Thermally driven phase transformation in shape-memory alloys

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(joint work with Alexander Mielke)

1. Mathematical formulation

We consider a body with reference configuration $\Omega \subset \mathbb{R}^d$. This body may undergo phase transformation and deformations $u : \Omega \to \mathbb{R}^d$. The phase transformation will be characterized by the internal variable $z : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{dev}}$ denoting the mesoscopic transformation strain where $\mathbb{R}^{d\times d}_{\mathrm{dev}}$ is the space of symmetric $d \times d$ tensors such that the trace of z vanishes. The set of admissible deformations $\mathcal{F}$ is chosen as a suitable subspace of $W^{1,2}(\Omega;\mathbb{R}^d)$ by describing Dirichlet data at the part Γ_{Dir} of $\partial\Omega$ and the internal variable z lives in $\mathcal{Z} = L^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$. We assume also that the material behavior depends on the temperature θ, which will be considered as a time dependent given parameter. Then we will not solve an associated heat equation but we will treat θ as an applied load and we denote it by $\theta_{\mathrm{appl}} : [0,T] \times \Omega \to [\theta_{\min}, \theta_{\max}]$. This approximation for the temperature is used in engineering models and we may justify it in the case where the changes of the loading are slow and the body is small in at least one direction such that excess of heat can be transported very fast to the surface and then radiated into the environmenent. We denote by $e(u) := \frac{1}{2}(\nabla u + \nabla u^T)$ and $\mathbb{C}(\theta)$ respectively the linearized strain tensor and the elasticity tensor that depends on the temperature θ. The potential energy takes then the following form

$$\mathcal{E}(t,u,z) := \int_\Omega W(e(u),z,\theta) + \frac{\sigma}{2}|\nabla z|^2 \,\mathrm{d}x - \langle l(t), u\rangle, \tag{1}$$

where $W(e(u),z,\theta) := \frac{1}{2}(e(u)-z) : \mathbb{C}(\theta) : (e(u)-z) + h(z,\theta)$. Here σ is positive coefficient that is expected to measure some nonlocal interaction effect for the internal variable z and $l(t)$ denotes the applied mechanical loading. In this work, we assume that $h(z,\theta) := c_1(\theta)|z|^2 + c_2(\theta)\sqrt{\delta^2+|z|^2} + \frac{1}{\delta}\big(|z|^2 - c_3(\theta)\big)_+^3$, where $c_i(\theta) > 0$, $i = 1,2,3$, are given and depending on the temperature θ. Observe that $c_1(\theta)$ measures the occurence of some hardening phenomenon with respect to the internal variable z, $c_2(\theta)$ is an activation threshold for initiation of martensitic phase transformations and $c_3(\theta)$ represents the maximum modulus of transformation strain that can be obtained by alignment of martensitic variants. We define the dissipation potential by

$$\mathcal{R}(\dot z) := \int_\Omega \rho|\dot z| \,\mathrm{d}x = \rho\|\dot z\|_{L^1(\Omega)},\ \rho > 0. \tag{2}$$

This model was initiated in [8] and further developed in [1, 2]. The original model is obtained in the limit $\delta \to 0$ and $\sigma \to 0$. For mathematical purposes we need to keep $\delta, \sigma > 0$ fixed. Finally our problem is assumed to be governed by the *energetic formulation* of rate-independent problems, for the details the reader is referred to [5, 6, 4, 3, 7]. A function $(u,z) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ is called an *energetic solution* of the rate-independent problem associated with $\mathcal{E}$ and $\mathcal{R}$ if for all $t \in [0,T]$, the *global stability condition* (S) and the *global energy conservation* (E) are satisfied, i.e.

$$(S)\ \ \forall (u,z) \in \mathcal{F} \times \mathcal{Z} : \mathcal{E}(t,u(t),z(t)) \leq \mathcal{E}(t,\bar{u},\bar{z}) + \mathcal{R}(\bar{z} - z(t)),$$

$$(E)\ \ \mathcal{E}(t,u(t),z(t)) + \int_0^t \mathcal{R}(\dot{z}(s))\,\mathrm{d}s = \mathcal{E}(0,u_0,z_0) + \int_0^t \partial_s \mathcal{E}(s,u(s),z(s))\,\mathrm{d}s.$$

Here we assume to be given initial data $(u(0),z(0)) = (u_0,z_0) \in \mathcal{F} \times \mathcal{Z}$.

2. Existence result

We clarify now the assumptions. The applied temperature θ_{appl} will extract or insert energy thanks to $\partial_\theta W(e(u),z,\theta_{\text{appl}})\dot{\theta}_{\text{appl}}$. One can prove that the derivatives $\partial_\theta^j W(e(u),z,\theta_{\text{appl}})$ exist for $j = 1,2$ and using Young's inequality that there exist $c_0^W, c_1^W > 0$ such that

$$|\partial_\theta^j W(e(u),z,\theta)| \leq c_1^W (W(e(u),z,\theta) + c_0^W). \tag{3}$$

Then $\partial_\theta^j W(e(u),z,\theta_{\text{appl}})\dot{\theta}_{\text{appl}}$ is controled if we assume that θ_{appl} is smooth enough. According to (3) with $j = 1$ and Gronwall's lemma, we have the following lemma.

Lemma 2.1. *If* (3) *holds, for all* $\theta_1 \in [\theta_{\min}, \theta_{\max}]$*, we have*

$$W(e(u),z,\theta_1) + c_0^W \leq \exp(c_1^W |\theta_1 - \theta|)(W(e(u),z,\theta) + c_0^W). \tag{4}$$

For a given temperature profile $\theta_{\text{appl}} \in C^1([0,T]; L^\infty(\Omega; [\theta_{\min}, \theta_{\max}]))$ and a given external loading $l \in C^1([0,T]; W^{1,2}(\Omega;\mathbb{R}^d)^*)$, we will study the potential energy $\mathcal{E}$ as defined in (1).

Proposition 2.2. *Under the above assumptions the following holds:*app

(i) If for some $(t_*,u,z) \in [0,T] \times \mathcal{F} \times \mathcal{Z}$ *we have* $\mathcal{E}(t_*,u,z) < +\infty$*, then* $\mathcal{E}(\cdot,u,z) \in C^1([0,T])$ *and* $\partial_t \mathcal{E}(t,u,z) = \int_\Omega \partial_\theta W(e(u),z,\theta_{\text{appl}}(t))\dot{\theta}_{\text{appl}}(t)\,\mathrm{d}x - \langle \dot{l}(t), u \rangle$.

(ii) There exist $c_0^E, c_1^E > 0$ *such that* $\mathcal{E}(t,u,z) < +\infty$ *implies* $|\partial_t \mathcal{E}(t,u,z)| \leq c_1^E(\mathcal{E}(t,u,z) + c_0^E)$.

(iii) For each $\varepsilon > 0$ *and* $E \in \mathbb{R}$ *there exists* $\delta > 0$ *such that* $\mathcal{E}(t_1,u,z) \leq E$ *and* $|t_1 - t_2| < \delta$ *imply* $|\partial_t \mathcal{E}(t_1,u,z) - \partial_t \mathcal{E}(t_2,u,z)| \leq \varepsilon$.

We prove now that the *energetic formulation* (S) and (E) has at least one solution $(u,z) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ for a given stable initial datum $(u_0,z_0) \in \mathcal{F} \times \mathcal{Z}$, i.e. (u_0,z_0) satisfies the *global stability condition* (S) at $t = 0$. The existence theory for (S) and (E) was developed [2, 1, 5] and it is based on the incremental minimization problem. More precisely, for a given partition $\Pi = \{0 = t < t_1 < \ldots < t_N = T\}$, we define the incremental problem as follows:

$$(IP)_\Pi \ \left\{ \begin{array}{l} \text{for } k = 1,\ldots,d \text{ find} \\ (u_k,z_k) \in \operatorname{Argmin}\{\mathcal{E}(t_k,\widetilde{u},\widetilde{z}) + \mathcal{R}(\widetilde{z} - z_k) : (\widetilde{u},\widetilde{z}) \in \mathcal{F} \times \mathcal{Z}\}. \end{array} \right.$$

One can observe that $(IP)_\Pi$ has always solutions. We define the piecewise constant interpolant $(u^\Pi, z^\Pi) : [0,T] \to \mathcal{F} \times \mathcal{Z}$ with $(u^\Pi(t), z^\Pi(t)) = (u_j, z_j)$ for $t \in [t_{j-1}, t_j)$

for $j = 0, \ldots, N$. Then we show that the limit function satisfies the *energetic formulation* (S) and (E) using Lemma 2.1, which gives the following Theorem:

Theorem 2.3. *Assume that $\mathcal{E}$, $\mathcal{R}$ and (u_0, z_0) satisfy the assumptions from above. Then there exists an energetic solution $(u, z) : [0, T] \to \mathcal{F} \times \mathcal{Z}$ such that $(u(0), z(0)) = (u_0, z_0)$ and*

$$u \in L^\infty([0,T]; W^{1,2}(\Omega; \mathbb{R}^d)),$$
$$z \in L^\infty([0,T]; W^{1,2}(\Omega; \mathbb{R}^{d\times d}_{\text{dev}})) \cap BV([0,T]; L^1(\Omega; \mathbb{R}^{d\times d}_{\text{dev}})).$$

In futur work we will investigate the question of uniqueness by using the theory developed in [5]. For this it is necessary to establish smoothness of $\mathcal{E}$ as a function of $(u, z) \in \mathcal{F} \times H^1(\Omega; \mathbb{R}^{d\times d}_{\text{dev}})$.

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