}|r(\lambda) - \bar{r}(\lambda)|^2 + h_{\mu,n}(\lambda_1, \lambda) \\ &\le c(\lambda_1) + 2\Big(\int_{\lambda_1}^{\lambda} |\varepsilon v'_{\mu_n}|^2\, d\xi\Big)^{1/2}\Big(\int_{\lambda_1}^{\lambda} |\bar{u}'|^2\, d\xi\Big)^{1/2}, \end{aligned} \tag{56}$$

where

$$h_{\mu,n}(\lambda_1,\lambda) = \int_{\lambda_1}^{\lambda} (\bar{r}' - \bar{u}'_x, r_\mu - r)\, d\xi + 2\int_{\lambda_1}^{\lambda} \big(\varepsilon(v'_{\mu_n} - v'_\mu), \bar{u}'\big)\, d\xi - g_\mu(\lambda).$$

Introducing (38), the estimate obtained in Lemma 4.3, in (56), we deduce that there exist $\gamma_i > 0$, $i = 1, 2$, such that

$$|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 + 2h_{\mu,n}(\lambda_1,\lambda) \le \gamma_1\big(||v_1||^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + |(u_{1xx} + r_{1x}) - (\bar{u}_{1xx} + \bar{r}_{1x})|^2\big) + \varepsilon\gamma_2.$$

The conclusion follows then from Lemma 4.2. □

4.4. *Stability of a quasi-static path*

In order to prove the stability result, it is convenient to compare the dynamical solution (v, u, r) with another dynamical solution $(\tilde{v}, \tilde{u}, \tilde{r})$ that solves (23) with the Dirichlet boundary conditions (12) and the initial conditions

$$\big(\tilde{v}(\lambda_1), \tilde{u}(\lambda_1), \tilde{r}(\lambda_1)\big) = (\varepsilon\bar{u}'_1, \bar{u}_1, \bar{r}_1) \in \mathcal{V}_0 \times \mathcal{W}. \tag{57}$$

Let us remark that the variational formulation of that problem is the following:

$$\begin{cases} \text{Find } (\tilde{u}, \tilde{r}) \in \mathcal{W} \text{ such that } \forall(\tilde{u}^*, \tilde{r}^*) \in \mathcal{W}, \\ (\varepsilon^2\tilde{u}'', \tilde{u}^*) + \dfrac{1}{2}(\tilde{u}_x, \tilde{u}^*_x) + \dfrac{1}{2}(\tilde{r}, \tilde{u}^*_x) = (f, \tilde{u}^*), \\ (\tilde{r}', \tilde{r} - \tilde{r}^*) - (\tilde{u}'_x, \tilde{r} - \tilde{r}^*) \le 0, \end{cases} \tag{58}$$

with the initial conditions (57).

Lemma 4.4. *Assume that* (25) *and* (57) *hold and that* f *belongs to* $L^2(\lambda_1, \lambda_2; \mathcal{H})$. *Then*

$$\begin{aligned} &2|v(\lambda) - \tilde{v}(\lambda)|^2 + |u_x(\lambda) - \tilde{u}_x(\lambda)|^2 + |r(\lambda) - \tilde{r}(\lambda)|^2 \\ &\le 2|v_1 - \tilde{v}(\lambda_1)|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2. \end{aligned} \tag{59}$$

Proof. Once again we use energy techniques to compare two elastic-plastic problems with hardening that have the same boundary conditions but different initial conditions. Choosing $u^* = u' - \tilde{u}'$ and $\tilde{u}^* = \tilde{u}' - u'$ in (17) and (58), respectively, we have

$$\big(\varepsilon^2(u'' - \tilde{u}''), u' - \tilde{u}'\big) + \frac{1}{2}(u_x - \tilde{u}_x, u'_x - \tilde{u}'_x) + \frac{1}{2}(r - \tilde{r}, u'_x - \tilde{u}'_x) = 0. \tag{60}$$

On the other hand, taking $r^* = \tilde{r}$ and $\tilde{r}^* = r$ in (17) and (58), respectively, we get

$$(r' - \tilde{r}', r - \tilde{r}) \le (r - \tilde{r}, u'_x - \tilde{u}'_x). \tag{61}$$

Carrying (61) into (60) and since $v = \varepsilon u'$ and $\tilde{v} = \varepsilon \tilde{u}'$, we obtain

$$\frac{d}{d\xi}\left(2|v-\tilde{v}|^2 + |u_x - \tilde{u}_x|^2 + |r-\tilde{r}|^2\right) \leq 0. \tag{62}$$

We integrate (62) over (λ_1, λ), $\lambda \in [\lambda_1, \lambda_2]$, and using the initial conditions (13) and (57), leads to the result in the Lemma. □

Proposition 4.3. *(Stability). Assume that* (25) *and* (57) *hold and that* f *belongs to* $L^2(\lambda_1, \lambda_2; \mathcal{H})$. *Then there exist* $\gamma > 0$ *such that for* $0 < \varepsilon < 1$,

$$\begin{aligned}&|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \\ &\leq \gamma\big(|v_1|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2 + \varepsilon\big).\end{aligned}$$

Proof. The stability result follows from the estimates obtained in Proposition 4.2 and Lemma 4.4. Let us remark that (59) leads to the following inequality

$$\begin{aligned}&\frac{1}{2}\big(|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2\big) \\ &\leq c(\lambda_1) + |\tilde{v}(\lambda)|^2 + |\tilde{u}_x(\lambda) - \bar{u}_x(\lambda)|^2 + |\tilde{r}(\lambda) - \bar{r}(\lambda)|^2,\end{aligned} \tag{63}$$

where

$$c(\lambda_1) = 2|v_1 - \tilde{v}(\lambda_1)|^2 + |u_{1x} - \bar{u}_{1x}|^2 + |r_1 - \bar{r}_1|^2.$$

On the other hand, choosing $u = \tilde{u}$, $v = \tilde{v}$ and $r = \tilde{r}$ in (50) and using the fact that $\tilde{u}(\lambda_1) = \bar{u}_1$ and $\tilde{r}(\lambda_1) = \bar{r}_1$, we obtain

$$|\tilde{v}(\lambda)|^2 + |\tilde{u}_x(\lambda) - \bar{u}_x(\lambda)|^2 + |\tilde{r}(\lambda) - \bar{r}(\lambda)|^2 \leq \gamma_1 ||\tilde{v}(\lambda_1)||^2 + \varepsilon\gamma_2. \tag{64}$$

Introducing (64) in (63), we get

$$|v(\lambda)|^2 + |u_x(\lambda) - \bar{u}_x(\lambda)|^2 + |r(\lambda) - \bar{r}(\lambda)|^2 \leq \gamma_1 ||\tilde{v}(\lambda_1)||^2 + 2c(\lambda_1) + \varepsilon\gamma_2.$$

Since $\bar{u}'(\lambda_1)$ and $\bar{u}'_x(\lambda_1)$ are bounded in $\mathcal{H}$ and $\tilde{v}(\lambda_1) = \varepsilon\bar{u}'_1$ then the Proposition follows. □

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References

1. B. Loret and a. M. J. A. C. Simões, F. M. F., Flutter instability and ill-posedness in solids and fluid-saturated porous media, in *Material Instabilities in Elastic and Plastic Solids (Udine, 1999)*, ed. H. Petryk, CISM Courses and Lectures, Vol. 414 (Springer, Vienna, 2000) pp. 109–207.
2. J. A. C. Martins, F. M. F. Simões, A. Pinto da Costa and I. Coelho (2004).
3. G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics* (Springer-Verlag, Berlin, 1976). Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.
4. C. Johnson, *J. Math. Pures Appl. (9)* **55**, 431 (1976).
5. C. Johnson, *J. Math. Anal. Appl.* **62**, 325 (1978).
6. J. A. C. Martins, M. D. P. Monteiro Marques and A. Petrov (2005).
7. H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert* (North-Holland Publishing Co., Amsterdam, 1973). North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).
8. R. E. Showalter and P. Shi, *J. Math. Anal. Appl.* **216**, 218 (1997).
9. R. E. Showalter and P. Shi, *Comput. Methods Appl. Engrg.* **151**, 501 (1998).
10. E. Zeidler, *Nonlinear functional analysis and its applications. III* (Springer-Verlag, New York, 1985). Variational methods and optimization, Translated from the German by Leo F. Boron.
11. J. A. C. Martins, M. D. P. Monteiro Marques and A. Petrov (2006).
12. J. A. C. Martins, N. V. Rebrova and V. A. Sobolev (2004).
13. R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques. Vol. 8*INSTN: Collection Enseignement. [INSTN: Teaching Collection], INSTN: Collection Enseignement. [INSTN: Teaching Collection] (Masson, Paris, 1988). Évolution: semi-groupe, variationnel. [Evolution: semigroups, variational methods], Reprint of the 1985 edition.
14. H. Brézis, *Analyse fonctionnelle*Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master’s Degree], Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master’s Degree] (Masson, Paris, 1983). Théorie et applications. [Theory and applications].
15. A. Kolmogorov, S. Fomine and V. M. Tihomirov, *Eléments de la théorie des fonctions et de l’analyse fonctionnelle* (Editions Mir, Moscow, 1974). Avec un complément sur les algèbres de Banach, par V. M. Tikhomirov, Traduit du russe par Michel Dragnev.