Stability analysis for CNSE - theory and numerics

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Part 1 : Concept and stability analysis of weak solutions in the continuous case based on joint works with E. Feireisl also with B. J. Jin, Y. Sun
Part 2 : Error estimates : joint work with T. Gallouet, R. Herbin, D. Maltese and later also with E. Feireisl, R. Hosek, D. Maltese

Compressible Navier-Stokes equations

We consider in $[0,T) \times \Omega$, $\Omega \subset R^3$ (a bounded Lipschitz domain) the following system of equations

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1}$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \tag{2}$$

Boundary conditions

$$\mathbf{u}\Big|_{(0,T)\times\partial\Omega} = 0 \tag{3}$$

Initial conditions

$$\varrho(0,x) = \varrho_0(x), \quad \varrho\mathbf{u}(0,x) = \varrho_0 \mathbf{u}_0(x). \tag{4}$$

Stress tensor

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}\mathbf{u}\mathbb{I}, \ \ \mu > 0; \ \ \eta = 0 \tag{5}$$

Pressure

$$p \in C^1[0,\infty) \quad p(0) = 0. \tag{6}$$

Helmholtz function H

$$\varrho H'(\varrho) - H(\varrho) = p(\varrho), \quad H(\varrho) = \varrho \int_1^{\varrho} \frac{p(s)}{s^2}\mathrm{d}s$$

Relative energy function E

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

Weak solutions

Functional spaces

$\varrho(t,x) \geq 0$ for a.a. $(t,x) \in (0,T) \times \Omega$, $\varrho \in L^\infty(0,T;L^1(\Omega))$,
$\varrho \mathbf{u} \in L^\infty(0,T;L^1(\Omega;R^3))$, $\varrho \mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$,
$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3))$, $p(\varrho) \in L^\infty(0,T;L^1(\Omega))$.

Continuity equation

$\varrho \in C_{\rm weak}([0,T];L^1(\Omega))$ and equation (1) is replaced by the family of integral identities

$$\int_\Omega \varrho\varphi \ {\rm d}x \Big|_0^\tau = \int_0^\tau \int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \Big) \ {\rm d}x \ {\rm d}t \tag{7}$$

for all $\tau \in [0,T]$ and for any $\varphi \in C^1([0,T] \times \overline{\Omega})$;

Momentum equation

$\varrho\mathbf{u} \in C_{\rm weak}([0,T];L^1(\Omega;R^3))$ and momentum equation (2) is satisfied in the sense of distributions, specifically,

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi \,{\rm d}x\Big|_0^\tau = \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi\Big)\,{\rm d}x{\rm d}t \tag{8}$$

$$+\int_0^\tau\int_\Omega \Big(p(\varrho){\rm div}_x\varphi - \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \nabla_x\varphi\Big)\,{\rm d}x\,{\rm d}t$$

for all $\tau\in[0,T]$ and for any $\varphi\in C^1_c([0,T]\times\Omega;R^3)$;

Energy inequality

$$\int_\Omega \Big(\frac{1}{2}\varrho\mathbf{u}^2 + E(\varrho,\overline{\varrho})\Big)\,{\rm d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\,{\rm d}x{\rm d}t \le 0, \tag{9}$$

for a.a. $\tau\in(0,T)$, where $\overline{\varrho}>0$.

Relative energy inequality

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho \,|\, r) \right) \ \mathrm{d}x \Big|_0^\tau \\
&+ \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u} - \mathbf{U})\Big) : \nabla_x \Big(\mathbf{u} - \mathbf{U}\Big) \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x \Big(\mathbf{U} - \mathbf{u}\Big) \ \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \Big(\varrho \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega p(\varrho) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left(\frac{r - \varrho}{r} \partial_t p(r) - \frac{\varrho}{r} u \cdot \nabla_x p(r) \right) \ \mathrm{d}x \ \mathrm{d}t
\end{aligned} \tag{10}
$$

for all

$$
r \in C^1_c([0,T] \times \overline{\Omega}), \ r > 0, \ \mathbf{U} \in C^1_c([0,T] \times \Omega).
$$

Dissipative solutions

Relative energy

$$\mathcal{E}(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}) = \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + E(\varrho \,|\, r) \right) \, \mathrm{d}x$$

Functional spaces

$\varrho(t,x) \geq 0$ for a.a. $(t,x) \in (0,T) \times \Omega$, $\varrho \in L^\infty(0,T;L^1(\Omega))$,
$\varrho \mathbf{u} \in L^\infty(0,T;L^1(\Omega;R^3))$, $\varrho \mathbf{u}^2 \in L^\infty(0,T;L^1(\Omega))$,
$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;R^3))$, $p(\varrho) \in L^\infty(0,T;L^1(\Omega))$.

Relative energy inequality

$$\mathcal{E}(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U})(\tau) + \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u} - \mathbf{U})\Big) : \nabla_x \Big(\mathbf{u} - \mathbf{U}\Big) \, \mathrm{d}x \, \mathrm{d}t$$

$$\leq \mathcal{E}(\varrho_0, \mathbf{u}_0 \,\Big|\, r(0), \mathbf{U}(0)) + \int_0^\tau \mathcal{R}\Big(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\Big) \mathrm{d}t$$

where the remainder $\mathcal{R}$ is given by the r.h.s. of formula (10) and the test functions are the same as in formula (10).

Existence of Weak solutions

Pressure for existence of weak solutions

$$p'(\varrho) \geq a_1 \varrho^{\gamma-1} - b, \quad \varrho > 0, \tag{11}$$

$$p(0) = 0, \quad p(\varrho) \leq a_2 \varrho^{\gamma} + b, \quad \varrho \geq 0,$$

with some $\gamma > 3/2$, $a_1 > 0$, $a_2, b \in R$.

Finite energy initial data

$$0 \neq \varrho_0 \geq 0, \int_{\Omega} \frac{1}{2} \varrho_0 \mathbf{u}_0^2 + E(\varrho_0 \,|\, \overline{\varrho}) \, \mathrm{d}x < \infty. \tag{12}$$

Existence of Weak solutions : Lions,98 ($\gamma \geq \frac{9}{5}$), Feireisl, Petzeltova, N., 02 ($\gamma > \frac{3}{2}$), Feireisl (02) non-monotone pressure as above, Bresch, Jabin "general" non monotone pressure ($\gamma \geq 9/5$), 15

Under assumptions on the initial data (12), and pressure $p \in C^1[0, \infty)$, (11), the compressible Navier-Stokes system (1–5) admits at least one weak solution.

Existence of Weak solutions

"More then polynomial growth"

In general it is an open problem. Treated in the case of "hard sphere model".

$$p \in C^1[0,\overline{\varrho}),\ p(0)=0,\ p'(\varrho)\geq 0, \tag{13}$$

$$p(\varrho) \approx_{\overline{\varrho}} (\overline{\varrho}-\varrho)^{-\beta},$$

with some $\beta \geq 5/2$. Feireisl, Zhang (2008) and Feireisl, Lu, Malek (2010)

Generalization to non-homogenous data

$$\mathbf{u}_{\partial\Omega} = \mathbf{u}_\infty,\ \varrho|_{\Gamma_{\text{in}}} = \varrho_\infty,$$

$$\Gamma_{\text{in}} = \{x \in \partial_\Omega \mid \mathbf{u}_\infty \cdot \mathbf{n} < 0\}$$

Novo(ball, $\mathbf{u}_\infty$ constant)-2002, Girinon(convex inflow/outflow and "no-reflux condition")-2010, Chang, Choe, Jin, Yang, N. (general piecewise C^2 boundaries) - 2017

Existence of Weak and dissipative solutions

Existence of disspative solutions Feireisl, Sun, N., 2012

Under assumptions on initial data(12) and pressure $p \in C^1[0,\infty)$, (11), there is at least one dissiptive solution.

Weak solutions are dissipative : Feireisl, Jin, N., 2012

Under assumptions on initial data(12) and pressure $p \in C^1[0,\infty)$, any weak solution of the compressible Navier-Stokes system (1–5) is a dissipative one.

Relative entropy with (r, U) strong solution of CNSE

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E(\varrho\,|\,r) \right)\ \mathrm{d}x \Big|_0^\tau \\
&+ \int_0^\tau \int_\Omega \mathbb{S}\Big(\nabla_x(\mathbf{u}-\mathbf{U})\Big) : \nabla_x\Big(\mathbf{u}-\mathbf{U}\Big)\,\mathrm{d}x\ \mathrm{d}t \\
&\le \int_0^\tau \int_\Omega (\varrho - r)\Big(\partial_t\mathbf{U} + \mathbf{U}\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \varrho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\Big)\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \Big(p(\varrho) - p'(r)(\varrho - r) - p(r)\Big)\mathrm{div}_x\mathbf{U}\ \mathrm{d}x\ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \frac{r-\varrho}{r}(\mathbf{u}-U)\cdot\nabla_x p(r)\ \mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{14}
$$

Thermodynamic stability conditions and relative energy function

Thermodynamic stability conditions

$$p'(\varrho) > 0, \quad \varrho > 0 \tag{15}$$

Relative energy function E under thermodynamic stability conditions

Under condition (15),

$$E(\varrho, r) = H(\varrho) - H'(r)(\varrho - r) - H(r)$$

$$E(\varrho, r) \geq 0, \quad E(\varrho, r) = 0 \iff \varrho = r$$

Relative energy and stability

Regularity of pressure

$$p \in C^1[0,\infty) \cap C^2(0,\infty), \quad p(0) = 0. \tag{16}$$

Weak strong uniqueness, stability I

Let $(\varrho, \mathbf{u})$ be a weak solution to the Navier-Stokes equations (1-6) with pressure obeying regularity (16) and thermodynamic stability conditions (15), emanating from finite energy initial data $(\varrho_0, \mathbf{u}_0)$ in the time interval $[0,T)$, $T > 0$ such that

$$0 < \underline{\varrho} < \varrho(t,x) < \overline{\varrho} < \infty. \tag{17}$$

Let $(r, \mathbf{U})$ be a strong solution of the same equations in the regularity class (19), with finite enrergy initial data $(r_0, \mathbf{U}_0)$. Then

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \leq c\mathcal{E}\left(\varrho_0, \mathbf{u}_0 \,\middle|\, r_0, \mathbf{U}_0\right).$$

No conditions on weak solution, minimum conditions on pressure

Weak strong uniqueness, stability II

Let the pressure verifies regularity (16), thermodynamic stability condition (15), and

$$p(\varrho) \le c_1 + c_2\varrho + H(\varrho). \tag{18}$$

Let $(\varrho, \mathbf{u})$ be a weak solution to the compressible Navier-Stokes equations (1-5) emanating from the initial data $(\varrho_0, \mathbf{u}_0)$, and let $(r, \mathbf{U})$ be a strong solution of the same system emanating from the initial data $(0 < r_0, \mathbf{U}_0)$ in class (19). Then there exists $c = c(\Omega, T, \|r^{-1}\|_{0,\infty}, \|r\|_{1,\infty}, \|\mathbf{U}\|_{1,\infty})$ such that

$$\mathcal{E}\Big(\varrho, \mathbf{u} \;\Big|\; r, \mathbf{U}\Big) \le c\mathcal{E}\Big(\varrho_0, \mathbf{u}_0 \;\Big|\; r_0, \mathbf{U}_0\Big).$$

Sufficient condition for (18)

$$0 < \frac{1}{p_\infty} \le \liminf_{\varrho\to\infty} \frac{p(\varrho)}{\varrho^\gamma} \le \limsup_{\varrho\to\infty} \frac{p(\varrho)}{\varrho^\gamma} \le p_\infty < \infty, \text{ where } \gamma > 0,$$

Eistence of strong solutions corresponding to situations I,II

$$0 < \underline{r} \le r \le \bar{r} < \infty; \quad \mathbf{U} \in L^\infty((0,T) \times \Omega), \tag{19}$$
$$\partial_t r,\ \partial_t \mathbf{U},\ \nabla_x r,\ \nabla_x \mathbf{U} \in L^2(0,T;L^\infty(\Omega)),$$

Local in time (in large) or global in time (in small) existence of such solutions under additional compatibility conditions on initial data follows from the theory developed in 80'th by in works of Matsumura, Nishida, Valli, Zajaczkowski

No conditions on weak solution, growth on pressure

Weak strong uniqueness, stability III

Let the pressure verifies regularity (16), thermodynamic stability condition (15), and (11) with $\gamma > 6/5$.
Let $(\varrho, \mathbf{u})$ be a weak solution to the compressible Navier-Stokes equations (1-5) emanating from the finite energy initial data $(\varrho_0, \mathbf{u}_0)$, and let $(r, \mathbf{U})$ be a strong solution in class (20) of the same system emanating from the initial data $(0 < r_0, \mathbf{U}_0)$. Then there exists $c > 0$ such that

$$\mathcal{E}\Big(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\Big) \leq c\mathcal{E}\Big(\varrho_0, \mathbf{u}_0 \,\Big|\, r_0, \mathbf{U}_0\Big).$$

Regularity of the strong solution II - corresponding to the theory of Cho, Choe, Kim, 90's

$$0 < \underline{r} \leq r \leq \overline{r} < \infty, \quad \mathbf{U} \in L^\infty((0,T) \times \Omega), \tag{20}$$

$$\nabla_x r \in L^2(0,T; L^q(\Omega; \mathbb{R}^3)),\ \nabla_x^2 \mathbf{U} \in L^2(0,T; L^q(\Omega)),\ q > \max\{(3, \frac{6\gamma}{5\gamma - 6})\}.$$

Regularity, Blow-up

Bow up criterion. Sun, Zhang, 2000

Let $(r, \mathbf{U})$ be a strong solution to CNSE in Cho,Choe, Kim's class on $[0, T_{\max})$. If $T_{\max} < \infty$ then

$$\lim_{T \to T_{\max}} \|\varrho\|_{L^\infty(Q_T)} \to \infty.$$

Regularity criterion

Let $(\varrho, \mathbf{u})$ be a weak solution to CNSE (on $(0, T)$) emanating from the Cho,Choe,Kim's initial data. Suppose that ϱ is bounded. Then $(\varrho, \mathbf{u})$ is a strong solution (on $(0, T)$ in the Cho, Choe, Kim's class.

Possible generalizations

- Stability generalizable to hard sphere pressure model ($\beta \geq 3$) so far only for the periodic boundary conditions : Feireisl, Lu, N. 2017. Other cases are open problems.
- It seems to be no way to generalize the stability results to non-homogenous boundary data (weak solutions seem to be too weak in order to provide weak strong uniqueness).
- Regularity criterion not proved for the hard sphere pressure model (probably difficult if not true)
- Regularity criterion might be generalizable for the non-homogenous data (open problem)

Approximating system of PDEs for weak solutions

Approximating system

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) - \varepsilon \Delta \varrho = 0,$$

$$\partial_n \varrho|_{\partial\Omega} = 0,$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x (p(\varrho) + \delta \varrho^4) + \varepsilon \nabla_x \varrho \cdot \nabla_x \mathbf{u}$$

$$= \mu \Delta \mathbf{u} + (\frac{\mu}{3} + \eta) \nabla_x \operatorname{div} \mathbf{u}$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

Weak solutions are obtained letting first $\varepsilon \to 0$ and then $\delta \to 0$. This is not exploitable neither in the numerical calculations nor in theoretical numerics !

Physical and numerical domain

The physical space is represented by a bounded domain $\Omega \subset R^3$. The numerical domains Ω_h are polyhedral domain, a union of tetrahedra K

$$\overline{\Omega}_h = \cup_{K \in \mathcal{T}_h} K.$$

If $K \cap L \neq \emptyset$, $K \neq L$, then $K \cap L$ is either a common face, or a common edge, or a common vortex. Either $\Omega = \Omega_h$ or

$$\sup_{x \in \partial\Omega} \mathrm{dist}(x, \partial\Omega_h) \leq ch, \tag{21}$$

or

$$\mathcal{V}_h \in \partial\Omega_h \text{ a vertex } \Rightarrow \mathcal{V}_h \in \partial\Omega. \tag{22}$$

Furthermore, we suppose that

$$\xi[K] \approx \mathrm{diam}[K] \approx h, \tag{23}$$

where $\xi[K]$ is the radius of the largest ball contained in K.

Discretisation

$K \in \mathcal{T}$- regular partition of Ω_h into tetrahedrons of size h.

$\sigma = K|L \in \mathcal{E}_{\text{int}}$ - set of internal faces, $\mathcal{E}$ - set of all faces.

$0 < t_1 < \ldots < t_n < \ldots < T$ - time discretisation of step Δt.

$\varrho(t_n, x) \approx \sum_{K \in \mathcal{T}} \varrho_K^n 1_K(x) \in Q_h(\Omega_h)$ - space of piecewise constants.

$$\mathbf{u}(t_n, x) \approx \mathbf{u}^n \in V_{h,0}(\Omega_h; R^3) := (V_{h,0}(\Omega_h))^3,$$

where

$$V_{h,0}(\Omega_h) := \{v|_K \in P^1,\ \int_\sigma v|_K = \int_\sigma v|_L \text{ if } \sigma = K|L,\ \int_\sigma v = 0 \text{ if } \sigma \in \mathcal{E}_{\text{ext}}\}$$

is the so called Crouzeix-Raviart space. There is a basis

$$\{\phi_\sigma\}_{\sigma \in \mathcal{E}_{\text{int}}} \subset V_{h,0}(\Omega_h),\ \int_\sigma \phi_\sigma \phi_{\sigma'} \mathrm{d}S = \delta_{\sigma,\sigma'}$$

so that

$$\mathbf{u}^n(x) = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \mathbf{u}_\sigma^n \phi_\sigma(x),\ \mathbf{u}_\sigma \in R^3.$$

Upwind, mean values

Upwind :

$$\varrho_\sigma^{\rm up} = \left\{ \begin{array}{c} \varrho_K \text{ if } \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma,K} > 0 \\ \\ \varrho_L \text{ otherwise} \end{array} \right\}, \quad \text{where } \sigma = K|L.$$

Mean values :

$$V_K = \frac{1}{|K|}\int_K V {\rm d}x, \quad \hat{V} = \sum_{K\in\mathcal{T}} v_K 1_K(x), \quad V_\sigma = \frac{1}{|\sigma|}\int_\sigma V {\rm d}S$$

Numerical scheme (Karper-2013)

$$\varrho^n \in Q_h(\Omega_h),\ \varrho^n \geq 0,\ \mathbf{u}^n \in V_{h,0}(\Omega_h; R^3), \quad n = 0, 1, \dots, N, \tag{24}$$

$$\sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}) = 0 \tag{25}$$

for any $K \in \mathcal{T}$ and $n = 1, \dots, N$,

$$\begin{aligned} &\sum_{K \in \mathcal{T}} \frac{|K|}{\Delta t} \left(\varrho_K^n \mathbf{u}_K^n - \varrho_K^{n-1} \mathbf{u}_K^{n-1} \right) \cdot \mathbf{v}_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} [\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K}] \cdot \mathbf{v}_K \\ &- \sum_{K \in \mathcal{T}} p(\varrho_K^n) \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \mathbf{v}_\sigma \cdot \mathbf{n}_{\sigma,K} + \mu \sum_{K \in \mathcal{T}} \int_K \nabla \mathbf{u}^n : \nabla \mathbf{v} \, \mathrm{d}x \\ &+ \frac{\mu}{3} \sum_{K \in \mathcal{T}} \int_K \mathrm{div} \mathbf{u}^n \mathrm{div} \mathbf{v} \, \mathrm{d}x = 0,\ \text{for any } \mathbf{v} \in V_{h,0}(\Omega; R^3) \text{ and } n = 1, \dots, N. \end{aligned} \tag{26}$$

$$\|(r,\mathbf{V})\|_{X_T(\mathbb{R}^3)} \equiv \|r\|_{C^1([0,T]\times R^3)} + \|\partial_t \nabla_x r\|_{C([0,T];L^6(R^3;R^3))} + \|\partial^2_{t,t} r\|_{C([0,T];L^6(R^3))}$$

$$\|\mathbf{V}\|_{C^1([0,T]\times R^3;R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(R^3;R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{C([0,T];L^6(R^3;R^{3\times 3}))}$$

$$+\|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^6(R^3))} \text{ and } r \geq \underline{r} > 0.$$

Case $\Omega = \Omega_h$, Gallouet, Herbin, Maltese, N., 2015

Let $(\varrho_h^n, \mathbf{u}_h^n) = (\varrho, \mathbf{u})$ be a family of numerical solutions of the numerical scheme (24–25) with $\gamma \geq 3/2$. Let $(r, \mathbf{V})$ be a classical solution of the compressible Navier-Stokes equations (1–6) in the class $X_T(\mathbb{R}^3)$. Then there exists $c > 0$ independent of h, Δt, ϱ, $\mathbf{u}$, such that

$$\mathcal{E}(\varrho^n, \mathbf{u}^n \Big| r, U) \leq c\Big(\mathcal{E}(\varrho_0, \mathbf{u}_0 \Big| r_0, \mathbf{U}_0) + h^\alpha + \sqrt{\Delta t}\Big),$$

where

$$\alpha = \frac{2\gamma - 3}{2} \text{ if } 3/2 \leq \gamma < 2,\ \alpha = \frac{1}{2} \text{ if } \gamma \geq 2,$$

$$\mathcal{E}(\varrho^n, \mathbf{u}^n \Big| r, \mathbf{V}) = \sum_{K\in\mathcal{T}} \int_K \Big(\frac{1}{2}\varrho_K^n(\mathbf{V}^n - \hat{\mathbf{u}}_K^n)^2 + E(\varrho_K^n | r^n)\Big)$$

Case $\Omega \neq \Omega_h$, Feireisl, Hosek, Maltese, N., 2016

Suppose that Ω is a bounded domain of class C^3 and Ω_h verifies condition (22) Let $(\varrho_h, \mathbf{u}_h) = (\varrho, \mathbf{u})$ be a family of numerical solutions of the numerical scheme (24–25) with $\gamma \geq 3/2$. Let $(r, \mathbf{V})$ be a weak solution solution of the compressible Navier-Stokes equations (1–6) with *bounded* density r emanating from initial data
$(0 < r_0, \mathbf{V}_0) \in C^3(\overline{\Omega})$ that satisfy the compatibility condition

$$\mathbf{V}_0|_{\partial\Omega} = 0, \ \nabla p(r_0)|_{\partial\Omega} = [\mu \Delta \mathbf{V}_0 + \frac{\mu}{3} \nabla \mathrm{div} \mathbf{V}_0]|_{\partial\Omega}.$$

Then there exists $c > 0$ independent of h, Δt, ϱ, $\mathbf{u}$, such that

$$\mathcal{E}\left(\varrho^n, \mathbf{u}^n \middle| r, \mathbf{V}\right) \leq c\left(\mathcal{E}\left(\varrho_0, \mathbf{u}_0 \middle| r_0, \mathbf{V}_0\right) + h^\alpha + \sqrt{\Delta t}\right),$$

where

$$\alpha = \frac{2\gamma - 3}{2} \text{ if } 3/2 \leq \gamma < 2, \ \alpha = \frac{1}{2} \text{ if } \gamma \geq 2.$$

Convergence of the n. solutions to w. solutions

Case $\Omega = \Omega_h$, Karper, 2013, complemented for $\Omega \neq \Omega_h$ Michalek, Feireisl, Karper

Let $(\varrho_h, \mathbf{u}_h)$ be a family of numerical solutions of the numerical scheme (24–25) with $\Delta t = h$ and $\gamma > 3$. Then for a suitable subsequence

$$\varrho_h \rightharpoonup_* \varrho \text{ in } L^\infty(0,T;L^\gamma(\Omega)),$$

$$\mathbf{u}_h \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;L^6(\Omega)),\ \nabla_h \mathbf{u}_h \rightharpoonup \nabla_x \mathbf{u} \text{ in } L^2(Q_T)$$

where $(\varrho, \mathbf{u})$ is a weak solution of problem (1–6).

Case $\Omega \neq \Omega_h$, Feireisl, Hosek, Maltese, N 2016

Let $\Omega \subset R^3$ be a bounded domain of class C^3 and and let Ω_h verify condition (21). Let the initial data $[\varrho_0, \mathbf{u}_0]$ belong to the regularity class

$$\varrho_0 \in C^3(\overline{\Omega}),\ \ \varrho_0 > 0 \text{ in } \overline{\Omega},\ \ \mathbf{u}_0 \in C^3(\overline{\Omega}; R^3),$$

and satisfy the compatibility conditions

$$\mathbf{u}_0|_{\partial\Omega} = 0,\ \ \nabla_x p(\varrho_0)|_{\partial\Omega} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_0)|_{\partial\Omega}.$$

Let $\{\varrho_h^n, \mathbf{u}_h^n\}_{h>0}$, $k = 0, 1, \ldots, [T/\Delta t]$, $h \approx \Delta t$, be a family of numerical solutions satisfying (24–26). Finally, suppose that

$$\varrho_h^n \leq \overline{\varrho} < \infty \text{ for all } h > 0,\ \ n = 0, 1, \ldots, [T/\Delta t]. \tag{27}$$

Then problem (1–6) admits a classical solution $[\varrho, \mathbf{u}]$ in $(0, T) \times \Omega$, and

$$\operatorname*{ess\ sup}_{t \in (0,T)} \int_{\Omega \cap \Omega_h} \left[\varrho_h |\widehat{\mathbf{u}}_h - \mathbf{u}|^2 + |\varrho_h - \varrho|^2 \right] (t, \cdot)\, \mathrm{d}x + \int_0^T \int_{\Omega \cap \Omega_h} |\nabla_h \mathbf{u}_h - \nabla_x \mathbf{u}|^2 \, \mathrm{d}x\, \mathrm{d}t$$

$$\leq c \left(h^{1/2} + \int_\Omega \left[\varrho^0 |\widehat{\mathbf{u}}^0 - \mathbf{u}_0|^2 + |\varrho^0 - \varrho_0|^2 \right] \, \mathrm{d}x \right). \tag{28}$$

Energy inequality - discrete case

$$\sum_K \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n|\hat{\mathbf{u}}_K^n|^2 - \varrho_K^{n-1}|\hat{\mathbf{u}}_K^{n-1}|^2\Big) + \sum_K \frac{|K|}{\Delta t}\Big(H(\varrho_K^n) - H(\varrho_K^{n-1})\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\varrho_K^{n-1}\frac{|\hat{\mathbf{u}}_K^n - \hat{\mathbf{u}}_K^{n-1}|^2}{2} + \sum_K \frac{|K|}{\Delta t}H''(\varrho_K^{n-1,n})\frac{|\varrho_K^n - \varrho_K^{n-1}|^2}{2}$$

$$+\sum_K \sum_{\sigma\in\mathcal{E}_K} \frac{1}{4}|\sigma|\varrho_\sigma^{n,\mathrm{up}}(\hat{\mathbf{u}}_K^n - \hat{\mathbf{u}}_L^n)^2\ |\mathbf{u}_\sigma\cdot\mathbf{n}_{\sigma,K}|$$

$$+\sum_K \sum_{\sigma\in\mathcal{E}_K} \frac{1}{4}|\sigma|H''(\varrho_{KL}^n)\Big(\varrho_K^n - \varrho_L^n\Big)^2\ |\mathbf{u}_\sigma\cdot\mathbf{n}_{\sigma,K}|$$

$$+\sum_K \Big(\mu\int_K |\nabla_x\mathbf{u}|^2\mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K |\mathrm{div}\mathbf{u}|^2\mathrm{d}x\Big) \le 0.$$

Discrete relative energy

$$\sum_K \frac{1}{2}\frac{|K|}{\Delta t}\Big(\varrho_K^n|\hat{\mathbf{u}}_K^n - \hat{\mathbf{U}}_{h,K}^n|^2 - \varrho_K^{n-1}|\hat{\mathbf{u}}_K^{n-1} - \hat{\mathbf{U}}_{h,K}^{n-1}|^2\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\Big(E(\varrho_K^n|\hat{r}_K^n) - E(\varrho_K^{n-1}|\hat{r}_K^{n-1})\Big)$$

$$+\sum_K \Big(\mu\int_K |\nabla_x(\mathbf{u}^n - \mathbf{U}_h^n)|^2 \mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K |\mathrm{div}(\mathbf{u}^n - \mathbf{U}_h^n)|^2 \mathrm{d}x\Big)$$

$$\preceq \sum_K \Big(\mu\int_K \nabla_x\mathbf{U}_h : \nabla_x(\mathbf{U}_h - \mathbf{u})\mathrm{d}x + (\frac{\mu}{3}+\eta)\int_K \mathrm{div}\mathbf{U}_h \mathrm{div}(\mathbf{U}_h - \mathbf{u})\mathrm{d}x\Big)$$

$$+\sum_K \frac{|K|}{\Delta t}\Big(\varrho_K^{n-1}(\hat{\mathbf{U}}_{h,K}^{n-1} - \hat{\mathbf{u}}_{h,K}^{n-1})\cdot(\hat{\mathbf{U}}_{h,K}^n - \hat{\mathbf{U}}_{h,K}^{n-1}) + (\hat{r}_K^n - \varrho_K)(H'(\hat{r}_K^n) - H'(\hat{r}_K^{n-1})\Big)$$

$$+\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\Big(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}}\Big)\cdot\hat{\mathbf{U}}_{h,K}(\mathbf{u}^n\cdot\mathbf{n}_{\sigma,K})$$

$$-\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| p(\varrho_K)(\hat{\mathbf{U}}_{h,\sigma}^n\cdot\mathbf{n}_{\sigma,K}) - \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma|\varrho_\sigma^{n,\mathrm{up}} H'(\hat{r}_K^n)(\mathbf{u}^n\cdot\mathbf{n}_{\sigma,K})$$

Sketch of the proof : Treatment of the Red term

$$\sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot \mathbf{U}_{h,K} (\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)(\mathbf{u}_\sigma^n \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)(\hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K})$$

$$\approx \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma) \hat{\mathbf{U}}_\sigma^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}$$

$$+ \sum_K \sum_{\sigma\in\mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \left(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \right) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma)(\hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} - \hat{\mathbf{U}}_\sigma^{n,\mathrm{up}}) \cdot \mathbf{n}_{\sigma,K}$$

$$\approx \sum_{K} \sum_{\sigma \in \mathcal{E}_K} |\sigma| \varrho_\sigma^{n,\mathrm{up}} \Big(\hat{\mathbf{U}}_{h,\sigma}^{n,\mathrm{up}} - \hat{\mathbf{u}}_\sigma^{n,\mathrm{up}} \Big) \cdot (\mathbf{U}_{h,K} - \mathbf{U}_\sigma) \hat{\mathbf{U}}_\sigma^{n,\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}$$

$$\sum_K \int_K r \partial_t \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}) + \sum_K \int_K r \mathbf{U} \cdot \nabla \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}) + \ldots = \ldots$$

$$\sum_K \int_K r \mathbf{U} \cdot \nabla \mathbf{U} \cdot (\mathbf{u} - \mathbf{U}_h) \approx \sum_K \int_K r_K \mathbf{U}_{h,K} \cdot \nabla \mathbf{U} \cdot (\mathbf{u}_K - \mathbf{U}_{h,K})$$

$$\approx \sum_K \sum_{\sigma \in \mathcal{E}_K} \int_\sigma r_K \mathbf{U}_{h,K} \cdot \mathbf{n}_{\sigma,K} (\mathbf{U} - \mathbf{U}_{h,K}) \cdot (\mathbf{u}_K - \mathbf{U}_{h,K})$$

$$\approx \sum_K \sum_{\sigma \in \mathcal{E}_K} |\sigma| \, r_K \mathbf{U}_{h,K} \cdot \mathbf{n}_{\sigma,K} (\mathbf{U}_\sigma - \mathbf{U}_{h,K}) \cdot (\mathbf{u}_K - \mathbf{U}_{h,K})$$

$$\approx \sum_K \sum_{\sigma \in \mathcal{E}_K} |\sigma| \, \hat{r}_\sigma^{\mathrm{up}} (\mathbf{U}_\sigma - \mathbf{U}_{h,K}) \cdot (\hat{\mathbf{u}}_\sigma^{\mathrm{up}} - \hat{\mathbf{U}}_{h,\sigma}^{\mathrm{up}}) \hat{\mathbf{U}}_\sigma^{\mathrm{up}} \cdot \mathbf{n}_{\sigma,K}$$

Projections

Projections

$$\Pi_h^V : W^{1,p}(\Omega) \to V_h, \quad \Pi_h^V(U) \equiv U_h \equiv \sum_{\sigma\in\mathcal{E}} \hat{U}_\sigma \phi_\sigma$$

$$\Pi_h^L : L^p(\Omega) \to L_h, \quad \Pi_h^L(r) \equiv r_h \equiv \sum_{K\in\mathcal{T}} \hat{r}_K 1_K$$

Estimates involving projections

Let $s = 1, 2$, $1 \leq p \leq \infty$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall r \in W^{1,p}(K), \quad \|r_h - r\|_{L^p(K)} \leq ch\|\nabla_x r\|_{L^p(K)},$$

$$\forall U \in W^{s,p}(K), \quad \|U_h - U\|_{L^p(K)} \leq ch^s\|\nabla_x^s U\|_{L^p(K)}$$

$$\forall U \in W^{s,p}(K), \quad \|\nabla_x U_h - \nabla_x U\|_{L^p(K)} \leq ch^{s-1}\|\nabla_x^s U\|_{L^p(K)}.$$

Some auxiliary estimates

Poincaré type inequalities

Let $1 \le p \le \infty$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall U \in W^{1,p}(K), \quad \|U_h - \hat{U}_{h,K}\|_{L^p(K)} \le ch\|\nabla_x U_h\|_{L^p(K)},$$

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_K\|_{L^p(K)} \le ch\|\nabla_x U\|_{L^p(K)}$$

$$\forall U \in W^{1,p}(K), \quad \|U_h - \hat{U}_\sigma\|_{L^p(K)} \le ch\|\nabla_x U\|_{L^p(K)}.$$

Sobolev type inequalities

Let $2 \le p \le 6$. There exists $c > 0$ independent of h such that for all $K \in \mathcal{T}$:

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_K\|_{L^p(K)} \le ch^{\frac{3}{p}-\frac{1}{2}}\|\nabla_x V\|_{L^2(K)},$$

$$\forall U \in W^{1,p}(K), \quad \|U - \hat{U}_\sigma\|_{L^p(K)} \le h^{\frac{3}{p}-\frac{1}{2}}\|\nabla_x U\|_{L^2(K)}$$

Compressible Navier-Stokes equations in low Mach number regime

We consider in $[0,T) \times \Omega$, $\Omega \subset R^3$ (a bounded Lipschitz domain or a periodic cell) the following system of equations

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \qquad (29)$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \qquad (30)$$

Boundary conditions

$$\mathbf{u}\Big|_{(0,T)\times\partial\Omega} = 0 \quad \text{or periodic b.c.} \qquad (31)$$

Initial conditions

$$\varrho(0,x) \equiv \varrho_{0,\varepsilon}(x) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}(x), \quad \mathbf{u}(0,x) = \mathbf{u}_{0,\varepsilon}(x). \qquad (32)$$

Expected target system as $\varepsilon \to 0$

Incompressible Navier-Stokes equations

$$\overline{\varrho}\Big(\partial_t \mathbf{V} + \mathbf{V}\cdot\nabla_x \mathbf{V}\Big) + \nabla\Pi = 0, \quad \mathrm{div}\mathbf{V} = 0.$$

$\mathbf{V}(0) = \mathbf{V}_0,$ homogenous Dirichlet or periodic b.c.

Ill and Well prepared initial data

Ill Prepared :

$$\mathcal{E}_\varepsilon(\varrho_0, \mathbf{u}_0 | \overline{\varrho}, \mathbf{V}_0) = \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{V}_0|^2 + \frac{1}{\varepsilon^2}E(\varrho_0|\overline{\varrho})\Big)\mathrm{d}x$$

is bounded as $\varepsilon \to 0$.

Well prepared :

$$\mathcal{E}_\varepsilon(\varrho_0, \mathbf{u}_0 | \overline{\varrho}, \mathbf{V}_0) = \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{V}_0|^2 + \frac{1}{\varepsilon^2}E(\varrho_0|\overline{\varrho})\Big)\mathrm{d}x$$

tends to 0 as $\varepsilon \to 0$.

A result of Lions/Masmoudi

Consider the sequence of weak solutions $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ of CNSE corresponding to ill prepared initial data $(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})$ such that $\mathbf{u}_{0,\varepsilon} \rightharpoonup \mathbf{u}_0$ in $W^{1,2}(\Omega)$. The there is a subsequence such that

$$\varrho_\varepsilon \to \overline{\varrho}, \quad \mathbf{u}_\varepsilon \rightharpoonup \mathbf{V}$$

and there is $\Pi \in \mathcal{D}'((0,T) \times \Omega)$ such that the couple $(\Pi, \mathbf{V})$ is a weak solution to the NSE with $\mathbf{V}_0 = \mathbf{H}(\mathbf{u}_0)$.

A result via the relative energy

Let $(\Pi, \mathbf{V})$ be a strong solution to the NSE with initial data $\mathbf{V}_0$ on interval $[0,T)$ and $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ a weak solution of CNSE with ill prepared initial data. Then there is c independent of ε such that

$$\mathcal{E}_\varepsilon(\varrho_\varepsilon, \mathbf{u}_\varepsilon | \overline{\varrho}, \mathbf{V})(\tau) \leq c\Big(\mathcal{E}_\varepsilon(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} | \overline{\varrho}, \mathbf{V}_0) + \varepsilon\Big)$$

$$\|(\Pi, \mathbf{V})\|_{X_T(\Omega)} \equiv \|\Pi\|_{C([0,T];C(\overline{\Omega}))} + \|\partial_t \Pi\|_{L^1(0,T;L^p(\Omega))} +$$
$$\|\mathbf{V}\|_{C^1([0,T]\times\overline{\Omega};R^3)} + \|\mathbf{V}\|_{C([0,T];C^2(\overline{\Omega};R^3))} + \|\partial_t \nabla_x \mathbf{V}\|_{L^2(0,T;L^{6/5}(\Omega;R^{3\times 3}))}$$
$$+\|\partial^2_{t,t}\mathbf{V}\|_{L^2(0,T;L^{6/5}(\Omega))} \text{ and } r \geq \underline{r} > 0.$$

FV/CR scheme Feireisl, Medvidova, Necasova, She, N., 2016

Let $(\varrho_h^n, \mathbf{u}_h^n) = (\varrho, \mathbf{u})$ be a family of numerical solutions of the FV/CR numerical scheme (24–25) with initial data $(\varrho^0, \mathbf{u}^0)$ obeying

$$\mathcal{E}_\varepsilon(\varrho^0, \mathbf{u}^0 | \overline{\varrho}, \mathbf{U}_0) \leq E_0 < \infty, \quad M/2 \leq \int_\Omega \varrho_0 \leq 2M, \quad M = \overline{\varrho}|\Omega|$$

and with $\gamma \geq 3/2$. Let $(\Pi, \mathbf{V})$ be a classical solution of the compressible Navier-Stokes equations (1–6) in the class $X_T(\Omega)$ corresponding to the initial velocity $\mathbf{U}_0$. Then there exists $c > 0$ independent of h, Δt, ε ϱ, $\mathbf{u}$, such that

$$\mathcal{E}_\varepsilon(\varrho^n, \mathbf{u}^n \Big| r, U) \leq c\Big(\mathcal{E}_\varepsilon(\varrho_0, \mathbf{u}_0 \Big| \overline{\varrho}, \mathbf{U}_0) + h^\alpha + \sqrt{\Delta t} + \varepsilon\Big),$$

where

$$\alpha = \min\left\{\frac{2\gamma - 3}{\gamma}, 1\right\}.$$

FV/FD - MAC scheme Gallouet, Herbin, Maltese, N., 2016

The same result holds for the FE/FD scheme :

$$\mathcal{E}_\varepsilon\left(\varrho^n, \mathbf{u}^n \middle| r, U\right) \leq c\left(\mathcal{E}_\varepsilon\left(\varrho_0, \mathbf{u}_0 \middle| \overline{\varrho}, \mathbf{U}_0\right) + h^\alpha + \sqrt{\Delta t} + \varepsilon\right).$$

Recall :

$$\mathcal{E}_\varepsilon\left(\varrho^n, \mathbf{u}^n \middle| r, \mathbf{V}\right) = \sum_{K\in\mathcal{T}} \int_K \left(\frac{1}{2}\varrho_K^n(\mathbf{V}^n - \hat{\mathbf{u}}_K^n)^2 + \frac{1}{\varepsilon^2}E(\varrho_K^n|r^n)\right)$$