Exercice 9

1) $f_n(x) = \dfrac{1}{n+x^n}$, $I = [1,2]$

$f_n \searrow \Rightarrow \|f_n\|_{\infty,[1,2]} = \dfrac{1}{n}$ pas de CVN.

2) $f_n(x) = \dfrac{\sin nx}{n}$, $I = \mathbb{R}$

$\|f_n\|_{\infty,\mathbb{R}} = \dfrac{1}{n}$ pas de CVN

3) $f_n(x) = e^{-nx^2}$, $I = \mathbb{R}$

$\|f_n\|_{\infty,\mathbb{R}} = f_n(0) = 1$, pas de CVN.

Exercice 10

1) $f_n(x) = \dfrac{1}{xn^2}$, $I =]0,+\infty[$

$f_n(1/n^2) = 1 \not\to 0$. Pas de CVN ni de CVU.

2) $f_n(x) = 2^{-n/x}$, $I =]0,+\infty[$

$f_n(n) = 1/2$ ✓

3) $f_n(x) = n^2 x^n$ $I = [0,1[$

$f_n(1-\frac{1}{n}) = n^2 \underbrace{(1-\frac{1}{n})^n}_{\to e^{-1}} \to +\infty$

4) $f_n(x) = e^{-nx}$, $I =]0,+\infty[$

$f_n(1/n) = e^{-1}$

5) $f_n(x) = \dfrac{n+x^2}{n^4+x^2}$, $I = [0,+\infty[$

$f_n(n^2) = \dfrac{n+n^4}{2n^4} = \dfrac{1}{2n^3} + 1$ ✓

Exercice 11

1) $f_n(x) = \dfrac{(-1)^n e^{-nx}}{n+1}$, $I =]0,+\infty[$

$|f_n(1/n)| = \dfrac{e^{-1}}{n+1}$

2) $f_n(x) = \dfrac{n+x^3}{n^4+x^4}$, $I = [0,+\infty[$

$f_n(n) = \dfrac{n+n^3}{2n^4} = \dfrac{1}{2n^3} + \dfrac{1}{2n}$.

Exercice 12

1) $f_n(x) = \dfrac{1}{n^2 x}$, $I = [1,2]$

$\|f_n\|_{\infty,[1,2]} = \dfrac{1}{n^2}$

2) $f_n(x) = \dfrac{(-1)^n e^{-nx}}{n+1}$, $I = [1,+\infty[$

$\|f_n(x)\|_{\infty,[1,+\infty[} = \dfrac{e^{-n}}{n+1}$, la série $\sum_n \dfrac{e^{-n}}{n+1}$ converge.

3) $f_n(x) = \dfrac{n+x^2}{n^4+x^2}$, $I = [0,1]$

$f_n'(x) = \dfrac{2x(n^4+x^2) - (n+x^2)2x}{(n^4+x^2)^2} = \dfrac{2x(n^4-n)}{(n^4+x^2)^2} \geq 0$

$f_n \uparrow$, $\|f_n\|_{\infty,[0,1]} = \dfrac{n+1}{n^4+1}$

Exercice 13

1) $f_n(x) = \dfrac{n+x}{n^4+x^2}$, $I = [0,+\infty[$

Si $x \leq n^2$ $f_n(x) \leq \dfrac{n+n^2}{n^4} = \dfrac{1}{n^3} + \dfrac{1}{n^2}$

Si $x \geq n^2$ $f_n(x) \leq \dfrac{n+x}{x^2} = \dfrac{n}{x^2} + \dfrac{1}{x} \leq \dfrac{n}{n^4} + \dfrac{1}{n^2} = \dfrac{1}{n^3} + \dfrac{1}{n^2}$.

2) $f_n(x) = e^{-n(x^2+1)} + e^{-n((x+1)^2+1)}$, $I = \mathbb{R}$

$f_n(x) \leq e^{-n} + e^{-n}$ ✓

3) $f_n(x) = \dfrac{x^n}{n^2}$, $I = [0,a]$
$a < 1$

$f_n(x) \leq \dfrac{a^n}{n^2}$.

Exercice 14

1) $f_n(x) = \dfrac{(-1)^n}{x+\sqrt{n}}$ $\quad I = \mathbb{R}^+$

- $\dfrac{1}{x+\sqrt{n}} \searrow_0$ et $\leq \dfrac{1}{\sqrt{n}}$ donc CV unif vers 0
- $\sum_{k=0}^{n} (-1)^k \leq 2$

2) $f_n(x) = \dfrac{\sin(nx)}{x^2+n+1}$, $I = [\pi/2, \pi]$

- $\dfrac{1}{x^2+n+1} \searrow_0$ et $\leq \dfrac{1}{n+1}$
- $\sum_{k=0}^{n} \sin(kx) = \dfrac{\sin^2\left(\frac{n+1}{2}\right)x}{\sin x/2} \leq \dfrac{1}{\sin \pi/4}$

3) $f_n(x) = \dfrac{e^{inx}}{nx}$, $I = [1, \pi]$

- $\dfrac{1}{nx} \searrow_0 \leq \dfrac{1}{n}$
- $\sum_{k=0}^{n} e^{ikx} = \dfrac{1-e^{i(n+1)x}}{1-e^{ix}} = \dfrac{e^{i\frac{n+1}{2}x}}{e^{ix/2}} \quad \dfrac{\sin\frac{n+1}{2}x}{\sin x/2}$

$$\left|\sum_{k=0}^{n} e^{ikx}\right| \leq \frac{1}{\sin 1/2}$$

4) $f_n(x) = \dfrac{(-1)^n e^{-nx}}{n+1}$, $I = \mathbb{R}^+$

- $\dfrac{e^{-nx}}{n+1} \searrow_0 \leq \dfrac{1}{n+1}$
- $\left|\sum_{k=0}^{n} (-1)^k\right| \leq 2$

Exercice 15

1) $\sum x^n$ CV sur $]-1,1[$ et $\sum_n n x^{n-1}$ CVN sur $[-a,a] \subset]-1,1[$

2) $\sum \dfrac{\sin nx}{n^2}$ cv sur $\mathbb{R}$ et $\sum \dfrac{\cos nx}{n}$ CVU sur $\mathbb{R}$ par Abel Uniforme.

3) $\sum \dfrac{\sin nx}{n^3}$ cv sur $\mathbb{R}$ et $\sum \dfrac{\cos nx}{n^2}$ CVN sur $\mathbb{R}$

4) $x \mapsto e^{-xn}$ continue
et $e^{-xn} \leq e^{-an}$ sur $[a,b]$.