ORTHOGONAL SPACES ASSOCIATED TO EXPONENTIALS OF INDICATOR FUNCTIONS ON FOCK SPACE

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A . — Parthasarathy and Sunder have proved in [P-S] that the set of coherent vectors associated to the indicator function of Borel sets is total in the boson Fock space $\Gamma(L^2(\mathbb{R}^+;\mathbb{C}))$. In this article we study the space generated by coherent vectors associated to the union of n intervals. We give a complete characterization of their orthogonal space in terms of their chaos expansion. By the way, we recover Parthasarathy -Sunder's result in a very simple way. In the cases of the Brownian motion or Poisson process interpretation of the Fock space, our result characterizes those random variables which are orthogonal to the exponential of any sum of n increments of the Brownian motion or Poisson process.

I – The Fock space and main results

Let $\mathcal{P}$ be the set of finite subsets of $\mathbb{R}^+$. Then $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ where $\mathcal{P}_n$ is the set of n-element subsets of $\mathbb{R}^+$ (with $\mathcal{P}_0 = \{\emptyset\}$). The set $\mathcal{P}_n$ can be clearly identified to the simplex $\Sigma_n = \{0 < t_1 < \cdots < t_n \in \mathbb{R}\}$ and thus inherits the Lebesgue measure structure of $\mathbb{R}^n$. By putting the Dirac mass $\delta_\emptyset$ on $\mathcal{P}_0$, we have finally equipped $\mathcal{P}$ with a measured space structure. Elements of $\mathcal{P}$ are denoted by small greek letters σ, α, β, $\cdots$ Any element f of $L^2(\mathcal{P};\mathbb{C})$ is thus of the form

$$f = \sum_{n\in\mathbb{N}} f_n$$

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with $f_0 \in \mathbb{C}$ and $f_n \in L^2(\Sigma_n; \mathbb{C})$, $n \geq 0$ and

$$\begin{aligned}
\|f\|^2_{L^2(\mathcal{P})} &= |f_0|^2 + \sum_{n=1}^{\infty} \|f_n\|^2_{L^2(\Sigma_n)} \\
&= |f_0|^2 + \sum_{n=1}^{\infty} \int_{0<t_1<\cdots<t_n} |f_n(t_1, \ldots, t_n)|^2 \, dt_1 \cdots dt_n \, .
\end{aligned}$$

The above expression is simply denoted, with obvious notations,

$$\|f\|^2_{L^2(\mathcal{P})} = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma \, .$$

The space $L^2(\mathcal{P})$ is denoted Φ and called the Fock space (it is the usual symmetric, or boson, Fock space over $L^2(\mathbb{R}^+)$).

The following lemma is very helpfull (cf. [L-P]).

Σ- . — *If f is a positive (resp. integrable) measurable function on $\mathcal{P} \times \mathcal{P}$, then the function*

$$g(\sigma) = \sum_{\alpha \subset \sigma} f(\alpha, \sigma \setminus \alpha)$$

is positive (resp. integrable) measurable on $\mathcal{P}$ and we have

$$\int_{\mathcal{P}} \int_{\mathcal{P}} f(\alpha, \beta) \, d\alpha d\beta = \int_{\mathcal{P}} g(\sigma) \, d\sigma \, .$$

Particular elements of Φ will be of interest for us: the *coherent* (or *exponential*) *vectors*: for any $u \in L^2(\mathbb{R}^+)$ define $\varepsilon(u)$ to be the element of Φ defined by

$$[\varepsilon(u)](\sigma) = \prod_{s \in \sigma} u(s) \, .$$

We then have $\|\varepsilon(u)\|^2_{\Phi} = e^{\|u\|^2_{L^2(\mathbb{R}^+)}}$.

The coherent vectors are linearly independent and total in Φ. If $\mathcal{M}$ is a dense subset of $L^2(\mathbb{R}^+)$, it is also easy to check that $\mathcal{E}(\mathcal{M}) = \{\varepsilon(u); u \in \mathcal{M}\}$ is total in Φ. The open problem is to characterise all those subsets $\mathcal{M} \subset L^2(\mathbb{R}^+)$ such that $\mathcal{E}(\mathcal{M})$ is total in Φ (this problem is far from being obvious even in the case where $L^2(\mathbb{R}^+)$ is replaced by $\mathbb{C}$!). In [P-S], Parthasarathy and Sunder have proved that if one take $\mathcal{M} = \{\mathbb{1}_B \, ; \, B$ is a bounded Borel subset of $\mathbb{R}^+\}$ then $\mathcal{E}(\mathcal{M})$ is total in Φ. Note that by the continuity of the mapping $u \mapsto \varepsilon(u)$, it suffices to take $\mathcal{M} = \{\mathbb{1}_B \, ; \, B$ is the union of disjoint bounded intervals of $\mathbb{R}^+\}$ to get the same conclusion.

In this article we consider, for all $n \in \mathbb{N}$, the space E_n generated by the $\{\varepsilon(u) \, ; \, u$ is the indicator of the union of n bounded intervals of $\mathbb{R}^+\}$. We first recover Parthasarathy

-Sunder result in a much simpler way. We characterise $E_n^\perp$, the orthogonal space of E_n in Φ, completely, showing that the typical element f of $E_n^\perp$ is as follows:

$$f = \sum_{q \in \mathbb{N}} f_q\,, \quad \text{where } f_q \in L^2(\mathcal{P}_q),\ q \in \mathbb{N}$$

with

- $f_0, f_1, \ldots, f_n$ null
- $f_{2n+1}, f_{2n+2}, \ldots$ can be chosen arbitrarily (modulo a small integrability condition)
- $f_{n+1}, f_{n+2}, \ldots, f_{2n}$ are uniquely determined by the choice of $f_{2n+1}, f_{2n+2}, \ldots$.

II – Probabilistic interpretations

It is interesting to note that our results have nice probabilistic interpretations in terms of the Brownian motion and Poisson process.

Let $(\Omega, \mathcal{F}, P)$ be the Wiener space and $(w_t)_{t\geq 0}$ be the canonical Brownian motion on Ω. It is well-known that every random variable $f \in L^2(\Omega, \mathcal{F}, P)$ admits a unique chaotic expansion

$$f = \mathbb{E}[f] + \sum_{q=1}^{\infty} \int_{0<t_1<\cdots<t_q} f_q(t_1, \ldots, t_q)\, d\omega_{t_1} \cdots d\omega_{t_q}$$

with $f_q \in L^2(\Sigma_q)$ for all $q \in \mathbb{N}^*$. We also have

$$\|f\|^2_{L^2(\Omega)} = |\mathbb{E}[f]|^2 + \sum_{q=1}^{\infty} \int_{0<t_1<\cdots<t_q} |f(t_1, \ldots, t_q)|^2\, dt_1 \cdots dt_q\,.$$

Thus the space $L^2(\Omega, \mathcal{F}, P)$ canonically identifies to our Fock space Φ (simply by identifying the coefficients $(f_q)_{q\in\mathbb{N}}$; the norm is then the same in both spaces).

If u belongs to $L^2(\mathbb{R}^+)$, the element of $L^2(\Omega, \mathcal{F}, P)$ which corresponds to $\varepsilon(u)$ is simply the Doleans' exponential

$$\varepsilon(u) = \exp\Big(\int_0^\infty u(s)\, d\omega_s - \frac{1}{2}\int_0^\infty u(s)^2\, ds\Big)\,.$$

If $u = \mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_n,t_n]}$ is the indicator of the union of n bounded intervals then

$$\varepsilon(u) = \exp(w_{t_1} - w_{s_1} + \cdots + w_{t_n} - w_{s_n}) \exp\big(-\frac{1}{2}(t_1 - s_1 + \cdots + t_n - s_n)\big)\,.$$

Thus our space E_n is exactly the space of random variables generated by exponentials of the sum of n increments of the Brownian motion.

What we have said for the Brownian motion, also holds for the compensated Poisson process $X_t = N_t - t$, $t \in \mathbb{R}^+$, on its canonical space. In this context, we have

$$\begin{aligned}\varepsilon(u) &= e^{\int_0^\infty u(s)\, dX_s} \prod_s (1 + u(s)\Delta X_s) e^{-u(s)\Delta X_s} \\ &= e^{\sum_s u(s)\Delta N_s - \int_0^\infty u(s)\, ds - \sum_s u(s)\Delta N_s} \prod_s (1 + u(s)\Delta N_s) \\ &= e^{-\int_0^\infty u(s)\, ds} \prod_s (1 + u(s)\Delta N_s)\,.\end{aligned}$$

In the case where $u = \mathbb{1}_B$ this gives, where λ denotes the Lebesgue measure,

$$\begin{aligned}\varepsilon(u) &= e^{-\lambda(B)} \prod_{s\in B} (1 + \Delta N_s) \\ &= e^{-\lambda(B)} \prod_{\substack{s\in B;\\ \Delta N_s \neq 0}} 2 \\ &= e^{-\lambda(B)} 2^{\#\{s\in B; \Delta N_s\neq 0\}}\end{aligned}$$

thus

$$\varepsilon\big(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_n,t_n]}\big) = e^{-(t_1-s_1+\cdots+t_n-s_n)} 2^{N_{t_1}-N_{s_1}+\cdots+N_{t_n}-N_{s_n}}\,.$$

Thus, in this case, our space E_n is the space generated by the exponentials of sums of n increments of the Poisson process (times $\ln 2$).

III – A simple proof of Parthasarathy -Sunder's result

We come back to the general setting of the Fock space Φ. For any $f \in \Phi$ we put for all $t \in \mathbb{R}^+$, all $\sigma \in \mathcal{P}$,

$$[D_t f](\sigma) = f(\sigma \cup \{t\}) \mathbb{1}_{\sigma \subset [0,t[}\,.$$

It is easy to check by the $\int$-lemma (cf. [Att]) that

$$\int_0^\infty \int_{\mathcal{P}} |[D_t f](\sigma)|^2 \, d\sigma \, dt = \|f\|_\Phi^2 - |f(\emptyset)|^2 < \infty\,.$$

Thus $D_t f$ is a well-defined element of Φ, for almost all t. Furthermore, for all $g \in \Phi$ we have (same reference as above)

$$\langle f, g\rangle = \overline{f(\emptyset)} g(\emptyset) + \int_0^\infty \langle D_t f, D_t g\rangle \, dt.$$

Finally note that

$$D_t \varepsilon(u) = u(t) \varepsilon(u \mathbb{1}_{[0,t]})$$

for a.a. t.

Let $n \in \mathbb{N}$ be fixed. Define E_n to be the subspace of finite linear combinations of $\varepsilon(u)$, with

$$u = \mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_n,t_n]}$$

for $s_1 \le t_1 \le s_2 \le t_2 \le \cdots \le s_n \le t_n \in \mathbb{R}^+$.

Let $E_n^\perp$ be the orthogonal space of E_n in Φ. As $\varepsilon(0) = \delta_\emptyset$, we clearly have $E_0 = L^2(\mathcal{P}_0) = \mathbb{C}\delta_\emptyset$ and $E_0^\perp = \{f \in \Phi\,;\, f(\emptyset) = 0\}$.

Note that $E_n \subset E_m$ and $E_m^\perp \subset E_n^\perp$ for all $n \le m$.

P 1. — *If f belongs to $E_n^\perp$ for some $n \ge 1$, then $D_t f$ belongs to $E_{n-1}^\perp$ for almost all t.*

Proof. — If f belongs to $E_n^\perp$ then f belongs to $E_0^\perp$ and thus $f(\emptyset) = 0$. Now, we have, for all $s_1 \le t_1 \le \cdots \le s_n \le t_n$

$$\begin{aligned} 0 &= \langle f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_n,t_n]})\rangle \\ &= \sum_{i=1}^n \int_{s_i}^{t_i} \langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_i,t]})\rangle \, dt\,. \end{aligned}$$

Deriving with respect to t_n at $t_n = t$ gives the following: for all $s_1 \le t_1 \le \cdots \le s_n$, for almost all $t \ge s_n$

$$\langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_n,t]})\rangle = 0\,.$$

The above expression is continuous in $s_1, t_1, \ldots, s_n$, thus we have: for almost all t, for all $s_1 \le t_1 \le \cdots \le s_n \le t$

$$\langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_n,t]})\rangle = 0\,.$$

Taking $s_n = t$ we get

$$\langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_{n-1},t_{n-1}]})\rangle = 0\,.$$

Now, if $s_1 \le t_1 \le \cdots \le s_{n-1} \le t_{n-1}$ and $t \in [s_i, t_i]$ we have

$$\begin{aligned} \langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_{n-1},t_{n-1}]})\rangle &= \langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_i,t]})\rangle \\ &= \lim_{\substack{t_i\to t\\ t_i>t}} \langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_i,t_i]})\rangle \\ &= 0\,. \end{aligned}$$

Finally, if $t \in [t_i, s_{i+1}]$, we have

$$\langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_{n-1},t_{n-1}]})\rangle = \langle D_t f, \varepsilon(\mathbb{1}_{[s_1,t_1]\cup\cdots\cup[s_i,t_i]})\rangle = 0\,. \qquad \blacksquare$$

We thus have an easy proof of Parthasarathy -Sunder's result.

T   1. — *The space* $\bigcup_n E_n$ *is dense in* Φ *.*

Proof. — If f belongs to $\bigcap_n E_n^\perp$, $f(\emptyset) = 0$ and $D_t f$ belongs to $\bigcap_n E_n^\perp$ for almost all t. Thus $[D_t f](\emptyset) = 0$ but $[D_t f](\emptyset)$ is equal to $f(\{t\})$. Thus the first chaos of f vanishes.

Furthermore, $D_s D_t t$ belongs to $\bigcap_n E_n^\perp$ for almost all (s,t) and thus $f(\{s,t\}) = [D_s D_t](\emptyset) = 0$ for almost all (s,t). And so on, by induction we get $f(\sigma) = 0$ for a.a. $\sigma \in \mathcal{P}$. ■

IV – A characterization of $E_n^\perp$

IV.1. Introduction.

For any $f = \sum_q f_q$ belonging to Φ we write $f_{q]} = f_1 + \cdots + f_q$, $f_{[q} = f_q + f_{q+1} + \cdots$, etc. The space $\Phi^{[q]}$ is the space $L^2(\mathcal{P}_q)$, the space $\Phi^{q]}$ is $L^2\big(\bigcup_{i=0}^{q} \mathcal{P}_i\big)$ and $\Phi^{[q}$ is $L^2\big(\bigcup_{i=q}^{\infty} \mathcal{P}_i\big)$.

The characterization we are going to prove implies that the space $\{h \in \Phi^{[2n+1}$; there exists $f \in E_n^\perp$ with $f_{[2n+1} = h\}$ is *dense* in $\Phi^{[2n+1}$ and that, given any such h there exists a *unique* $f \in E_n^\perp$ such that $f_{[2n+1} = h$. Furthermore, we will make explicit the so announced bijection between this dense subspace and $E_n^\perp$.

IV.2. The enlarged Fock space.

In order to state our result we need to introduce a family $(A_q)_{q\in\mathbb{N}}$ of projectors and this will be easier on an enlargement of the Fock space. Let $\widetilde{\Phi}$ be defined by

$$\widetilde{\Phi} = \Big\{\text{measurable functions } f \text{ on } \mathcal{P} \text{ such that, for all } N \geq 1, \text{ all } T \geq 0,$$
$$\int_{\mathcal{P}} N^{\#\sigma} |f(\sigma)| \mathbb{1}_{\sigma\subset[0,T]}\, d\sigma < \infty \Big\}\,.$$

The following remarks and notations will be of constant use in the sequel:

Remark 1. — Let $q \in \mathbb{N}$. For each $f \in \widetilde{\Phi}$, the vectors f_q, $f_{q]}$ and $f_{[q}$ are in $\widetilde{\Phi}$ (obvious).

Notation. — For each $q \in \mathbb{N}$, let $\widetilde{\Phi}^{[q]}$, $\widetilde{\Phi}^{[q}$, $\widetilde{\Phi}^{q]}$ be defined in the obvious way, in the same way as the corresponding definitions for Φ.

Remark 2. — The space Φ is a subspace of $\widetilde{\Phi}$ (left to the reader).

Remark 3. — For any $f \in \widetilde{\Phi}$ and any $t \in \mathbb{R}^+$, we put

$$[\nabla_t f](\sigma) = f(\sigma \cup \{t\}) .$$

Then $\nabla_t f$ belongs to $\widetilde{\Phi}$ for almost all t. Indeed, take $N \geq 1$ and $T \geq 0$, for any $S \leq T$ we have

$$\int_0^S \int_{\mathcal{P}} |[\nabla_t f](\sigma)| \mathbb{1}_{\sigma \subset [0,T]} N^{\#\sigma} d\sigma \, dt \leq \int_0^S \int_{\mathcal{P}} |f(\sigma \cup \{t\})| \mathbb{1}_{\sigma \cup \{t\} \subset [0,S]} N^{\#\sigma \cup \{t\}} \, d\sigma \, dt$$
$$= \int_{\mathcal{P}} (\#\beta) |f(\beta)| \mathbb{1}_{\beta \subset [0,S]} N^{\#\beta} d\beta \ \text{ (by the } \textstyle\sum\!\!\!\!\int\text{-lemma)}$$

which is finite for $\#\beta N^{\#\beta} \leq (N+1)^{\#\beta}$.

Remark 4. — If f belongs to $\widetilde{\Phi}$, if B is a bounded Borel set in $\mathbb{R}^+$, then $f\varepsilon(\mathbb{1}_\beta)$ is integrable on $\mathcal{P}$. Indeed, take $T \in \mathbb{R}^+$ such that $\beta \subset [0,T]$, then $|f(\sigma)\varepsilon(\mathbb{1}_\beta)(\sigma)| = |f(\sigma)| \mathbb{1}_{\sigma \subset \beta} \leq |f(\sigma)| \mathbb{1}_{\sigma \subset [0,T]}$ which is integrable.

Notation. — If f belongs to $\widetilde{\Phi}$ and if β is a bounded Borel set in $\mathbb{R}^+$, we write $\langle f, \varepsilon(\mathbb{1}_\beta) \rangle$ for

$$\int f(\alpha) \mathbb{1}_{\alpha \subset B} d\alpha .$$

IV.3. The characterising projectors.

For $\sigma \in \mathcal{P}_q$, we write $\sigma_1, \sigma_2, \ldots, \sigma_q$ the elements of σ, arranged in the increasing order.

If $\#\sigma = 2p$, we write $[\sigma]$ for union of n intervals

$$[\sigma_1, \sigma_2] \cup \cdots \cup [\sigma_{2p-1}, \sigma_{2p}] ;$$

if $\#\sigma = 2p + 1$, then $[\sigma]$ denotes the union of intervals

$$[\sigma_1, \sigma_2] \cup \cdots \cup [\sigma_{2p-1}, \sigma_{2p}] .$$

In any case, we write $\varepsilon_{[\sigma]}$ for $\varepsilon(\mathbb{1}_{[\sigma]})$. Note that if $\#\sigma$ is odd the $\varepsilon_{[\sigma]}$ does not depend on $\max \sigma$.

For any $\sigma \in \mathcal{P}$, we write ∇_σ for $\nabla_{\sigma_1} \cdots \nabla_{\sigma_{\#\sigma}}$, and $\nabla_\emptyset = I$. Note that for all s, t, $\nabla_s \nabla_t = \nabla_t \nabla_s$.

L 3. — *Let $f \in \widetilde{\Phi}$, then Af defined on $\mathcal{P}$ by*

$$[Af](\sigma) = \langle \nabla_\sigma f, \varepsilon_{[\sigma]} \rangle$$

belongs to $\widetilde{\Phi}$.

Proof. — Let $N \geq 1$ and $T \geq 0$ be fixed. Write

$$\varphi(\alpha,\sigma) = f(\alpha \cup \sigma)\mathbb{1}_{\alpha \subset [\sigma]}\mathbb{1}_{\sigma \subset [0,T]} N^{\#\sigma}$$

defined on $\mathcal{P} \times \mathcal{P}$. Then φ is measurable and satisfies

$$\begin{aligned}\int_{\mathcal{P}\times\mathcal{P}} |\varphi(\alpha,\sigma)|\, d\alpha\, d\sigma &\leq \int_{\mathcal{P}\times\mathcal{P}} |f(\alpha\cup\sigma)|\mathbb{1}_{\alpha\cup\sigma\subset[0,T]} N^{\#(\alpha\cup\sigma)}\, d\alpha\, d\sigma \\ &= \int_{\mathcal{P}} 2^{\#\beta} |\mathbb{1}_{\beta\subset[0,T]} N^{\#\beta}\, d\beta < \infty\,.\end{aligned}$$

Thus, by Fubini's theorem, we have that the mapping

$$\sigma \mapsto \int_{\mathcal{P}} \varphi(\alpha,\sigma)\, d\alpha = \langle \nabla_\sigma f, \varepsilon_{[\sigma]}\rangle \mathbb{1}_{\sigma\subset[0,T]} \mathbb{N}^{\#\sigma}$$

is measurable and integrable on $\mathcal{P}$. ■

For any $q \in \mathbb{N}$, we define A_q as the operator from $\widetilde{\Phi}$ to $\widetilde{\Phi}$ such that

$$[A_q f](\sigma) = [Af](\sigma)\mathbb{1}_{\#\sigma=q}\,.$$

P 4. — *Let* $q \in \mathbb{N}$. *The operator* A_q *is a projector from* $\widetilde{\Phi}$ *onto* $\widetilde{\Phi}^{[q]}$. *Moreover* $A_q A_r = 0$ *if* $r < q$. *If* $q = 0$ *then* $A_q f = f_0$. *If* $q = 1$ *then* $A_q f = f_1$. *If* $q = 2p+1$ *then*

$$[A_{2p+1} f](\{\sigma_1,\ldots,\sigma_{2p+1}\}) = [A_{2p} D_{\sigma_{2p+1}} f](\{\sigma_1,\ldots,\sigma_{2p}\})\,,$$

for a.a. $\sigma = \{\sigma_1,\ldots,\sigma_{2p+1}\}$.

Proof. — All these results are simple verifications from the definitions. ■

P 5. — *Let* $p \in \mathbb{N}$ *and* $f \in \widetilde{\Phi}$, *then the mapping*

$$(\sigma_1,\sigma_2,\ldots,\sigma_{2p}) \longmapsto \langle f, \varepsilon(\mathbb{1}_{[\sigma_1,\sigma_2]\cup\cdots\cup[\sigma_{2p-1},\sigma_{2p}]})\rangle$$

admits a.e. on Σ_{2p} *a* $\frac{\partial^{2p}}{\partial\sigma_1\cdots\partial\sigma_{2p}}$ *derivative and one has*

$$[A_{2p} f](\sigma) = (-1)^p \frac{\partial^{2p}}{\partial\sigma_1\cdots\partial\sigma_{2p}} \langle f, \varepsilon_{[\sigma]}\rangle \mathbb{1}_{\#\sigma=2p} \text{ a.e. on } \mathcal{P}\,,$$

and

$$[A_{2p+1} f](\sigma) = (-1)^p \frac{\partial^{2p}}{\partial\sigma_1\cdots\partial\sigma_{2p}} \langle \nabla_{\sigma_{2p+1}} f, \varepsilon_{[\sigma]}\rangle \mathbb{1}_{\#\sigma=2p+1} \text{ a.e. on } \mathcal{P}\,.$$

The above proposition is an easy consequence of the following lemma, which will be of constant use in the sequel.

L 6. — *Let $f \in \widetilde{\Phi}$, let B_1, B_2 be two bounded Borel sets of $\mathbb{R}^+$. Let $[a, b]$ be an interval of $\mathbb{R}^+$ with*

$$\max B_1 \leq a < b \leq \min B_2 \ .$$

Then

i) the mapping $(s,t) \mapsto \langle f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle$ is continuous on $[a, b] \times [a, b]$;

ii) the mapping $(s,t) \mapsto \langle f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle$ is derivable in s and $t \in]a, b[$ with derivatives given by

$$\frac{\partial}{\partial t}\langle f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle = \langle \nabla_t f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle \ \text{a.e.}$$
$$\frac{\partial}{\partial s}\langle f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle = \langle \nabla_s f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle \ \text{a.e..}$$

Proof.

i) comes from Lebesgue's theorem:
when (s_n, t_n) tends to (s, t), then $f(\sigma)\mathbb{1}_{\sigma\subset B_1\cup[s_n,t_n]\cup B_2}$ tends to $f(\sigma)\mathbb{1}_{\sigma\subset B_1\cup[s,t]\cup B_2}$ a.e. on $\mathcal{P}$ and $|f(\sigma)|\mathbb{1}_{\sigma\subset B_1\cup[s_n,t_n]\cup B_2}$ is dominated by $|f(\sigma)|\mathbb{1}_{\sigma\subset[0,T]}$ (where T satisfies $T \geq \max B_2$) which is integrable.

ii) Let us prove the formula for t:
we have

$$\begin{aligned}
\int_s^t \langle \nabla_x f, \varepsilon(\mathbb{1}_{\sigma\subset B_1\cup[s,x]\cup B_2})\rangle \, dx &= \int_s^t \int_{\mathcal{P}} f(\sigma\cup\{x\})\mathbb{1}_{\sigma\subset B_1\cup[s,x]\cup B_2}\, d\sigma\, dx \\
&= \int_{\mathcal{P}} f(\beta)\mathbb{1}_{\beta\subset B_1\cup[s,t]\cup B_2}\mathbb{1}_{\#(\beta\cap[s,t])\geq 1}\, d\beta \\
&= \int f(\beta)\mathbb{1}_{\beta\subset B_1\cup[s,t]\cup B_2}\, d\beta - \int f(\beta)\mathbb{1}_{\beta\subset B_1\cup B_2}\, d\beta \\
&= \langle f, \varepsilon(\mathbb{1}_{B_1\cup[s,t]\cup B_2})\rangle - \langle f, \varepsilon(\mathbb{1}_{B_1\cup B_2})\rangle
\end{aligned}$$

which gives the desired formula. ■

For $n \in \mathbb{N}$, let us define

$$B_n = I - (I - A_0)(I - A_1)\cdots(I - A_{2n}) \ .$$

P 7. — *The operator B_n is a projector from $\widetilde{\Phi}$ onto $\widetilde{\Phi}^{2n]}$.*

Proof. — The fact that B_n is a projector comes immediately from $A_q^2 = A_q$ and $A_s A_r = 0$ for $r < q$. As each A_q, for $q \leq n$, has its range included in $\widetilde{\Phi}^{2n]}$, the same holds for B_n. To show that B_n is onto, just notice that if $f \in \widetilde{\Phi}^{[q]}$ for some $q \leq 2n$, then $(I - A_0)(I - A_1)\cdots(I - A_{2n})f = (I - A_0)\cdots(I - A_q)f = 0$ and thus $B_n f = f$. ■

Let $\mathcal{B}_n$ be the set of $f \in \Phi$ such that $B_n f$ belongs to Φ.

P 8. — *The space $\mathcal{B}_n$ is dense in Φ.*

Proof. — Let $\Phi_{0,0}$ be $\{f \in \Phi\,;\ \exists k \geq 1$ and $\tau \geq 0$ with

$$f(\sigma) = f(\sigma)\mathbb{1}_{\#\sigma=k}\mathbb{1}_{\sigma\subset[0,T]}\}\,.$$

Then $\Phi_{0,0}$ is dense in Φ and is contained in $\mathcal{B}_n$ (for $A_q f \in \Phi_{0,0}$ for all $q \in \mathbb{N}$ and all $f \in \Phi_{0,0}$). ■

Remark. — The above proof also implies that $\mathcal{B}_n \cap \Phi^{[2n+1}$ is dense in $\Phi^{[2n+1}$.

IV.4. Characterization of $E_n^\perp$.

We finally come to our characterization.

T 8. — *Let $n \in \mathbb{N}$. Let $f \in \Phi$. The following assertions are equivalent.*

i) $f \in E_n^\perp$.

ii) $A_q f = 0$ *for all* $q \leq 2n$.

iii) $B_n f = 0$.

iv) There exists $h \in \Phi$ such that

$$f = (I - A_0)(I - A_1)\cdots(I - A_{2n})h\,.$$

v) $f_{[2n+1} \in \mathcal{B}_n$ *and* $f_{2n]} = -B_n f_{[2n+1}$.

Proof of Theorem 8.

ii) ⇒ iii) is obvious since $A_q f = 0$ for each $q \leq 2n$ implies $(I-A_0)(I-A_1)\cdots(I-A_{2n})f = f$.

iii) ⇒ iv) comes from the fact that B_n is a projector.

iv) ⇒ ii) comes from $A_q A_r = 0$ if $r < q$ and $A_q^2 = A_q$.

v) ⇔ iv) since *iv)* is equivalent,by definition of B_n, to "$\exists h \in B_n\,;\ f = (I - B_n)h$", and since B_n projects $\mathcal{B}_n$ onto $\Phi^{2n]}$.

We are just left to prove *i) ⇒ ii)*.

The case $n = 0$ is obvious. So by induction our result is equivalent to proving

$$f \in E_n^\perp \iff f \in E_{n-1}^\perp\,,\ \ A_{2n} f = 0\,,\ \ A_{2n-1} f = 0\,,$$

knowing that $f \in E_p^\perp \iff f \in E_{p-1}^\perp$, $A_{2p}f = 0$, $A_{2p-1}f = 0$ for each $p < n$.

$\Rightarrow$: Take $f \in E_n^\perp$. Of course f belongs to $E_{n-1}^\perp$. By Proposition 1, we know that $D_t f \in E_{n-1}^\perp$ for almost all $t \in \mathbb{R}^+$. Thus by induction hypothesis $A_{2n-2}D_t f = 0$. This exactly means $[A_{2n-1}f](\sigma) = 0$, *a.e.* on $\mathcal{P}$, *i.e.* $A_{2n-1}f = 0$. We are left to prove $A_{2n}f = 0$, nnnn *i.e.* $\langle \nabla_\sigma f, \varepsilon_{[\sigma]} \rangle = 0$ for almost all σ such that $\#\sigma = 2n$. This comes immediately from $\langle f, \varepsilon_{[\sigma]} \rangle = 0$ (since $f \in E_n^\perp$) and $\langle \nabla_\sigma f, \varepsilon_{[\sigma]} \rangle = (-1)^n \frac{\partial^{2n}}{\partial_{\sigma_1} \cdot \partial_{\sigma_{2n}}} \langle f, \varepsilon_{[\sigma]} \rangle$ a.e. on $\mathcal{P}_{2n}$.

$\Leftarrow$: Take $f \in E_{n-1}^\perp$, with $A_{2n}f = 0$ and $A_{2n-1}f = 0$. We want to prove $\langle f, \varepsilon_{[\sigma]} \rangle = 0$ for every σ such that $\#\sigma = 2n$. By a continuity argument it is enough to do it a.e. This will come step by step as follows.

First step: From $A_{2n}f = 0$, *i.e.* $\langle \nabla_\sigma f, \varepsilon_{[\sigma]} \rangle = 0$ a.e. we deduce, by Lemma 6, that $\langle \nabla_{\sigma_1 \cdots \sigma_{2n-1}} f, \varepsilon_{[\sigma]} \rangle$ does not depend on σ_{2n} and is equal (making σ_{2n} tend to σ_{2n-1}) to $\langle \nabla_{\sigma_1 \cdots \sigma_{2n-1}} f, \varepsilon_{[\sigma^{2n-1,2n}]} \rangle$, where $\sigma^{2n-1,2n} = \sigma \setminus \{\sigma_{2n-1}, \sigma_{2n}\}$. But this last equality is $[A_{2n-1}f](\sigma_1, \ldots, \sigma_{2n-1})$ which is null by hypothesis. We have proved that

$$\langle \nabla_{\sigma_1 \cdots \sigma_{2n-1}} f, \varepsilon_{[\sigma]} \rangle = 0 \text{ a.e. on } \mathcal{P}_{2n} .$$

General step: Suppose we have $\langle \nabla_{\sigma_1 \cdots \sigma_j} f, \varepsilon_{[\sigma]} \rangle = 0$ a.e. on $\mathcal{P}_{2n}$, with $j \leq 2n-1$. Then, as above, $\langle \nabla_{\sigma_1 \cdots \sigma_{j-1} f, \varepsilon_{[\sigma]}} \rangle$ does not depend on σ_{j-1} and is equal (making σ_j tend to σ_{j+1}) to $\langle \nabla_{\sigma_1 \cdots \sigma_{j-1}} f, \varepsilon_{[\sigma^{j,j+1}]} \rangle$ (where $\sigma^{j,j+1} = \sigma \setminus \{\sigma^j, \sigma^{j+1}\}$). But this quantity is $\frac{\partial^{j-1}}{\partial \sigma_1 \cdots \partial \sigma_{j-1}} \langle f, \varepsilon_{[\sigma^{j,j+1}]} \rangle$ a.e. and $\langle f, \varepsilon_{\sigma^{j,j+1}]} \rangle = 0$ since $\#\sigma^{j,j+1} = 2n-2$ and $f \in E_{n-1}^\perp$ by hypothesis. So we have

$$\langle \nabla_{\sigma_1 \cdots \sigma_{j-1}} f, \varepsilon_{[\sigma]} \rangle = 0 \text{ a.e. on } \mathcal{P}_{2n} .$$

Last step: With $j = 1$, *i.e.* $\langle \nabla_{\sigma_1} f, \varepsilon_{[\sigma]} \rangle = 0$ a.e. on $\mathcal{P}_{2n}$, we arrive to $\langle f, \varepsilon_{[\sigma]} \rangle = 0$ a.e. on $\mathcal{P}_{2n}$ and so $f \in E_n^\perp$. ■

The following consequence of Theorem 8 together with Proposition 1 gives more details on the construction of the elements of $E_n^\perp$.

P 9. — *Let $n \in \mathbb{N}$. The orthogonal projection of $E_n^\perp$ on $\Phi^{[2n+1}$ is equal to $\mathcal{B}_n \cap \Phi^{[2n+1}$. For each h belonging to this* dense *subspace of $\Phi^{[2n+1}$, there exists a* unique *$f \in E_n^\perp$ such that $f_{[2n+1} = h$. The coefficients $f_0, f_1, \ldots, f_{2n}$ of f are given by*

$$f_q = -(B_n)_q h \,, \;\; 0 \leq q \leq 2n$$

where $(B_n)_q$ is the q-th coefficient of B_n, given by

$$(B_n)_q = A_q \times \prod_{q+1 \leq j \leq 2n} (I - A_j) \,, \;\; 0 \leq q \leq 2n \,.$$

Moreover, *the $n+1$ first coefficients $f_0, f_1, \ldots, f_n$ of f all vanish.*

Proof. — The equivalence between *i)* and *v)* in Theorem 8 shows that the space $\{h \in \Phi^{[2n+1}; \exists f \in E_n^\perp$ such that $f_{[2n+1} = h\}$ is equal to $\mathcal{B}_n \cap \Phi^{[2n+1}$ (which is dense in $\Phi^{[2n+1}$, see remark at the end of IV.3), and that any $f \in E_n^\perp$ is determined by its projection $h = f_{[2n+1}$ with $f_{2n]} = -B_n h$.

To compute the coefficients $(B_n)_q$, $0 \leq q < 2n$, write down, for $g \in \widetilde{\Phi}$:

$$\begin{aligned}(B_n g)_q &= \big[I - (I - A_0)(I - A_1)\cdots(I - A_{2n})g\big]_q \\ &= g_q - \big[(I - A_q)\cdots(I - A_{2n})g\big]_q \\ &= A_q(I - A_{q+1})\cdots(I - A_{2n})g\end{aligned}$$

(the result for $q = 2n$ is evident).

The fact that the $n+1$ first coefficients $f_0, f_1, \ldots, f_n$ of any $f \in E_n^\perp$ must vanish comes from repeated applications of Proposition 1. ■

Remark. — The fact that each element of $E_n^\perp$ is determined by its projection $f_{[2n+1}$ is equivalent, by linearity, to $E_n^\perp \cap \Phi_{2n]} = \{0\}$. This last equality is an immediate consequence of the following lemma, which we think could be of independent interest.

L 10. — *Let $u_1, \ldots, u_n$ be the indicators of n disjoint bounded intervals of $\mathbb{R}^+$. Then one has, for a.a. $\sigma \in \mathcal{P}$*

$$\sum_{0\leq p\leq n}\ \sum_{1\leq j_1<j_2<\cdots<j_p\leq n} (-1)^p \varepsilon(u_{j_1} + \cdots + u_{j_p})(\sigma) = \sum_{k\geq n}\ \sum_{\substack{k_1+\cdots+k_n=k \\ k_j\geq 1,\ \forall j=1,\ldots,n}} \big[u_1^{0k_1} \circ u_2^{0k_2} \circ \cdots \circ u_n^{0k_n}\big](\sigma)$$

where $\big[u_1^{0k_1} \circ u_2^{0k_2} \circ \cdots \circ u_n^{0k_n}\big](\sigma) = 1$ if the k_1 first elements of σ lie in u_1, the k_2 following ones lie in $u_2, \ldots,$ the k_n last elements of σ lie in u_n, and $\big[u_1^{0k_1} \circ u_2^{0k_2} \circ \cdots \circ u_n^{0k_n}\big](\sigma) = 0$ otherwise.

Proof. — Writing down the left hand side for a fixed σ, the formula above is just an expression of the usual inclusion-exclusion principle. ■

Proposition 9 shows that for each $0 \leq q \leq n$, the coefficient $(B_n)_q$ is equal to the natural projector I_q of $\widetilde{\Phi}$ onto $\widetilde{\Phi}^q$. We have not been able to derive a direct proof of this fact. That is, to prove that

$$A_q(I - A_{q+1})\cdots(I - A_{2n}) = I_q \ \text{ for } \ q \leq n\,.$$

Another challenge is to develop B_n in order to give more explicit formulas for $f_{n+1}, \ldots, f_{2n}$. The coefficient $(B_n)_{2n} = A_{2n}$ is evident. We give in Propositions 11 and 14 below expressions for $(B_n)_{2n-1}$ and $(B_n)_{2n-2}$, just to put in evidence a certain underlying complexity.

IV.5. Some additional computations.

P 11. — *Let $n \geq 1$. One has, for each $h \in \widetilde{\Phi}$, for a.a. $\sigma \in \mathcal{P}_{2n-1}$,*

$$\big[A_{2n-1}(I - A_{2n})h\big](\sigma) = \sum_{1\leq i\leq 2n-1} (-1)^{i+1}\big[A_{2n-2}\nabla_{\sigma_i}h\big](\sigma^i)\,.$$

The proof of Proposition 11 is based on two lemmas.

L 12. — *Let $g \in \widetilde{\Phi}$ be such that $A_{2n}g = 0$. Then for almost all $\sigma \in \mathcal{P}_{2n-1}$, the quantity $[A_{2n-2}\nabla_{\sigma_i}f](\sigma^i)$ is independent of $i \in \{1,\ldots,2n-1\}$.*

Proof of Lemma 12. — Let $i \in \{1,\ldots,2n-2\}$. By hypothesis we have, for a.a. $\sigma \in \mathcal{P}_{2n-1}$ and a.a. $t \in]\sigma_i,\sigma_{i+1}[$, that $\langle \nabla_{\sigma\cup t}g, \varepsilon_{[\sigma\cup t]}\rangle = 0$. By Lemma 6 this implies $\langle \nabla_{\sigma}g, \varepsilon_{[\sigma\cup t]}\rangle$ is independent of t. Making t tend to σ_i or σ_{i+1} gives $\langle \nabla_{\sigma}g, \varepsilon_{[\sigma^i]}\rangle = \langle \nabla_{\sigma}g, \varepsilon_{[\sigma^{i+1}]}\rangle$. ■

L 13. — *Let $\sigma \in \mathcal{P}_{2n-1}$, then*

$$\sum_{1\leq i\leq 2n-1} (-1)^{i+1}\mathbb{1}_{[\sigma^i]} = 0\,.$$

Proof of Lemma 13. — Evaluating the left hand side on any $x \in \mathbb{R}^+$, gives the same number of sign $+$ and sign $-$ in the sum. ■

Proof of Proposition 11. — Define $C_{n,2n-1}$ on $\widetilde{\Phi}$ by

$$(C_{n,2n-1}h)(\sigma) = \sum_{1\leq i\leq 2n-1} (-1)^{i+1}(A_{2n-2}\nabla_{\sigma_i}h)(\sigma^i)\mathbb{1}_{\#\sigma=2n-1}$$

for each $h \in \widetilde{\Phi}$ and a.a. $\sigma \in \mathcal{P}$. By Lemma 12, if $g \in \widetilde{\Phi}$ is such that $A_{2n}g = 0$ then $C_{n,2n-1}g = A_{2n-1}g$. Thus for each $h \in \widetilde{\Phi}$. $C_{n,2n-1}(I - A_{2n})h = A_{2n-1}(I - A_{2n})h$. But by Lemma 8 we have $C_{n,2n-1}A_{2n} = 0$. Finally $C_{n,2n-1} = A_{2n-1}(I - A_{2n})$. ■

P 14. — *Let $n \geq 1$. One has for each $h \in \widetilde{\Phi}$ and a.a. $\sigma \in \mathcal{P}_{2n-2}$*

$$\big[A_{2n-2}(I - A_{2n-1})(I - A_{2n})h\big](\sigma) = \sum_{1\leq i<j\leq 2n-2} (-1)^{i+j+1}[A_{2n-4}\nabla_{\sigma_i,\sigma_j}h](\sigma^{i,j}) - (n-2)[A_{2n-2}h](\sigma)\,.$$

We again need two lemmas.

L 15. — *Let $g \in \widetilde{\Phi}$ be such that $A_{2n}g = 0$ and $A_{2n-1}g = 0$. Then for a.a. $\sigma \in \mathcal{P}_{2n-2}$, the quantity $[A_{2n-4}\nabla_{\sigma_i\sigma_j}h](\sigma^{i,j})$ is independent of i, j with $1 \leq i < j \leq 2n-2$.*

Proof of Lemma 15. — Apply Lemma 12 to $[\nabla_{\sigma_j} f](\sigma^j)$ and to $[\nabla_{\sigma_i} f](\sigma^i)$. ■

L 16. — *Let $\sigma \in \mathcal{P}_{2n-2}$. Then*

$$\sum_{1\le i<j\le 2n-2} \varepsilon_{[\sigma^{i,j}]}(\alpha) = (n-2)\varepsilon_{[\sigma]}(\alpha) \ \textit{ if } \#\alpha = 1 \ \textit{ or } \#\alpha = 2\,.$$

Proof of Lemma 16. — For each $k \in \{1, \dots, 2n-1\}$ define

$$J_k = \big\{(i,j)\,;\ 1 \le i < j \le 2n-2 \ \text{ and } \ [\sigma_k, \sigma_{k+1}] \subset [\sigma^{i,j}]\big\}\,.$$

Note that:

- if k is odd then $(i,j) \in J_k$ if and only if i and j are on the same side of $k + \frac{1}{2}$;
- if k is even then $(i,j) \in J_k$ if and only if i and j are not on the same side of $k + \frac{1}{2}$.
- for $p \le q$ we have

$$\begin{aligned}\sum_{2p\le i<j\le 2q} (-1)^{i+j+1} &= \sum_{2p+1\le i<j\le 2q} (-1)^{i+j+1} = \sum_{2p+1\le i<j\le 2q+1} (-1)^{i+j+1}\\ &= \sum_{2p+2\le i<j\le 2q+1} (-1)^{i+j+1} = q-p\,.\end{aligned}$$

To get Lemma 16 for $\#\alpha = 1$ we have to prove that for $1 \le k \le 2n-3$ the quantity $\sum_{(i,j)\in J_k} (-i)^{i+j+1}$ equals $n-2$ if k is odd and equals 0 if k is even. This comes immediately from the remarks above:

- if $k = 2p+1$ then

$$\begin{aligned}\sum_{(i,j)\in J_k} (-1)^{i+j+1} &= \sum_{1\le i<j\le 2p+1} (-1)^{i+j+1} + \sum_{2p+2\le i<j\le 2n-2} (-1)^{i+j+1}\\ &= p + (n-1) - (p+1) = n-2\end{aligned}$$

- if $k = 2p$ then

$$\sum_{(i,j)\in J_k} (-1)^{i+j+1} = \sum_{\substack{1\le i<j\le 2p\\ 2p+1\le j\le 2n-2}} (-1)^{i+j+1} = 0\,.$$

Finally, to get Lemma 16 for $\#\alpha = 2$ we have to prove that if h and k are between 1 and $2n-1$, we have $\sum_{(i,j)\in J_h\cap J_k} (-1)^{i+j+1} = n-2$ if h and k are odd, 0 otherwise. That comes again in the same way as above. ■

Proof of Proposition 14. — Define $C_{n,2n-2}$ on $\widetilde{\Phi}$ by

$$\begin{aligned}[C_{n,2n-2}h](\sigma) = \Big[&\sum_{1\le i<j\le 2n-2} (-1)^{i+j+1} (A_{2n-4} \nabla_{\sigma_i\sigma_j} h)(\sigma^{i,j})\\ &- (n-2)(A_{2n-2}h)(\sigma)\Big]\, \mathbb{1}_{\#\sigma=2n-2}\end{aligned}$$

for each $h \in \widetilde{\Phi}$ and a.a. $\sigma \in \mathcal{P}$.

By Lemma 15, if $g \in \widetilde{\Phi}$ is such that $A_{2n}g = 0$ and $A_{2n-1}g = 0$ then $C_{n,2n-2}g = A_{2n-2}g$ (use $\sum_{1 \le i < j \le 2n-1} (-1)^{i+j+1} = n-1$). So for each $h \in \widetilde{\Phi}$, we have

$$C_{n,2n-2}(I - A_{2n-1})(I - A_{2n})h = A_{2n-2}(I - A_{2n-1})(I - A_{2n})h\,.$$

But by Lemma 16, we have $C_{n,2n-2}A_{2n} = 0$ and $C_{2n,2n-2}A_{2n-1} = 0$. Thus $C_{n,2n-2} = A_{2n-2}(I - A_{2n-2})(I - A_{2n})$. ■

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