Expansion of an inviscid isentropic fluid in vacuum

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or

Why do we care about T.-P. Liu's

Physical singularity at the boundary ?

Euler equations

Density ρ, velocity u, pressure p, internal energy e :

$$\begin{aligned} \partial_t \rho + \mathrm{div}(\rho u) &= 0, & (1) \\ \partial_t(\rho u) + \mathrm{Div}(\rho u \otimes u) + \nabla p(\rho) &= 0. & (2) \end{aligned}$$

Equation of state (ideal gas)

$$p = A\rho^{\gamma}, \qquad e = \frac{p}{(\gamma - 1)\rho} \qquad (\gamma > 1).$$

For smooth flows, (1,2) imply

$$\partial_t \left(\frac{1}{2}\rho|u|^2 + \rho e \right) + \mathrm{div} \left((\frac{1}{2}\rho|u|^2 + \rho e + p)u \right) = 0. \qquad (3)$$

Initial data. $\rho u(x,0) = m_0(x)$ and $\rho(x,0) = \rho_0(x)$. Equivalently,

$$c(x,0) = c_0(x), \qquad c := \sqrt{\frac{dp}{d\rho}} = a\rho^\kappa \quad \text{the sound speed.}$$

Cauchy problem with vacuum around.

$$\Omega(t) = \{x \in \mathbb{R}^d \,|\, \rho(x,t) > 0\} \qquad \text{bounded, but unknown.}$$

Match the vacuum. Rankine–Hugoniot gives

$$\rho(u \cdot \nu - \sigma) = \mathbf{0}, \qquad \rho(u \cdot \nu - \sigma)u + p(\rho)\nu = \mathbf{0},$$

with ν the unit normal to $\partial\Omega(t)$, σ the normal velocity of the boundary.

$$\Longrightarrow \quad \rho = 0 \qquad \text{on} \quad \partial\Omega(t).$$

Boundary shocks are impossible.

Classical solutions. Local theory

Needs a symmetrization (Godunov, ...)

Makino, Ukai & Kawashima (1986) found a non-singular symmetrization:

$$\begin{pmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{pmatrix} (\partial_t + u \cdot \nabla) \begin{pmatrix} c \\ u \end{pmatrix} + c \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0_d \end{pmatrix} \begin{pmatrix} c \\ u \end{pmatrix} = 0.$$

with $\kappa := \frac{\gamma-1}{2}$.

> Implies locally well-posedness of the Cauchy problem in variables (c, u), even in presence of vacuum !!

$\ldots \longrightarrow$

Theorem 1 (MUK, Chemin.) *Let* $s > 1 + \frac{d}{2}$ *be given. Let the initial data* $(u_0, c_0) \in H^s(\mathbb{R}^d)$ *be such that* $c_0 \geq 0$. *Then there exists* $T^* > 0$ *and a unique solution*

$$(u, c) \in \mathcal{C}([0, T^*); H^s(\mathbb{R}^d)) \cap \mathcal{C}^1([0, T^*); H^{s-1}(\mathbb{R}^d))$$

of the Cauchy problem.

♣

These are classical solutions (Sobolev: $H^s \subset \mathcal{C}^1$, $H^{s-1} \subset \mathcal{C}^0$),

$$(u, c) \in \mathcal{C}^1([0, T^*) \times \mathbb{R}^d).$$

Linear growth:

$$\Omega(t) \subset \Omega(0) + B_{|t|M}, \qquad \text{where} \quad M := \sup_{x,t} |u|.$$

Questions $\longrightarrow \ldots$

- **Can the H^s-solution be a global one ?** (Is $T^* = +\infty$ possible?)

- **When it is not, is it sufficient to incorporate shock wave theory ?**

A positive result

Warm-up: an academic case. If $c_0 \equiv 0$ (no gas at all !) then $c \equiv 0$, and the system reduces to

$$\partial_t u + (u \cdot \nabla) u = 0,$$

a *vector*-Burgers equation.

Method of characteristics: $\frac{dX}{dt} = u(X,t)$, $\frac{du}{dt} = 0$ yields

$$\frac{d\nabla u}{dt} + (\nabla u)^2 = 0,$$

a Ricatti equation in $\mathbf{M}_d(\mathbb{R})$!

The Ricatti ODE

$$M' + M^2 = 0, \qquad M(0) = M_0$$

yields

$$Y' + MY = 0 \qquad \text{for} \qquad Y(t) := M(t)(I_d + tM_0) - M_0,$$

whence $Y \equiv 0_d$:

$$M(t) = M_0(I_d + tM_0)^{-1}.$$

Its solution is global (up to $t = +\infty$) *iff*

$$\sigma(M_0) \bigcap (-\infty, 0) = \emptyset.$$

Lemma 1 *Let* $\nabla u_0 \in H^{s-1}(\mathbb{R}^d)$ *be given, with* $s > 1 + \frac{d}{2}$ *.The classical solution of the vector-Burgers equation exists for all* $t > 0$ *iff*

$$\sigma(\nabla u_0) \bigcap (-\infty, 0) = \emptyset, \qquad \forall x \in \mathbb{R}^d.$$

Typically, $\nabla u(X(t), t) \sim \frac{1}{t} I_d$.

A perturbation argument shows that the dispersion may dominate the non-linearity:

Theorem 2 (M. Grassin 1988.) *Suppose*

$$\min_x \mathrm{dist}(\sigma(\nabla u_0(x)); \mathbb{R}^-) > 0.$$

There exists $\delta > 0$ such that if $\|c_0\|_{H^s} < \delta$, then the classical solution exists for all $t > 0$.

◇

A negative result.

The boundary $\partial\Omega$ is convected by the flow. For a classical solution, trajectories along $\partial\Omega$ satisfy

$$\frac{dX}{dt} = u(X,t), \qquad \frac{du}{dt} = -\frac{c}{\kappa}\nabla c = \mathbf{0}.$$

Hence the flow coincides, at the boundary, with that of the free transport:

$$\phi^t(a) \equiv a + tu_0(a) =: \psi^t(a), \qquad \forall a \in \partial\Omega_0.$$

Theorem 3 (*à la* J. Milnor.) *The volume $Q(t)$ of $\Omega(t)$ is a polynomial function, of degree $\leq d$.*

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Proof
$$\Omega(t) = \phi^t(\Omega_0).$$

$$\begin{aligned}
\mathrm{vol}\,\Omega(t) &= \int_{\Omega_0} \mathrm{Jac}(\phi^t)\,dx = \int_{\Omega_0} \mathrm{div}(\phi_1^t \nabla\phi_2^t \wedge \cdots \wedge \nabla\phi_d^t)\,dx \\
&= \int_{\partial\Omega_0} \phi_1^t(\nabla\phi_2^t \wedge \cdots \wedge \nabla\phi_d^t)\cdot\nu_x\,ds(x) \\
&= \int_{\partial\Omega_0} \phi_1^t \det(\nabla\phi_2^t, \cdots, \nabla\phi_d^t, \nu_x)\,ds(x)
\end{aligned}$$

But the restrictions of ϕ^t and ψ^t to $\partial\Omega_0$ coincide :

$$\nabla\phi_j^t = \nabla\psi_j^t + a_j(t,x)\nu_x.$$

Hence

$$\begin{aligned}
\mathrm{vol}\,\Omega(t) &= \int_{\partial\Omega_0} \psi_1^t \det(\nabla\psi_2^t, \cdots, \nabla\psi_d^t, \nu_x)\,ds(x) \\
&= \int_{\partial\Omega_0} \psi_1^t(\nabla\psi_2^t \wedge \cdots \wedge \nabla\psi_d^t)\cdot\nu_x\,ds(x).
\end{aligned}$$

The integrand is a polynomial in t, of degree $\leq d$.

Q.E.D.

Dominant term

$$Q(t) = Jt^d + \text{ l.o.t.}\,,$$

with

$$\begin{aligned} J \;&:=\; \int_{\partial\Omega_0} u_{01}(\nabla u_{02} \wedge \cdots \wedge \nabla u_{0d}) \cdot \nu_x \, ds(x) \\ &=\; \int_{\Omega_0} \mathsf{Jac}(u_0)\, dx. \end{aligned}$$

Recalling that $Q(t) \geq 0$, we infer

Theorem 4 *Suppose*

$$\int_{\Omega_0} \mathrm{Jac}(u_0)\, dx < 0.$$

Then the classical solution is only local-in-time:

$$T^* < \infty.$$

Remark. Thm 4 needs only that (u, c) be smooth at the boundary $\partial\Omega(t)$. Shocks are welcome in $\Omega(t)$!

The limit case $J = 0$

Theorem 5 ($\gamma \le 1 + \frac{2}{d-1}$.) *Let (u, ρ) be an admissible* solution of the isentropic Euler system , with (u, c) smooth at the boundary. Suppose*

$$\int_{\Omega_0} \mathrm{Jac}(u_0)\, dx \le 0.$$

Then the solution is only local: $T^* < \infty$.

♣

Allowing shock waves is not sufficient to have global solutions !!

Proof.

The case $J < 0$ has been treated above. Suppose instead $J = 0$. Then

$$Q(t) = |\Omega(t)| \in \mathbb{R}[t], \qquad \deg Q \le d - 1.$$

$\cdots \longrightarrow$

*In the sense of the energy inequality.

Lemma 2 (Dispersion.) If $\gamma \leq 1 + \frac{2}{d}$**,** *then*

$$\int \rho^\gamma dx \leq C(1+t)^{-2d\kappa}.$$

If instead $1 + \frac{2}{d} < \gamma$**,** *then*

$$\int_0^{+\infty} t\,dt \int \rho^\gamma dx < \infty.$$

Proof based upon

$$\begin{aligned} \frac{d}{dt}\int \left(\frac{1}{2}\rho|tu-x|^2 + t^2\rho e\right)\,dx &= \frac{d}{dt}\Big[t^2\int\left(\frac{1}{2}\rho|u|^2+\rho e\right)\,dx \\ &\quad -t\int \rho u\cdot x\,dx + \int \frac{|x|^2}{2}\rho\,dx\Big] \\ &\leq t\int (2\rho e - dp)\,dx = 2t(1-d\kappa)\int \rho e\,dx. \end{aligned}$$

Use $\rho e = \mathrm{cst}\cdot\rho^\gamma$, plus the Gronwall inequality.

Proof of Theorem 5 (say $\gamma \leq 1 + \frac{2}{d}$).

Let M be the total mass. Apply Hölder inequality and $\deg Q \leq d-1$:

$$M = \int \rho(t,x)\,dx \;\leq\; |\Omega(t)|^{1-1/\gamma} \left(\int \rho^\gamma \, dx \right)^{1/\gamma}$$
$$\leq\; C(1+t)^{(d-1)(1-1/\gamma)} \left(\int \rho^\gamma \, dx \right)^{1/\gamma}.$$

Dispersion gives

$$M \;\leq C(1+t)^{(d-1)(1-1/\gamma)-2d\kappa/\gamma} = \; C(1+t)^{\frac{1}{\gamma}-1}$$

$$\stackrel{t\to+\infty}{\longrightarrow} 0.$$

But $M > 0$ is a constant ; *contradiction* ! Q.E.D.

The case of eternal flows

Definition 1 *A solution $U(t)$ of a dynamical system is eternal if it is defined for all $t \in \mathbb{R}$.*

... thus not only for all $t > 0$.

Examples of eternal solutions:

- Steady, periodic, homoclinic, heteroclinic solutions,
- (For PDEs) travelling waves and solitons, $U(x,t) = \phi(x - ct)$.
- Multi-solitons (KdV),

- Interacting fronts (Hamel & Nadirashvili, 1999) in reaction-diffusion equations,

- Interacting viscous shock fronts (D. S. 1998).

- Global Maxwellians of the Boltzmann equation (D. Levermore 2014).

- (Averaging the former) Eternal isothermal flow with Gaussian density.

Question : *Do there exist eternal solutions of the isentropic Euler equations, compactly supported in space ?*

A dichotomy d even/odd

d **even (existence).** One may choose $\nabla u_0 \in H^{s-1}_{\text{loc}}(\mathbb{R}^d)$ such that

$$\sigma(\nabla u_0(x)) \cap \mathbb{R} = \emptyset, \qquad \forall\, x. \tag{4}$$

If $\|c_0\|_{H^s} << 1$, then Grassin's Theorem provides an eternal smooth solution (u, c). *Example* : $u_0(x) = x^\perp$.

d **odd (non-existence).** The choice (4) is not possible. Instead, let (ρ, u) be admissible, eternal, with (u, c) smooth at the boundary $\partial\Omega$. The volume $Q \in \mathbb{R}[t]$ satisfies $Q \geq 0$, hence has an *even* degree.

But deg $Q \leq d$, hence

$$\deg Q \leq d - 1.$$

Again, dispersion and $t \to +\infty$ yields a contradiction.

Resolving a paradox

We have seen that the regularity of c at the boundary, in the sense that $c\nabla c$ vanishes, is an obstacle to the existence of global solutions

... because it does not always let the volume of the domain grow as t^d,

... even if we allow shock waves.

However, ...

The ~~show~~ flow must go on !

Whence the need of $\cdots \longrightarrow$

Physical singularities at the boundary (after T.-P. Liu)

The volume $\Omega(t)$ has to grow like t^d (at least) to cope with the dispersion estimate and the conservation of mass ...

... even if $J < 0$.

Some kind of singularity may/must form at the front, sooner or later ...

... but we know that boundary shocks don't exist. So what ?

Revisit the particle trajectories

$$\frac{dX}{dt} = u(X(t), t), \qquad \frac{d^2X}{dt^2} = u_t + (u \cdot \nabla)u = -\frac{1}{\gamma - 1}\nabla(c^2).$$

If c is only $\frac{1}{2}$-Hölder (instead of Lipschitz), the front experiences a normal acceleration

$$g := -\frac{1}{\gamma - 1}\frac{\partial(c^2)}{\partial\nu} \geq 0.$$

Notice the sign: c^2 is > 0 in the interior and vanishes at the boundary:

the front is *accelerated.*

Local existence of shock-free solutions with physical singularity:

$d = 1$**.** Juhi Jang, N.Masmoudi. *Commun. Pure Appl. Math.*, **62** (2009), pp 1327–1385.

D. Coutand, S. Shkoller. *Commun. Pure Appl. Math.*, **64** (2011), pp 328–366.

$d \geq 2$**.** Juhi Jang, N. Masmoudi. *Commun. Pure Appl. Math.*, **68** (2015), pp 61–111.

D. Coutand, S. Shkoller. *Archive Rat. Mech. Anal.*, **206** (2012), pp 515–616.

Spherical symmetry : Tao Luo, Zhouping Xin & Huihui Zeng. *Archive Rat. Mech. Anal.*, **213** (2014), pp 763–831.

Physical singularity $(d = 1)$

Domain

$$\Omega(t) = (a(t), b(t)).$$

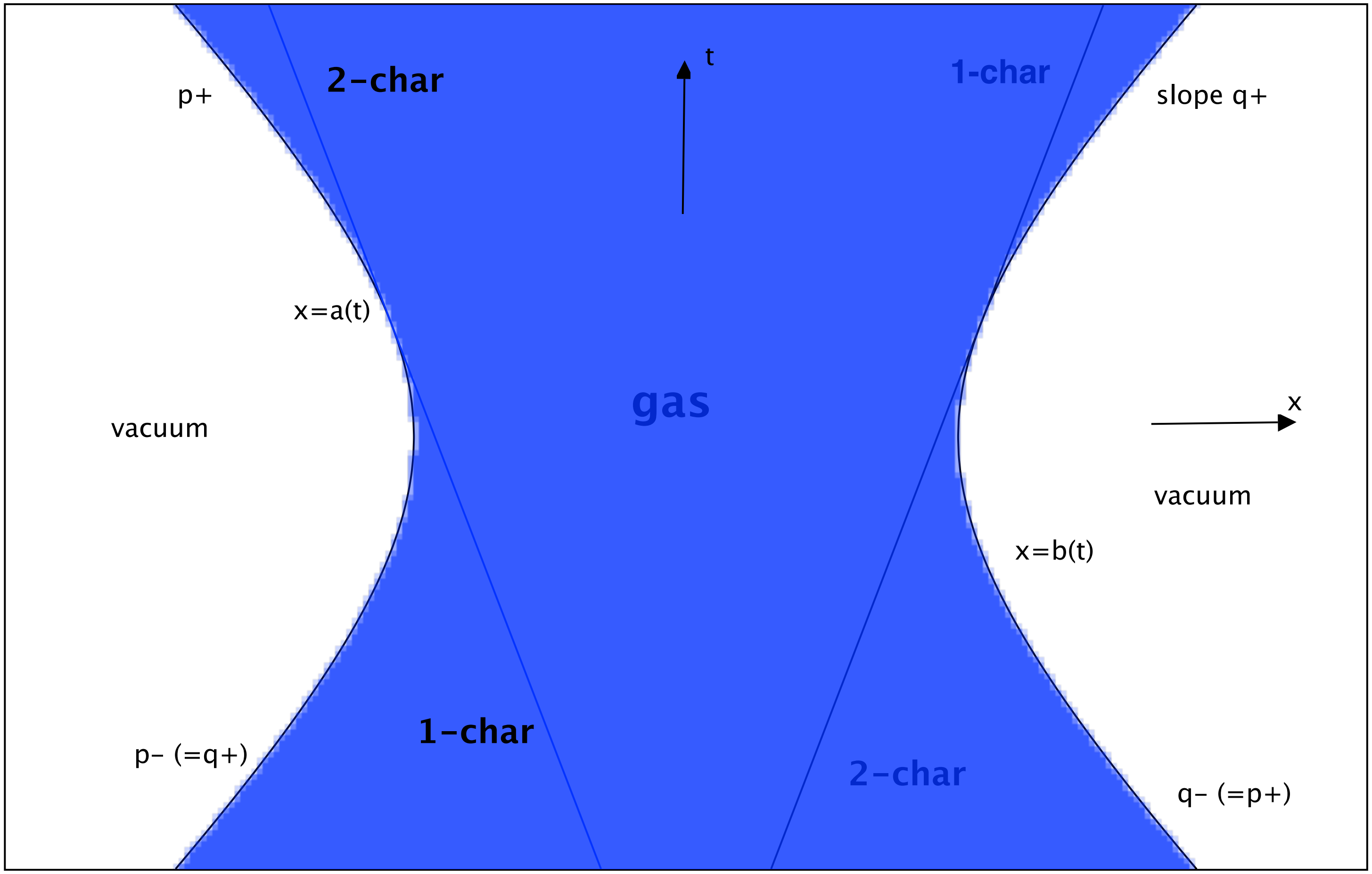
Due to acceleration, $t \mapsto b(t)$ is convex ; a is concave.

In Ω, (ρ, u) is an entropy solution of the Euler system.

At almost every $P \in \partial\Omega$,

$$\lim_{(x,t)\to P} \rho(x,t) = 0.$$

A 2-characteristic may emanate from $(a(t), t)$ or may terminate at $(b(t), t)$, but not the converse. Switch $a \longleftrightarrow b$ for 1-characteristics.



Shock-free 1-D gas bubble with physical singularity at the boundary.

The (space-time) domain is hyperbola-shaped.

One-dimensional scattering

Let (ρ, u) be an eternal flow.

Define (remember the convexity)

$$p_\pm = \lim_{t \to \pm\infty} a'(t), \qquad q_\pm = \lim_{t \to \pm\infty} b'(t).$$

Theorem 6 ($d = 1$, $\gamma > 1$.) *If the flow is shock-free in Ω and has physical singularity at the front, then*

$$p_+ = q_-, \qquad p_- = q_+.$$

$\heartsuit$

In other words

$$\Omega(t) \sim t(q_-, q_+), \qquad |t| \to +\infty.$$

Similar to flows that are smooth at the boundary (d even) : $\Omega(t) \sim t u_0(\Omega(0))$.

Mono-atomic gas

For $\gamma = 3$, the system is made of two Burgers equations

$$\partial_t r_\pm + r_\pm \partial_x r_\pm = 0, \qquad r_\pm := u \pm c.$$

Coupled only at shocks and at the vacuum front !

Example. Domain

$$|x| < \sqrt{1+t^2}$$

with flow

$$\rho = \frac{\sqrt{1+t^2-x^2}}{1+t^2}, \qquad u = \frac{tx}{1+t^2}\,.$$

Theorem 7 ($d = 1$, $\gamma = 3$.) *For the solution with physical boundary to be eternal and shock-free, it is necessary and sufficient that the initial data $r_0 = u_0 + c_0$ be concave and $s_0 = u_0 - c_0$ convex.*

Given an equation $F(x, v) < 0$ (F convex) of the domain

$$\{(x, v) \mid s_0(x) < v < r_0(x)\},$$

the solution is given by

$$F(x - vt, v) = 0, \qquad v = u \pm c.$$

♠

Thank you for attention

Très bon anniversaire Jérôme !