

Stochastic Master Equations in Thermal Environment

Stéphane ATTAL

*Université de Lyon, Université de Lyon 1
Institut Camille Jordan, U.M.R. 5208
21 av Claude Bernard, 69622 Villeurbanne cedex, France
attal@math.univ-lyon1.fr*

Clément PELLEGRINI *

*Institut de Mathématiques de Toulouse
Laboratoire de Statistique et de Probabilité
Université Paul Sabatier (Toulouse III)
118 Route de Narbonne, 31062 Toulouse Cedex 9, France
clement.pellegrini@math.univ-toulouse.fr*

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Abstract. We derive the stochastic master equations which describe the evolution of open quantum systems in contact with a heat bath and undergoing indirect measurements. These equations are obtained as a limit of a quantum repeated measurement model where we consider a small system in contact with an infinite chain at positive temperature. At zero temperature it is well-known that one obtains stochastic differential equations of jump-diffusion type. At strictly positive temperature, we show that only pure diffusion type equations are relevant.

1. Introduction

The theory of *Open Quantum Systems* aims to study the time evolution of a small system \mathcal{H}_0 interacting with an environment \mathcal{E} , cf [1, 2, 24, 26]. Starting from an *Hamiltonian* description of the coupled system [2, 24, 26], the evolution of the reduced system \mathcal{H}_0 is obtained by tracing over the degree of freedom of the environment.

In the *Markovian* approach of open systems, the time evolution of the state of the reduced system is characterized by a semigroup of completely positive maps, with a typical generator called *Lindblad generator*, which gives rise to an ordinary differential equation called *master equation* [2, 24, 28].

In this framework, an active line of research, motivated by recent experimental applications in quantum optics and quantum communications, is focused on the description of quantum measurement [7, 8, 9, 10, 11, 12, 13, 14, 24, 27, 28, 26]. Basically, in order to avoid *Zeno effect* [24], the measurement is performed on the environment. According to the postulates of quantum mechanics, this involves a random perturbation of the evolution of the state of \mathcal{H}_0 . The dynamics of \mathcal{H}_0 is then described by classical stochastic differential equations, which are perturbations of the master equation in terms of white noise [7, 22, 23, 8, 9, 10, 11, 12, 21, 30, 32, 37, 38]. Usually, these equations are called *Stochastic Schrödinger Equations* or *Stochastic Master Equations* and their solutions are called *Quantum Trajectories* (the name “stochastic Schrödinger equation” is usually reserved for the evolution of the state of \mathcal{H}_0 in terms of pure states whereas stochastic master equations concerns evolution of density matrices).

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In the literature, most of the results concern models where the environment is at zero temperature and a lot of investments are still in progress to get right descriptions for model at positive temperature (even without measurement). Typically, at zero temperature, for the random evolution of density matrices we get two typical stochastic master equations of the following form.

1. A diffusive equation

$$d\rho_t = \mathcal{L}(\rho_t) dt + \left(C\rho_t + \rho_t C^* - \text{Tr} \left[\rho_t (C + C^*) \right] \rho_t \right) dW_t, \quad (1)$$

where W_t is a one dimensional Brownian motion.

2. A jump equation

$$d\rho_t = \mathcal{L}(\rho_t) dt + \left(\frac{C \rho_t C^*}{\text{Tr} [C \rho_t C^*]} - \rho_t \right) \left(d\tilde{N}_t - \text{Tr} [C \rho_t C^*] dt \right), \quad (2)$$

where (\tilde{N}_t) is a counting process with stochastic intensity $t \rightarrow \int_0^t \text{Tr} [C \rho_s C^*] ds$.

In the above expressions \mathcal{L} is the Lindblad operator, the operator C is an arbitrary operator (which depends actually on the interaction between the system and the environment, see Section 2.4). Note that we can recognize from the form of these equations that they all are *simulations* of the master equation, for they are stochastic differential equations valued in the set of states of \mathcal{H}_0 and on average they satisfy the master equation.

More complicated models use jump-diffusion stochastic differential equations which are mixing of equations (1) and (2) [9, 11, 38] (see Section 2.).

Mathematically, there are three usual ways to justify these equations. A first approach is based on *Instrumental Operator Processes* connected with the notion of *Operator Valued Measures* and *Quantum Markov Semi-groups* analogy [2, 7]. The second one is based on classical stochastic differential equation theory and on the concept of a *posteriori state* [7, 8, 9, 10, 11, 30]. The third approach often called *Quantum Filtering* is based on the formalism of *Quantum Stochastic Calculus* and the notion of input and output field in quantum optics [2, 12, 13, 14, 21, 22, 23]. In this last setup, the evolution of the small system and the environment is modeled by *Quantum Langevin Equations* (also called *Quantum Stochastic Differential Equations* or *Hudson-Parthasarathy Equations*) [1, 2, 31]. These equations are namely driven by *Quantum Noises* [1, 2]. Next, by adapting the classical framework of stochastic filtering, one can obtain appropriate stochastic differential equations driven by classical noises, which take into account the “incomplete” information of indirect observations.

In [32, 37, 38], an alternative discrete way has been developed. The approach is based on the model of *Quantum Repeated Interactions* which provides a “useful” discrete approximation model of quantum Langevin equations [3, 4]. The setup of quantum repeated interactions is the one of the interaction of a small system \mathcal{H}_0 with an environment represented by an infinite chain $\bigotimes_{k=0}^{\infty} \mathcal{H}_k$. Moreover, the pieces of the chain are identical and independent quantum system, that is $\mathcal{H}_k = \mathcal{H}$ for all k . Each copy \mathcal{H} interacts with \mathcal{H}_0 , one after the other, during a time τ . In this framework, an appropriate language of *discrete quantum noise* and *discrete quantum stochastic differential equations* is constructed. Next, it is shown that the time continuous limit ($\tau \rightarrow 0$) gives rise to continuous models of quantum Langevin equations

and continuous quantum noises. In this context, a discrete time model of indirect quantum measurements, called model of *Quantum Repeated Measurements* has been developed in [32, 37, 38]. It consists in performing a measurement of an observable of \mathcal{H}_k after each interaction with \mathcal{H}_0 (similar models of successive measurements has been considered in [16, 34]). As in the continuous case, the measurement introduces a random perturbation of the evolution of \mathcal{H}_0 , described by *discrete stochastic master equations* and *discrete quantum trajectories*. In the same spirit as [4], by considering the continuous time limit ($\tau \rightarrow 0$), the continuous models of stochastic master equations and quantum trajectories are recovered. Much beyond the approximation result, these discrete time models (without measurement [4] and with measurement [32, 37, 38]) provide a concrete and intuitive physical justification of the continuous models of quantum Langevin and stochastic master equations. For other applications and results regarding quantum repeated interactions, we refer to [17, 18, 19, 20, 35, 36, 39] where ‘return to equilibrium’, ‘large time behaviour’ or ‘thermalization’ problems are considered. We refer also to [25, 29, 42, 43, 44] for an approach based on collision-like models in physics, where properties such as ‘entanglement’ and ‘decoherence’ are studied.

It is important to notice that these four different approaches of stochastic master equations are crucially based on the ‘zero temperature’ assumption. For example in the approach based on quantum stochastic calculus, one the main obstacle to consider positive temperature model is to describe the action of the heat bath on the small system and to derive adequate Langevin equations.

Recently, in [5], the discrete approach of quantum repeated interactions [4] has been adapted to models with heat bath at positive temperature. This way, thermal quantum Langevin equations have been obtained and a clear justification of the action of the thermal bath has been presented. It is then natural to combine this approach with quantum repeated measurements in order to derive stochastic master equations for heat baths.

In order to introduce temperature, we consider that each copy \mathcal{H} is in a thermal Gibbs state at inverse temperature β . The crucial point in all the different approaches at zero temperature is the fact the state of the environment is a pure state. This is clearly not the case for a Gibbs state. In order to get around this difficulty, we apply the G.N.S. representation of that state. This way, the Gibbs state of each copy \mathcal{H} can be considered as a pure state in an enlarged Hilbert space. Hence, with this representation, the convergence result of [38] can be applied and stochastic master equations for heat bath are derived. Surprisingly, models of the form (2) with counting processes disappears and only diffusive models remains.

This article is structured as follows.

In Section 2., we remind the discrete models of quantum repeated interactions and then quantum repeated measurements. Next, we recall the main result of [38] which gives the stochastic Master equations as continuous limits of these discrete models.

In Section 3., we adapt the result of [38] for model with positive temperature. This is achieved by describing the G.N.S representation of the heat bath. Hence we obtain the complete description of stochastic master equations for a small system in contact with a heat bath and undergoing indirect quantum measurements.

2. From Discrete to Continuous Quantum Trajectories at Zero Temperature

In this section we recall the mathematical description of the quantum repeated measurement model, as developed in [38].

2.1. QUANTUM REPEATED INTERACTIONS

The model of quantum repeated interactions consists in studying the interaction of a finite dimensional quantum system \mathcal{H}_0 in contact with an infinite chain $T\Phi = \bigotimes_{\mathbb{N}^*} \mathcal{H}_k$, where $\mathcal{H}_k = \mathcal{H} = \mathbb{C}^{N+1}$ for all k . One can imagine that the pieces \mathcal{H} represent for example “a measurement apparatus” or an “incoming photon”...

We first need to make precise the definition of the countable tensor product $T\Phi$. To this end, let us define an explicit orthonormal basis of $T\Phi$. Let $\{X_0, X_1, \dots, X_N\}$ be an orthonormal basis of $\mathcal{H} = \mathbb{C}^{N+1}$. Actually note that only the choice of X_0 is relevant in our construction: it represents a reference state of \mathcal{H} (for example a ground state). Put X_i^n to be the copy of the basis vector X_i but acting on the n -th copy of \mathcal{H} . The orthonormal basis of $T\Phi$ is then made of those tensor products

$$X_{i_1}^1 \otimes \dots \otimes X_{i_n}^n \otimes \dots$$

such that all the i_n 's, but a finite number, are all equal to 0.

On \mathcal{H} consider the basic operators $a_j^i, i, j = 0, \dots, N$ defined by $a_j^i X_k = \delta_{ik} X_j$ (in Dirac notation $a_j^i = |X_j\rangle\langle X_i|$). We dilate them as operators $a_j^i(k)$ on $T\Phi$ by asking them to act as a_j^i on the k -th copy of \mathcal{H} and as the identity operator on the rest of the chain.

The model of repeated interactions [4] is now described as follows. Each copy of \mathcal{H} is supposed to interact, one after the other, with \mathcal{H}_0 during a time duration τ . Each elementary interaction between \mathcal{H}_0 and \mathcal{H} is described by a total Hamiltonian

$$H_{\text{tot}} = H_0 \otimes I + I \otimes H_R + \lambda H_I. \quad (3)$$

The operator H_0 corresponds to the free Hamiltonian of the system \mathcal{H}_0 , the operator H_R is the free Hamiltonian of the system \mathcal{H} , the operator H_I is the interaction Hamiltonian and λ is the coupling constant.

The basis $\{X_0, \dots, X_n\}$ is chosen to be the basis of eigenvectors of H_R , that is, with our notations:

$$H_R = \sum_{i=1}^N \gamma_i a_i^0 a_0^i. \quad (4)$$

The interaction Hamiltonian H_I is chosen to be of so-called “dipole-type”:

$$H_I = \sum_{i=1}^N (C_i \otimes a_i^0 + C_i^* \otimes a_0^i). \quad (5)$$

After a time duration τ of interaction, the evolution of $\mathcal{H}_0 \otimes \mathcal{H}$ is governed by the unitary operator

$$U = e^{-i\tau H_{\text{tot}}}.$$

That is, in the Schrödinger picture, the evolution of states on $\mathcal{H}_0 \otimes \mathcal{H}$ is given by

$$\rho \mapsto U \rho U^*$$

and in the Heisenberg picture, the observables evolve as

$$X \mapsto U^* \rho U.$$

Now, in order to describe the repeated interactions on the whole space $\mathcal{H}_0 \otimes T\Phi$, we consider, for each $k \in \mathbb{N}^*$, the unitary operator U_k which acts like the operator U on the tensor product $\mathcal{H}_0 \otimes \mathcal{H}_k$ and like the identity operator on the rest of the space. For k being fixed, the operator U_k describes the k -th interaction. The whole procedure is then described by the sequence of unitary operators (V_k) defined by $V_k = U_k U_{k-1} \dots U_1$. For example the evolution of an initial state ρ on $\mathcal{H}_0 \otimes T\Phi$ after k interactions is given by

$$\rho \mapsto V_k \rho V_k^*.$$

2.2. QUANTUM REPEATED MEASUREMENTS

Now, we are in position to describe the model of quantum repeated measurements [32]. To this end, we need to specify the reference states of \mathcal{H}_0 and \mathcal{H} . Let ρ denote the initial state of \mathcal{H}_0 . For each copy of \mathcal{H} , we consider the usual thermal Gibbs state at inverse temperature β :

$$\rho_\beta = \frac{e^{-\beta H_R}}{\text{Tr}[e^{-\beta H_R}]}, \quad (6)$$

where H_R is defined in expression (4). In particular ρ_β is diagonal, with diagonal elements that we shall denote by $\{\beta_0, \beta_1, \dots, \beta_N\}$.

In order to describe the indirect measurement of an observable A of \mathcal{H} , we come back to the description of the first interaction in the space $\mathcal{H}_0 \otimes \mathcal{H}$. After the interaction the new state of $\mathcal{H}_0 \otimes \mathcal{H}$ is $\mu = U(\rho \otimes \rho_\beta)U^*$. Now if A owns the spectral decomposition

$$A = \sum_{i=0}^p \lambda_i P_i,$$

the measurement of A gives a random result in the set of eigenvalues $\lambda_0, \dots, \lambda_p$. In particular, the value λ_i is obtained with probability

$$P[\text{to observe } \lambda_i] = \text{Tr}[\mu I \otimes P_i]. \quad (7)$$

After having observed the eigenvalue λ_i , the state μ is projected and becomes

$$\tilde{\rho}_1(i) = \frac{(I \otimes P_i) \mu (I \otimes P_i)}{\text{Tr}[\mu (I \otimes P_i)]}.$$

For i being fixed, the state $\tilde{\rho}_1(i)$ represents the new state of $\mathcal{H}_0 \otimes \mathcal{H}$ after the first interaction and the first measurement. Usually, we are only interested in the reduced system \mathcal{H}_0 , that is, we shall consider only the partial trace

$$\rho_1(i) = \text{Tr}_{\mathcal{H}}(\tilde{\rho}_1(i)).$$

The state ρ_1 is a random state which takes the values

$$\frac{\mathcal{L}_i(\rho)}{\text{Tr}[\mathcal{L}_i(\rho)]}$$

with probability $\text{Tr}[\mathcal{L}_i(\rho)]$ respectively, where

$$\mathcal{L}_i(\rho) = \text{Tr}_{\mathcal{H}}[(I \otimes P_i) U(\rho \otimes \rho_\beta) U^* (I \otimes P_i)].$$

This way, the state ρ_1 is the new reference state of \mathcal{H}_0 and we can consider a new interaction with a copy of \mathcal{H} and a new measurement of A . By reducing on \mathcal{H}_0 , we then get a new random state ρ_2 which satisfies similar properties as ρ_1 . Iterating this procedure we obtain a random sequence of state (ρ_k) on \mathcal{H}_0 . This sequence, called *discrete quantum trajectory* describes the random modifications of the state of \mathcal{H}_0 undergoing quantum repeated interactions and measurements. More precisely, we have the following proposition (cf [32]).

PROPOSITION 1. *The sequence (ρ_k) is a Markov chain. More precisely, if $\rho_k = \theta$, the state ρ_{k+1} can take the values*

$$\frac{\mathcal{L}_i(\theta)}{\text{Tr} [\mathcal{L}_i(\theta)]}, \quad i = 0, \dots, p$$

with probability $p_i(\theta) = \text{Tr} [\mathcal{L}_i(\theta)]$, where $\mathcal{L}_i(\theta) = \text{Tr}_{\mathcal{H}} [I \otimes P_i U(\theta \otimes \rho_\beta) U^* I \otimes P_i]$.

In other words, the above proposition can be summarized as follows. Let $\mathbf{1}_i^k$ denote the random variable which takes the value 1 if we observe the eigenvalue λ_i during the k -th measurement and 0 otherwise, we then have

$$\rho_{k+1} = \sum_{i=0}^p \frac{\mathcal{L}_i(\rho_k)}{p_i(\rho_k)} \mathbf{1}_i^{k+1}. \quad (8)$$

2.3. QUANTUM TRAJECTORIES AT ZERO TEMPERATURE

The equation above is a discrete-time stochastic master equation. In the articles [32, 37, 38] the author computes explicitly the continuous-time limit of (8) at zero temperature. In the limit, he obtains the usual stochastic master equations describing continuous measurement experiments. In general, these equations are stochastic differential equations mixing diffusive and jump noises. We shall now recall these results.

We now focus on the case where the temperature is zero, that is $\beta = +\infty$. This way, we have $\beta_i = 0$ for all $i = 1, \dots, N$ and $\rho_\beta = a_0^0 = |X_0\rangle\langle X_0|$.

If K is an operator on $\mathcal{H}_0 \otimes \mathcal{H}$ and for the choice $\{X_0, \dots, X_N\}$ of an orthonormal basis for \mathcal{H} , the operator K can be written as a $(N+1) \times (N+1)$ -block-matrix, with coefficients

$$K_j^i = \text{Tr}_{\mathcal{H}} \left[(I \otimes |X_j\rangle\langle X_i|) K \right],$$

being operators on \mathcal{H}_0 . With these notations, we have

$$\text{Tr}_{\mathcal{H}}[K] = \sum_{i=0}^N K_i^i.$$

In particular the $\mathcal{L}_i(\rho)$ are easy to compute explicitly: let $U = (U_k^l)_{0 \leq k, l \leq N}$ be the block-matrix representation of the unitary evolution U and let $P_i = (p_{kl}^i)_{0 \leq k, l \leq N}$ in the basis $\{X_0, \dots, X_1\}$ (in block form we have $I \otimes P_i = (p_{kl}^i I)_{0 \leq k, l \leq N}$), we get

$$\mathcal{L}_i(\rho) = \text{Tr}_{\mathcal{H}} [I \otimes P_i U(\rho \otimes \rho_\beta) U^* I \otimes P_i] = \sum_{k, l=0}^N p_{kl}^i U_k^0 \rho (U_l^0)^*. \quad (9)$$

The convergence result of discrete quantum trajectories is based on the asymptotic assumptions described in [4]. These assumptions concern the unitary operator U . If we consider

the time of interaction τ being $\tau = 1/n$, the unitary operator U depends on the parameter n , that is, $U = U(n) = (U_k^l(n))_{0 \leq k, l \leq N}$. In [4], it is shown that the operator process $(V_{[nt]})$ satisfying

$$V_{[nt]} = U_{[nt]}(n) \dots U_1(n)$$

converges, non trivially, to a process (V_t) , only if the coefficients $U_j^i(n)$ obey certain normalizations. The limit process (V_t) then satisfies a quantum Langevin equation describing the evolution of a small system coupled with a Fock space.

Their asymptotic conditions concern the existence of operators L_j^i such that for all $(i, j) \in \{0, \dots, N\}^2$ we have

$$\lim_{n \rightarrow \infty} n^{\epsilon_{ij}} (U_j^i(n) - \delta_{ij} I) = L_j^i, \quad (10)$$

where $\epsilon_{ij} = \frac{1}{2}(\delta_{0i} + \delta_{0j})$.

In the context of measurement, the expression (9) implies that only the asymptotic of the terms $U_j^0(n)$ are relevant. In terms of total Hamiltonian (3), in [4] it is shown that these asymptotics for the U_j^i 's can be obtained by considering interaction Hamiltonian H_I of type (5) and by considering the coupling constant $\lambda = \sqrt{n}$. In that case, they obtain

$$L_k^0 = -iC_k.$$

Remark In other words, this corresponds to a renormalization in \sqrt{n} of the field operator a_i^0 and a_0^i in order to strengthen the force of the interaction while the interaction-time $\tau = 1/n$ decreases (when n goes to infinity). Such normalization is essential in order to obtain non trivial limit (see discussion in Section 4.2 in [4] and [6]). We refer also to [34], where scaling time and appearance of noises are discussed.

Now with (10), we are in position to express the main result of [38] which links discrete and continuous quantum trajectories. To this end, we introduce functions (when it has a meaning) defined on the set of states:

$$g_i(\rho) = \frac{\sum_{k,l=1}^N p_{kl}^i L_k^0 \rho (L_l^0)^*}{\text{Tr} \left[\sum_{k,l=1}^N p_{kl}^i L_k^0 \rho (L_l^0)^* \right]} - \rho$$

$$v_i(\rho) = \text{Tr} \left[\sum_{k,l=1}^N p_{kl}^i L_k^0 \rho (L_l^0)^* \right]$$

$$h_i(\rho) = \frac{1}{\sqrt{p_{00}^i}} \left[\sum_{k=1}^N (p_{k0}^i L_k^0 \rho + p_{0k}^i \rho (L_k^0)^*) - \text{Tr} \left[\sum_{k=1}^N (p_{k0}^i L_k^0 \rho + p_{0k}^i \rho (L_k^0)^*) \right] \rho \right]$$

$$\mathcal{L}(\rho) = L_0^0 \rho + \rho (L_0^0)^* + \sum_{k=1}^N L_k^0 \rho (L_k^0)^*.$$

THEOREM 2. Let $A = \sum_{i=0}^p \lambda_i P_i$ be an observable of \mathcal{H} . As $\sum_i P_i = I$, without restriction, we can assume that $p_{00}^0 \neq 0$. Let $I = \{i \in \{1, \dots, p\} / p_{00}^i = 0\}$ and $J = \{1, \dots, p\} \setminus I$. Let ρ_0

be a state on \mathcal{H}_0 and let $(\rho_n(t))$ be the stochastic process defined from the discrete quantum trajectory (ρ_k) by $\rho_n(t) = \rho_{[nt]}$. We then have the following convergence result.

- If $J = \emptyset$, the process $(\rho_n(t))$ converges in distribution to the solution of the stochastic differential equation

$$\rho_t = \rho_0 + \int_0^t \mathcal{L}(\rho_{s-}) ds + \sum_{i=1}^p \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-}) \mathbf{1}_{0 < x < v_i(\rho_{s-})} [N_i(dx, ds) - dx ds], \quad (11)$$

where $(N_i)_{1 \leq i \leq N}$ are N independent Poisson processes on \mathbb{R}^2 .

- If $J \neq \emptyset$, the process $(\rho_n(t))$ converges in distribution to the solution of the stochastic differential equation

$$\begin{aligned} \rho_t = & \rho_0 + \int_0^t \mathcal{L}(\rho_{s-}) ds + \sum_{i \in J \cup \{0\}} \int_0^t h_i(\rho_{s-}) dW_i(s) \\ & + \sum_{i \in I} \int_0^t \int_{\mathbb{R}} g_i(\rho_{s-}) \mathbf{1}_{0 < x < v_i(\rho_{s-})} [N_i(dx, ds) - dx ds] \end{aligned} \quad (12)$$

where $(W_i(t))_{0 \leq i \leq N}$ are $N+1$ independent Brownian motions independent of the Poisson processes $(N_i)_{1 \leq i \leq N}$.

Remark: In Theorem 2, the notion of convergence in distribution concerns the convergence in distribution for stochastic processes, that is, in the Skorohod topology (this corresponds also to the notion of the weak convergence, see [15, 33] for a complete introduction on this theory).

Remark: Let us stress that Theorem 2 concerns model in finite dimension. In infinite dimension, there exist results concerning stochastic Schrödinger equations, we refer to [11] for Lindblad bounded operators and [30] for non-bounded operators. We refer also to [20] for quantum repeated interactions setup in infinite dimension related to "one-atom maser" model.

2.4. THE 2-DIMENSIONAL CASE

In order to illustrate this theorem, we investigate the case where $\mathcal{H} = \mathbb{C}^2$. In this situation, we get two different behaviours depending on the fact that $p_{00}^0 = 1$ or not.

Indeed, in the case $p_{00}^0 = 1$ we have $J = \emptyset$ and the case $p_{00}^0 \neq 1$ corresponds to $J \neq \emptyset$. Furthermore the case $p_{00}^0 = 1$ corresponds to a case where the observable A is diagonal in the basis $\{X_0, X_1\}$, that is, of the form

$$A = \lambda_0 a_0^0 + \lambda_1 a_1^1.$$

The limit equation is then

$$\begin{aligned} \rho_t = & \rho_0 + \int_0^t \mathcal{L}(\rho_{s-}) ds + \int_0^t \int_{\mathbb{R}} \left(\frac{L_1^0 \rho_{s-} (L_1^0)^*}{\text{Tr}[L_1^0 \rho_{s-} (L_1^0)^*]} - \rho_{s-} \right) \times \\ & \times \mathbf{1}_{0 < x < \text{Tr}[L_1^0 \rho_{s-} (L_1^0)^*]} [N(dx, ds) - dx ds]. \end{aligned} \quad (13)$$

By putting $L_1^0 = C$ and by considering the process $\tilde{N}_t = \int_0^t \int_{\mathbb{R}} \mathbf{1}_{0 < x < \text{Tr}[L_1^0 \rho_s - (L_1^0)^*]} N(dx, ds)$, we obtain the jump equation (2) mentioned in Introduction. Indeed, the process (\tilde{N}_t) is a counting process with stochastic intensity $\int_0^t \text{Tr}[L_1^0 \rho_s - (L_1^0)^*] ds$. Actually, the expression (13) is a rigorous way to consider jump stochastic Schrödinger equations (see [37]). Typically this equation describes the experiment of *photon-detection* in quantum optics [24, 27, 40, 41, 12].

The other case $p_{00}^0 \neq 1$ gives rise to a diffusive equation. For example, consider the case $p_{00}^0 = 1/2$ (the other situations are similar). The observable A has then to be of the form

$$A = \frac{\lambda_0}{2} (a_0^0 + a_1^0 + a_0^1 + a_1^1) + \frac{\lambda_1}{2} (a_0^0 - a_1^0 - a_0^1 + a_1^1).$$

Hence, we get the limit equation

$$\begin{aligned} \rho_t &= \rho_0 + \int_0^t \mathcal{L}_0(\rho_s) ds + \int_0^t \left(L_1^0 \rho_s + \rho_s (L_1^0)^* - \text{Tr}[L_1^0 \rho_s + \rho_s (L_1^0)^*] \rho_s \right) \frac{\sqrt{2}}{2} dW_1(s) \\ &\quad + \int_0^t \left(L_1^0 \rho_s + \rho_s (L_1^0)^* - \text{Tr}[L_1^0 \rho_s + \rho_s (L_1^0)^*] \rho_s \right) \frac{-\sqrt{2}}{2} dW_2(s). \end{aligned} \quad (14)$$

Note that by defining a Brownian motion $W_t = (\sqrt{2}/2) W_1(t) - (\sqrt{2}/2) W_2(t)$, we recover the diffusive equation (1). In quantum optics, this equation describes *heterodyne/homodyne* detection experiments [7, 40, 41].

In [32, 37], the equations (13, 14) are studied in details and the convergence from discrete to continuous trajectories is obtained (with different techniques than the more general result [38]).

3. From Discrete to Continuous Quantum Trajectories at Positive Temperature

All the results mentioned in previous section are based on the construction of [4] which makes heavy use of the fact that the reference state of \mathcal{H} is a pure state. Indeed, this condition is strongly needed in order to define the countable tensor product $\otimes_{n \in \mathbb{N}^*} \mathcal{H}$ and its continuous limit, the continuous tensor product $\otimes_{t \in \mathbb{R}^+} \mathcal{H}$.

When considering that the environment is made of a chain of systems \mathcal{H} each of which in thermal equilibrium state

$$\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H_R}$$

we cannot directly apply their results. The idea here follows the one developed in [5], that is, we take the G.N.S. representation of the state ρ_β . This way, the state ρ_β becomes a pure state, but on a larger state space.

3.1. THE G.N.S REPRESENTATION OF THE HEAT BATH

The G.N.S representation of $(\mathcal{H}, \rho_\beta)$, also called cyclic representation, is described as follows. At positive temperature, since the state ρ_β is faithful, it defines a scalar product on $\mathcal{H}' = \mathcal{B}(\mathcal{H})$ by

$$\langle A, B \rangle = \text{Tr} [\rho_\beta A^* B], \quad (15)$$

for all $(A, B) \in \mathcal{H}'$.

For all $A \in \mathcal{H}'$, we denote by $\pi(A)$ the linear map from \mathcal{H}' to \mathcal{H}' defined by

$$\pi(A)B = AB,$$

for all $B \in \mathcal{H}'$. The linear map π from \mathcal{H}' into $\mathcal{B}(\mathcal{H}')$ is a representation of $(\mathcal{H}, \rho_\beta)$, the so-called ‘‘G.N.S. representation’’. In particular we have

$$\langle I, \pi(A)I \rangle = \text{Tr} [\rho_\beta A],$$

for all $A \in \mathcal{H}'$. This means that transported by π the action of the state ρ_β on a observable A is the same that the one of the pure state $|I\rangle\langle I|$ on $\pi(A)$.

In order to compute the matrix coefficients of the operator $\pi(U)$, we need to specify an orthonormal basis of \mathcal{H}' . The only restriction on this basis is that it has to contain the reference state $|I\rangle\langle I|$.

The Hilbert space \mathcal{H}' is a $(N + 1)^2$ dimensional space. We denote by X_0^0 the identity operator. Next for $i = 1, \dots, N$, we denote by X_i^i the diagonal matrix with diagonal elements $\{\nu_i^0, \dots, \nu_i^N\}$ such that

$$\langle X_i^i, X_j^j \rangle = \delta_{ij},$$

for all $i, j = 0, \dots, N$. Such operators can be constructed by extending the vector $(1, \dots, 1)$ into an orthonormal basis of \mathbb{C}^{N+1} for the scalar product

$$\sum_{i=0}^N \beta_i \bar{x}_i y_i.$$

In order to complete the basis, we define X_j^i for $i \neq j \in \{0, \dots, N\}$ by

$$X_j^i = \frac{1}{\sqrt{\beta_i}} a_j^i.$$

Thus, we have construct an orthonormal basis $\{X_j^i, i, j = 0, \dots, N\}$ of \mathcal{H}' for the scalar product (15).

In this basis an operator K on $\mathcal{H}_0 \otimes \mathcal{H}$ is transported by π as an operator $\pi(K) = (K_{kl}^{ij})_{0 \leq i, j, k, l \leq N}$ where the coefficients K_{kl}^{ij} are operator on \mathcal{H}_0 . These coefficients are given by

$$K_{kl}^{ij} = \text{Tr}_{\mathcal{H}} [(I \otimes \rho_\beta)(I \otimes X_l^k)^* K (I \otimes X_j^i)]. \quad (16)$$

3.2. ASYMPTOTICS OF U IN THE G.N.S. REPRESENTATION

Recall that we have defined the basic unitary interaction U as

$$U = \exp \left(-i \frac{1}{n} (H_0 \otimes I + I \otimes H_R + \sqrt{n} H_I) \right). \quad (17)$$

We are in position to describe $\pi(U)$ in the asymptotic way. Here, the translation of condition (10) is the existence of operators L_{kl}^{ij} such that

$$\lim_{n \rightarrow \infty} n^{\epsilon_{kl}^{ij}} \left(U_{kl}^{ij} - \delta_{(i,j), (k,l)} I \right) = L_{kl}^{ij}, \quad (18)$$

where $\epsilon_{00}^{00} = 1$, $\epsilon_{kl}^{00} = \epsilon_{00}^{kl} = 1/2$ and the others are equal to zero.

We need to check that the unitary operator (17) provides good asymptotic. Actually in our context of indirect quantum measurement, according to Theorem 2.3., we only need the expression of L_{kl}^{00} .

PROPOSITION 3. *The expression of the final relevant limit operators L_{kl}^{00} is given by*

$$\begin{aligned} L_{00}^{00} &= iH_0 + \sum_{i=1}^N \beta_i \gamma_i I + \frac{1}{2} \sum_{i=0}^N (\beta_0 C_i^* C_i + \beta_i C_i C_i^*), \\ L_{k0}^{00} &= -i\sqrt{\beta_k} C_k^*, \quad L_{0l}^{00} = -i\sqrt{\beta_0} C_l \end{aligned} \quad (19)$$

Proof: In block form we have

$$H_{\text{tot}} = \begin{pmatrix} H_0 & \sqrt{n} C_1^* & \sqrt{n} C_2^* & \dots & \sqrt{n} C_N^* \\ \sqrt{n} C_1 & H_0 + \gamma_1 I & 0 & \dots & 0 \\ \sqrt{n} C_2 & 0 & H_0 + \gamma_2 I & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \vdots \\ \sqrt{n} C_N & 0 & 0 & \dots & H_0 + \gamma_N I \end{pmatrix}$$

and hence the operator $U(n)$ can be shown to be of the form (cf [4])

$$\begin{pmatrix} I - \frac{1}{n} i H_0 - i \frac{1}{n} \gamma_0 I & -i \frac{1}{\sqrt{n}} C_1^* + o\left(\frac{1}{n^{3/2}}\right) & \dots & -i \frac{1}{\sqrt{n}} C_1^* + o\left(\frac{1}{n^{3/2}}\right) \\ -\frac{1}{2} \frac{1}{n} \sum_{i=1}^N C_i^* C_i + o\left(\frac{1}{n^2}\right) & & & \\ -i \frac{1}{\sqrt{n}} C_1 + o\left(\frac{1}{n^{3/2}}\right) & I - \frac{1}{n} i H_0 - i \frac{1}{n} \gamma_1 I & \dots & -\frac{1}{2} \frac{1}{n} C_1 C_N^* \\ & -\frac{1}{2} \frac{1}{n} C_1 C_1^* + o\left(\frac{1}{n^2}\right) & & \\ \vdots & \vdots & \ddots & \vdots \\ -i \frac{1}{\sqrt{n}} C_N + o\left(\frac{1}{n^{3/2}}\right) & -\frac{1}{2} \frac{1}{n} C_N C_1^* & \dots & I - \frac{1}{n} i H_0 - i \frac{1}{n} \gamma_N I \\ & & & -\frac{1}{2} \frac{1}{n} C_N C_N^* + o\left(\frac{1}{n^2}\right) \end{pmatrix}.$$

Now we can compute the asymptotic form of $U_{kl}^{00}(n)$. Keeping in mind that in the appropriate basis \tilde{B} , the partial trace of an operator is the operator obtained by summing the diagonal blocks, we get

$$\begin{aligned} U_{00}^{00}(n) &= \text{Tr}_{\mathcal{H}}[I \otimes \rho_\beta X_0^0 U X_0^0] = \text{Tr}_{\mathcal{H}}[I \otimes \rho_\beta U] \\ &= \beta_0 \left(I - \frac{1}{n} \left(i H_0 + \frac{1}{2} \sum_{i=1}^N C_i^* C_i \right) \right) \\ &\quad + \sum_{i=1}^N \beta_k \left(I - \frac{1}{n} \left(i H_0 + \gamma_i I + \frac{1}{2} C_i C_i^* \right) \right) + o\left(\frac{1}{n}\right) \\ &= I - \frac{1}{n} \left(i H_0 + \sum_{i=1}^N \beta_i \gamma_i I + \frac{1}{2} \sum_{i=0}^N (\beta_0 C_i^* C_i + \beta_i C_i C_i^*) \right) + o\left(\frac{1}{n}\right). \end{aligned} \quad (20)$$

Let us stress that we have used $\sum_{i=0}^N \beta_i = 1$ to get the expression (20). Now if $k = 0$ and $l \neq 0$, we have

$$\begin{aligned} U_{0,l}^{00} &= \frac{1}{\sqrt{\beta_l}} \text{Tr}_{\mathcal{H}} [(I \otimes \rho_\beta) (I \otimes a_0^l) U] \\ &= -i \frac{1}{\sqrt{n}} \sqrt{\beta_0} C_l + o\left(\frac{1}{\sqrt{n}}\right). \end{aligned} \quad (21)$$

In the same way if $l = 0$ and $k \neq 0$, we get

$$U_{k,0}^{00} = -i \frac{1}{\sqrt{n}} \sqrt{\beta_k} C_k^* + o\left(\frac{1}{\sqrt{n}}\right). \quad (22)$$

Actually the terms (20, 21, 22) are the only terms which remain when considering the limit by applying (18). Indeed the other terms are expressed as

$$U_{kk}^{00} = -\frac{1}{n} \left(\sum_{i=1}^N \beta_i \overline{\nu_k^i} (\gamma I + \frac{1}{2} C_i C_i^*) + \beta_0 \overline{\nu_k^0} \left(i\gamma_0 I + \frac{1}{2} \sum_{i=1}^N C_i C_i^* \right) \right) + o\left(\frac{1}{n}\right)$$

and if $k \neq 0, l \neq 0$ and $k \neq l$

$$U_{kl}^{00} = -\frac{1}{n} \frac{1}{2} \sqrt{\beta_k} C_l C_k^* + o\left(\frac{1}{n}\right).$$

Hence, by (18), they do not contribute in the limit. \square

3.3. QUANTUM TRAJECTORIES AT POSITIVE TEMPERATURE

Before stating the equivalent of Theorem 2.3. with positive temperature, we need to be clear on how observables are transformed by the G.N.S representation. In particular, we have to describe $\pi(I \otimes P)$ when P is a projector. By the rule (16), the coefficients P_{kl}^{ij} of $\pi(I \otimes P)$ are given by

$$\begin{aligned} P_{kl}^{ij} &= \text{Tr}_{\mathcal{H}} [I \otimes \rho_\beta (I \otimes X_l^k)^* I \otimes P(I \otimes X_j^i)] \\ &= \text{Tr}_{\mathcal{H}} [I \otimes (\rho_\beta (X_l^k)^* P X_j^i)] \\ &= \text{Tr} [\rho_\beta (X_l^k)^* P X_j^i] I. \end{aligned} \quad (23)$$

Define $p_{kl}^{ij} = \text{Tr} [\rho_\beta (X_l^k)^* P X_j^i]$ and $P' = (p_{kl}^{ij})$. Equation (23) means

$$\pi(P) = I \otimes P'. \quad (24)$$

We are then in a similar situation as for zero temperature, but one has to notice a very important fact: the first coefficient p_{00}^{00} is now always strictly positive. Indeed, we have

$$p_{00}^{00} = \text{Tr} [\rho_\beta (X_0^0)^* P X_0^0] = \text{Tr} [\rho_\beta P] = \text{Tr} [\rho_\beta P^* P] = \langle P, P \rangle > 0. \quad (25)$$

This simple remark has an important consequence: **there will be no jump contribution in the stochastic master equation for a heat bath.**

Indeed, let us consider an observable $A = \sum_{i=0}^p \lambda_i P_i$ and $\pi(A) = \sum_{i=0}^p \lambda_i I \otimes P'_i$. In Theorem 2.3., the jump contribution is directly connected to the set $I = \{i \in \{1, \dots, p\} / p_{00}^i =$

0}. In positive temperature, with the notation $P'_m = (p_{kl}^{ij}(m))$ the analogue of set I is the set $I' = \{i \in \{1, \dots, p\} / p_{00}^{00}(i) = 0\}$. Hence, the property (25) implies that $I' = \emptyset$, which implies that there is no jump contribution.

Consider the following functions defined on the set of the states:

$$\begin{aligned} \tilde{h}_i(\rho) = & \frac{1}{\sqrt{p_{00}^{00}(i)}} \left[\sum_{k=1}^N \left(p_{k0}^{00}(i) L_{k0}^{00} \rho + p_{0k}^{00}(i) L_{0k}^{00} \rho + p_{00}^{k0}(i) \rho (L_{k0}^{00})^* + p_{00}^{0k}(i) \rho (L_{0k}^{00})^* \right) \right. \\ & \left. - \text{Tr} \left[p_{k0}^{00}(i) L_{k0}^{00} \rho + p_{0k}^{00}(i) L_{0k}^{00} \rho + p_{00}^{k0}(i) \rho (L_{k0}^{00})^* + p_{00}^{0k}(i) \rho (L_{0k}^{00})^* \right] \rho \right], \end{aligned} \quad (26)$$

$$\tilde{\mathcal{L}}(\rho) = L_{00}^{00} \rho + \rho L_{00}^{00} + \sum_{k=1}^N \left(L_{k0}^{00} \rho + \rho (L_{k0}^{00})^* + L_{0k}^{00} \rho + \rho (L_{0k}^{00})^* \right). \quad (27)$$

THEOREM 4. *Let $A = \sum_{i=0}^p \lambda_i P_i$ be an observable. Let ρ_0 be a state on \mathcal{H}_0 . Let (ρ_k) be the discrete quantum trajectory describing the quantum repeated measurements at positive temperature. Let $(\rho_n(t))$ be the sequence of stochastic processes defined for all t and all n by $\rho_n(t) = \rho_{[nt]}(t)$. Then $(\rho_n(t))$ converges in distribution, when n goes to infinity, to the solution of the stochastic differential equation*

$$\rho_t = \rho_0 + \int_0^t \tilde{\mathcal{L}}(\rho_s) ds + \sum_{i=0}^p \int_0^t \tilde{h}_i(\rho_s) dW_i(s), \quad (28)$$

where $(W_i(t))$ are $N + 1$ independent Brownian motions.

Proof: With the expression (24), with the fact that the state ρ_β is a pure state in the G.N.S. representation, we can apply directly Theorem 2.3., with the particular restriction we have mentioned above. This gives easily Equation (28). \square

With the explicit expression of the coefficients L_{kl}^{00} , Equation (28) can be made more explicit:

$$\begin{aligned} \rho_t = & \rho_0 + \int_0^t \left(-i[H_0, \rho_s] - \frac{1}{2} \sum_{k=0}^N \left(\beta_0 (C_k^* C_k \rho_s + \rho_s C_k^* C_k - 2C_k \rho_s C_k^*) \right) \right. \\ & \left. - \frac{1}{2} \sum_{k=0}^N \left(\beta_k (C_k C_k^* \rho_s + \rho_s C_k C_k^* - 2C_k^* \rho_s C_k) \right) \right) ds \\ & - \sum_{m=0}^p \int_0^t \frac{1}{\sqrt{p_{00}^{00}(m)}} \left[\sum_{k=1}^N i\sqrt{\beta_0} \left(p_{0k}^{00}(m) C_k \rho_s - p_{00}^{0k}(m) \rho_s C_k^* - \text{Tr} \left[p_{0k}^{00}(m) C_k \rho_s - p_{00}^{0k}(m) \rho_s C_k^* \right] \rho_s \right) \right. \\ & \left. + \sum_{k=1}^N i\sqrt{\beta_k} \left(p_{k0}^{00}(m) C_k^* \rho_s - p_{00}^{k0}(m) \rho_s C_k - \text{Tr} \left[p_{k0}^{00}(m) C_k^* \rho_s - p_{00}^{k0}(m) \rho_s C_k \right] \rho_s \right) \right] dW_m(s). \end{aligned} \quad (29)$$

In this equation the expression

$$\begin{aligned} \tilde{\mathcal{L}}(\rho) = & -i[H_0, \rho] - \frac{1}{2} \sum_{k=0}^N \left(\beta_0 (C_k^* C_k \rho + \rho C_k^* C_k - 2C_k \rho C_k^*) \right) \\ & - \frac{1}{2} \sum_{k=0}^N \left(\beta_k (C_k C_k^* \rho + \rho C_k C_k^* - 2C_k^* \rho C_k) \right) \end{aligned} \quad (30)$$

corresponds to the usual Lindblad operator describing the evolution of a small system in contact with a heat bath at positive temperature.

3.4. THE 2-DIMENSIONAL CASE

As in Section 2.4., we want now to specialize the equation (29) when $\mathcal{H} = \mathbb{C}^2$ and when considering particular observables.

The first case is when the observable is diagonal, that is $A = \lambda_0 a_0^0 + \lambda_1 a_1^1$. In this case we have to compute $\pi(a_0^0)$ and $\pi(a_1^1)$. As we have $a_0^0 + a_1^1 = I$ and $\pi(I) = I$, we just have to compute $\pi(a_0^0)$. Since $p_{00}^{01}(0) = p_{01}^{00}(0)$ and $p_{00}^{10}(0) = p_{10}^{00}(0)$, we have only three terms to determine: $p_{00}^{00}(0)$, $p_{01}^{00}(0)$ and $p_{10}^{00}(0)$.

We get

$$\begin{aligned} p_{00}^{00}(0) &= \text{Tr}[\rho_\beta a_0^0] = \beta_0 \\ p_{01}^{00}(0) &= \text{Tr}[\rho_\beta (X_1^0)^* a_0^0 X_0^0] = \frac{1}{\sqrt{\beta_0}} \text{Tr}[\rho_\beta a_0^1 a_0^0] = 0 \\ p_{10}^{00}(0) &= \text{Tr}[\rho_\beta (X_0^1)^* a_0^0 X_0^0] = \frac{1}{\sqrt{\beta_1}} \text{Tr}[\rho_\beta a_1^0 a_0^0] = 0. \end{aligned} \quad (31)$$

As a consequence the equation (29) for a diagonal observable becomes

$$\rho_t = \rho_0 + \int_0^t \tilde{\mathcal{L}}(\rho_s) ds$$

which is just the master equation for a heat bath, with no noise contribution. Hence, at positive temperature, the repeated measurements of the observable A gives rise to a deterministic limit behavior. Recall that at zero temperature, the limit behavior was described by a jump equation.

At positive temperature, we had already seen that the limit behavior should not involve jump contribution, but here, in addition we have no randomness at all in the limit.

Let us now consider the observable $A = \lambda_0/2(a_0^0 + a_1^0 + a_0^1 + a_1^1) + \lambda_1/2(a_0^0 - a_1^0 - a_0^1 + a_1^1)$. As we have $1/2(a_0^0 + a_1^0 + a_0^1 + a_1^1) + 1/2(a_0^0 - a_1^0 - a_0^1 + a_1^1) = I$, we just need to compute $\pi(1/2(a_0^0 + a_1^0 + a_0^1 + a_1^1))$. With the previous computations, we only have to determine $\pi(a_1^0)$ and $\pi(a_0^1)$. In order to simplify the notations put $R = a_1^0$ and $S = a_0^1$. We have

$$\begin{aligned} R_{00}^{00} &= R_{10}^{00} = S_{00}^{00} = S_{01}^{00} = 0 \\ R_{01}^{00} &= \sqrt{\beta_0}, \quad S_{10}^{00} = \sqrt{\beta_1}. \end{aligned} \quad (32)$$

Hence by (31, 32), for $P_0 = 1/2(a_0^0 + a_1^0 + a_0^1 + a_1^1)$, we get

$$p_{00}^{00}(0) = \frac{1}{2}, \quad p_{01}^{00}(0) = \frac{\sqrt{\beta_0}}{2}, \quad p_{10}^{00}(0) = \frac{\sqrt{\beta_1}}{2}. \quad (33)$$

The equation (29) becomes

$$\begin{aligned} \rho_t = \rho_0 + \int_0^t \tilde{\mathcal{L}}(\rho_s) ds - \int_0^t \frac{i\sqrt{2}}{2} & \left(\beta_0 (C_1 \rho_s - \rho_s C_1^* - \text{Tr}[C_1 \rho_s - \rho_s C_1^*] \rho_s) + \right. \\ & \left. + \beta_1 (C_1^* \rho_s - \rho_s C_1 - \text{Tr}[C_1^* \rho_s - \rho_s C_1] \rho_s) \right) dW_0(s) \\ & + \int_0^t \frac{i\sqrt{2}}{2} \left(\beta_0 (C_1 \rho_s - \rho_s C_1^* - \text{Tr}[C_1 \rho_s - \rho_s C_1^*] \rho_s) + \right. \\ & \left. + \beta_1 (C_1^* \rho_s - \rho_s C_1 - \text{Tr}[C_1^* \rho_s - \rho_s C_1] \rho_s) \right) dW_1(s). \end{aligned} \quad (34)$$

By defining a new Brownian motion $W_t = \sqrt{2}/2(W_0(t) - W_1(t))$ and by putting $C = -iC_1$, the above equation becomes

$$\begin{aligned} \rho_t = \rho_0 + \int_0^t \tilde{\mathcal{L}}(\rho_s) ds + \int_0^t & \left(\beta_0 (C \rho_s + \rho_s C^* - \text{Tr}[C \rho_s + \rho_s C^*] \rho_s) + \right. \\ & \left. + \beta_1 (C^* \rho_s + \rho_s C - \text{Tr}[C^* \rho_s + \rho_s C] \rho_s) \right) dW_s. \end{aligned} \quad (35)$$

The equation (35) is then the equivalent of diffusive equation (1) (see Introduction) for the model with positive temperature.

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