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A pseudodifferential linear complementarity problem related to a one dimensional viscoelastic model with Signorini conditions

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Abstract

The simplified viscoelastic problem

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x \in \mathbb{R}^-, \quad t \in \mathbb{R}^+, \quad \alpha > 0,$$

with boundary condition

$$u(0,\cdot) \geq 0, \quad (u_x + \alpha u_{xt})(0,\cdot) \geq 0, \quad \big(u(u_x + \alpha u_{xt})\big)(0,\cdot) = 0,$$

is reduced to pseudodifferential linear complementarity problem (LCP)

$$\lambda_1 * w = g + b, \quad 0 \leq w \perp b \geq 0.$$

where λ_1 is the inverse Fourier transform of the causal determination of $\widehat{\lambda}_1(\omega) = i\omega\sqrt{1 + i\alpha\omega}$. We prove the existence of a solution of this LCP; the energy relation for the original problem is equivalent to

$$\langle \dot{w}, b \rangle = 0.$$

This relation is formally and rigorously true, but highly non trivial since *a priori* b is a measure and $\dot{w}$ is defined almost everywhere.

1. Introduction and notations

We consider a solid body in motion in a landscape, it may happen that this solid body hit a rigid obstacle then forces are transmitted by the contact domain. These forces have two components; a normal component created by the friction and the tangential component preventing the interpenetration of materials. In the mathematical point of view, we model a contact problem by a system of differential equations describing the motion, the deformations of the solid body and taken into account the contact and the friction forces at the boundary.

The contact problems are found in various domains of solid mechanics as for example the machining process which is the cause of this study.

Most of mathematicians model the contact and the friction respectively by the Signorini conditions and the Coulomb friction law. The researchers have quickly obtained the first results for the static contact problems less realistic than dynamic one but easier to study. Fichera [12] solved indeed the static contact problem with Signorini boundary conditions but without friction. The problem is more complicated if we replace the Signorini conditions by the Coulomb friction law at the boundary since the research is confined to the contact problems with given friction, where the normal traction in the friction law is a known function. Observe that the first result of the solvability using the convex analysis is proved for the static problem with given friction in [10]. Some important results are also obtained by the Czech school in [21], [11].

Regarding dynamic problems of contact for elastic material, there are hyperbolic problems. The contact condition formulated in displacement terms velocities for the hyperbolic problem, which is an approximation of the impenetrability condition, is more convenient than the original condition formulated in displacement. Then the research is confined to that formulation for thirty years. Let us remark that the contact problem can be approximated by an auxiliary problem which has a simpler structure. It was quite usual to approximate the contact problem by the penalty method for the Signorini contact conditions and by the contact problem with given friction for the Coulomb friction law. Therefore according to classical methods, it is easy to prove the solvability of the auxiliary problem. The transition between the auxiliary contact problem and the original one is done by the passage to the limit for the penalty parameter in the case of the penalty approximation and by a fixed point argument in the case of the contact problem with given friction.

The theory of vibrations of continuous media with unilateral conditions at the boundary purports to understand the mathematical description of the so-called dynamical Signorini problem; when the medium is elastic and satisfies the assumptions of the theory of small deformations, the one-dimensional medium is fairly well understood. In the case of continuous obstable, Amerio and Prouse [1,4] obtained the first result for a string and Schatzman [22] treated the case of concave obstacle. In the case of a point-shaped obstacle, there are works of Amerio [2,3], Schatzman [23], Citrini [8], Citrini and Marchionna [9] and the theory can be considered as complete. In the multidimensional case, despite important efforts of mathematicians, the theory is still quite poor and there are deep functional analytic reasons for our ignorance. The only case where an energy relation is proved is that of is a wave equation in a half-space with unilateral conditions at the boundary [19]. It was only in this work that the equivalence between codimension one obstacle and constraint at the boundary was clearly stated, though it was probably understood before that article appeared. Observe that there is no friction in this research. The weaker results are also obtained by Uhn [15].

The researchers are also interested in the contact models for the material with viscous damping. The existence theorems for different models of viscoelasticity with contact and with or witout a given friction at the boundary are established in [13], [14]. These results do not define a trace space, nor do they give any information on the balance of energy. There are also several remarkable results obtained by Shillor et al. in [18] and [7].

Since the contact problem with Signorini conditions and Coulomb friction law is very difficult to solve, some researchers have proposed other mathematical models leading to the simpler mathematical formulations as for example the normal compliance model developed in [20]. Many papers, in particular these treating on quasistatic problems [5], [6], [16] and [17] used this method. The normal compliance is entirely justified by an analysis of the behaviour of contacting field on a microscale. This analysis leads also to observe that the model parameters must have a comparable scale to the non-smooth boundaries. We remark that the Signorini conditions and Coulomb friction law are recovered if the model parameters tends to zero. On the other hand, the normal compliance enables an unlimited penetration to the support and if we fix a bound for the penetration, we lose the compactness informations on the friction term and we obtain the problem more complicate than in the case of original Signorini conditions and Coulomb friction law. This approach is especially adapted to study the influences of asperities.

The present work is dedicated to the study of a one-dimensional viscoelastic contact problem in the particular case of a Kelvin-Voigt material. The existence is easy to establish by the penalty method and was already known in [13]. A characterization of the trace spaces is given and enables to show that the weak solution is indeed strong. The main result of this paper is the balance of energy leading to the conclusion that the energy loss is purely viscous.

In the most general situation, the dynamical evolution system for Kelvin-Voigt material is

$$\rho u_{tt} = Au + Bu_t + f, \quad x \in \Omega, \quad t \in (0,T),$$

where A and B are elasticity operators defined with the help of Hooke tensors a_{ijkl} and b_{ijkl} and Ω is the part of the space occupied by the material. Define the strain and respective stress tensors:

$$\varepsilon_{ij}(u) = \frac{\partial_i u_j + \partial_j u_i}{2}, \quad \sigma^A_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u), \quad \sigma^B_{ij}(u) = b_{ijkl}\varepsilon_{kl}(u);$$

the normal displacement at the boundary is

$$u_N = u_i \nu_i,$$

where we have chosen ν_i to be unit normal pointing inward; the normal and the tangential components of the stress vectors at the boundary are

$$\sigma_N^A = \sigma_{ij}^A \nu_i \nu_j, \quad \sigma_N^B = \sigma_{ij}^B \nu_i \nu_j,$$
$$\left(\sigma_T^A\right)_j = \sigma_{ij}^A \nu_i - \sigma_N^A \nu_j, \quad \left(\sigma_T^B\right)_j = \sigma_{ij}^B \nu_i - \sigma_N^B \nu_j.$$

With these notations, the boundary condition on that part of the boundary where contact may take place is written:

$$\sigma_T^A + \dot\sigma_T^B = 0,$$
$$\sigma_N^A + \dot\sigma_N^B \geq 0, \quad u_N \geq 0, \quad u_N\left(\sigma_N^A + \dot\sigma_N^B\right) = 0.$$

If we consider the very particular case where $\Omega = \mathbb{R}^- \times \mathbb{R}^{d-1}$ and if we seek a solution u which depends only on x_1 and t, while the material under consideration is homogeneous and isotropic, we are led to the following boundary problem for u_1:

$$\rho \frac{\partial^2 u_1}{\partial t^2} = \left(\lambda^A + 2\mu^A\right) \frac{\partial^2 u_1}{\partial x_1^2} + \left(\lambda^B + 2\mu^B\right) \frac{\partial^3 u_1}{\partial x_1^2 \partial t} + f_1, \tag{1}$$

with the boundary conditions written in the fashion of a linear complementarity problem (LCP):

$$0 \leq \left(\lambda^A + 2\mu^A\right) \frac{\partial u_1}{\partial x_1} + \left(\lambda^B + 2\mu^B\right) \frac{\partial^2 u_1}{\partial x_1 \partial t} \perp u_1 \geq 0;$$

here the orthogonality has the natural meaning: an appropriate duality product between the two terms of the relation vanishes. The problem for the second and third components of u is linear, viz. for $j = 2$, 3,

$$\rho \frac{\partial^2 u_j}{\partial t^2} = \mu^A \frac{\partial^2 u_j}{\partial x_1^2} + \mu^B \frac{\partial^3 u_j}{\partial x_1^2 \partial t} + f_j,$$

with boundary conditions given by

$$\mu^A \frac{\partial u_j}{\partial x_1} + \mu^B \frac{\partial^2 u_j}{\partial x_1 \partial t} = 0.$$

Therefore we concentrate our efforts on (1), which becomes after appropriate adimensionalization the one-dimensional viscously damped wave equation on a half-line:

$$u_{tt} - u_{xx} - \alpha u_{xxt} = f, \quad x < 0, \quad t > 0, \quad \alpha > 0, \tag{2}$$

with initial data

$$u(\cdot, 0) = u_0 \quad \text{and} \quad u_t(\cdot, 0) = u_1, \tag{3}$$

and boundary conditions

$$0 \leq u(0, \cdot) \perp (u_x + \alpha u_{xt})(0, \cdot) \geq 0. \tag{4}$$

If $u_0(0)$ is strictly positive, we may solve the linear problem (2) with initial conditions (3) and boundary condition

$$(u_x + \alpha u_{xt})(0, \cdot) = 0. \tag{5}$$

Then by energy estimates we conclude easily that if $u_{0,x}$, u_1 and f are square integrable respectively on $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R} \times (0, T)$, u is continuous over $\mathbb{R} \times [0, T]$; it suffices therefore to solve (2)-(3)-(5) on the maximum time interval over which $u_0(0, \cdot)$ is strictly positive, to be reduced to the case

$$u_0(0) = 0.$$

Denote by $\bar{u}$ the solution of

$$\bar{u}_{tt} - \bar{u}_{xx} - \alpha \bar{u}_{xxt} = f, \quad x < 0, \quad t > 0, \quad \alpha > 0, \tag{6}$$

with initial data (3) and Dirichlet boundary data at $x = 0$. Define

$$g = -\bar{u}_x(0, \cdot) - \alpha \bar{u}_{xt}(0, \cdot). \tag{7}$$

Then $v = u - \bar{u}$ solves

$$v_{tt} - v_{xx} - \alpha v_{xxt} = 0, \tag{8a}$$

$$v(0, \cdot) \geq 0, \quad (v_x + \alpha v_{xt})(0, \cdot) \geq g, \quad \big(v(v_x + \alpha v_{xt})\big)(0, \cdot) = 0 \tag{8b}$$

$$v(\cdot, t) = 0 \quad \text{if} \quad t \leq 0. \tag{8c}$$

Call λ_1 the distribution whose Fourier transform in time is the causal determination of $i\omega\sqrt{1 + \alpha i\omega}$; we will show in Section 3 that v solves (8) iff $w = v(0, \cdot)$ solves

$$\lambda_1 * w = g + b, \quad 0 \leq w \perp b \geq 0. \tag{9}$$

In order to construct a solution of (9), we require that g be a half-integral of a measure ϕ with support in $\mathbb{R}^+$, i.e. for almost every $t > 0$:

$$g(t) = \int_{[0,t[} \frac{1}{\sqrt{\pi(t-s)}}\, \phi(s). \tag{10}$$

We will show in Section 2 that if u_0 and u_1 belong to $H^2(-\infty, 0) \cap H^1_0(-\infty, 0)$ and if f and f_t belong to $L^2_{\text{loc}}\big([0, \infty); L^2(-\infty, 0)\big)$, then g belongs to $H^{1/2}_{\text{loc}}(\mathbb{R})$ and is supported in $\mathbb{R}^+$, so that our theory can be applied.

In Section 4, we define a penalized problem associated to (9), for which we prove the existence and uniqueness of a solution. It is not difficult to extract weakly convergent subsequences, to pass to the limit and to obtain therefore the existence of a solution of (9) belonging to $H^{5/4}_{\text{loc}}(\mathbb{R})$.

Let us write at least formally an energy relation for (2): we multiply this equation by u_t, we integrate by parts over $(-\infty, 0) \times (0, \tau)$, and we get

$$\frac{1}{2} \int_{-\infty}^0 \big(u_x^2 + u_t^2\big)(\cdot, \tau)\, dx + \alpha \int_{-\infty}^0 \int_0^\tau u_{xt}^2\, dx\, dt = \frac{1}{2} \int_{-\infty}^0 \big(|u_1|^2 + |u_{0,x}|^2\big)\, dx$$
$$+ \int_0^\tau \big((u_x + \alpha u_{xt}) u_t\big)(0, \cdot)\, dt + \int_0^\tau \int_{-\infty}^0 f u_t\, dx\, dt.$$

The energy loss is purely viscous, iff

$$\int_0^\tau \big((u_x + \alpha u_{xt})u_t\big)(0,\cdot)\,dt = 0. \tag{11}$$

By construction, $(u_x + \alpha u_{xt})(0,\cdot)$ is equal to $\lambda_1 * w - g$; therefore, (11) is equivalent to

$$\langle b, \dot{w}\rangle = 0. \tag{12}$$

A priori, b belongs to $H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$ and is non negative and $\dot{w}$ belongs to $H^{1/4}_{\mathrm{loc}}(\mathbb{R})$: this is not enough to conclude that (12) holds.

The standard methods used for variational inequalities break down here: the non local character of the convolution by λ_1 seems to preclude any kind of argument based on the signs of functions. While local estimates using sign cannot work, we construct a global estimate which will work. This is where we need regularity on data, i.e. (10).

Let us sketch the principle of the construction of Section 5: define H to be the Heaviside function and

$$\nu(t) = e^{-t/\alpha}\frac{H(t)}{\alpha}.$$

Let ρ be the inverse Fourier transform of the causal determination of $\sqrt{1+\alpha i\omega}$, and μ the inverse Fourier transform of $1/\widehat{\rho}$. Then ν is equal to $\mu * \mu$ and $\lambda_1 = \rho'$; the convolution inverse of λ_1 is $H * \mu = \mu_1$.

More information on these distributions is given in the Appendix.

Assume first that our data ϕ is such that $H * \nu * \phi$ is positive on some interval $[\sigma,\tau)$ and then negative on some interval (τ,σ'); an explicit calculation shows that a good candidate for a solution on $[\sigma,\tau]$ is $1_{[\sigma,\tau]}\big(H*\nu*\phi\big)$; then, we calculate a candidate for b, hoping that the support of b will be included in $[\tau,\sigma']$, and we find $b = -\mu * \psi$, where ψ is obtained from ϕ by a linear operation; however, since after time σ', we expect the solution to be positive again, the really good candidate for b is rather $-\big(\mu*\psi\big)1_{[\tau,\sigma')}$, and, lo and behold, there exists a measure ζ such that

$$-\big(\mu*\psi\big)1_{[\tau,\sigma']} = \mu * \zeta;$$

the calculation of this measure is the object of Lemma 4, where we also give the formula for w on the next interval where it is non zero. Once we have a formula for two intervals, we generalize to any number of intervals (Corollary 3). Moreover, we are able to give an estimate on the left derivative of w over any interval where w is positive (Lemma 6), and this estimate leads to the above mentioned global sign argument.

However, this construction has a very significant defect: we do not know that it is possible to extend it to an interval of finite length; therefore, the next idea is to modify ϕ so as to realize the locally finite construction; this is the recursion of Section 6.

A number of estimates are given in that Section, and they lead easily to the extraction of a convergent subsequence in Section 7; but this is not enough to show the desired energy equality; the function $\dot w$ cannot vanish everywhere on the support of b, since w may have a strictly negative left derivative at the right end of intervals where w is strictly positive. The requirement that w vanish almost everywhere on the support of b does not suffice, since we do not know that b is absolutely continuous with respect to Lebesgue's measure. Therefore, we show that $\dot w$ vanishes on the support of b except on a countable set; on the other hand, as $\mu * b$ is a locally essentially bounded function, the atomic part of b vanishes; thus $\dot w$ is b-integrable and we are able to conclude.

2. Regularity results for the damped wave equation

Theorem 1. *If u_0 and u_1 belong to $H^2(-\infty,0)\cap H^1_0(-\infty,0)$ and if f and f_t belong to $L^2_{\rm loc}\big([0,\infty),L^2(-\infty,0)\big)$, then $\bar u$ has the following functional properties:*

$$\bar u \in W^{2,\infty}_{loc}\big([0,\infty);L^2(-\infty,0)\big), \tag{13a}$$

$$\bar u_x \in W^{2,\infty}_{loc}\big([0,\infty);L^2(-\infty,0)\big)\cap H^2_{loc}\big([0,\infty);L^2(-\infty,0)\big), \tag{13b}$$

$$\bar u_{xx} \in L^{\infty}_{loc}\big([0,\infty);L^2(-\infty,0)\big). \tag{13c}$$

Proof. We sketch here the proof of (13), using a straightforward energy inequality. The proof could be easily completed by a Galerkin method, but since it is quite routine, we leave the verification to the reader.

Multiply (6) by $\bar u_t$, and integrate by parts in x; then we find

$$\frac12\int_{-\infty}^0\big(|\bar u_t(\cdot,\tau)|^2+|\bar u_x(\cdot,\tau)|^2\big)\,dx+\alpha\int_0^\tau\int_{-\infty}^0|\bar u_{xt}|^2\,dx\,dt$$
$$=\int_0^\tau\int_{-\infty}^0 f\bar u_t\,dx\,dt+\frac12\int_{-\infty}^0\big(|u_1|^2+|u_{0,x}|^2\big)\,dx.$$

A straightforward application of Gronwall's lemma yields: $\bar u_t$, $\bar u_x$ are bounded in $L^\infty_{\rm loc}\big([0,\infty);L^2(-\infty,0)\big)$, $\bar u_{xt}$ is bounded in $L^2_{\rm loc}\big([0,\infty);L^2(-\infty,0)\big)$ under the assumptions u_0 belongs to $L^2(-\infty,0)\cap H^1_0(-\infty,0)$, u_1 belongs to $L^2(-\infty,0)$ and f belongs to $L^2_{\rm loc}\big([0,\infty);L^2(-\infty,0)\big)$.

If we multiply (6) by $\bar u_{xxt}$, an application of Cauchy-Schwarz inequality shows that $\bar u_{xxt}$ belongs to $L^2_{\rm loc}\big([0,\infty);L^2(-\infty,0)\big)$, $\bar u_{xt}$ and $\bar u_{xx}$ belong to $L^\infty_{\rm loc}([0,\infty);L^2_{\rm loc}(-\infty,0))$.

Similarly, after differentiating (6) with respect to t, and multiplying it by $\bar u_{tt}$, we find that $\bar u_{xxt}$ belongs to $L^2_{\rm loc}\big([0,\infty);L^2(-\infty,0)\big)$.

Corollary 1. *Under the hypotheses of Theorem 1, $(\bar u_x+\alpha\bar u_{xt})(0,\cdot)$ belongs to the space $H^{1/2}_{\rm loc}\big([0,\infty)\big)$.*

Proof. This proof is a consequence of the classical theory of traces of Sobolev spaces.

Remark 1. The conclusion of Corollary 1 is much stronger than needed; its purpose is only to show that our theory is not empty. Obtaining estimates in Banach spaces for the traces of the solution of the viscoelastodynamic equation (6) is much more difficult than in Hilbert spaces, which is the reason why we have opted for a simple approach of the regularity theory.

3. Reduction to a problem at the boundary

Our convention for the Fourier transform is

$$\widehat{z}(\omega) = \int_{\mathbb{R}} e^{-i\omega t} z(t)\, dt.$$

Let us apply a partial Fourier transform in time to (8a), calling ω the dual variable of t; then equation (8a) becomes

$$\widehat{v}_{xx} = -\frac{\omega^2}{1 + i\alpha\omega}\, \widehat{v}. \tag{14}$$

The general solution of (14) is given by

$$\widehat{v}(x, \omega) = \widehat{a}(\omega) \exp\bigl(\widehat{\lambda}(\omega) x\bigr) + \widehat{b}(\omega) \exp\bigl(-\widehat{\lambda}(\omega) x\bigr); \tag{15}$$

since we performed a Fourier transform on ν, we assumed implicitly that v and $\widehat{v}$ are tempered respectively in t and ω.

Intuitively the term $\widehat{b} \exp\bigl(-\widehat{\lambda} x\bigr)$ can be tempered only if $\widehat{b}$ decays at infinity very fast, and since this must be true for all x, it will imply that b vanishes as proved in next lemma:

Lemma 1. *If v is of finite energy, the coefficient b vanishes.*

Proof. We eliminate a by performing a linear combination of v and v_x:

$$-\widehat{v}_x(x, \cdot) + \widehat{\lambda}\, \widehat{v}(x, \cdot) = 2\widehat{b}\, \widehat{\lambda} \exp\bigl(-\widehat{\lambda} x\bigr).$$

The Paley-Wiener-Schwartz theorem implies that λ is a tempered distribution on $\mathbb{R}$, with support included in $\mathbb{R}^+$, i.e. a causal distribution. Let us define

$$\widehat{w}(x, \cdot) = \widehat{b}\, \widehat{\lambda} \exp\bigl(-\widehat{\lambda} x\bigr).$$

Since v is of finite energy, it is tempered, and therefore w and $\widehat{w}$ are tempered. Therefore there exists φ_γ, $|\gamma| \le m$, which is continuous and polynomially increasing such that

$$\widehat{w}(x, \cdot) = \sum_{|\gamma| \le m} \partial^\gamma \varphi_\gamma(x, \cdot) \text{ in the distributions sense.}$$

Here γ is multi-index (γ_1, γ_2) and $|\gamma| = \gamma_1 + \gamma_2$.

Let ψ and $\widehat{\varphi}$ belong respectively to $C_0^\infty(\mathbb{R})$ and $C_0^\infty(0, \infty)$; assume that the support of $\widehat{\varphi}$ is included in $\{\omega : \omega_1 \le |\omega| \le \omega_2\}$ with $\omega_1 > 0$, and call

$[x_1, x_2]$ an interval containing the support of ψ. The distribution $\widehat{b}$ restricted to $(0, \infty) \times \big(\mathbb{R} \setminus \{0\}\big)$ is equal to

$$\widehat{w}(x, \cdot) \exp\big(\widehat{\lambda} x\big) / \widehat{\lambda}.$$

If we assume that

$$\int_{x_1}^{x_2} \psi(x)\, dx = 1,$$

and if y is an arbitrary negative number, we have the equality in the sense of distributions:

$$\begin{aligned} \big\langle \widehat{b}, \widehat{\varphi} \big\rangle &= \big\langle \widehat{b}, \widehat{\varphi} \otimes \psi \big\rangle \\ &= \sum_{|\gamma| \le m} \int_{\mathbb{R}} \int_{-\infty}^{0} (-1)^{|\gamma|} \varphi_\gamma(x, \cdot) \partial^\gamma \left(\frac{\exp\big(\widehat{\lambda} x\big)}{\widehat{\lambda}}\, \widehat{\varphi} \psi(x - y) \right) dx\, d\omega. \end{aligned} \tag{16}$$

The reader will check that all the derivatives of $\exp\big(\widehat{\lambda} x\big) \widehat{\varphi} \psi(x - y) / \widehat{\lambda}$ are finite sums of expressions of the form

$$\widehat{a}(\omega) \widehat{\varphi}^{(j)}(\omega) \psi^{(k)}(x - y) x^m \exp\big(\widehat{\lambda}(\omega) x\big)$$

where $\widehat{a}$ is the quotient of a polynomial in $\widehat{\lambda}$ and a finite number of its derivatives, and of a power of $\widehat{\lambda}$. Since 0 is excluded from the support of ω, we have the estimate

$$\big|\widehat{a} \widehat{\varphi}^{(j)}\big| \le C.$$

Let $k > 0$ be a lower bound of $\Re \widehat{\lambda}$ over $\omega_1 \le |\omega| \le \omega_2$. Then there exists C_1 such that for ω verifying $\omega_1 \le |\omega| \le \omega_2$ and for $x < 0$,

$$|\varphi_\gamma(x, \cdot)| \le C_1 \exp(-kx/2).$$

We may estimate each term in the right hand side of (16) by

$$2(\omega_2 - \omega_1) \int_{-\infty}^{0} C C_1 \exp(kx/2) \big|\psi^{(k)}(x - y)\big| |x|^m\, dx.$$

As y tends to $-\infty$, it is clear that this integral tends to 0, and therefore the restriction of $\widehat{b}$ to $\mathbb{R} \setminus \{0\}$ vanishes. This means that $\widehat{b}$ can be a finite combination of the derivatives of a Dirac mass at 0; but these terms can be included in $\widehat{a}$, proving thus the lemma.

We deduce from Lemma 1 that if v is tempered in the neighborhood of $x = -\infty$, it must be of the form

$$\widehat{v}(x, \cdot) = \widehat{a} \exp\big(\widehat{\lambda} x\big).$$

In particular

$$\widehat{v}_x(0, \cdot) + \alpha \widehat{v_{xt}}(0, \cdot) = \widehat{\lambda}_1 \widehat{v}(0, \cdot). \tag{17}$$

If we let w be the trace $v(0,\cdot)$, (8) can be written now

$$\lambda_1 * w = g + b, \quad w \geq 0, \quad b \geq 0, \quad b \perp w. \tag{18}$$

Of course, w vanishes for all negative times, and g has been defined at (7). At this point, $b \perp w$ is a formal statement, and a good part of this article aims to turn this formal statement into a *bona fide* mathematical relation — which implies in particular that we are able to assign a coherent sense to all the quantities involved.

4. Existence and uniqueness of the solution of the penalized equation

In (7), the rigid constraint is defined by a set of linear complementary conditions. We approximate this constraint by a very stiff response which vanishes when the constraint is not active and is linear when the constraint is active. More precisely, if $r^- = -\min(r,0)$, we consider the problem

$$\lambda_1 * w^\varepsilon = g + (w^\varepsilon)^-/\varepsilon, \tag{19}$$

where w^ε vanishes for all $t < 0$. We recall that μ_1 is the integral of μ which vanishes at 0, where $\widehat{\mu}(\omega)$ is the causal determination of $1/\sqrt{1+i\alpha\omega}$.

We establish now the existence and uniqueness of the solution of (19).

Theorem 2. *Assume that g belongs to $L^1_{\mathrm{loc}}(\mathbb{R}) \cap H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$ and vanishes for $t < 0$. Let h be a uniformly Lipschitz continuous function; then there exists a unique solution vanishing for $t < 0$ of the convolution equation:*

$$w = \mu_1 * (g - h(w)). \tag{20}$$

Moreover w is continuous.

Proof. We see from (89) that on any interval $[0,T]$, μ_1 satisfies the estimate

$$0 \leq \mu_1(t) \leq \frac{2\sqrt{T}}{\sqrt{\pi\alpha}}.$$

Define an integral operator $\mathcal{T}$ by

$$\mathcal{T}w = \mu_1 * (g + h(w)).$$

It is clearly equivalent to find a solution of (20) or a fixed point of $\mathcal{T}$. Therefore, it is suffices to find an integer k such that $\mathcal{T}^k$ will be a strict contraction in an appropriate functional space.

Let us check first that $\mathcal{T}$ maps $C^0\big([0,T]\big)$ to itself: since μ_1 is an integral of the integrable function μ, it is continuous and therefore $\mu_1 * g$ is also continuous. Since the composition $h \circ w$ is continuous, it is plain that $\mathcal{T}w$ is a continuous function.

We estimate now the Lipschitz constant of $\mathcal{T}$ restricted to $C^0\big([0,T]\big)$, denoting by L the Lipschitz constant of h:

$$\big|\big(\mathcal{T}w_2 - \mathcal{T}w_1\big)(t)\big| \le \int_0^t \mu_1(t-\cdot)L|w_2-w_1|\,ds \le \frac{2L\sqrt{T}}{\sqrt{\pi\alpha}}\int_0^t |w_2-w_1|\,ds.$$

Since this estimate is completely analogous to the classical estimate of Picard iterations, we obtain by induction the estimate

$$\big|\big(\mathcal{T}^k w_2 - \mathcal{T}^k w_1\big)(t)\big| \le \left(\frac{2L\sqrt{t}}{\sqrt{\pi\alpha}}\right)^k \frac{\|w_2-w_1\|_\infty}{k!}.$$

Therefore, for all $T\in(0,\infty)$, we can find an integer k such that the restriction of $\mathcal{T}^k$ to $C^0\big([0,T]\big)$ is a strict contraction. As T is arbitrary the theorem is proved.

Remark 2. We could have obtained the stronger estimate:

$$\big|\big(\mathcal{T}^k w_2 - \mathcal{T}^k w_1\big)(t)\big| \le \left(\frac{L}{\sqrt{\alpha}}\right)^k \chi^{1+3k/2}(t)\,\|w_2-w_1\|_\infty\,,$$

where $\chi^a(t) = (t^+)^{a-1}/\Gamma(a)$, which leads to the same conclusion, but with a smaller k for each T.

Remark 3. The same proof works if instead of $h(w)$ we introduce a continuous function $h(t,w)$ which is Lipschitz continuous with respect to its second argument.

Remark 4. If g_1 and g_2 coincide over $(-\infty,T]$, then the corresponding solutions w_1 and w_2 of $w_1 = \mu_1 * (g_1 - h(w_1))$ and $w_2 = \mu_1 * (g_2 - h(w_2))$ coincide over $(-\infty,T)$, thanks to the causal character of μ_1.

We would like to estimate w^ε in appropriate functional spaces independently of ε. We will assume that g belongs to $H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$. Formally we multiply (19) by $\dot{w}^\varepsilon$, and we estimate the pseudodifferential term in the Fourier variables. We obtain

$$\begin{aligned}&\frac{1}{2\pi}\,\Re\bigg(\int_{\mathbb{R}} \widehat{\lambda}_1 \widehat{w}^\varepsilon \overline{i\omega\widehat{w}^\varepsilon}\,d\omega\bigg) + \int_0^\infty \frac{1}{2\varepsilon}\frac{d}{dt}\big((w^\varepsilon)^-\big)^2\,dt \\ &= \frac{1}{2\pi}\,\Re\bigg(\int_{\mathbb{R}} \overline{i\omega\widehat{w}^\varepsilon}\widehat{g}\,d\omega\bigg).\end{aligned} \tag{21}$$

We infer from the estimate

$$\big|\widehat{\lambda}_1 i\omega\big| \ge \frac{1}{C}\,|\omega|^2\big(1+|\omega|\big)^{1/2}$$

that

$$\int_{\mathbb{R}} |\omega|^{5/2}|\widehat{w}^\varepsilon|^2\,d\omega \le C\int_{\mathbb{R}} |\omega|^2\big(1+|\omega|\big)^{1/2}|\widehat{w}^\varepsilon|\big(1+|\omega|\big)^{-1/4}|\widehat{g}|\,d\omega, \tag{22}$$

and therefore if $\bigl(1+|\omega|\bigr)^{-1/4}|\widehat{g}(\omega)|$ is bounded in $L^2(\mathbb{R})$, we see that $|\omega|\bigl(1+|\omega|\bigr)^{1/4}|\widehat{w}^\varepsilon(\omega)|$ is bounded in $L^2(\mathbb{R})$ independently of ε. As it stands, this calculation is insane, and the aim of the present section is turn it into a valid result. The essential idea is to use the causality: it enables us to modify g for large times, to validate the desired result on a time interval for which g has not been modified, and then to conclude for $\mathbb{R}^+$, since the modification time has been arbitrarily chosen.

The first step consists in proving the following lemma:

Lemma 2. *Assume that g belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ and vanishes for $t<0$. Then there exists for all $T>0$ an $S>T$ and a compactly supported function G which coincides with g over $[-\infty,T]$ such that for all $\varepsilon>0$ the solution W^ε of*

$$\lambda_1 * W^\varepsilon = G + (W^\varepsilon)^-/\varepsilon, \tag{23}$$

is non negative over $[S,\infty)$.

Proof. We choose ψ to be a $\mathcal{C}^\infty$ function from $\mathbb{R}$ to $\mathbb{R}$, which takes its values in $[0,1]$, and satisfies

$$\psi(t)=\begin{cases}0 & \text{if} \quad t<0,\\ 1 & \text{if} \quad t\geq 1.\end{cases}$$

We define then

$$G(t)=\psi(T+1-t)g(t)+\beta\psi(t-T-2)\psi(T+3-t),$$

where β is a number to be defined later. Then, for $t\geq T+3$, we may write

$$W^\varepsilon=\mu_1 * G+\mu_1 * (W^\varepsilon)^-/\varepsilon. \tag{24}$$

The first term on the right hand side of (24) can be written also as

$$\begin{aligned}&\int_0^{T+1}\mu_1(t-s)g(s)\psi(T+1-s)\,ds \\ &+\beta\int_{T+2}^{T+3}\mu_1(t-s)\psi(s-T-2)\psi(T+3-s)\,ds.\end{aligned} \tag{25}$$

The function $\mu_1(t)$ is increasing and tends to 1 as t tends to infinity, since μ is non negative and $\widehat{\mu}(0)=1$. Therefore, the limit for t going to infinity of the second term of (25) is

$$\lim_{t\to\infty}\beta\int_{T+2}^{T+3}\mu_1(t-s)\psi(s-T-2)\psi(T+3-s)\,ds=\beta\int_0^1\psi(s)\psi(1-s)\,ds.$$

We estimate from below the first term of (25) as follows:

$$\int_0^{T+1}\mu_1(t-s)g(s)\psi(T+1-s)\,ds\geq-\int_0^{T+1}|g(s)|\,ds.$$

We choose β so large that

$$\beta \int_0^1 \psi(t)\psi(1-t)\,dt > \int_0^{T+1} |g(s)|\,ds.$$

Then there exists S such that for all $t \geq S$, $(\mu_1 * G)(t) \geq 0$, and thanks to (24), the conclusion is clear.

This Lemma yields a Corollary:

Corollary 2. *The function $\lambda_1 * W^\varepsilon$ is compactly supported and $\dot{W}^\varepsilon$ belongs to $H^{1/4}(\mathbb{R})$; moreover $\dot{W}^\varepsilon$ decays exponentially to 0 as time goes to infinity.*

Proof. As a consequence of Lemma 2, the supports of $(W^\varepsilon)^-$ and G are included in the interval $[0, S]$. Define

$$g_1 = G + (W^\varepsilon)^-/\varepsilon.$$

We infer from (89) and the identity $\dot{W}^\varepsilon = \mu * g_1$ that for $t > S$

$$|\dot{W}^\varepsilon| \leq \frac{\exp(-t/\alpha)}{\sqrt{\pi\alpha(t-S)}} \int_0^S |g_1(s)| \exp(s/\alpha)\,ds$$

which implies the exponential decay of $\dot{W}^\varepsilon$. On the other hand, G belongs to $H^{-1/4}(\mathbb{R})$ and $(W^\varepsilon)^-$ is square integrable and therefore belongs also to $H^{-1/4}(\mathbb{R})$. Therefore, by Fourier transformation,

$$(\dot{W}^\varepsilon)\widehat{\ }(\omega) = \big(\widehat{\mu}\,\widehat{g}_1\big)(\omega) = \frac{\widehat{g}_1(\omega)}{\sqrt{1+i\alpha\omega}},$$

and it is plain that $\dot{W}^\varepsilon$ belongs to $H^{1/4}(\mathbb{R})$, thus concluding the proof of corollary.

Lemma 3. *The following estimate holds:*

$$\sup_{\varepsilon>0} \int_{\mathbb{R}} |\omega|^2 \sqrt{1+|\omega|}\,\big|\widehat{W}^\varepsilon\big|^2\,d\omega < +\infty. \tag{26}$$

Proof. Since G belongs to $H^{-1/4}(\mathbb{R})$ and $\dot{W}^\varepsilon$ belongs to $H^{1/4}(\mathbb{R})$, we perform the duality product of (23) with $\dot{W}^\varepsilon$. By standard properties of the Fourier transform,

$$2\pi\big\langle \lambda_1 * W^\varepsilon, \dot{W}^\varepsilon\big\rangle_{H^{-1/4},H^{1/4}} = \Re \int_{\mathbb{R}} \widehat{\lambda}_1 \widehat{W}^\varepsilon \overline{i\omega\widehat{W}^\varepsilon}\,d\omega;$$

thanks to the definition of λ_1, there exists a constant C such that $\Re\widehat{\lambda}_1 \geq C|\omega|\big(1+|\omega|\big)^{1/2}$, so that

$$2\pi\big\langle \lambda_1 * W^\varepsilon, \dot{W}^\varepsilon\big\rangle_{H^{-1/4},H^{1/4}} \geq C \int_{\mathbb{R}} |\omega|^2 \big(1+|\omega|\big)^{1/2} \big|\widehat{W}^\varepsilon\big|^2\,d\omega.$$

The duality product of $(W^\varepsilon)^-$ with $\dot W^\varepsilon$ can be identified with the L^2 scalar product of these two quantities, since $(W^\varepsilon)^-$ is continuous with compact support. In consequence, as follows from the classical results on the derivative of the negative part of an $H^1_{\mathrm{loc}}(\mathbb{R})$ function,

$$\big\langle (W^\varepsilon)^-, \dot W^\varepsilon \big\rangle_{H^{-1/4},H^{1/4}} = -\frac{1}{2}\int_{\mathbb{R}} \frac{d}{dt}\big((W^\varepsilon)^-\big)^2\,dt$$

which vanishes since $(W^\varepsilon)^-$ is compactly supported. The last term is

$$\begin{aligned}\big\langle G, \dot W^\varepsilon \big\rangle_{H^{-1/4},H^{1/4}} &= \frac{1}{2\pi}\,\Re \int_{\mathbb{R}} \widehat{G}\,\overline{i\omega\widehat{W^\varepsilon}}\,d\omega \\ &\le \frac{1}{2\pi}\left(\int_{\mathbb{R}} \frac{\big|\widehat{G}\big|^2}{\big(1+|\omega|\big)^{1/2}}\,d\omega\right)^{1/2}\left(\int_{\mathbb{R}} |\omega|^2\big(1+|\omega|\big)^{1/2}\big|\widehat{W^\varepsilon}\big|^2\,d\omega\right)^{1/2}.\end{aligned}$$

By construction $\big|\widehat{G}\big|\big(1+|\omega|\big)^{-1/4}$ is bounded in $L^2(\mathbb{R})$, and the conclusion is clear.

Theorem 3. *Assume that g belongs to $L^1_{\mathrm{loc}}(\mathbb{R})\cap H^{-1/4}_{\mathrm{loc}}(\mathbb{R})$ and vanishes for $t<0$. Then there exists a function $w\in H^{5/4}_{\mathrm{loc}}(\mathbb{R})$ which vanishes for $t<0$ and a measure b supported in $\mathbb{R}^+$ such that*

$$\lambda_1 * w = g + b,\quad w\ge 0,\quad b\ge 0,\quad \langle w, b\rangle = 0.$$

Proof. Define

$$N(T) = \int_0^{T+1} |g(t)|\,dt + \|\psi(T+1-\cdot)g\|_{H^{-1/4}}.$$

The construction of Lemma 2 and Corollary 2 shows that there exists a constant C such that

$$\|\dot W^\varepsilon\|_{H^{1/4}} \le CN(T).$$

Let us estimate the mass of $(W^\varepsilon)^-/\varepsilon$ over a finite interval: by definition of ρ and λ_1, we have

$$\lambda_1 * W^\varepsilon = \rho * \dot W^\varepsilon;$$

but the distribution ρ can be described precisely since

$$\rho * \rho = \delta + \alpha\delta'$$

and therefore

$$\rho = \mu + \alpha\mu' \text{ in the sense of distributions.}$$

We convolve with H the identity

$$\rho * \dot W^\varepsilon = G + \frac{(W^\varepsilon)^-}{\varepsilon},$$

obtaining therefore

$$H * (\mu + \alpha\mu) * \dot W^\varepsilon = H * \left(G + \frac{(W^\varepsilon)^-}{\varepsilon}\right).$$

As μ is non negative and of integral 1, we know that $0 \le H*\mu \le 1$; therefore for any $T_0 > 0$,

$$|(H * \mu * \dot W^\varepsilon)(T_0)| \le \left(\int_0^{T_0} |\dot W^\varepsilon|^2 \, dt\right)^{1/2} \sqrt{T_0} \le \|\dot W^\varepsilon\|_{H^{1/4}} \sqrt{T_0}.$$

On the other hand,

$$(\mu * \dot W^\varepsilon)\hat{}(\omega) = (\dot W^\varepsilon)\hat{}(\omega)/\sqrt{1 + i\alpha\omega}$$

and therefore $\mu * \dot W^\varepsilon$ belongs to $H^{3/4}(\mathbb{R})$, with the estimate

$$\|\mu * \dot W^\varepsilon\|_{H^{3/4}} \le C\|\dot W^\varepsilon\|_{H^{1/4}},$$

and by Sobolev injections,

$$\|\mu * \dot W^\varepsilon\|_{L^\infty} \le C\|\dot W^\varepsilon\|_{H^{1/4}}.$$

Thus, we have obtained the estimate

$$\begin{aligned}\int_0^{T_0} \frac{(W^\varepsilon)^-}{\varepsilon} \, dt &\le \|\dot W^\varepsilon\|_{H^{1/4}}\left(C + \sqrt{T_0}\right) + \|G\|_{L^1(0,T_0)} \\ &\le CN(T)\left(1 + \sqrt{T_0}\right).\end{aligned}$$

We set up now a diagonal process; denote by g_n the function associated to $T = n$, and by S_n the corresponding number constructed at Lemma 2. The solution of (23) is now called w_n^ε. For each n, let $(\varepsilon_m)_{m\in\mathbb{N}}$ be a sequence of positive numbers decreasing to 0 as m tends to infinity. There exists a subset J of $\mathbb{N}$ such that as m tends to infinity in J,

$$\left(\dot w_n^{\varepsilon_m}\right)_{m\in J} \rightharpoonup \dot w_n \text{ in } H^{1/4}(\mathbb{R}) \text{ weak}, \tag{27}$$

and

$$\left((w_n^{\varepsilon_m})^-/\varepsilon_m\right)_{m\in J} \rightharpoonup b_n \text{ in } \mathcal{M}^1(\mathbb{R}) \text{ weak } *. \tag{28}$$

Then, in the limit we have

$$\lambda_1 * w_n = g_n + b_n \text{ in the sense of distributions.}$$

It is plain that $b_n \ge 0$. Condition (27) implies in particular that $w_n^{\varepsilon_m}$ tends to w_n uniformly on compact subsets of $\mathbb{R}$; condition (28) implies that $(w_n^{\varepsilon_m})^-$ tends to 0 strongly in $L^1_{\text{loc}}(\mathbb{R})$ and therefore $w_n \ge 0$. If $w_n(x) > 0$, we can find $\gamma > 0$ such that for all large enough m, and all y such that $|y - x| \le \gamma$,

$$w_n^{\varepsilon_m}(y) \ge \frac{1}{2}\, w_n(x),$$

and therefore, the support of $(w_n^{\varepsilon_m})^-$ does not intersect $(x-\gamma, x+\gamma)$; in the limit, the support of b_n does not intersect $(x-\gamma, x+\gamma)$. Thus we have obtained

$$\operatorname{supp} b_n \subset \{w_n = 0\}.$$

Take now $(\varepsilon_m)_{m\in\mathbb{N}}$ to be any sequence decreasing to 0 which will be fixed henceforth. We define $J_1 \subset \mathbb{N}$ as an infinite set such that $(w_1^{\varepsilon_m})_{m\in J_1}$ converges in the sense (27), (28). Given J_n, we take $J_{n+1} \subset J_n$ such that $(w_{n+1}^{\varepsilon_m})_{m\in J_{n+1}}$ converges in the sense (27), (28). Let $\bar{J}$ be the set made out of the first element of J_1, the second of J_2, the n-th of J_n and so forth. Thanks to Remark 4, we have also for all $m \in \mathbb{N}$, and all n, all $p \geq n$: $w_p^{\varepsilon_m}\big|_{[0,n]} = w_n^{\varepsilon_m}\big|_{[0,n]}$. As $(w_p^{\varepsilon_m})_{m\in\bar{J}}$ converges to w_p, we see immediately that for all n and all $p \geq n$: $w_p\big|_{[0,n]} = w_n\big|_{[0,n]}$. In particular, we may define w by $w\big|_{[0,n]} = w_n$, and therefore w is the desired solution.

Remark 5. We know nothing about uniqueness — alas!

5. Solutions whose support is included in a locally finite union of intervals

If ψ is a measure over $\mathbb{R}$ and h a function which is ψ-measurable, we shall write either

$$\int h\psi = \int h(s)\psi(s)$$

or

$$\langle \psi, h \rangle$$

for the integral of h against ψ. If h is ψ-measurable, for all interval I, the function $h1_I$ is also ψ-measurable, and its integral against ψ can be written

$$\int h1_I\psi = \int_I h\psi.$$

We shall keep the traditional notation dt for the Lebesgue measure, though it is not entirely coherent with the above notation; nevertheless, the meaning of these notations will always be clear from the context.

We denote by $\mathcal{M}$ the space of Radon measures on $\mathbb{R}$ and by $\mathcal{M}^1(\mathbb{R})$ the space of bounded measures on $\mathbb{R}$ with norm given by

$$\|\lambda\|_{\mathcal{M}^1} = \big\langle |\lambda|, 1 \big\rangle.$$

We assume from now on that there exists $\phi \in \mathcal{M}^1(\mathbb{R})$ with support in $\mathbb{R}^+$ such that

$$g = \mu * \phi. \tag{29}$$

We recall that a measure ψ on $\mathbb{R}$ has a positive and a negative part, denoted respectively by ψ^+ and ψ^-; the following identities hold:

$$\psi^+ = \max(\psi, 0), \quad \psi^- = -\min(\psi, 0), \quad \psi = \psi^+ - \psi^-, \quad |\psi| = \psi^+ + \psi^-.$$

The first step is to prove an identity for which we need the function:

$$\omega(t) = \frac{1_{(0,\infty)}(t)}{\pi(t+1)\sqrt{t}}. \tag{30}$$

Later, we shall need an integral of ω:

$$\Omega(t) = \int_0^t \omega(s)\,ds = \frac{2}{\pi} 1_{(0,\infty)}(t) \arctan \sqrt{t}. \tag{31}$$

In the following proof and in the remainder of this section, the prime symbol will never denote a derivative; the distributions μ, ρ, μ_1, λ, λ_1 have been defined previously and the reader is referred to the Appendix for formulas.

Lemma 4. *Assume that g satisfies the hypothesis* (29) *and that w satisfies the relation*

$$\lambda_1 * w = g + b; \tag{32}$$

assume moreover that b is a measure belonging to $\mathcal{M}(\mathbb{R}^+)$ and that there exist four numbers $\sigma < \tau < \sigma' < \tau'$ for which w and b satisfy the support conditions:

$$\operatorname{supp} w \subset [\sigma, \tau] \cup [\sigma', \tau'], \tag{33a}$$

$$\operatorname{supp} b \subset [\tau, \sigma'] \cup [\tau', \infty]. \tag{33b}$$

Then the following identities hold:

$$w 1_{[\sigma',\tau']} = \big(H * \nu * \phi'\big) 1_{[\sigma',\tau']}, \tag{34a}$$

where ϕ' is a measure given by

$$\begin{aligned} &\phi' 1_{[\sigma',\infty)} = \phi 1_{[\sigma',\infty)} \\ &+ 1_{[\sigma',\infty)} \int_{[\tau,\sigma']} \exp\Big(-\frac{\cdot - s}{\alpha}\Big) \omega\left(\frac{\cdot - \sigma'}{\sigma' - s}\right) \frac{\psi(s)}{\sigma' - s}. \end{aligned} \tag{34b}$$

Proof. Thanks to the support condition (33a) and to equation (32), w satisfies the relation

$$w 1_{[\sigma,\sigma']} = \big(H * \nu * \phi\big) 1_{[\sigma,\tau]}. \tag{35}$$

Relation (88) enables us to observe that $\big((H * \nu * \phi) 1_{]\tau,\infty)}\big)(t)$ can be decomposed as

$$\begin{aligned} &\Big[\int_{[\sigma,\tau]} \Big(1 - \exp\Big(-\frac{\tau - s}{\alpha}\Big)\Big) \phi(s) \\ &+ \int_{[\sigma,\tau]} \Big(\exp\Big(-\frac{\tau - s}{\alpha}\Big) - \exp\Big(-\frac{t - s}{\alpha}\Big)\Big) \phi(s) \\ &+ \int_{]\tau,t]} \Big(1 - \exp\Big(-\frac{t - s}{\alpha}\Big)\Big) \phi(s)\Big] 1_{]\tau,\infty)}. \end{aligned}$$

But

$$\int_{[\sigma,\tau]} \left(1 - \exp\left(-\frac{\tau - s}{\alpha}\right)\right)\phi(s) = w(\tau) \tag{36}$$

which vanishes, and

$$\begin{aligned}&\int_{[\sigma,\tau]} \left(\exp\left(-\frac{\tau - s}{\alpha}\right) - \exp\left(-\frac{t - s}{\alpha}\right)\right)\phi(s)\\ &= \left(1 - \exp\left(-\frac{t - \tau}{\alpha}\right)\right) \int_{[\sigma,\tau]} \exp\left(-\frac{\tau - s}{\alpha}\right)\phi(s);\end{aligned}$$

therefore

$$\begin{aligned}&(H * \nu * \phi)1_{]\tau,\infty)}\\ &= H * \nu * \left(\delta(\cdot - \tau) \int_{[\sigma,\tau]} \exp\left(-\frac{\tau - s}{\alpha}\right)\phi(s) + \phi 1_{]\tau,\infty)}\right),\end{aligned}$$

which implies that

$$\begin{aligned}w1_{[\sigma,\sigma']} &= (H * \nu * \phi)1_{[\sigma,\tau]}\\ &= H * \nu * \left(\phi 1_{[\sigma,\tau]} - \delta(\cdot - \tau) \int_{[\sigma,\tau]} \exp\left(-\frac{\tau - s}{\alpha}\right)\phi(s)\right).\end{aligned} \tag{37}$$

We infer from this identity that

$$\begin{aligned}b1_{[\tau,\sigma')} &= -\big(\dot\rho * \big(w1_{[\sigma,\tau)}\big) - g\big)1_{[\tau,\sigma')}\\ &= -\left(\rho * \frac{d}{dt}\left(w1_{[\sigma,\tau)}\right) - g\right)1_{[\tau,\sigma')},\end{aligned}$$

and therefore

$$b1_{[\tau,\sigma')} = -\big(\mu * \psi\big)1_{[\tau,\sigma')},$$

with ψ given by

$$\psi = \phi 1_{(\tau,\infty)} + \delta(\cdot - \tau) \exp\left(-\frac{\tau}{\alpha}\right) \int_{[\sigma,\tau]} \exp\left(\frac{s}{\alpha}\right)\phi(s).$$

Define now

$$w' = w1_{[\sigma',\infty)}, \quad g' = \big(g - \lambda_1 * (w1_{[\sigma,\tau]})\big)1_{[\sigma',\infty)}, \quad b' = b1_{[\sigma',\infty)}.$$

Then it is immediate that w' satisfies

$$\lambda_1 * w' = g' + b'.$$

Our purpose now is to identify a measure ϕ' such that

$$\big(\mu_1 * g'\big)1_{[\sigma',\tau']} = \big(H * \nu * \phi'\big)1_{[\sigma',\tau']}.$$

If this identity holds and ϕ' is supported in $[\sigma', \infty)$, we must have over $[\sigma', \tau']$

$$\mu_1 * g' = H * \nu * \phi',$$

or in other words

$$\phi' 1_{[\sigma',\tau']} = \bigl(\rho * g'\bigr) 1_{[\sigma',\tau']}.$$

We proceed now to calculate

$$\rho * g' = \rho * \Bigl(1_{[\sigma',\infty)}\bigl(g - \lambda_1 * \bigl(w 1_{[\sigma,\tau]}\bigr)\bigr)\Bigr)\,;$$

relation (37) implies

$$1_{[\sigma',\infty)}\bigl(g - \lambda_1 * \bigl(w 1_{[\sigma,\tau]}\bigr)\bigr) = 1_{[\sigma',\infty)}(\mu * \psi).$$

Therefore, we need the value of the expression

$$\rho * \bigl(1_{[\sigma',\infty)}(\mu * \psi)\bigr),$$

which is equal to $\rho * \rho * \mu * \bigl(1_{[\sigma',\infty)}(\mu * \psi)\bigr)$, or in other words to

$$\left(1 + \alpha \frac{d}{dt}\right)\Bigl(\mu * \bigl(1_{[\sigma',\infty)}(\mu * \psi)\bigr)\Bigr).$$

But, for $t > \sigma'$, we have the formula

$$\mu * \bigl(1_{[\sigma',\infty)}(\mu * \psi)\bigr)(t) = \int_{[\sigma',t]} \mu(t-s) \left(\int_{[\tau,s]} \mu(s-r)\psi(r)\right) ds.$$

We exchange the order of the integrations and we write $r \vee \sigma' = \max(r, \sigma')$, obtaining thus

$$\mu * \bigl(1_{[\sigma',\infty)}(\mu * \psi)\bigr)(t) = \int_{[\tau,t]} \psi(r) \int_{r\vee\sigma'}^{t} \mu(t-s)\mu(s-r)\, ds.$$

It is plain that

$$\begin{aligned}&\int_{r\vee\sigma'}^{t} \mu(t-s)\mu(s-r)\, ds\\ &= \frac{1}{\pi\alpha} \exp\left(-\frac{t-r}{\alpha}\right)\left(\frac{\pi}{2} - \arcsin\frac{2(r\vee\sigma') - t - r}{t-r}\right),\end{aligned}$$

so that

$$\begin{aligned}\mu * \bigl(1_{[\sigma',\infty)}(\mu * \psi)\bigr)(t) &= \frac{1}{\alpha} \int_{]\sigma',t]} \exp\left(-\frac{t-r}{\alpha}\right)\psi(r)\\ &+ \frac{1}{\alpha\pi} \int_{[\tau,\sigma')} \exp\left(-\frac{t-r}{\alpha}\right)\left(\frac{\pi}{2} - \arcsin\frac{2\sigma' - t - r}{t-r}\right)\psi(r);\end{aligned}$$

now, we have to prove that we can exchange differentiation with respect to t and integration with respect to the measure $\psi(r)$. For $\sigma' \in]r, t[$, the function $t \mapsto \arcsin\bigl((2\sigma' - t - r)/(t - r)\bigr)$ is analytic in t; moreover, its derivative with respect to t is equal to

$$-\frac{\pi}{\sigma' - r}\omega\left(\frac{t - \sigma'}{\sigma' - r}\right),$$

as can be immediately checked; it is a bounded function with respect to r when t is bounded away from σ'; therefore, we have the pointwise equality for $t > \sigma'$

$$\begin{aligned}&\left(1 + \alpha\frac{d}{dt}\right)\int_{[\tau,\sigma')}\frac{\psi(r)}{\pi\alpha}\left(\frac{\pi}{2} - \arcsin\frac{2\sigma' - t - r}{t - r}\right)\exp\left(-\frac{t - r}{\alpha}\right)\\&= \int_{[\tau,\sigma')}\omega\left(\frac{t - \sigma'}{\sigma' - r}\right)\frac{\psi(r)}{\sigma' - r}\exp\left(-\frac{t - r}{\alpha}\right).\end{aligned} \tag{38}$$

The right hand side of (38) is integrable with respect to Lebesgue measure on every compact subinterval of $[\sigma', \infty)$: it is clearly measurable for $t > \sigma'$, and, exchanging the order of integrations, we have the following estimate

$$\int_{[\sigma',\infty]}\left|\int_{[\tau,\sigma']}\omega\left(\frac{t - \sigma'}{\sigma' - r}\right)\frac{\psi(r)}{\sigma' - r}\exp\left(-\frac{t - r}{\alpha}\right)\right| dt \le \int_{[\tau,\sigma')}|\psi(r)|,$$

since $\Omega(+\infty)$ is equal to 1. Then a plain application of general theorems on the differentiation of integral expressions shows indeed that the expression on the right hand side of (34b) is a measure on $[\sigma', \tau']$.

This identity can be made recursive:

Corollary 3. *Assume that g satisfies assumption* (29)*, that w, g and the measure b are related by condition* (32)*; assume moreover that there exist two sequences $(\tau_j)_{j\in J}$, $(\sigma_j)_{j\in J}$ where J is a finite or infinite interval of $\mathbb{N}$ starting at 0, satisfying*

$$0 \le \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \dots, \tag{39}$$

such that w and b satisfy the following support conditions:

$$\operatorname{supp} w \subset \bigcup_{j\in J}[\sigma_j, \tau_j], \quad \operatorname{supp} b \subset \bigcup_{j\in J}[\tau_j, \sigma_{j+1}].$$

Then, if we define

$$\phi_0 = \phi$$

and for all $j \in J \setminus \{\max J\}$

$$\psi_j = \phi_j 1_{[\tau_j,\sigma_{j+1})} + \delta(\cdot - \tau_j)\exp\left(-\frac{\tau_j}{\alpha}\right)\int_{[\sigma_j,\tau_j)}\exp\left(\frac{s}{\alpha}\right)\phi_j, \tag{40}$$

$$\begin{aligned}\phi_{j+1} &= 1_{[\sigma_{j+1},\infty)}\phi_j\\&+ 1_{[\sigma_{j+1},\infty)}\int_{[\tau_j,\sigma_{j+1})}\exp\left(-\frac{\cdot - s}{\alpha}\right)\omega\left(\frac{\cdot - \sigma_{j+1}}{\sigma_{j+1} - s}\right)\frac{\psi_j(s)}{\sigma_{j+1} - s},\end{aligned} \tag{41}$$

then w is given by

$$w1_{[\sigma_j,\tau_j]} = \big(H * \nu * \phi_j\big)1_{[\sigma_j,\tau_j]}. \tag{42}$$

Moreover, the expression for ϕ_j can be rewritten

$$\begin{aligned}\phi_j &= 1_{[\sigma_j,\infty)}\phi_0 \\ &+ 1_{[\sigma_j,\infty)}\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} \exp\Big(-\frac{\cdot - s}{\alpha}\Big)\omega\Big(\frac{\cdot-\sigma_i}{\sigma_i - s}\Big)\frac{\psi_{i-1}(s)}{\sigma_i - s}.\end{aligned} \tag{43}$$

Proof. The proof of this Corollary is a simple induction which is left to the reader.

Let us obtain now estimates on the negative part of the derivative $\dot{w}(t)$, assuming now that w is non negative. These estimates depend on some elementary inequalities relative to Ω and to ω, which were defined at (30) and (31).

Lemma 5. *Let H and H_1 be defined for $\sigma > \sigma' > 0$ and $r > 0$ by*

$$H(r,\sigma,\sigma') = \frac{\sigma-\sigma'}{\alpha}\exp\Big(-\frac{\sigma r}{\alpha}\Big)\big(\Omega(\sigma)-\Omega(\sigma')\big),$$

and

$$\begin{aligned}H_1(r,\sigma,\sigma') =& \frac{1}{\alpha}\int_{\sigma'}^{\sigma}\exp\Big(-\frac{rt}{\alpha}\Big)\Omega(t)\,dt \\ &+ \frac{1}{r}\bigg(\exp\Big(-\frac{r\sigma}{\alpha}\Big) - \exp\Big(-\frac{r\sigma'}{\alpha}\Big)\bigg)\Omega(\sigma').\end{aligned}$$

Then the following inequalities hold:

$$\frac{1}{2}H(r,\sigma,\sigma') \le H_1(r,\sigma,\sigma') \le \exp\Big(r\,\frac{\sigma-\sigma'}{\alpha}\Big)H(r,\sigma,\sigma'). \tag{44}$$

Proof. We rewrite H_1 with the help of an integration by parts:

$$H_1(r,\sigma,\sigma') = \frac{1}{r}\int_{\sigma'}^{\sigma}\omega(t)\bigg(\exp\Big(-\frac{rt}{\alpha}\Big) - \exp\Big(-\frac{r\sigma}{\alpha}\Big)\bigg)\,dt.$$

Then the first inequality in (44) is equivalent to

$$\begin{aligned}\int_{\sigma'}^{\sigma}\omega(t)\bigg[&\frac{K}{\rho}\bigg(\exp\Big(-\frac{rt}{\alpha}\Big) - \exp\Big(-\frac{r\sigma}{\alpha}\Big)\bigg) \\ &- \frac{\sigma-\sigma'}{\alpha}\exp\Big(-\frac{r\sigma}{\alpha}\Big)\bigg]\,dt \ge 0,\end{aligned} \tag{45}$$

with $K \le 2$. We choose $K > 0$ so that the factor of ω in the above integral vanishes at $t = (\sigma+\sigma')/2$; hence

$$K = r(\sigma-\sigma')\bigg(\alpha\exp\Big(r\,\frac{\sigma-\sigma'}{2\alpha}\Big) - \alpha\bigg)^{-1},$$

and clearly $K \leq 2$. The factor of ω in (45) is decreasing function of t which is positive in the first half of the interval $[\sigma', \sigma]$ and negative in its second. As ω is a positive and decreasing function, it is clear that (45) holds.

For the second inequality in (44), we use the inequality $\exp(-rt/\alpha) - \exp(-rt/\alpha) \leq r\alpha^{-}(\sigma - \sigma')\exp(-r\sigma'/\alpha)$ so that

$$
\begin{aligned}
H_1(r,\sigma,\sigma') &\leq \frac{\sigma-\sigma'}{\alpha}\,\exp\biggl(-\frac{r\sigma'}{\alpha}\biggr)\int_{\sigma'}^{\sigma}\omega(t)\,dt \\
&\leq \frac{\sigma-\sigma'}{\alpha}\,\exp\biggl(-\frac{r\sigma'}{\alpha}\biggr)\exp\biggl(-r\,\frac{\sigma-\sigma'}{\alpha}\biggr)\bigl(\Omega(\sigma)-\Omega(\sigma')\bigr) \\
&\leq \exp\biggl(-r\,\frac{\sigma-\sigma'}{\alpha}\biggr)H(r,\sigma,\sigma').
\end{aligned}
$$

In consequence, we have the

Corollary 4. *Let*

$$
F(\sigma,s,i,j) = \frac{1}{\alpha}\,\exp\biggl(-\frac{\sigma-s}{\alpha}\biggr)\biggl(\Omega\biggl(\frac{\sigma-\sigma_i}{\sigma_i-s}\biggr) - \Omega\biggl(\frac{\sigma_j-\sigma_i}{\sigma_i-s}\biggr)\biggr), \tag{46}
$$

and

$$
\begin{aligned}
F_1(\sigma,s,i,j) &= \frac{1}{\alpha(\sigma-\sigma_j)}\int_{[\sigma_j,\sigma)}\exp\biggl(-\frac{t-s}{\alpha}\biggr)\Omega\biggl(\frac{t-\sigma_i}{\sigma_i-s}\biggr)\,dt \\
&+ \frac{1}{\sigma-\sigma_j}\biggl(\exp\biggl(-\frac{\sigma-s}{\alpha}\biggr) - \exp\biggl(-\frac{\sigma_j-s}{\alpha}\biggr)\biggr)\Omega\biggl(\frac{\sigma_j-\sigma_i}{\sigma_i-s}\biggr).
\end{aligned} \tag{47}
$$

Then for all $i \leq j$, all $\sigma \in (\sigma_j, \tau_j]$ and all $s \in [\tau_{i-1}, \sigma_i)$, the following inequalities hold:

$$
\frac{1}{2}\,F(\sigma,s,i,j) \leq F_1(\sigma,s,i,j) \leq \exp\biggl(\frac{\sigma-\sigma_j}{\alpha}\biggr)F(\sigma,s,i,j). \tag{48}
$$

Proof. After multiplying them by

$$
\exp\biggl(\frac{\sigma_i-s}{\alpha}\biggr)\frac{\sigma-\sigma_j}{\sigma_i-s},
$$

the inequalities (48) are equivalent to

$$
\begin{aligned}
&\frac{1}{2}\,H\biggl(\sigma_i-s,\frac{\sigma-\sigma_i}{\sigma_i-s},\frac{\sigma_j-\sigma_j}{\sigma_i-s}\biggr) \leq H_1\biggl(\sigma_i-s,\frac{\sigma-\sigma_i}{\sigma_i-s},\frac{\sigma_j-\sigma_j}{\sigma_i-s}\biggr) \\
&\leq \exp\biggl(\frac{\sigma-\sigma_j}{\alpha}\biggr)H\biggl(\sigma_i-s,\frac{\sigma-\sigma_i}{\sigma_i-s},\frac{\sigma_j-\sigma_j}{\sigma_i-s}\biggr).
\end{aligned}
$$

The conclusion is immediate.

These inequalities will allow us to estimate the negative part of the left derivatives $\dot{w}(\sigma - 0)$, for $\sigma \in (\sigma_j, \sigma]$ as is explained in next Lemma:

Lemma 6. *Assume that the condition of Corollary 3 are satisfied. For all $j \in J$, the following estimate holds if σ belongs to $(\sigma_j, \tau_j]$ and $w(\sigma) \geq 0$:*

$$\big(\dot w(\sigma-0)\big)^- \leq \frac{1}{\alpha}\int_{[\sigma_j,\sigma)} |\phi_0(s)| \tag{49}$$

$$+\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} \left(2\exp\left(\frac{\sigma-\sigma_j}{\alpha}\right)-1\right) G(\sigma,s,i,j)\psi^+_{i-1}(s),$$

where

$$G(\sigma,s,i,j) = \left(\Omega\left(\frac{\sigma-\sigma_i}{\sigma_i-s}\right) - \Omega\left(\frac{\sigma_j-\sigma_i}{\sigma_i-s}\right)\right)\psi^+_{i-1}(s).$$

Proof. In virtue of (42), for $\sigma \in [\sigma_j, \tau_j]$, $w(\sigma)$ can be rewritten as

$$\begin{aligned} w(\sigma) = &\int_{[\sigma_j,\sigma)} \left(1-\exp\left(-\frac{\sigma-s}{\alpha}\right)\right)\phi_0(s) \\ &+\sum_{i=1}^{j}\int_{[\sigma_j,\sigma)}\left[\int_{[\tau_{i-1},\sigma_i)}\left(\exp\left(-\frac{t-s}{\alpha}\right)\right.\right. \\ &\left.\left.-\exp\left(-\frac{\sigma-s}{\alpha}\right)\right)\omega\left(\frac{t-\sigma_i}{\sigma_i-s}\right)dt\right]\frac{\psi_{i-1}(s)}{\sigma_i-s}. \end{aligned}$$

We exchange the order of the integrations in the double integral, we divide by $\sigma-\sigma_j$ and we get with the help of (47):

$$\begin{aligned} \frac{w(\sigma)}{\sigma-\sigma_j} = &\int_{[\sigma_j,\sigma)} \frac{1}{\sigma-\sigma_j}\left(1-\exp\left(-\frac{\sigma-s}{\alpha}\right)\right)\phi_0(s) \\ &+\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F_1(\sigma,s,i,j)\psi_{i-1}(s). \end{aligned}$$

Under the assumption $w(\sigma) \geq 0$, we obtain the following estimate:

$$\begin{aligned} &\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F_1(\sigma,s,i,j)\psi^-_{i-1}(s) \leq \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F_1(\sigma,s,i,j)\psi^+_{i-1}(s) \\ &+\int_{[\sigma_j,\sigma)}\frac{1}{\sigma-\sigma_j}\left(1-\exp\left(-\frac{\sigma-s}{\alpha}\right)\right)\phi_0(s). \end{aligned} \tag{50}$$

According to (46), we consider the expression for $\dot w(\sigma-0)$, which is given by

$$\dot w(\sigma-0) = \frac{1}{\alpha}\int_{[\sigma_j,\sigma)}\exp\left(-\frac{\sigma-s}{\alpha}\right)\phi_0(s) + \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\psi_{i-1}(s).$$

We have now the inequality

$$\big(\dot w(\sigma-0)\big)^- \le \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\psi_{i-1}^-(s) \tag{51}$$
$$-\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\psi_{i-1}^+(s) - \frac{1}{\alpha}\int_{[\sigma_j,\sigma)} \exp\Big(-\frac{\sigma-s}{\alpha}\Big)\phi_0(s).$$

We substitute the inequality (48) into the factors of ψ_{i-1}^- in (51), and then, thanks to (50), we get

$$\big(\dot w(\sigma-0)\big)^- \le \sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} \big(2F_1(\sigma,s,i,j)-F(\sigma,s,i,j)\big)\psi_{i-1}^+(s) \tag{52}$$
$$+\int_{[\sigma_j,\sigma)}\bigg(\frac{2}{\sigma-\sigma_j}\Big(1-\exp\Big(-\frac{\sigma-s}{\alpha}\Big)\Big)-\frac{1}{\alpha}\exp\Big(-\frac{\sigma-s}{\alpha}\Big)\bigg)\phi_0(s).$$

Since the value of the factor of ϕ_0 in (52) is comprised between $-1/\alpha$ and $1/\alpha$ and according to inequality (48), we infer that

$$\begin{aligned}\big(\dot w(\sigma-0)\big)^- \le{}& \frac{1}{\alpha}\int_{[\sigma_j,\sigma)}|\phi_0(s)| \\ &+\sum_{i=1}^{j}\int_{[\tau_{i-1},\sigma_i)} F(\sigma,s,i,j)\bigg(2\exp\Big(\frac{\sigma-\sigma_j}{\alpha}\Big)-1\bigg)\psi_{i-1}^+(s),\end{aligned}$$

which is exactly relation (49).

6. Construction of the approximate solution

The principle of the construction of an approximate solution is the following: we do not know *a priori* whether for a given g satisfying (29), there is a solution of (32) which has the locally finite structure determined by the conditions of Corollary 3, and most probably there is no such solution; however, we shall choose a parameter $n \gg 1$ and construct a slightly different ϕ^n and a solution w^n which has the structure described at Corollary 3 and which approximates well a solution of (32). The construction is recursive.

Let us start by a lemma which tells us that the lower bound of the support of ϕ can be taken equal to the lower bound of the support of w:

Lemma 7. *Let τ_{-1} be the lower bound of the support of ϕ and let w be a solution of* (32)*; assume that the lower bound of the support of the positive part of $H*\rho*\phi$ is $\sigma_0 > \tau_{-1}$; define*

$$\phi_0 = 1_{[\sigma_0,\infty)}\bigg(\phi + \int_{[\tau_{-1},\sigma_0)} \exp\Big(-\frac{\cdot - s}{\alpha}\Big)\omega\Big(\frac{\cdot-\sigma_0}{\sigma_0-s}\Big)\frac{\phi(s)}{\sigma_0-s}\bigg); \tag{53}$$

then the functions

$$w_0 = w1_{[\sigma_0,\infty)}, \quad g_0 = \mu * \phi_0, \quad b_0 = b1_{[\sigma_0,\infty)}$$

solve the problem

$$\lambda_1 * w_0 = g_0 + b_0, \quad 0 \le b_0 \perp w_0 \ge 0. \tag{54}$$

Moreover, the lower bound of the support of ϕ_0 is equal to the lower bound σ_0 of the support of w_0.

Proof. Relation (54) holds as a corollary of the calculation performed at Lemma 3. The definition of σ_0 implies immediately that the lower bound of the support of w_0 is indeed σ_0. There remains to prove that the lower bound of the support of ϕ_0 is also σ_0. Definition (53) implies that this lower bound of the support of ϕ_0 is at least equal to σ_0. If the lower bound of the support of ϕ_0 were strictly larger than the lower bound of the support of w_0, $\mu * \phi_0$ would vanish identically on some interval $[\sigma_0, \sigma_1]$. Then, w_0 would coincide on $[\sigma_0, \sigma_1]$ with $H * \mu * b_0$ which is strictly positive at every point of $(\sigma_0, \sigma_1]$. But w_0 could not be orthogonal to b_0 unless it vanished on that interval, which is a contradiction.

6.1. Initialization of the recursion

Let us describe now formally the construction of the approximate solution. If $\tau_{-1} < \sigma_0$, we start with ϕ_0 defined by (53); the number σ_0 is the lower bound of the support of $(H * \nu * \phi_0)^+$. If $\tau_{-1} = \sigma_0$, we let $\phi_0 = \phi$. We let

$$\widetilde{\phi}_0^n = \phi_0, \quad \sigma_0^n = \sigma_0.$$

Call $\widetilde{\tau}_0^n$ the lower bound of the support of the negative part of $H * \nu * \widetilde{\phi}_0^n$; if $\widetilde{\tau}_0^n > \sigma_0^n$, we let $\phi_0^n = \widetilde{\phi}_0^n$ and $\tau_0^n = \widetilde{\tau}_0^n$.

If the lower bound of the support of the negative part of $H * \nu * \widetilde{\phi}_0^n$ is equal to σ_0^n, this means that we can find, arbitrarily close to σ_0^n, times t for which $H * \nu * \widetilde{\phi}_0^n$ is of either sign. In particular, we can find, arbitrarily close to σ_0^n, times t for which $\big(H * \nu * \widetilde{\phi}_0^n\big)(t)$ vanishes, while $\big(\nu * \widetilde{\phi}_0^n\big)(t-0)$ is less than or equal to 0. We choose any time t in the interval $(\sigma_0^n, \sigma_0^n + 1/n]$ which satisfies all these conditions, we call it τ_0^n, we let

$$a = \int_{[\sigma_0^n, \tau_0^n)} \big(\widetilde{\phi}_0^n\big)^+ \quad \text{and} \quad b = \int_{[\sigma_0^n, \tau_0^n)} \big(\widetilde{\phi}_0^n\big)^-, \tag{55}$$

and we define

$$\phi_0^n = a\delta(\cdot - \sigma_0^n) - b\delta(\cdot - \tau_0^n) + \phi_0^n 1_{[\tau_0^n, \infty)}. \tag{56}$$

It is plain that the lower bound of the support of the negative part of $H * \nu * \phi_0^n$ is equal to τ_0^n, and that the respective mass of the positive and the negative parts of ϕ_0^n on $[\sigma_0^n, \tau_0^n)$ coincide with their counterparts for $\widetilde{\phi}_0^n$, in particular, $b > 0$.

6.2. The recursion

The construction will now be described inductively.

We start from a measure ϕ_j^n such that

$$\inf\operatorname{supp}\phi_j^n = \inf\operatorname{supp}\big(H*\nu*\phi_j^n\big)^+. \tag{57}$$

It is important to observe that if ϕ is a measure, with support bounded on the left, $H*\nu*\phi$ is a continuous function. Let

$$\tau_j^n = \inf\operatorname{supp}\big(H*\nu*\phi_j^n\big)^-.$$

If $\tau_j^n=\infty$, the construction stops; we shall show later that τ_j^n is always strictly larger than σ_j^n for $j\geq 1$; it is already true by construction for $j=0$.

With reference to (40), we define

$$\widetilde{\psi}_j^n = 1_{[\tau_j^n,\infty)}\phi_j^n + \delta(\cdot-\tau_j^n)\exp\left(-\frac{\tau_j^n}{\alpha}\right)\int_{[\sigma_j^n,\tau_j^n)}\exp\left(\frac{s}{\alpha}\right)\phi_j^n(s), \tag{58}$$

and

$$\widetilde{\sigma}_{j+1}^n = \inf\operatorname{supp}\big(\widetilde{\psi}_j^n\big)^+. \tag{59}$$

We have two cases to consider.

Case 1. If $\widetilde{\sigma}_{j+1}^n=\infty$, the construction stops.

Case 2. If $\infty>\widetilde{\sigma}_{j+1}^n\geq\tau_j^n$, we choose any time σ_{j+1}^n which does not carry an atom of $\widetilde{\psi}_j^n$ and which satisfies

$$\max\left(\widetilde{\sigma}_{j+1}^n,\tau_j^n+\frac{1}{2n}\right)\leq\sigma_{j+1}^n\leq\max\left(\widetilde{\sigma}_{j+1}^n,\tau_j^n+\frac{1}{2n}\right)+\frac{1}{2n}, \tag{60}$$

and we define

$$\begin{aligned}\psi_j^n &= -1_{[\tau_j^n,\sigma_{j+1}^n)}\big(\widetilde{\psi}_j^n\big)^- \\ &\quad+\delta(\cdot-\sigma_{j+1}^n)\int_{[\tau_j^n,\sigma_{j+1}^n)}\big(\widetilde{\psi}_j^n\big)^+ + 1_{[\sigma_{j+1}^n,\infty)}\widetilde{\psi}_j^n.\end{aligned} \tag{61}$$

Since the lower bound of the support of $\big(\widetilde{\psi}_j^n\big)^+$ is equal to $\widetilde{\sigma}_{j+1}^n$, the mass of the atom of ψ_j^n at σ_{j+1}^n is strictly positive.

The last step of the construction is the construction of ϕ_{j+1}^n; with reference to (41), it is given by

$$\begin{aligned}\phi_{j+1}^n &= 1_{[\sigma_{j+1}^n,\infty)}\psi_j^n \\ &\quad+1_{[\sigma_{j+1}^n,\infty)}\int_{[\tau_j^n,\sigma_{j+1}^n)}\exp\left(-\frac{\cdot-s}{\alpha}\right)\omega\left(\frac{\cdot-\sigma_{j+1}^n}{\sigma_{j+1}^n-s}\right)\frac{\psi_j^n(s)}{\sigma_{j+1}^n-s}.\end{aligned} \tag{62}$$

In order to validate this process, we have to prove the

Lemma 8. *For all $j \geq 0$, there exists a non empty interval $(\sigma^n_{j+1}, \tau^n_{j+1})$ on which $H * \nu * \phi^n_{j+1}$ is strictly positive.*

Proof. Write for simplicity

$$\tau = \tau^n_j, \quad \sigma' = \sigma^n_{j+1}, \quad \psi = \psi^n_j,$$
$$\phi' = 1_{[\sigma',\infty)}\left(\psi + \int_{[\tau,\sigma')} \exp\left(-\frac{\cdot - s}{\alpha}\right)\omega\left(\frac{\cdot - \sigma'}{\sigma' - s}\right)\frac{\psi(s)}{\sigma' - s}\right).$$

By construction, $\psi\big|_{[\tau,\sigma')} \leq 0$, and ψ has a positive atom at σ', whose measure will be denoted by $\beta > 0$. We have the following identity for $t > \sigma'$:

$$\begin{aligned}
&(H * \nu * \phi')(t) = \left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right)\beta \\
&+ \int_{(\sigma',t)} \left(1 - \exp\left(-\frac{t-s}{\alpha}\right)\right)\psi(s) \\
&+ \int_{[\sigma',t)} \int_{[\tau,\sigma')} \exp\left(-\frac{s-r}{\alpha}\right)\omega\left(\frac{s-\sigma'}{\sigma'-r}\right)\frac{\psi(r)}{\sigma'-r}\,ds \\
&- \int_{(\sigma',t)} \exp\left(-\frac{t-s}{\alpha}\right)\int_{[\tau,\sigma')} \exp\left(-\frac{s-r}{\alpha}\right)\omega\left(\frac{s-\sigma'}{\sigma'-r}\right)\frac{\psi(r)}{\sigma'-r}\,ds.
\end{aligned}$$

We can find $t_1 > \sigma'$ such that

$$\int_{(\sigma',t_1)} |\psi(s)| \leq \frac{\beta}{4};$$

then, for $t \in [\sigma', t_1]$, we will have

$$\left|\int_{(\sigma',t)} \left(1 - \exp\left(-\frac{t-s}{\alpha}\right)\right)\psi(s)\right| \leq \left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right)\frac{\beta}{4};$$

we cut the interval $[\tau, \sigma')$ into two pieces, $[\tau, \sigma' - \varepsilon)$ and $[\sigma' - \varepsilon, \sigma')$, thus, we have the estimate

$$\begin{aligned}
&\left|\int_{[\sigma',t)} \left(1 - \exp\left(-\frac{t-s}{\alpha}\right)\right)\int_{[\tau,\sigma')} \exp\left(-\frac{s-r}{\alpha}\right)\omega\left(\frac{s-\sigma'}{\sigma'-r}\right)\frac{\psi(r)}{\sigma'-r}\,ds\right| \\
&\leq \left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right)\left(\int_{[\tau,\sigma'-\varepsilon)} |\psi(r)| + \int_{[\sigma'-\varepsilon,\sigma')} |\psi(r)|\right).
\end{aligned}$$

We choose ε so small that

$$\int_{[\sigma'-\varepsilon,\sigma')} |\psi(r)| \leq \frac{\beta}{4};$$

then

$$\left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right)\int_{[\sigma'-\varepsilon,\sigma')} |\psi(r)| \leq \left(1 - \exp\left(-\frac{t-\sigma'}{\alpha}\right)\right)\frac{\beta}{4};$$

we fix ε and we choose $t_2 \in (\sigma', t_1]$ so small that

$$\left(1-\exp\left(-\frac{t_2-\sigma'}{\alpha}\right)\right)\int_{[\tau,\sigma'-\varepsilon)}|\psi(r)| \le \left(1-\exp\left(-\frac{t_2-\sigma'}{\alpha}\right)\right)\frac{\beta}{4};$$

then, for $t \in (\sigma', t_2)$, $(H * \nu * \phi')(t) \ge \bigl(1-\exp(-(t-\sigma')/\alpha)\bigr)\beta/4 > 0$ and the lemma is proved.

6.3. Mass and order properties of the measures ϕ_j^n and ψ_j^n

The measures defined in this part have some important order properties, which are summarized in next lemma:

Lemma 9. *The following inequalities hold:*

$$\bigl(\widetilde{\psi}_j^n\bigr)^+ \le \bigl(\phi_j^n\bigr)^+, \tag{63}$$

$$\phi_{j+1}^n \le 1_{[\sigma_{j+1}^n,\infty)}\psi_j^n, \tag{64}$$

$$\int|\phi_{j+1}^n| \le \int|\psi_j^n| \le \int|\phi_j^n|, \tag{65}$$

$$\int_{[\tau_j^n,\sigma_{j+1}^n)}\bigl(\widetilde{\psi}_j^n\bigr)^+ \le \int_{[\tau_j^n,\sigma_{j+1}^n)}\phi^+, \tag{66}$$

$$\bigl(\phi_{j+1}^n\bigr)^+ \le 1_{[\sigma_{j+1}^n,\infty)}\bigl(\phi_j^n\bigr)^+ + \delta(\cdot-\sigma_{j+1}^n)\int_{[\tau_j^n,\sigma_{j+1}^n)}\bigl(\phi_j^n\bigr)^+. \tag{67}$$

Proof. For relation (63), we just take the definition (61) of $\widetilde{\psi}_j^n$, observing that the quantity

$$\frac{1}{\alpha}\exp\left(-\frac{\tau_j^n}{\alpha}\right)\int_{[\sigma_j^n,\tau_j^n)}\exp\left(\frac{s}{\alpha}\right)\phi_j^n(s)$$

is simply equal to the velocity $\dot{w}(\tau_j^n-0)$ and hence less than or equal to 0. Relation (64) can be read on formula (62) with upper indices n thrown in and the sign condition $\psi_j^n \le 0$ over $[\tau_j^n, \sigma_{j+1}^n)$ coming from the construction. In order to obtain estimate (65), we integrate

$$\int_{[\tau_j^n,\sigma_{j+1}^n)}\exp\left(-\frac{t-s}{\alpha}\right)\omega\left(\frac{t-\sigma_{j+1}^n}{\sigma_{j+1}^n-s}\right)\frac{|\psi_j^n(s)|}{\sigma_{j+1}^n-s} \tag{68}$$

over $[\sigma_{j+1}^n,\infty)$, we exchange the order of the integrations, we find that

$$\int_{[\sigma_{j+1}^n,\infty)}\left(\int_{[\tau_j^n,\sigma_{j+1}^n)}\exp\left(-\frac{t-s}{\alpha}\right)\omega\left(\frac{t-\sigma_{j+1}^n}{\sigma_{j+1}^n-s}\right)\frac{|\psi_j^n(s)|}{\sigma_{j+1}^n-s}\right)dt$$
$$\le \int_{[\tau_j^n,\sigma_{j+1}^n)}\varOmega\left(\frac{t-\sigma_{j+1}^n}{\sigma_{j+1}^n-s}\right)\Bigg|_{t=\sigma_{j+1}^n}^{t=\infty}\bigl|\psi_j^n(s)\bigr| = \int_{[\tau_j^n,\sigma_{j+1}^n)}\bigl|\psi_j^n(s)\bigr|,$$

where we have estimated $\exp(-(t-s)/\alpha)$ by 1; the conclusion is clear. Relation (66) is a consequence of (58) and of the relations (63) and (64) with an induction on j; finally (67) is an immediate consequence of (61)-(63).

6.4. The approximate problem

We define now

$$\bar{\phi}^n = \phi + \left(\phi_0^n - \phi\right) 1_{[\sigma_0^n, \tau_0^n)} + \sum_{j \geq 0} \left(\delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n, \sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n\right)^+ - 1_{[\tau_j^n, \sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n\right)^+ \right). \tag{69}$$

The above sum must be understood as extended to the set of indices for which the recursion is defined: it is a finite sum if one of the τ_j^n or σ_j^n is infinite. As the length of the intervals $[\tau_j^n, \sigma_{j+1}^n)$ is at least equal to $1/(2n)$, we know that this sum is locally finite.

Theorem 4. *Let $g^n = \mu * \bar{\phi}^n$. Then, the function w^n given by*

$$w^n = \sum_{j \geq 0} 1_{[\sigma_j^n, \tau_j^n]} \left(H * \nu * \phi_j^n \right) \tag{70}$$

is continuous, non negative, and it is a solution of

$$\lambda_1 * w^n = g^n + b^n, \tag{71a}$$

$$w^n \geq 0 \tag{71b}$$

$$b^n \geq 0 \tag{71c}$$

$$\left\langle w^n, b^n \right\rangle = 0. \tag{71d}$$

Proof. The function w^n is non negative on the intervals $[\sigma_j^n, \tau_j^n]$ by construction, i.e. (71b) holds.

The definition of g^n comes from the fact that we modify the data in each interval $[\tau_j^n, \sigma_{j+1}^n)$, according to (61); in particular,

$$\psi_j^n - \widetilde{\psi}_j^n = \delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n, \sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n\right)^+ - 1_{[\tau_j^n, \sigma_{j+1}^n)} \left(\widetilde{\psi}_j^n\right)^+; \tag{72}$$

therefore w^n satisfies (71a), with b^n given by

$$b^n = - \sum_{j \geq 0} 1_{[\tau_j^n, \sigma_{j+1}^n)} \left(\mu * \psi_j^n \right),$$

and we just have to check (71c): but it is a result of the construction performed in Subsection 6.2 that the measure ψ_j^n is negative on $[\tau_j^n, \sigma_{j+1}^n)$. Moreover, $\mu * \psi_j^n$ belongs to the space L^p_{loc} for all $p \in [1, 2)$, and therefore, the duality product $\langle w^n, b^n \rangle$ makes sense locally as a Lebesgue integral of the product of two functions, and therefore (71d) is true.

6.5. Estimates on the approximation

We prove first a result on the convergence of the measure $\bar{\phi}^n$:

Lemma 10. *The norm of the measure $\bar{\phi}^n$ is bounded by $3\|\phi\|_{\mathcal{M}^1}$ and converges weakly $*$ to ϕ as n tends to infinity.*

Proof. We estimate the norm in $\mathcal{M}^1$ of all the terms in (69): thanks to (55) and (56), we have the estimate

$$\|(\phi_0^n - \phi)1_{[\sigma_0^n, \tau_0^n)}\|_{\mathcal{M}^1} \leq 2\|\phi 1_{[\sigma_0^n, \tau_0^n)}\|_{\mathcal{M}^1}.$$

Similarly thanks to (72)

$$\|(\psi_j^n - \widetilde{\psi}_j^n)1_{[\tau_j^n, \sigma_{j+1}^n)}\|_{\mathcal{M}^1} \leq 2\int_{[\tau_j^n, \sigma_{j+1}^n)} \big(\widetilde{\psi}_j^n\big)^+$$

and in virtue of (63) and (67)

$$\|(\psi_j^n - \widetilde{\psi}_j^n)1_{[\tau_j^n, \sigma_{j+1}^n)}\|_{\mathcal{M}^1} \leq 2\int_{[\tau_j^n, \sigma_{j+1}^n)} \phi^+;$$

therefore, we find the inequality

$$\|\bar{\phi}^n\|_{\mathcal{M}^1} \leq 3\|\phi\|_{\mathcal{M}^1}. \tag{73}$$

Let h be a continuous function on $\mathbb{R}$ with compact support and let ρ be its modulus of continuity: ρ is a continuous increasing function from $\mathbb{R}^+$ to itself such that

$$\forall x, y \in \mathbb{R}, \quad |h(x) - h(y)| \leq \rho|x - y|;$$

moreover, ρ vanishes at 0. According to (61), we rewrite as follows the duality product between h and $\psi_j^n - \widetilde{\psi}_j^n$:

$$\langle \psi_j^n - \widetilde{\psi}_j^n, h\rangle = \int_{[\tau_j^n, \sigma_{j+1}^n)} \big(h(\sigma_{j+1}^n) - h(s)\big)\big(\widetilde{\psi}_j^n(s)\big)^+;$$

but we have the straightforward estimate

$$\left|\int_{[\tau_j^n, \sigma_{j+1}^n)} \big(h(\sigma_{j+1}^n) - h(s)\big)\big(\widetilde{\psi}_j^n(s)\big)^+\right| \leq \rho\big(\sigma_{j+1}^n - \widetilde{\sigma}_{j+1}^n\big)\int_{[\tau_j^n, \sigma_{j+1}^n)} \big(\widetilde{\psi}_j^n(s)\big)^+;$$

thanks to the choice (60) of σ_{j+1}^n, we know that

$$\sigma_{j+1}^n - \widetilde{\sigma}_{j+1}^n \leq \frac{1}{n},$$

and therefore in virtue of (61) and (66),

$$\big|\langle \psi_j^n - \widetilde{\psi}_j^n, h\rangle\big| \leq \frac{\rho}{n}\int_{[\tau_j^n, \sigma_{j+1}^n)} \phi^+;$$

we have an analogous estimate for the initial term, and the lemma is proved.

The next result gives an estimate of $g^n - g$:

Lemma 11. *For all $p \in [1, 2)$, there exists a constant C such that the following inequality holds:*

$$\|g^n - g\|_{L^p} \le C n^{1/2-1/p} \exp\left(\frac{1}{2\alpha n}\right)\left(\int_{[\sigma_0^n, \tau_0^n)} |\phi| + \sum_{j\ge 0} \int_{[\tau_j^n, \sigma_{j+1}^n)} (\phi)^+\right).$$

Proof. The difference $g^n - g$ a sum of terms of the form

$$\mu * \left(\delta(\cdot - \sigma_{j+1}^n) \int_{[\tau_j^n, \sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+ - 1_{[\tau_j^n, \sigma_{j+1}^n)} (\widetilde{\psi}_j^n)^+\right), \tag{74}$$

and possibly of an initial term given by

$$\mu * \left(\delta(\cdot - \sigma_0^n) \int_{[\sigma_0^n, \tau_0^n)} \phi^+ + \delta(\cdot - \bar{\tau}) \int_{[\sigma_0^n, \tau_0^n)} \phi^- - \phi 1_{[\sigma_0^n, \tau_0^n)}\right). \tag{75}$$

Let us start by the terms of the form (74): they can be rewritten as

$$\int_{[\tau_j^n, \sigma_{j+1}^n)} \big(\mu(t - \sigma_{j+1}^n) - \mu(t - s)\big)\big(\widetilde{\psi}_j^n(s)\big)^+,$$

which we estimate in $L^p(0, T)$ by appealing to Minkowski inequality for integrals:

$$\begin{aligned}
&\left(\int_{\tau_j^n}^T \left|\int_{[\tau_j^n, \sigma_{j+1}^n)} \big(\mu(t - \sigma_{j+1}^n) - \mu(t - s)\big)\big(\widetilde{\psi}_j^n(s)\big)^+\right|^p dt\right)^{1/p} \\
&\le \int_{[\tau_j^n, \sigma_{j+1}^n)} \big(\widetilde{\psi}_j^n(s)\big)^+ \left(\int_{\tau_j^n}^T \big|\mu(t - \sigma_{j+1}^n) - \mu(t - s)\big|^p \, dt\right)^{1/p}.
\end{aligned}$$

But we observe the following inequality

$$\begin{aligned}
&\int_{[\tau_j^n, \sigma_{j+1}^n)} \big|\mu(t - \sigma_{j+1}^n) - \mu(t - s)\big|^p \, dt \\
&\le \frac{(\sigma_{j+1}^n - \tau_j^n)^{1-p/2}}{(\alpha\pi)^{p/2}(1 - p/2)} \exp\left(p\, \frac{\sigma_{j+1}^n + \tau_j^n}{\alpha}\right);
\end{aligned}$$

We cut the interval $[\sigma_{j+1}^n, T]$ into two pieces, one from σ_{j+1}^n to $\sigma_{j+1}^n + \varepsilon$ and the second one on the remainder of the interval, and we will adjust ε so as to obtain the best possible result. On the first piece, we have the estimate

$$\begin{aligned}
&\int_{\sigma_{j+1}^n}^{\sigma_{j+1}^n + \varepsilon} \big|\mu(t - \sigma_{j+1}^n) - \mu(t - s)\big|^p \, dt \\
&\le \int_{\sigma_{j+1}^n}^{\sigma_{j+1}^n + \varepsilon} \big|\mu(t - \sigma_{j+1}^n)\big|^p \, dt = \frac{\varepsilon^{1-p/2}}{(\alpha\pi)^{p/2}(1 - p/2)}.
\end{aligned}$$

On the third piece, we use the derivative of μ, this derivative is equal to $-(1/(2t^{3/2}\sqrt{\alpha\pi}) + 1/(\alpha t^{1/2}\sqrt{\alpha\pi}))\exp(-t/\alpha)$ for $t > 0$, and we obtain the estimate

$$\begin{aligned}
&\int_{\sigma_{j+1}^n+\varepsilon}^{T} \left|\mu(t-\sigma_{j+1}^n) - \mu(t-s)\right|^p \, dt \\
&\le \int_{\sigma_{j+1}^n+\varepsilon}^{T} \left|\frac{(s-\sigma_{j+1}^n)(\alpha + 2(t-\sigma_{j+1}^n))}{2\alpha\sqrt{\alpha\pi}\,(t-\sigma_{j+1}^n)^{3/2}}\right|^p \exp\Bigl(-\frac{p\varepsilon}{\alpha}\Bigr)\, dt.
\end{aligned}$$

Since $s \in [\tau_j^n, \sigma_{j+1}^n)$, we estimate this integral by

$$\frac{(\sigma_{j+1}^n - \tau_j^n)^p(\alpha + 2(T-\sigma_{j+1}^n))^p}{\varepsilon^{3p/2-1}(3p/2-1)(2\alpha)^p(\alpha\pi)^{p/2}} \exp\Bigl(-\frac{p\varepsilon}{\alpha}\Bigr).$$

We choose $\varepsilon = \sigma_{j+1}^n - \tau_j^n$, and we see that there is a constant C such that for all n and all j:

$$\begin{aligned}
&\Bigl(\int_{\tau_j^n}^{T} \left|\mu(t-\sigma_{j+1}^n) - \mu(t-s)\right|^p\Bigr)^{1/p} dt \\
&\le C\bigl(\sigma_{j+1}^n - \tau_j^n\bigr)^{1/p-1/2} \exp\Bigl(\frac{\sigma_{j+1}^n - \tau_j^n}{\alpha}\Bigr).
\end{aligned}$$

But $\sigma_{j+1}^n - \tau_j^n$ is at most equal to $1/(2n)$, and we obtain finally the estimate

$$\|\mu * (\widetilde{\psi}_j^n - \psi_j^n)\|_{L^p} \le C n^{1/2-1/p} \exp\Bigl(\frac{1}{2\alpha n}\Bigr) \int_{[\tau_j^n, \sigma_{j+1}^n)} \bigl(\widetilde{\psi}_j^n\bigr)^+.$$

Inequality (66) enables us to estimate the integral of $\bigl(\psi_j^n\bigr)^+$ over $[\tau_j^n, \sigma_{j+1}^n]$, by the integral of ϕ^+ over the same interval. Let us pass now to estimates on (75). Arguing as above, we observe that

$$\Bigl(\int_{[\sigma_0^n, T]} \left|\mu * \bigl(\phi_0^n - \phi\bigr)\right|^p \, dt\Bigr)^{1/p} \le C n^{1/2-1/p} \exp\Bigl(\frac{1}{2\alpha n}\Bigr) \int_{[\sigma_0^n, \tau_0^n]} |\phi|.$$

The assertion of the lemma is proved.

Let us obtain now some estimates on w^n and its derivatives:

Lemma 12. *The time derivative $\dot{w}^n$ belongs to $L^\infty(\mathbb{R})$ and the bound on w^n is independent of n; the measure b^n is a function which is locally integrable on $\mathbb{R}^+$, with bound independent of n.*

Proof. By the definition (70) of w^n on the interval $[\sigma_j^n, \tau_j^n)$ we have the estimate

$$\Bigl\|\dot{w}^n\big|_{[\sigma_j^n, \tau_j^n)}\Bigr\|_{L^\infty} \le \frac{1}{\alpha} \int_{[\sigma_j^n, \tau_j^n)} |\phi_j^n|,$$

and thanks to (65), we obtain immediately the estimate

$$\|\dot{w}^n\|_{L^\infty} \leq \frac{1}{\alpha}\|\phi\|_{\mathcal{M}^1}. \tag{76}$$

Since $\lambda_1 = \dot{\rho}$, the convolution of (71a) with H yields the identity

$$\rho * w^n = H * (g^n + b^n). \tag{77}$$

But $\rho = \mu + \alpha\dot{\mu}$, hence (77) can be rewritten

$$\big(\mu * w^n + \alpha\mu * \dot{w}^n\big)(t+0) = \int_{[0,t]} g^n(s)\, ds + \int_{[0,t]} b^n(s)\, ds.$$

Relation (76) implies that the left hand side of the above relation is bounded by $(1+t)\|\phi\|_{\mathcal{M}^1}/\alpha$; similarly, we use (73) to find that the integral of g^n over $[0,T]$ is bounded by $3\|\phi\|_{\mathcal{M}^1}$. This shows the desired estimate.

7. Passage to the limit

We start by an easy convergence result:

Lemma 13. *There exists a subsequence, still denoted by w^n which has the following convergence properties:*

w^n converges to w uniformly on compact sets;

$\dot{w}^n$ converges to $\dot{w}$ in $L^\infty(0,T)$ weakly $*$ for all positive t;

b^n converges to b in $\mathcal{M}^1(0,T)$ weakly $*$ for all positive t.

Moreover, w and b satisfy (32)*.*

Proof. The possibility of extracting a subsequence is an immediate consequence of Lemma 12. It is clear that w and b are non negative; the duality product $\langle w^n, b^n\rangle$ converges to its limit which is $\langle w, b\rangle$. Thus we have constructed a solution of (18).

We infer from this result an important information on the measure b:

Lemma 14. *The measure b has no atoms.*

Proof. The derivative $\dot{w}$ is equal in the sense of distributions to $\nu*\phi+\mu*b$. As ϕ is a measure, $\nu*\phi$ is locally essentially bounded; since $\dot{w}$ is essentially bounded, this means that $\mu * b$ is essentially bounded. Denoting by b^a its atomic part, we infer from the positivity of b that $\mu * b^a$ is also essentially bounded; but this means clearly that b^a must vanish, which concludes the proof.

Now comes the essential result of this article:

Theorem 5. *Let N be the set of atoms of ϕ: $1_N\phi$ is a purely atomic measure and $(1-1_N)\phi$ is a diffuse measure. For any solution of* (18) *defined by the above convergence process, let U be the open set*

$$U = \{t \in \mathbb{R} : w(t) > 0\}, \tag{78}$$

which is a countable union of connected components:

$$U = \bigcup_{\kappa \in K} (\sigma_\kappa, \tau_\kappa).$$

The set composed of all the points σ_κ and τ_κ is a countable set called N_1. Then for all $t \notin N \cup N_1 \cup U$, w is differentiable at t and its derivative vanishes.

Proof. Assume thus that $w(t)$ vanishes, that t is not an end point σ_κ or τ_κ, and that t does not belong to N. We have to deal with derivatives on the left and on the right, and we use different strategies for each of them. If there exist respectively a non empty interval $[t, t+\varepsilon)$ or $(t-\varepsilon, t]$ included in the complement of U, it is clear that the derivative on the right or on the left of w at t vanishes.

Assume that there is no interval of the form $[t, t+\varepsilon)$ included in the complement of U. Since t is not an end point σ_κ or τ_κ, this means that there exists decreasing subsequences $\sigma_{\kappa(m)}$ and $\tau_{\kappa(m)}$ converging to t.

If it is not true that the right derivative of w at t vanishes, we can find a subsequence t_m decreasing to t and a number $\beta > 0$ such that

$$\frac{w(t_m)}{t_m - t} \geq \beta > 0, \tag{79}$$

and in particular, for all m, $w(t_m)$ is strictly positive. Possibly extracting subsequences, we may assume that the following situation holds for all large enough m:

$$\sigma_{\kappa(m)} < t_m < \tau_{\kappa(m)} < \sigma_{\kappa(m-1)}$$

For all m, there exists a number $n(m)$ such that for all $n \geq n(m)$, $w^n(t_m)$ is strictly positive, thanks to the uniform convergence of w^n to its limit. Let $(\sigma^n_{j(n)}, \tau^n_{j(n)})$ be the connected component of t_m in the open set

$$U^n = \{t \in \mathbb{R} : w^n(t) > 0\}.$$

We infer from (42) the identity

$$w^n(t_m) = \int_{[\sigma^n_{j(n)}, t_m)} \left(1 - \exp\left(-\frac{t_m - s}{\alpha}\right)\right) \phi^n_{j(n)}(s).$$

At this point, we observe that relation (67) implies

$$\int_{[\sigma^n_{j(n)}, t_m)} \phi^n_{j(n)} \leq \int_{[\tau^n_{j(n)-1}, t_m)} \phi^+;$$

therefore, we have the estimate

$$\frac{w^n(t_m) - w(t)}{t_m - t} \le \frac{1}{t_m - t} \int_{[\tau^n_{j(n)-1}, t_m)} \phi^+. \tag{80}$$

If the inferior limit of $\tau^n_{j(n)-1}$ as n tends to infinity is $\bar\sigma < t$, this means that there exists a subsequence of w^n, still denoted by w^n, such that for n large enough

$$\forall s \in [(t+\bar\sigma)/2, t_m], \quad w^n(s) > 0,$$

and hence the support of b does not meet $((t+\bar\sigma)/2, t_m)$; in particular, on this interval, $\dot w$ is of bounded variation, and as t does not carry an atom of ϕ, $\dot w$ is continuous at t; the sign condition implies then that $\dot w(t)$ vanishes. If the inferior limit of $\tau^n_{j(n)-1}$ is at least equal to t, then we estimate for n large enough the right hand side of (80) by

$$\frac{1}{t_m - t} \int_{(t-\varepsilon, t_m)} \phi^+(s),$$

with ε an arbitrary positive number. We pass to the limit in n and then in ε and we obtain the inequality

$$\frac{w(t_m) - w(t)}{t_m - t} \le \int_{[t, t_m)} \phi^+(s); \tag{81}$$

since t is not an atom of ϕ, we may choose m so large that the right hand side of (81) is less than or equal to $\beta/2$, contradicting thus the assumption (79). Therefore, the derivative of w on the right at t exists and vanishes.

Let us turn now to the other side of the estimates. This is where estimate (49) will prove useful. Assume then that there is no interval of the form $(t-\varepsilon, t]$ included in the complement of U. Since t is not an end point σ_κ or τ_κ, this means that there exists increasing subsequences $\sigma_{\kappa(m)}$ and $\tau_{\kappa(m)}$ converging to t.

If it is not true that the derivative on the left of w at t vanishes, we can find a number $\beta > 0$ and a sequence of times t_m increasing to t such that

$$\frac{w(t_m) - w(t)}{t - t_m} \ge \beta. \tag{82}$$

We assume also that for all m, b charges a neighborhood of $\sigma_{\kappa(m)}$ and a neighborhood of $\tau_{\kappa(m)}$: if this were not true, we could always take a smaller $\sigma_{\kappa(m)}$ and a larger $\tau_{\kappa(m)}$.

As t does not carry an atom of ϕ, we may choose m and $\varepsilon > 0$ such that

$$\frac{1}{\alpha} \int_{[\sigma_{\kappa(m)}-\varepsilon, \tau_{\kappa(m)}]} |\phi| \le \frac{\beta}{4};$$

as above, we denote by $(\sigma^n_{j(n)}, \tau^n_{j(n)})$ the connected component of t_m in U^n; relation (82) implies that for all large enough n,

$$w^n(t_m) - w^n(\tau^n_{j(n)}) \ge \frac{3\beta}{4}(t - t_m),$$

and therefore

$$w^n(t_m) - w^n(\tau^n_{j(n)}) \geq \frac{3\beta}{4}(t - \tau^n_{j(n)});$$

therefore, there exists in $[\sigma^n_{j(n)}, \tau^n_{j(n)}]$ a set M of positive measure on which the derivative of w is negative enough:

$$\dot{w}^n(\sigma) \leq -\frac{3\beta}{4}, \quad \forall \sigma \in M.$$

We apply inequality (49), observing that the terms ψ^n_i are all non positive on $[\tau^n_{i-1}, \sigma^n_i)$ and therefore

$$\left(\dot{w}^n(\sigma - 0)\right)^- \leq \frac{1}{\alpha} \int_{[\sigma^n_{j(n)}, \sigma]} |\phi(s)|,$$

which is at most equal to

$$\frac{1}{\alpha} \int_{[\sigma_{\kappa(m)} - \varepsilon, \tau_{\kappa(m)}]} |\phi|,$$

since $\sigma^n_{j(n)}$ tends to $\sigma_{\kappa(m)}$ under the assumption that b charges a neighborhood of $\sigma_{\kappa(m)}$. Therefore, on the set M, we have the estimate

$$\left(\dot{w}^n(\sigma)\right)^- \leq \frac{\beta}{4},$$

which is clearly a contradiction.

We have another expression for the derivative of w in the sense of distributions:

$$\dot{w} = \mu * (g + b).$$

Under assumption (29), this relation can be rewritten

$$\dot{w} = \nu * \phi + \mu * b \text{ in the sense of distributions.} \tag{83}$$

Except at the atoms of ϕ, $\nu * \phi$ is a continuous function. On the other hand, $\mu * b$ is defined everywhere on $\mathbb{R}$, as proved in next Lemma:

Lemma 15. *If $\mu * b$ is locally essentially bounded, the function $\mu * b$ is defined for all $t \in \mathbb{R}$, lower semi-continuous and locally bounded on $\mathbb{R}$.*

Proof. The function $\mu(t - \cdot)$ is continuous except at 0; therefore, it is b-measurable, since b has no atoms, thanks to Lemma 14. Therefore, the expression

$$\int \mu(t - s) b(s)$$

is defined as an element of $[0, \infty]$ and can be obtained as a limit of integral of continuous functions with respect to the measure b. Take for instance

$$\rho_h(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq h, \\ (t - h)/h & \text{if } h \leq t \leq 2h, \\ 1 & \text{if } 2h \leq t; \end{cases}$$

then it is plain that

$$\big(\mu * b\big)(t) = \lim_{h\downarrow 0} \int \mu(t-s)\rho_h(t-s)b(s).$$

Moreover, $\mu * b$ is lower semi-continuous: if t_n is a sequence converging to t, the inferior limit of $\mu(t_n - \cdot)$ is greater than or equal to $\mu(t-\cdot)$ and thanks to Fatou's lemma

$$\liminf \int \mu(t_n - s)b(s) \geq \int \mu(t-s)b(s).$$

Finally, the function $\mu * b$ is locally bounded if it is locally essentially bounded: suppose indeed that there exists a time t such that

$$\int \mu(t-s)b(s) = \infty.$$

Let M be the essential bound of $\mu * b$ over $[0,T]$; for all $M' > M$ there exists $\varepsilon > 0$ such that

$$\int_0^{t-\varepsilon} \mu(t-s)b(s) \geq M'.$$

Then, for all $t' \in (t-\varepsilon, t]$, and for all $s \in [0, t-\varepsilon]$ we have the inequality

$$\mu(t'-s) \geq \mu(t-s)$$

which we integrate over $[0, t-\varepsilon]$ with respect to b, obtaining thus

$$\int_{[0,t-\varepsilon]} \mu(t'-s)b(s) \geq \int_{[0,t-\varepsilon]} \mu(t-s)b(s),$$

so that there is a set of measure ε on which $\mu * b$ is at least equal to M', which contradicts the assumption on the essential bound of $\mu * b$ over $[0,T]$.

We have now two expressions for the derivative of w, which are known to coincide in the sense of distributions and therefore almost everywhere. We wish to show that they coincide everywhere, except at a countable number of point; this will be a consequence of next Lemma. Write $\Phi = \nu * \phi$ and observe that *a priori*, thanks to the lower semi-continuity, we expect the inequality

$$\dot{w}(t) \geq \Phi(t) + \big(\mu * b\big)(t),$$

if we are able to prove that $\dot{w}$ is continuous at the points t which are not atoms of ϕ.

Lemma 16. *At all the points where w is differentiable and Φ continuous we have the relation*

$$\dot{w}(t) = \Phi(t) + \big(\mu * b\big)(t).$$

Proof. Let t be a point which is not an atom of ϕ and at which w is differentiable. Let us examine the differentiation of $H*\mu*b$ at t from either side. For $t+h>0$ and $t>0$, we use the identity

$$\frac{1}{h}\left(2\sqrt{t+h}-2\sqrt{t}-h\frac{1}{\sqrt{t}}\right)=-\frac{1}{\sqrt{t}}\frac{\sqrt{t+h}-\sqrt{t}}{\sqrt{t+h}+\sqrt{t}}; \tag{84}$$

We decompose the expression used for defining the derivative as follows for $h>0$:

$$\begin{aligned}
&\frac{w(t+h)-w(t)}{h}-(\mu*b)(t)-\Phi(t)\\
=&\frac{1}{h}\int_0^{t+h}\big((H*\nu)(t+h-s)-(H*\nu)(t-s)-h\nu(t-s)\big)\phi(s)\\
&+\frac{1}{h}\int_0^{t}\big((H*\mu)(t+h-s)-(H*\mu)(t-s)-h\mu(t-s)\big)b(s)\\
&+\frac{1}{h}\int_{[t,t+h]}\big((H*\nu)(t-s)-h\nu(t-s)\big)\phi(s)\\
&+\frac{1}{h}\int_{[t,t+h]}(H*\nu)(t+h-s)b(s).
\end{aligned}$$

Passing to the limit and using Lebesgue's dominated convergence theorem we obtain, thanks to (84)

$$\begin{aligned}
\dot w(t+0)=&\Phi(t)+(\mu*b)(t)-\lim_{h\downarrow 0}\frac{1}{h}\int_{[t,t+h]}(H*\nu)(t+h-s)b(s)\\
&-\lim_{h\downarrow 0}\frac{1}{h}\int_{[t,t+h]}\big((H*\nu)(t-s)-h\nu(t-s)\big)\phi(s).
\end{aligned} \tag{85}$$

Similarly, for $h<0$, we write

$$\begin{aligned}
&\frac{w(t+h)-w(t)}{h}-(\mu*b)(t)-\Phi(t)\\
=&\frac{1}{h}\int_0^{t+h}\big((H*\nu)(t+h-s)-(H*\nu)(t-s)-h\nu(t-s)\big)\phi(s)\\
&+\frac{1}{h}\int_0^{t+h}\big((H*\mu)(t+h-s)-(H*\mu)(t-s)-h\mu(t-s)\big)b(s)\\
&-\lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^{t}(H*\nu)(t-s)\phi(s)-\lim_{h\uparrow 0}\int_{t+h}^{t}\nu(t-s)\phi(s)\\
&-\lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^{t}(H*\mu)(t-s)b(s)-\lim_{h\uparrow 0}\int_{t+h}^{t}\mu(t-s)b(s).
\end{aligned}$$

Using as above (84) and Lebesgue's dominated convergence theorem, we see that

$$\begin{aligned}\dot{w}(t-0) =&\Phi(t) + \int_0^t \mu(t-s)b(s)\\ &+ \lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^t (H*\nu)(t-s)\phi(s) + \lim_{h\uparrow 0}\int_{t+h}^t \nu(t-s)\phi(s)\\ &+ \lim_{h\uparrow 0}\frac{1}{h}\int_{t+h}^t (H*\mu)(t-s)b(s) + \lim_{h\uparrow 0}\int_{t+h}^t \mu(t-s)b(s).\end{aligned} \tag{86}$$

Relation (85) implies that

$$\dot{w}(t) \geq \Phi(t) + \big(\mu * b\big)(t),$$

and symmetrically, relation (86) implies that

$$\dot{w}(t) \leq \Phi(t) + \big(\mu * b\big)(t).$$

These two inequalities enable us to conclude the proof.

We are able now to conclude the article by proving the last result:

Proposition 1. *Let w be the solution constructed at Lemma 13; then, for all $T>0$, we have the identity*

$$\int_0^T \big(\dot{\mu} * w\big)\dot{w}\,dt = \int_0^T g\dot{w}\,dt.$$

Proof. The convolution $\dot{\mu}*w$ is the sum of the measure b and the function g; therefore, the duality product $\langle 1_{[0,T]}, g+b\rangle$ is well defined; the function $\dot{w}(t)$ is bounded, and it vanishes b-almost everywhere on the support of b; therefore, for all $T>0$, the integral $\int_0^T b\dot{w}$ vanishes, which proves the proposition.

Now, we can drop the requirement that ϕ be a bounded measure:

Corollary 5. *Let ϕ be a Radon measure with support included in $\mathbb{R}^+$; then there exists a function w such that* (54) *and hold.*

Proof. For each m, the measure $\phi 1_{[0,m]}$ is bounded and we may construct $w^{n,m}(t)$ so that it coincides with $w^{n,k}(t)$ for $t \leq \min(m,k)(t)$. Therefore, a diagonal process lets us extract a solution possessing the required properties.

Therefore the duality product $\langle w, b\rangle$ makes sense and vanishes which enables us to deduce immediately that the energy loss is purely viscous.

A possible extension to this work is to investigate if the techniques developed in this paper for the balance of energy remain still valid for any geometrical obstacle. On the other hand, Jarušek proved by the penalty method an existence result for the full viscoelasticity including the Signorini conditions and a given friction at the boundary, his result published in [14] do not define a trace of the stress at the boundary, nor do they give information on the balance of energy. We may examine if the Fourier analysis enables us to characterize the trace spaces and then to deduce that the weak solution is also strong one. Here, we mean by strong solution that all the traces can be defined. We remark that the question of uniqueness for the dynamic contact problems are open problems.

Appendix

Define the following complex-valued functions:

$$\begin{aligned}
\widehat{\nu}(\omega) &= \frac{1}{1+i\alpha\omega},\\
\widehat{\mu}(\omega) &= \sqrt{\widehat{\nu}(\omega)} = \frac{1}{\sqrt{1+i\alpha\omega}}, \quad \Re\widehat{\mu} \geq 0,\\
\widehat{\rho}(\omega) &= \frac{1}{\widehat{\mu}(\omega)} = \sqrt{1+i\alpha\omega}, \quad \Re\widehat{\rho} \geq 0,\\
\widehat{\lambda}_1(\omega) &= i\omega\widehat{\rho}(\omega) = i\omega\sqrt{1+i\alpha\omega},\\
\widehat{\lambda}(\omega) &= i\omega\widehat{\mu}(\omega) = \frac{i\omega}{\sqrt{1+i\alpha\omega}},\\
\widehat{\mu}_1(\omega) &= \frac{\widehat{\mu}(\omega)}{i(\omega - i0)},
\end{aligned}$$

where we have used the notation

$$\frac{1}{\omega - i0} = \lim_{\varepsilon \downarrow 0} \frac{1}{\omega - i\varepsilon}.$$

The inverse Fourier transform of $\widehat{\nu}$ is given by

$$\nu(t) = \frac{\exp(-t/\alpha)}{\alpha}\, 1_{\mathbb{R}^+}(t). \tag{87}$$

Let H denote the Heaviside function. We observe that

$$H * \nu = \big(1 - \exp(-\,\cdot\,/\alpha)\big)H. \tag{88}$$

The functions $\widehat{\mu}$, $\widehat{\rho}$, μ_1 and $\widehat{\lambda}_1$ are analytic in the upper half-plane, so that their inverse Fourier transform, μ, ρ and λ_1 are supported in $\mathbb{R}^+$.

We have an explicit expression of μ:

Lemma 17. *The inverse Fourier transform of* $\widehat{\mu}$ *is*

$$\mu(t) = \frac{\exp(-t/\alpha)}{\sqrt{2\pi\alpha t}}\, 1_{(0,\infty)}(t). \tag{89}$$

Proof. It is plain that $\widehat{\mu}$ is holomorphic in $\mathbb{C} \setminus i[1/\alpha, +\infty)$; thanks to Paley-Wiener-Schwartz theorem, the support of μ is included in $[0, +\infty)$. In order to calculate the inverse Fourier transform of $\widehat{\mu}$, choose the integration path Γ pictured at Figure 1; the part on the arcs of circle AB and EF converges to 0 as R tends to infinity; thanks to Cauchy's theorem and an obvious passage to the limit

$$\int_{\mathbb{R}} \exp(i\omega t)\widehat{\mu}\, d\omega = 2\int_{1/\alpha}^{+\infty} \frac{\exp(-rt)}{\sqrt{\alpha r - 1}}\, dr.$$

The change of variable $s = (r - 1/\alpha)t$ yields the desired conclusion.

PSfrag replacements

A
B
C
D
E
F
R
0
$\Re(\omega)$
$\Im(\omega)$
Γ
$1/\alpha$
θ
2ε

Fig. 1. The path Γ in the complex plane.

The function μ_1 is given in physical variables by

$$\mu_1(t) = \int_0^t \mu(s)\, ds = \int_0^t \frac{\exp(-s/\alpha)}{\sqrt{\pi\alpha s}}\, ds. \tag{90}$$

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