Mathematical investigation of the coupling between dry friction and linear elasticity

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The mathematical problem raised by the frictionless equilibrium of a linear elastic body pressed against some rigid obstacle (the so-called Signorini problem) was first investigated by Fichera, Lions and Stampacchia in the sixties. The underlying mathematical structure of the minimization problem of a coercive continuous strictly convex energy over a closed convex cone provided the existence and uniqueness of the solution, which is characterized by an optimality condition under the form of a so-called variational inequality.

The above nice picture immediately suggested to investigate the more involved situation where the contact between the elastic body and the obstacle is no longer frictionless but obeys the Coulomb law of dry friction. Such an investigation was first undertaken by Duvaut and Lions in the early seventies and faced great mathematical difficulties. They suggested a fixed point approach that they were unable to solve. It was finally solved in the early eighties by Nečas and Jarušek who were able to prove the existence of a solution for a sufficiently small friction coefficient by a compactness argument, leaving the question of uniqueness completely open. Their almost forty-years-old result is still the best available up to now. Yet, the question of uniqueness of the solution has drawn increasing interest, as there are clues that uniqueness could hold true for small friction coefficients and could be lost for larger friction coefficients. Such a bifurcation (with respect to the friction coefficient as a control parameter) could account for the qualitatively different responses that are commonly observed in frictional contact (steady sliding versus judder), and advocates for a new investigation of the mathematical structure of the problem raised by the quasi-static evolution of a linear elastic body in frictional contact with a rigid obstacle.

Introducing a time discretization, the problem to solve at each time step is put under the form of a minimization problem. The study of convexity in this minimization problem raises a new simpler problem which is seen of physical interest on its own: the so-called steady sliding frictional contact problem. This is the problem raised by the equilibrium of a linear elastic body

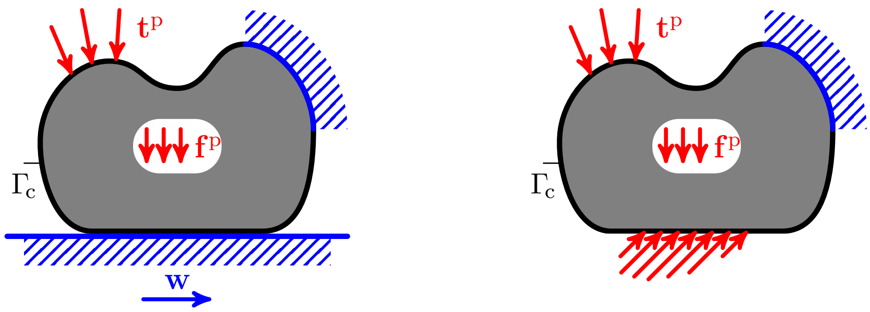

Figure 1: The steady sliding frictional contact problem for a continuum.

in frictional contact with a moving obstacle, the geometry of which remaining invariant along time (see figure 1).

The steady sliding frictional contact problem is first investigated in the case of a homogeneous friction coefficient $\mathcal{F}$. The main result which is proved is that there exists a critical friction coefficient $0 < \mathcal{F}_c \leq +\infty$, such that, for all $0 \leq \mathcal{F} < \mathcal{F}_c$, the steady sliding frictional contact problem admits one and only one equilibrium solution. An example is provided of a steady sliding frictional contact problem admitting infinitely many solutions, showing that $\mathcal{F}_c$ is necessarily finite in certain cases. An example is also provided in which $\mathcal{F}_c$ is infinite. Hence, taking the friction coefficient $\mathcal{F}$ as a control parameter in the steady sliding frictional contact problem, it is observed that the steady sliding frictional contact problem may display a bifurcation, or not.

To pass from the steady sliding frictional contact problem to the original quasi-static frictional contact problem, it is shown that it is necessary to understand the steady sliding frictional contact problem with *heterogeneous* friction coefficient. The steady sliding frictional contact problem is then completely studied in the case of the geometry of a half-space and an arbitrary piecewise constant friction coefficient. It is proved that the corresponding steady sliding frictional contact problem has always one and only one solution. This solution exhibits unintuitive universal singularities, as represented on figure 2.

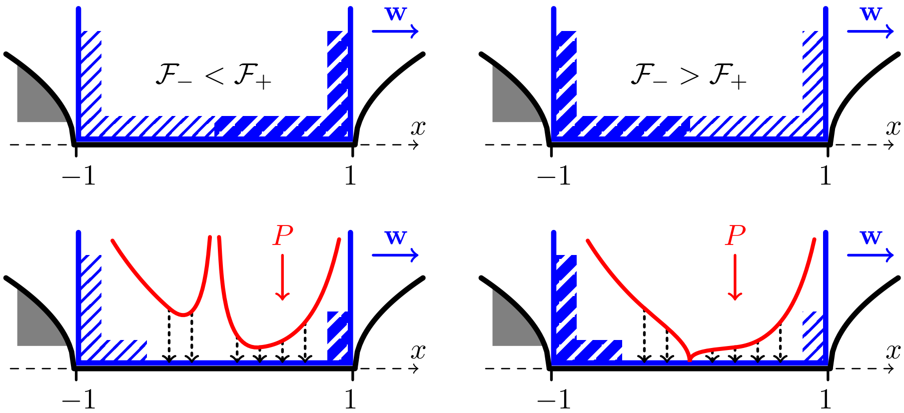

Figure 2: Normal component of the surface traction when the larger friction coefficient is front (left) or rear (right).

References

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