ON THE BEST CONSTANT IN GAFFNEY INEQUALITY

Bernard DACOROGNA

EPFL - Lausanne - Suisse

avec G. CSATO et S. SIL

à paraître dans Journal of Functional Analysis (2018)

En l'honneur des 60 ans de Jérôme Pousin

GAFFNEY inequality for vector fields states that there exists a constant $C = C(\Omega) > 0$ such that for every vector field $u \in W^{1,2}(\Omega; \mathbb{R}^n)$

$$\|\nabla u\|_{L^2}^2 \leq C \left(\|\operatorname{curl} u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2 + \|u\|_{L^2}^2 \right)$$

where, on $\partial\Omega$, either

$$\nu \wedge u = 0$$

(i.e. u is parallel to ν and we write then $u \in W_T^{1,2}(\Omega; \mathbb{R}^n)$) or

$$\nu \lrcorner u = 0$$

(i.e. u is orthogonal to ν and we write then $u \in W_N^{1,2}(\Omega; \mathbb{R}^n)$).

GAFFNEY inequality Let $0 \leq k \leq n$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Then there exists a constant $C = C(\Omega, k) > 0$ such that

$$\|\nabla \omega\|_{L^2}^2 \leq C\left(\|d\omega\|_{L^2}^2 + \|\delta\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right)$$

for every $\omega \in W_T^{1,2}\left(\Omega; \Lambda^k\right) \cup W_N^{1,2}\left(\Omega; \Lambda^k\right)$.

Define ($\|\cdot\|$ stands for the L^2-norm)

$$C_T(\Omega, k) = \sup_{\omega \in W_T^{1,2} \setminus \{0\}} \left\{ \frac{\|\nabla \omega\|^2}{\|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2} \right\}$$

$$C_N(\Omega, k) = \sup_{\omega \in W_N^{1,2} \setminus \{0\}} \left\{ \frac{\|\nabla \omega\|^2}{\|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2} \right\}$$

It is easy to see that

$$C_T(\Omega, k), C_N(\Omega, k) \geq 1.$$

The cases $k = 0$ and $k = n$ are trivial and $C_T = C_N = 1$.

The discussion deals with the case $1 \leq k \leq n-1$ ($k = 1$ is the case of vector fields).

Bibliography

- Gaffney (1951) for manifolds without boundary

- For manifolds with boundary: Friedrichs (1955), Morrey and Morrey-Eells (1956).

- Bolik (in L^p and in Hölder spaces), Csató-Dacorogna-Kneuss, Iwaniec-Martin, Iwaniec-Scott-Stroffolini (in L^p), Mitrea , Mitrea-Mitrea, Schwarz, Taylor, von Wahl.

- For vector fields in dimension 2 and 3 : Amrouche-Bernardi-Dauge-Girault, Costabel, Dautray-Lions, Desvillettes-Villani.

A stronger version (in fact a regularity result) of Gaffney inequality is

$$\|\nabla\omega\|^2 \leq C\left(\|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2\right), \quad \forall\, \omega \in W_T^{d,\delta,2}\left(\Omega;\Lambda^k\right)$$

(and similarly with T replaced by N) where

$$\omega \in W_T^{d,\delta,2}\left(\Omega;\Lambda^k\right) = \left\{\omega \in L^2\left(\Omega;\Lambda^k\right) : \left\{\begin{array}{c} d\omega \in L^2\left(\Omega;\Lambda^{k+1}\right) \\ \delta\omega \in L^2\left(\Omega;\Lambda^{k-1}\right) \\ \nu \wedge \omega = 0 \text{ on } \partial\Omega \end{array}\right.\right\}.$$

For smooth or convex Lipschitz domains

$$W_T^{d,\delta,2}\left(\Omega;\Lambda^k\right) = W_T^{1,2}\left(\Omega;\Lambda^k\right).$$

For non-convex Lipschitz domains, in general,

$$W_T^{1,2} \underset{\neq}{\subset} W_T^{d,\delta,2}.$$

Bibliography (for the stronger version)

- Mitrea, Mitrea-Mitrea

- For vector fields in dimension 2 and 3 : Amrouche-Bernardi-Dauge-Girault, Ben Belgacem-Bernardi-Costabel-Dauge, Ciarlet-Hazard-Lohrengel, Costabel, Costabel-Dauge, Girault-Raviart.

Plan of the talk

I) The main theorem

II) Some examples

III) The case of polytopes

IV) Idea of proof

I) The main theorem

Definition Let $\Omega \subset \mathbb{R}^n$ be an open smooth set and $\boldsymbol{\Sigma} = \partial\Omega$ be the associated $(n-1)$ −surface. Let $\gamma_1, \cdots, \gamma_{n-1}$ be the principal curvatures of $\boldsymbol{\Sigma}$. Let $1 \leq k \leq n-1$. We say that Ω is k-convex if

$$\gamma_{i_1} + \cdots + \gamma_{i_k} \geq 0 \quad \text{for every } 1 \leq i_1 < \cdots < i_k \leq n-1.$$

Remark *(i)* When $k = 1$: Ω is convex if and only if Ω is 1-convex, i.e.

$$\gamma_i \geq 0 \quad \text{for every } 1 \leq i \leq n-1.$$

The result is due to Hadamard and to Chern-Lashof.

(ii) When $k = n-1$: Ω is (n-1)-convex if and only if the mean curvature of $\Sigma = \partial\Omega$ is non-negative, i.e.

$$\gamma_1 + \cdots + \gamma_{n-1} \geq 0.$$

Theorem Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and $1 \leq k \leq n-1$. Then the following statements are equivalent.

(i) $C_T(\Omega, k) = 1$ (i.e. $\|\nabla\omega\|^2 \leq \|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2$)

(ii) The sharper version of Gaffney inequality holds, namely

$$\|\nabla\omega\|^2 \leq \|d\omega\|^2 + \|\delta\omega\|^2, \quad \forall\, \omega \in W_T^{1,2}\left(\Omega; \Lambda^k\right).$$

(iii) Ω is $(n-k)$ $-$convex.

(iv) The supremum is not attained.

(v) C_T is scale invariant , namely, for every $t > 0$

$$C_T(t\,\Omega, k) = C_T(\Omega, k)\,.$$

Remark *(i)* Similar results for N, since

$$C_T(\Omega, k) = C_N(\Omega, n-k).$$

(ii) When $k = 1$, we have the equivalence between

- $C_T(\Omega, 1) = 1$ (i.e. $\|\nabla\omega\|^2 \le \|\operatorname{curl}\omega\|^2 + \|\operatorname{div}\omega\|^2 + \|\omega\|^2$)

- The sharper version of Gaffney inequality

$$\|\nabla\omega\|^2 \le \|\operatorname{curl}\omega\|^2 + \|\operatorname{div}\omega\|^2, \quad \forall\, \omega \in W_T^{1,2}(\Omega; \mathbb{R}^n)$$

- Ω is $(n-1)-$convex i.e. the mean curvature of $\Sigma = \partial\Omega$ is non-negative, meaning that

$$\gamma_1 + \cdots + \gamma_{n-1} \ge 0.$$

(iii) Still when $k = 1$, since

$$C_T(\Omega, 1) = C_N(\Omega, n-1)$$

- $C_N(\Omega, 1) = 1$ (i.e. $\|\nabla\omega\|^2 \leq \|\mathbf{curl}\,\omega\|^2 + \|\mathbf{div}\,\omega\|^2 + \|\omega\|^2$)

- The sharper version of Gaffney inequality

$$\|\nabla\omega\|^2 \leq \|\mathbf{curl}\,\omega\|^2 + \|\mathbf{div}\,\omega\|^2, \quad \forall\, \omega \in W_N^{1,2}(\Omega; \mathbb{R}^n)$$

- Ω is convex i.e.

$$\gamma_1, \cdots, \gamma_{n-1} \geq 0.$$

II) Some examples

Proposition Let $1 \leq k \leq n-1$. Then there exists a set $\Omega_k \subset B$ (a fixed ball of $\mathbb{R}^n$) such that $C_T\left(\Omega_k, k\right)$ is arbitrarily large (and similarly for C_N).

Proof $(k = 1)$ Choose Ω_1 an annulus with very small inner radius. Then choose

$$\omega(x) = \frac{x}{|x|^n}$$

and observe that $\nu \wedge \omega = 0$ on $\partial \Omega_1$ and it is a harmonic field i.e.

$$d\omega = 0 \quad \text{and} \quad \delta\omega = 0.$$

Then $\left[\|\nabla\omega\|^2 / \|\omega\|^2\right] \to \infty$ as the inner radius $\to 0$.

III) The case of Polytopes

Definition $\Omega \subset \mathbb{R}^n$ is said to be a generalized polytope, if there exist

$$\Omega_0\,, \Omega_1\,, \cdots\,, \Omega_M$$

bounded open polytopes such that, for every $i, j = 1, \cdots, M$ with $i \neq j$,

$$\overline{\Omega}_i \subset \Omega_0\,, \quad \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad \text{and} \quad \Omega = \Omega_0 \setminus \left(\bigcup_{i=1}^{M} \overline{\Omega}_i \right).$$

In this case $\overline{\Omega}_i\,, i = 1, \cdots, M$, are called the *holes*.

Theorem Let $\Omega \subset \mathbb{R}^n$ be a generalized polytope. Then the following identity holds

$$\|\nabla \omega\|^2 = \|d\omega\|^2 + \|\delta\omega\|^2\,, \quad \forall\, \omega \in C_T^1\left(\overline{\Omega}; \Lambda^k\right) \cup C_N^1\left(\overline{\Omega}; \Lambda^k\right).$$

Remark In particular if $\omega \in C^1_T \cup C^1_N$ is a harmonic field i.e.

$$d\omega = 0 \quad \text{and} \quad \delta\omega = 0,$$

then $\omega \equiv 0$ independently of the topology !!!

IV) Idea of proof

We discuss only the case $k = 1$ (i.e. $d \sim$ **curl** and $\delta \sim$ **div**). Recall that

(i) $C_T(\Omega, 1) = 1$ (i.e. $\|\nabla\omega\|^2 \leq \|d\omega\|^2 + \|\delta\omega\|^2 + \|\omega\|^2$)

(ii) The sharper version of Gaffney inequality holds, namely

$$\|\nabla\omega\|^2 \leq \|d\omega\|^2 + \|\delta\omega\|^2, \quad \forall\, \omega \in W_T^{1,2}(\Omega; \mathbb{R}^n).$$

(iii) Ω is $(n-1)$ $-$convex i.e.

$$\gamma_1 + \cdots + \gamma_{n-1} \geq 0.$$

The implication $(ii) \Rightarrow (i)$ is trivial and we discuss only

$$(iii) \quad \Rightarrow \quad (ii).$$

For $k = 1$ we have

$$|\mathrm{curl}\,\omega|^2 + |\mathrm{div}\,\omega|^2 - |\nabla\omega|^2 = 2\sum_{i<j}\left(\frac{\partial\omega^i}{\partial x_i}\frac{\partial\omega^j}{\partial x_j} - \frac{\partial\omega^j}{\partial x_i}\frac{\partial\omega^i}{\partial x_j}\right)$$

Integrating the above we get, for every $\omega \in W_T^{1,2}\left(\Omega; \Lambda^k\right)$,

$$\int_\Omega \left(|\mathrm{curl}\,\omega|^2 + |\mathrm{div}\,\omega|^2 - |\nabla\omega|^2\right) = \int_{\partial\Omega} |\langle\omega;\nu\rangle|^2 \left(\gamma_1 + \cdots + \gamma_{n-1}\right)$$