ELEMENTS OF OPERATOR ALGEBRAS AND MODULAR THEORY

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1. C^*-ALGEBRAS

1.1 First definitions

A *C^*-algebra* is an algebra $\mathcal{A}$ equipped with an involution $A \mapsto A^*$ and a norm $||\cdot||$ satisfying:

i) $A^{**} = A$

ii) $(\lambda A + \mu B)^* = \overline{\lambda} A^* + \overline{\mu} B^*$

iii) $(AB)^* = B^* A^*$

i') $||A|| \geq 0$ and $||A|| = 0$ if and only if $A = 0$

ii') $||\lambda A|| = |\lambda|\,||A||$

iii') $||A + B|| \leq ||A|| + ||B||$

iv') $||AB|| \leq ||A||\,||B||$

i") $\mathcal{A}$ is complete for $||\cdot||$

ii") $||AA^*|| = ||A||^2$.

An algebra with an involution as above satisfying i), ii) and iii) is called a *$*$-algebra*.

An algebra satisfying all the conditions above but where ii") is replaced by

ii") $||A^*|| = ||A||$

is called a *Banach algebra*.

The basic examples of C^*-algebras are:

1) $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on a Hilbert space $\mathcal{H}$. The involution is the usual adjoint mapping and the norm is the usual operator norm:

$$||A|| = \sup_{||f||=1} ||Af||$$

.

2) $\mathcal{A} = \mathcal{K}(\mathcal{H})$, the algebra of compact operators on $\mathcal{H}$. It is a sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$.

3) $\mathcal{A} = C_0(X)$, the space of continuous functions vanishing at infinity on a locally compact space X. Recall that a function f is *vanishing at infinity* if for every $\varepsilon > 0$ there exists a compact $K \subset X$ such that $|f| < \varepsilon$ outside of K. The involution on $\mathcal{A}$ is the complex conjugation $\overline{f}$ and the norm is

$$||f|| = \sup_{x \in X} |f(x)|\,.$$

We will see later that examples i) and ii) contain all the cases of C^*-algebras.

Proposition 1.1 – *On a C^*-algebra $\mathcal{A}$ we have $||A^*|| = ||A||$ for all $A \in \mathcal{A}$.*

Proof

We have $||A||^2 = ||A^*A|| \leq ||A^*||\,||A||$ and thus $||A|| \leq ||A^*||$. Inverting the role of A and A^* gives the result. ■

An element I of a C^*-algebra $\mathcal{A}$ is a *unity* if

$$IA = AI = A$$

for all $A \in \mathcal{A}$.

If a unity exists it is unique and norm 1 (except if $\mathcal{A} = \{0\}$). But it may not always exists. Indeed, in the example $\mathcal{K}(\mathcal{H})$ there is a unity if and only if $\mathcal{H}$ is finite dimensional. In the example $C_0(X)$ there exists a unity if and only if X is compact.

But if a C^*-algebra does not contain a unity one can easily add one as follows. Consider the vector space $\mathcal{A}' = \mathcal{A} \oplus \mathbb{C}$ and provide it with the product

$$(A, \lambda)(B, \mu) = (AB + \lambda B + \mu A, \lambda\mu),$$

the involution

$$(A, \lambda)^* = (A^*, \overline{\lambda})$$

and the norm

$$||(A, \lambda)|| = \sup_{||B||=1} ||AB + \lambda B|| \,.$$

Equipped this way $\mathcal{A}'$ is a C^*-algebra. It admits a unity $(0, 1)$. The algebra $\mathcal{A}$ identifies to the subset of elements of the form $(A, 0)$. The only delicate point is to check that $||(A, \lambda)|| = 0$ if and only if $A = 0$ and $\lambda = 0$. One can assume that $\lambda \neq 0$ for if not we are in $\mathcal{A}$. Thus one can assume that $\lambda = 1$. We have

$$||B - AB|| \leq ||B|| \, ||(-A, 1)|| \,.$$

Thus if $||(-A, 1)|| = 0$ then $B = AB$ for all $B \in \mathcal{A}$. Applying the involution gives $B = BA^*$ for all $B \in \mathcal{A}$. In particular $A^* = AA^* = A$ and thus $B = AB = BA$. This means A is a unity. Contradiction.

Note that the above definition of the norm in $\mathcal{A}'$ comes from the fact that in any C^*-algebra we have

$$||A|| = \sup_{||B||=1} ||AB|| \,.$$

Indeed, there is obviously an inequality $\geq$ between the two terms above. The equality is obtained by considering $B = A^* / ||A||$.

1.2 Spectral analysis

Let $\mathcal{A}$ be a C^*-algebra with unity I. An element A of $\mathcal{A}$ is *invertible* if there exists an element A^{-1} of $\mathcal{A}$ such that

$$A^{-1}A = AA^{-1} = I.$$

One calls *resolvant set* of A the set

$$\rho(A) = \{\lambda \in \mathbb{C}; \lambda I - A \text{ is invertible}\}.$$

We put

$$\sigma(A) = \mathbb{C} \setminus \rho(A)$$

and call it the *spectrum* of A.

If $|\lambda| > ||A||$ then the series

$$\frac{1}{\lambda}\sum_n \left(\frac{A}{\lambda}\right)^n$$

is normally convergent and equals $(\lambda I - A)^{-1}$. This implies that $\sigma(A)$ is included in $B(0, ||A||)$.

Furthermore, if λ_0 belongs to $\rho(A)$ and if $\lambda \in \mathbb{C}$ is such that $|\lambda - \lambda_0| < ||\lambda_0 I - A||$, then the series

$$(\lambda_0 I - A)^{-1} \sum_n \left(\frac{\lambda_0 - \lambda}{\lambda_0 I - A}\right)^n$$

normally converges to $(\lambda I - A)^{-1}$. In particular we have proved that:

1) the set $\rho(A)$ is open

2) the mapping $\lambda \mapsto (\lambda I - A)^{-1}$ is analytic on $\rho(A)$

3) the set $\sigma(A)$ is compact.

We define

$$r(A) = \sup\{|\lambda|\,;\lambda \in \sigma(A)\}$$

the *spectral radius* of A.

Theorem 1.2– *We have for all* $A \in \mathcal{A}$

$$r(A) = \lim_n ||A^n||^{1/n} = \inf_n ||A^n||^{1/n} \leq ||A||\,.$$

In particular the above limit always exists and $\sigma(A)$ *is never empty.*

Proof

Let n be fixed and let $|\lambda| > ||A^n||^{1/n}$. Every integer m can be written $m = pn + q$ with p, q integers and $q < n$. Thus we have

$$\begin{aligned}\sum_m \left|\left|\left(\frac{A}{\lambda}\right)^m\right|\right| = \sum_m \left|\left|\left(\frac{A}{\lambda}\right)^{pn+q}\right|\right| &\leq \sum_m \left(\frac{||A^n||}{|\lambda|^n}\right)^p \left(\frac{||A||}{|\lambda|}\right)^q \\ &\leq \left(1 + \frac{||A||}{|\lambda|} + \ldots + \left(\frac{||A||}{|\lambda|}\right)^{n-1}\right) \sum_p \left(\frac{||A^n||}{|\lambda|^n}\right)^p < \infty.\end{aligned}$$

Thus the series

$$\frac{1}{\lambda}\sum_m \left(\frac{A}{\lambda}\right)^m$$

converges and is equal to $(\lambda I - A)^{-1}$. This proves that $r(A) \leq ||A^n||^{1/n}$ and thus $r(A) \leq \liminf_n ||A^n||^{1/n}$.

Let us prove that $r(A) \geq \limsup_n ||A^n||^{1/n}$. If we have

$$r(A) < \limsup_n ||A^n||^{1/n}$$

then consider the open set

$$\mathcal{O} = \{\lambda \in \mathbb{C}; r(A) < |\lambda| < \limsup_n ||A^n||^{1/n}\}.$$

On $\mathcal{O}$ all the operators $\lambda I - A$ are invertible, thus so are the operators $I - \frac{1}{\lambda}A$. The mapping $\lambda \mapsto (I - \frac{1}{\lambda}A)^{-1}$ is analytic on $\mathcal{O}$ and its Taylor series $\sum_n (\frac{A}{\lambda})^n$ converges. But the convergence radius of the series $\sum_n z^n A^n$ is exactly $(\limsup_n ||A^n||^{1/n})^{-1}$. This would mean

$$\frac{1}{|\lambda|} < (\limsup_n ||A^n||^{1/n})^{-1}$$

which contradicts the fact that $\lambda \in \mathcal{O}$. We have proved the first part of the theorem.

If $r(A) > 0$ then it is clear that $\sigma(A)$ is not empty. It remains to consider the case $r(A) = 0$. But note that if 0 belongs to $\rho(A)$ this means that A is invertible and $1 = ||A^n A^{-n}|| \leq ||A^n||\,||A^{-n}||$. In particular, passing to the limit, we get $r(A) > 0$. Thus if $r(A) = 0$ we must have $0 \in \sigma(A)$. In any case $\sigma(A)$ is non empty. ■

Corollary 1.3 – *A C^*-algebra $\mathcal{A}$ with unity and all of which elements, except 0, are invertible is isomorphic to $\mathbb{C}$.*

Proof

If $A \in \mathcal{A}$ its spectrum $\sigma(A)$ is non empty. Thus there exists a $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not invertible. This means $\lambda I - A = 0$ and $A = \lambda I$. ■

An element A of a C^*-algebra $\mathcal{A}$ with unity is
normal if $A^*A = AA^*$,
self-adjoint if $A = A^*$,
isometric if $A^*A = I$,
unitary if $A^*A = AA^* = I$.

Theorem 1.4 –

a) If A is normal then $r(A) = ||A||$.
b) If A is self-adjoint then $\sigma(A) \subset [-\,||A||\,,\,||A||]$.
c) If A is isometric then $r(A) = 1$.
d) If A is unitary then $\sigma(A) \subset \{\lambda \in \mathbb{C}; |\lambda| = 1\}$.
e) For all $A \in \mathcal{A}$ we have $\sigma(A^*) = \overline{\sigma(A)}$ and $\sigma(A^{-1}) = \sigma(A)^{-1}$.
f) For every polynomial function P we have

$$\sigma(P(A)) = P(\sigma(A)).$$

g) For any two $A, B \in \mathcal{A}$ then

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

Proof

a) If A is normal than

$$\left|\left|A^{2^n}\right|\right|^2 = \left|\left|A^{2^n} A^{*2^n}\right|\right| = \left|\left|(AA^*)^{2^n}\right|\right| = \left|\left|(AA^*)^{2^{n-1}}(AA^*)^{2^{n-1}}\right|\right|$$

$$= \left|\left|(AA^*)^{2^{n-1}}\right|\right|^2 = \ldots = ||AA^*||^{2^n} = ||A||^{2^{n+1}}.$$

One now concludes easily with Theorem 1.2.

b) We only have to prove that the spectrum of any self-adjoint element of $\mathcal{A}$ is a subset of $\mathbb{R}$. Let $\lambda = x + iy$ be an element of $\sigma(A)$, with x, y real. We have $x + i(y+t) \in \sigma(A + itI)$. But

$$||A + itI||^2 = ||(A + itI)(A - itI)|| = \left|\left|A^2 + t^2 I\right|\right| \leq ||A||^2 + t^2.$$

This implies

$$|x + i(y+t)|^2 = x^2 + (y+t)^2 \leq ||A||^2 + t^2$$

or else

$$2yt \leq ||A||^2 - x^2 - y^2$$

for all t. This means $y = 0$.

c) If A is isometric then

$$||A^n||^2 = ||A^{*n} A^n|| = \left|\left|A^{*n-1} A^{n-1}\right|\right| = \ldots = ||A^* A|| = ||I|| = 1.$$

d) Assume e) is proved. Then if A is unitary we have

$$\sigma(A) = \overline{\sigma(A^*)} = \overline{\sigma(A^{-1})} = \overline{\sigma(A)}^{-1}.$$

This and c) imply that $\sigma(A)$ is included in the unit circle.

e) The property $\sigma(A^*) = \overline{\sigma(A)}$ is obvious. For the other identity we write $\lambda I - A = \lambda A(A^{-1} - \lambda^{-1} I)$ and $\lambda^{-1} I - A^{-1} = \lambda^{-1} A^{-1}(A - \lambda I)$.

f) Note that if $B = A_1 \ldots A_n$ in $\mathcal{A}$, where all the A_i are two by two commuting, we have that B is invertible if and only if each A_i is invertible. Now choose α and $\alpha_1, \ldots \alpha_n$ in $\mathbb{C}$ such that

$$P(x) - \lambda = \sigma \prod_i (x - \alpha_i).$$

In particular we have

$$P(A) - \lambda I = \alpha \prod_i (A - \alpha_i I).$$

As a consequence $\lambda \in \sigma(P(A))$ if and only if $\alpha_i \in \sigma(A)$ for a i. But as $P(\alpha_i) = \lambda$ this exactly means that λ belongs to $\sigma(P(A)$ if and only if λ belongs to $P(\sigma(A))$.

g) If λ belong to $\rho(BA)$ then

$$(\lambda I - AB)(I + A(\lambda I - BA)^{-1} B) = \lambda I.$$

This proves that $\lambda I - AB$ is invertible, with possible exception of $\lambda = 0$. This proves one inclusion. The converse inclusion is obtained exchanging the role of A and B. ∎

Theorem 1.5 – *The norm which makes a $*$-algebra being a C^*-algebra, when it exists, is unique.*

Proof

By the above results we have

$$||A||^2 = ||AA^*|| = r(AA^*)$$

for AA^* is always normal. But $r(AA^*)$ depends only on the algebraic structure of $\mathcal{A}$. ■

Proposition 1.6 – *The set of invertible elements of a C^*-algebra $\mathcal{A}$ is open and the mapping $A \mapsto A^{-1}$ is continuous on this set.*

Proof

If A is invertible and if B is such that $||B - A|| < \left|\left|A^{-1}\right|\right|^{-1}$ then $B = A(I - A^{-1}(A - B))$ is invertible for

$$r(A^{-1}(A - B)) \leq \left|\left|A^{-1}(A - B)\right|\right| < 1$$

and thus $I - A^{-1}(A - B)$ is invertible. The open character is proved. Let us now show the continuity. If $||B - A|| < 1/2 \left|\left|A^{-1}\right|\right|^{-1}$ then

$$\begin{aligned}
\left|\left|B^{-1} - A^{-1}\right|\right| &= \left|\left|\sum_{n=0}^{\infty} \left(A^{-1}(A - B)\right)^n A^{-1} - A^{-1}\right|\right| \\
&\leq \sum_{n=1}^{\infty} \left|\left|A^{-1}(A - B)\right|\right|^n \left|\left|A^{-1}\right|\right| \\
&\leq \frac{\left|\left|A^{-1}\right|\right| ||A - B||}{1 - ||A^{-1}(A - B)||} \\
&\leq 2\left|\left|A^{-1}\right|\right|^2 ||A - B|| \,.
\end{aligned}$$

This proves the continuity. ■

Theorem 1.7 [Functional calculus] – *Let $\mathcal{A}$ be a C^*-algebra with unity. Let A be a self-adjoint element in $\mathcal{A}$. Let $C(\sigma(A))$ be the C^*-algebra of continuous functions on $\sigma(A)$. Then there is a unique morphism of C^*-algebra*

$$\begin{array}{ccc} C(\sigma(A)) & \longrightarrow & \mathcal{A} \\ f & \longmapsto & f(A) \end{array}$$

which sends the function $\mathbb{1}$ on I and the function $id_{\sigma(A)}$ on A.

Furthermore we have

$$\sigma(f(A)) = f(\sigma(A)) \tag{3}$$

for all $f \in C(\sigma(A))$.

Proof

When f is a polynomial function the application $f \mapsto f(A)$ is well-defined and isometric for

$$||f(A)|| = \sup\{|\lambda|\,;\lambda \in \sigma(f(A))\} = \sup\{|\lambda|\,;\lambda \in f(\sigma(A))\} = ||f|| \,.$$

Thus it extends to an isometry on $C(\sigma(A))$ by Weierstrass theorem. The extension is easily seen to be a morphism also. The only delicate point to check is the identity (3). Let $\mu \in f(\sigma(A))$, with $\mu = f(\lambda)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of polynomial functions converging to f. The sequence $(f_n(\lambda)I - f_n(A))_{n\in\mathbb{N}}$ converges to $\mu I - f(A)$. As none of the $f_n(\lambda)I - f_n(A)$ is invertible then $\mu I - f(A)$is not either (Proposition 1.6). Thus $f(\sigma(A)) \subset \sigma(f(A))$. Finally, if $\mu \in \mathbb{C} \setminus f(\sigma(A))$ then let $g(t) = (\mu - f(t))^{-1}$. Then g belongs to $C(\sigma(A))$ and $g(A) = (\mu I - f(A))^{-1}$. Thus μ belongs to $\mathbb{C} \setminus \sigma(f(A))$. ■

An element A of a C^*-algebra $\mathcal{A}$ is *positive* if it is self-adjoint and its spectrum is included in $\mathbb{R}^+$.

Theorem 1.8 – *Let A be an element of $\mathcal{A}$. The following assertions are equivalent.*

i) A is positive.

ii) A is self-adjoint and $||tI - A|| \leq t$ for some $t \geq ||A||$.

iii) A is self-adjoint and $||tI - A|| \leq t$ for all $t \geq ||A||$.

iv) $A = B^*B$ for a $B \in \mathcal{A}$.

v) $A = C^2$ for a self-adjoint $C \in \mathcal{A}$.

Proof

Let us first prove that i) implies iii). If i) is assumed then $tI - A$ is a normal operator and

$$||tI - A|| = \sup\{|\lambda|\,;\lambda \in \sigma(tI - A)\} = \sup\{|\lambda - t|\,;\lambda \in \sigma(A)\} \leq t.$$

This gives iii).

Obviously iii) implies ii). Let us prove that ii) implies i). If ii) is satisfied and if $\lambda \in \sigma(A)$ then $t - \lambda \in \sigma(tI - A)$ and with the same computation as above $|t - \lambda| \leq ||tI - A|| \leq t$. But as $\lambda \leq t$ we must have $\lambda \geq 0$. This proves i).

We have proved the equivalence of the first 3 assertions.

We have that v) implies iv) obviously. In order to show that i) implies v) it suffices to consider $C = \sqrt{A}$ (using the functional calculus of Theorem 1.7 and identity (3)). It remains to prove that iv) implies i). Let $f_+(t) = t \vee 0$ and $f_-(t) = (-t) \vee 0$. Let $A_+ = f_+(A)$ and $A_- = f_-(A)$ (note that when ii) holds then A is automatically self-adjoint and thus accepts the functional calculus of Theorem 1.7). We have $A = A_+ - A_-$ and the elements A_+ and A_- are positive (by (3)). Furthermore the identity $f_+f_- = 0$ implies $A_+A_- = 0$. We have

$$(BA_-)^*(BA_-) = A_-(A_+ - A_-)A_- = -A_-^3.$$

In particular $-(BA_-)^*(BA_-)$ is positive.

Writing $BA_- = S + iT$ with S and T self-adjoint gives

$$(BA_-)(BA_-)^* = -(BA_-)^*(BA_-) + 2(S^2 + T^2).$$

In particular, as the equivalence established between i), ii) and iii) proves it easily (exercise), the set of positive elements of $\mathcal{A}$ is a cone, thus the element $(BA_-)(BA_-)^*$ is positive. As a consequence $\sigma((BA_-)(BA_-)^*) \subset [0, ||B||\,||A_-||]$.

But by Theorem 1.4 g) we must also have $\sigma((BA_-)^*(BA_-)) \subset [0, ||B||\,||A_-||]$. In particular $\sigma(-A_-^3) \subset [0, ||B||^2 ||A_-||^2]$. This implies $\sigma(A_-^3) = \{0\}$ and $\left|\left|A_-^3\right|\right| = 0 = ||A_-||^3$. That is $A_- = 0$. ■

This notion of positivity defines an order on elements of $\mathcal{A}$, by saying that $U \geq V$ in $\mathcal{A}$ if $U - V$ is a positive element of $\mathcal{A}$.

Proposition 1.9 – *Let U, V be self-adjoint elements of $\mathcal{A}$ such that $U \geq V \geq 0$. Then*

i) $W^*UW \geq W^*VW \geq 0$ for all $W \in \mathcal{A}$;

ii) $(V + \lambda I)^{-1} \geq (U + \lambda I)^{-1}$ for all $\lambda \geq 0$.

Proof

i) is obvious from Theorem 1.8.

ii) As we have $U + \lambda I \geq V + \lambda I$, then by i) we have

$$(V + \lambda I)^{-1/2}(U + \lambda I)V + \lambda I)^{-1/2} \geq I.$$

Now, note that if W is self-adjoint and $W \geq I$ then $\sigma(W) \subset [1, +\infty[$ and $\sigma(W^{-1}) \subset [0, 1]$. In particular $W^{-1} \leq I$. This argument applied to the above inequality shows that

$$(V + \lambda I)^{1/2}(U + \lambda I)^{-1}V + \lambda I)^{1/2} \leq I.$$

Multiplying both sides by $(V + \lambda I)^{-1/2}$ gives the result. ■

1.3 Representations and states

A $*$-*algebra morphism* is a linear mapping $\Pi : \mathcal{A} \to \mathcal{B}$, between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$, such that $\Pi(A^*B) = \Pi(A)^*\Pi(B)$ for all $A, B \in \mathcal{A}$.

Such a morphism is always positive, that is it maps positive elements of $\mathcal{A}$ on positive elements of $\mathcal{B}$. Indeed we have $\Pi(A^*A) = \Pi(A)^*\Pi(A)$.

Theorem 1.10 – *If Π is a morphism between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ then Π is continuous, with norm smaller than 1. Furthermore the range of Π is a sub-C^*-algebra of $\mathcal{B}$.*

Proof

If A is self-adjoint then so is $\Pi(A)$ and thus

$$||\Pi(A)|| = \sup\{|\lambda|\,; \lambda \in \sigma(\Pi(A))\}.$$

But it is easy to see that $\sigma(\Pi(A))$ is included in $\sigma(A)$ and consequently

$$||\Pi(A)|| \leq \sup\{|\lambda|\,; \lambda \in \sigma(A)\} = ||A||\,.$$

For a general A now, we have

$$||\Pi(A)||^2 = ||\Pi(A^*A)|| \leq ||A^*A|| = ||A||^2.$$

We have proved the first part of the theorem.

For proving the second part we reduce the problem to the case where $\ker \Pi = \{0\}$. If this is not the case, following Annexe 1.1, we consider the quotient of $\mathcal{A}$ by the two-sided closed ideal $\ker \Pi$: $\mathcal{A}_\Pi = \mathcal{A}/\ker \Pi$ which is a C^*-algebra. We can thus assume $\ker \Pi = \{0\}$. Let $\mathcal{B}_\Pi$ be the image of Π, it is sufficient to prove that it is closed. Consider the inverse morphism Π^{-1} from $\mathcal{B}_\Pi$ onto $\mathcal{A}$. As previously, for A self-adjoint in $\mathcal{A}$ we have

$$||A|| = \left|\left|\Pi^{-1}(\Pi(A))\right|\right| \leq ||\Pi(A)|| \leq ||A|| \,.$$

Thus Π^{-1} and Π are isometric and one concludes easily. ■

A *representation* of a C^*-algebra $\mathcal{A}$ is a couple $(\mathcal{H}, \Pi)$ made of a Hilbert space $\mathcal{H}$ and a morphism Π from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. The representation is *faithful* if $\ker \Pi = \{0\}$.

Proposition 1.11 – *Let $(\mathcal{H}, \Pi)$ be a representation of a C^*-algebra $\mathcal{A}$. Then the following assertions are equivalent.*

i) Π is faithful.

ii) $||\Pi(A)|| = ||A||$ for all $A \in \mathcal{A}$.

iii) $\Pi(A) > 0$ if $A > 0$.

Proof

We have already seen that i) implies ii), in the proof above. Let us prove that ii) implies iii). If $A > 0$ then $||A|| > 0$ and thus $||\Pi(A)|| > 0$ and $\Pi(A) \neq 0$. As we already know that $\Pi(A) \geq 0$, we conclude that $\Pi(A) > 0$. Finally, assume iii) is satisfied. If B belongs to $\ker \Pi$ and $B \neq 0$ then $\Pi(B^* B) = 0$. But $||B^* B|| = ||B||^2 > 0$ and thus $B^* B > 0$. Which is contradictory and ends the proof. ■

Clearly we have not yet discussed the existence of representations for C^*-algebras. The key tool for this existence theorem is the notion of *state*.

A linear form ω on $\mathcal{A}$ is *positive* if $\omega(A^* A) \geq 0$ for all $A \in \mathcal{A}$.

Note that for such positive linear form one can easily prove a Cauchy-Schwarz inequality:

$$|\omega(B^* A)|^2 \leq \omega(B^* B)\omega(A^* A).$$

Proposition 1.12 – *Let ω be a linear form on $\mathcal{A}$. Then the following assertions are equivalent.*

i) ω is positive.

ii) ω is continuous with $||\omega|| = \omega(I)$.

Proof

By Theorem 1.8 ii), recall that a self-adjoint element A of $\mathcal{A}$, with $||A|| = 1$ is positive if and only if $||(I - A)|| \leq 1$. In particular, for any $A \in \mathcal{A}$, we have that $||A^* A||\, I - A^* A$ is positive.

If i) is satisfied then $\omega(A^* A) \leq ||A^* A||\, \omega(I)$. By Cauchy-Schwarz we have

$$|\omega(A)| \leq \omega(I)^{1/2} \, |\omega(A^* A)|^{1/2} \leq ||A^* A||^{1/2} \, \omega(I) = ||A|| \, \omega(I). \tag{1.2}$$

This proves ii).

Conversely, if ii) is satisfied. One can assume $\omega(I) = 1$. Let A be a self-adjoint element of $\mathcal{A}$. Write $\omega(A) = \alpha + i\beta$ for some α, β real. For every $\lambda \in \mathbb{R}$ we have

$$||A + i\lambda I||^2 = \left|\left|A^2 + \lambda^2 I\right|\right| = ||A||^2 + \lambda^2.$$

Thus we have

$$\beta^2 + 2\lambda\beta + \lambda^2 \leq \left|\alpha^2 + i(\beta + \lambda)\right|^2 = |\omega(A + i\lambda I)|^2 \leq ||A||^2 + \lambda^2.$$

This implies that $\beta = 0$ and $\omega(A)$ is real. Consider now A positive, with $||A|| = 1$. We have

$$|1 - \omega(A)| = |\omega(I - A)| \leq ||I - A|| \leq I.$$

Thus $\omega(A)$ is positive. ■

We call *state* any positive linear form on $\mathcal{A}$ such that $\omega(I) = 1$.

We need an existence theorem for states.

Theorem 1.13 – *Let A be any element of $\mathcal{A}$. Then there exists a state ω on $\mathcal{A}$ such that $\omega(A^*A) = ||A||^2$.*

Proof

On the space $\mathcal{B} = \{\alpha I + \beta A^*A; \alpha, \beta \in \mathbb{C}\}$ we define the linear form

$$f(\alpha I + \beta A^*A) = \alpha + \beta ||A||^2.$$

One easily checks that $||f|| = 1$. By Hahn-Banach we extend f to the whole of $\mathcal{A}$ into a norm 1 continuous linear form ω. By the previous proposition ω is a state. ■

We now turn to the construction of a representation which is going to be fundamental for us, the so called Gelfand-Naimark-Segal construction (G.N.S. construction). Indeed, note that if $(\mathcal{H}, \Pi)$ is a representation of a C^*-algebra $\mathcal{A}$ and if Ω is any norm 1 vector of $\mathcal{H}$, then the mapping

$$\omega(A) = <\Omega\,,\, \Pi(A)\Omega>$$

clearly defines a state on $\mathcal{A}$. The G.N.S. construction proves that any C^*-algebra with a state can be represented this way.

Theorem 1.14 (G.N.S. representation) – *Let $\mathcal{A}$ be a C^*-algebra with unit and ω be a state on $\mathcal{A}$. Then there exists a Hilbert space $\mathcal{H}_\omega$, a representation Π_ω of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_\omega)$ and a unit vector Ω_ω of $\mathcal{H}_\omega$ such that*

$$\omega(A) = <\Omega_\omega\,,\, \Pi_\omega(A)\Omega_\omega>$$

for all A. Furthermore the space $\{\Pi_\omega(A)\Omega_\omega; A \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.

Such a representation is unique up to unitary isomorphism.

Proof

Let $V_\omega = \{A \in \mathcal{A}; \omega(A^*A = 0)\}$. By Cauchy-Schwarz inequality one easily sees that V_ω is a closed two-sided ideal. We consider the quotient C^*-algebra $\mathcal{A}/V_\omega$. On $\mathcal{A}/V_\omega$ we define

$$<[A]\,,\,[B]> = \omega(B^*A).$$

It is a positive sesquilinear form which makes $\mathcal{A}/V_\omega$ a pre-Hilbert space. Let $\mathcal{H}_\omega$ be the its closure. We put

$$\begin{array}{rccc} L_A : & \mathcal{A}/V_\omega & \to & \mathcal{A}/V_\omega \\ & [B] & \mapsto & [AB]. \end{array}$$

We have

$$< L_A[B]\,,\, L_A[B] > = \omega(B^*A^*AB) \leq ||A||^2 \omega(B^*B)$$

for $C \mapsto \omega(B^*CB)$ is a positive linear form equal to $\omega(B^*B)$ on $C = I$. In particular $< L_A[B]\,,\, L_A[B] > \leq ||A||^2 < [B]\,,\, [B] >$. One can extend L_A into a bounded operator $\Pi_\omega(A)$ on $\mathcal{H}_\omega$. If we put $\Omega_\omega = [I]$ then the construction is achieved.

Let us check uniqueness. If $(\mathcal{H}', \Pi', \Omega')$ is another such triple, we have

$$\begin{aligned} < \Pi_\omega(B)\Omega_\omega\,,\, \Pi_\omega(A)\Omega_\omega > &= < \Omega_\omega\,,\, \Pi_\omega(B^*A)\Omega_\omega > = \omega(B^*A) \\ &= < \Omega'\,,\, \Pi'(B^*A)\Omega' > = < \Pi'(B)\Omega'\,,\, \Pi'(A)\Omega' >. \end{aligned}$$

The unitary isomorphism is thus defined by $U : \Pi_\omega(A)\Omega_\omega \mapsto \Pi'(A)\Omega'$. ■

Theorem 1.15 – *Let $\mathcal{A}$ be a C^*-algebra . Then $\mathcal{A}$ is isomorphic to a sub-C^*-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Proof

For every state ω we have the G.N.S. representation $(\mathcal{H}_\omega, \Pi_\omega, \Omega_\omega)$. Put $\mathcal{H} = \oplus_\omega \mathcal{H}_\omega$ and $\Pi = \oplus_\omega \Pi_\omega$ where the direct sums run over the set of all states on $\mathcal{A}$.

For every $A \in \mathcal{A}$ there exists a state ω_A such that $||\Pi_{\omega_A}(A)|| = ||A||$ (Theorem 1.12). But we have $||\Pi(A)|| \geq ||\Pi_{\omega_A}(A)|| = ||A||$. Thus we get $||\Pi(A)|| = ||A||$. This means that Π is faithful. ■

1.4 Commutative C^*-algebras

We have shown the very important characterization of C^*-algebras, namely they are exactly the closed $*$-sub-algebras of bounded operators on Hilbert space. We dedicate this last section to prove the (not very useful for us but) interesting characterization of commutative C^*-algebras .

Let $\mathcal{A}$ be a commutative C^*-algebra . A *character* on $\mathcal{A}$ is a linear form χ on $\mathcal{A}$ satisfying

$$\chi(AB) = \chi(A)\chi(B)$$

for all $A, B \in \mathcal{A}$. On then calls *spectrum* of $\mathcal{A}$ the set $\sigma(\mathcal{A})$ of all characters on $\mathcal{A}$.

Proposition 1.16 – *Every character is positive.*

Proof

Let $A \in \mathcal{A}$ and $\lambda \notin \sigma(A)$. Then there exists $B \in \mathcal{A}$ such that $(\lambda I - A)B = I$. Thus $\chi(\lambda I - A)\chi(B) = (\lambda\chi(I) - \chi(A))\chi(B) = \chi(I) = 1$. This implies in particular that $\lambda \neq \chi(A)$. We have proved that $\chi(A)$ always belong to $\sigma(A)$. In particular $\chi(A^*A)$ is always positive. ■

As a corollary every character is a state and thus is continuous. The set $\sigma(\mathcal{A})$ is a subset of $\mathcal{A}^*$, the dual of $\mathcal{A}$.

Theorem 1.17 – *Let $\mathcal{A}$ be a commutative C^*-algebra and X be the spectrum of $\mathcal{A}$ endowed with the $*$-weak topology of $\mathcal{A}^*$. Then X is a Haussdorf locally compact set; it is compact if and only if $\mathcal{A}$ admits a unit.*

Furthermore $\mathcal{A}$ is isomorphic to the C^*-algebra $C_0(X)$ of continuous functions on X which vanish at infinity.

Proof

Let $\omega_0 \in X$. Let A positive be such that $\omega_0(A) > 0$. One can assume $\omega_0(A) > 1$. Let $K = \{\omega \in X; \omega(A) > 1\}$. It is an open neighborhood of ω_0. Its closure $\overline{K}$ is included into $\{\omega \in X; \omega(A) \geq 1\}$. The latest set is closed and included in the unit ball of $\mathcal{A}^*$ which is compact. Thus X is locally compact.

If $\mathcal{A}$ contains a unit I, then the same argument applied to $A = 2I$ shows that X is compact.

Now, for all $A \in \mathcal{A}$ we put $\widehat{A}(\omega) = \omega(A)$. Then $\widehat{A}$ is a continuous complex function and $A \mapsto \widehat{A}$ is a morphism. Furthermore

$$\left|\left|\widehat{A}\right|\right|^2 = \sup_{\omega \in X} \left|\widehat{A}(\omega)\right|^2 = \sup_{\omega \in X} \left|\widehat{A^*A}(\omega)\right| = ||A||^2$$

for it exists an ω such that $|\omega(A^*A)| = ||A||$. Thus $A \mapsto \widehat{A}$ is an isomorphism.

The set $K_\varepsilon = \{\omega \in X; \omega(A) > \varepsilon\}$ is $*$-weakly compact and thus $\widehat{A}$ belong to $C_0(X)$. Finally $\widehat{A}$ separates the points of X, thus by Stone-Weierstrass theorem, the mapping $\widehat{A}$ gives the whole of $C_0(X)$. ■

Appendix: Quotient algebras and approximate identities

A subspace $\mathcal{J}$ of a C^*-algebra $\mathcal{A}$ is a *left ideal* if for all $J \in \mathcal{J}$ and all $A \in \mathcal{A}$ then JA belongs to $\mathcal{J}$. In the same way one obviously defines *right ideals* and *two-sided ideals*.

If $\mathcal{J}$ is a two-sided, self-adjoint ideal of $\mathcal{A}$, one can easily define the quotient algebra $\mathcal{A}/\mathcal{J}$ by the usual rules:

$$\begin{aligned} \text{i)}&\lambda[X] + \mu[Y] = [\lambda X + \mu Y], \\ \text{ii)}&[X]\,[Y] = [XY], \\ \text{iii)}&[X]^* = [X^*], \end{aligned}$$

where $[X] = \{X + J; J \in \mathcal{J}\}$ is the equivalence class of $X \in \mathcal{A}$ modulo $\mathcal{J}$. We leave to the reader to check the consistency of the above definitions.

We now define a norm on $\mathcal{A}/\mathcal{J}$ by

$$||[X]|| = \inf\{||X + J||\,; J \in \mathcal{J}\}.$$

The true difficulty is to check that the above norm is a C^*-algebra norm. For this aim we need the notion of approximate identity.

If $\mathcal{J}$ is a left ideal of $\mathcal{A}$ then an *approximate identity* in $\mathcal{J}$ is a generalized sequence $(e_\alpha)_a$ of positive elements of $\mathcal{J}$ satisfying

i) $||e_\alpha|| \leq 1$,

ii) $\alpha \leq \beta$ implies $e_\alpha \leq e_\beta$,

iii) $\lim_\alpha ||Xe_\alpha - X|| = 0$ for all $X \in \mathcal{J}$.

Proposition 1.18 – *Every left ideal $\mathcal{J}$ of a C^*-algebra $\mathcal{A}$ possesses an approximate unit.*

Proof

Let $\mathcal{J}_+$ be the set of positive elements of $\mathcal{J}$. For each $J \in \mathcal{J}_+$ put

$$e_J = J(I+J)^{-1} = I - (I+J)^{-1}.$$

It is a generalized sequence, it is increasing by Proposition 1.9 and $||e_J|| \leq 1$. Let us now fix $X \in \mathcal{J}$. For every $n \in \mathbb{N}$ there exists a $J \in \mathcal{J}_+$ such that $J \geq nX^*X$. Thus

$$(X - Xe_J)^*(X - Xe_J) = (I - e_J)X^*X(I - e_J) \leq \frac{1}{n}(I - e_J)J(I - e_J)$$

by Proposition 1.9. It suffices to prove that

$$\sup_{J \in \mathcal{J}_+} \left|\left|J(I - e_J)^2\right|\right| < \infty.$$

But note that $J(I - e_J)^2 = J(I+J)^{-2}$ and using the functional calculus this reduces to the obvious remark that $\lambda/(1+\lambda^2)$ is bounded on $\mathbb{R}^+$. ■

We can now prove the main result of the appendix.

Theorem 1.19 – *If $\mathcal{J}$ is a closed, self-adjoint, two-sided ideal of a C^*-algebra $\mathcal{A}$, then the quotient algebra $\mathcal{A}/\mathcal{J}$, equiped with the quotient norm, is a C^*-algebra.*

Proof

Let us first show that

$$||[X]|| = \lim_\alpha ||e_\alpha X - X||$$

for all $X \in \mathcal{J}$. By definition of the quotient we obviously have

$$||[X]|| \leq \lim_\alpha ||e_\alpha X - X|| .$$

As $\sigma(e_\alpha) \subset [0,1]$ we have $\sigma(I - e_\alpha) \subset [0,1]$ and $||I - e_\alpha|| \leq 1$. This implies

$$||(X + e_\alpha X) + (Y + e_\alpha Y)|| = ||(I - e_\alpha)(X+Y)|| \leq ||X+Y|| .$$

In particular $\limsup_\alpha ||(X + e_\alpha X)|| \leq ||X+Y||$ for every $Y \in \mathcal{J}$. This proves our claim.

Now we have

$$\begin{aligned} ||[X]||^2 &= \lim_\alpha ||X - e_\alpha X||^2 = \lim_\alpha ||(X^* - X^* e_\alpha)(X - e_\alpha X)|| \\ &= \lim_\alpha ||(I - e_\alpha)(X^*X + Y^*)(I - e_\alpha)|| \\ &\leq ||X^*X + \psi|| \end{aligned}$$

for every $Y \in \mathcal{J}$. This implies

$$||[X]||^2 \leq ||[X]^*[X]||$$

and thus the result. ■

2. VON NEUMANN ALGEBRAS

2.1 Topologies on $\mathcal{B}(\mathcal{H})$

As every C^*-algebra is a sub-$*$-algebra of some $\mathcal{B}(\mathcal{H})$, closed for the operator norm topology (or *uniform topology*), then it inherits new topologies, which are weaker.

On $\mathcal{B}(\mathcal{H})$ we define the *strong topology* to be the locally convex topology defined by the semi-norms $P_x(A) = ||Ax||$, $x \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$. This is to say that a base of neighborhood is formed by the sets

$$V(A; x_1, \dots, x_n; \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}); ||(A - B)x_i|| < \varepsilon, i = 1, \dots, n\}.$$

On $\mathcal{B}(\mathcal{H})$ we define the *weak topology* to be the locally convex topology defined by the semi-norms $P_{x,y}(A) = |< x\,,\, Ay >|$, $x, y \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$. This is to say that a base of neighborhood is formed by the sets

$$V(A; x_1, \dots, x_n; y_1, \dots, y_n; \varepsilon) = \{B \in \mathcal{B}(\mathcal{H}); |< x_i\,,\, Ay_j >| < \varepsilon, i, j = 1, \dots, n\}.$$

Proposition 2.1 –

i) The weak topology is weaker than the strong topology which is itself weaker than the uniform topology. Once $\mathcal{H}$ is infinite dimensional then these comparisons are strict.

ii) A linear form on $\mathcal{B}(\mathcal{H})$ is strongly continuous if and only if it is weakly continuous.

iii) The strong and the weak closure of any convex subset of $\mathcal{B}(\mathcal{H})$ coincide.

Proof

i) All the comparisons are obvious in the large sense. To make the difference in infinite dimension assume that $\mathcal{H}$ is separable with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. The sequence $(P_n)_{n\in\mathbb{N}}$ of orthogonal projections onto the space generated by $e_1, \dots, e_n$ converges strongly to I but not uniformly. Furthermore, consider the unilateral shift $S : e_i \mapsto e_{i+1}$. Then S^k converges weakly to 0 when k tends to $+\infty$ but not strongly.

ii) Let $\Psi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a strongly continuous linear form. Then there exists $x_1, \dots, x_n \in \mathcal{H}$ such that

$$|\Psi(B)| \leq \sum_{i=1}^{n} ||Bx_i||$$

for all $B \in \mathcal{B}(\mathcal{H})$ (classical result on locally convex topologies, not proved here). On $\mathcal{B}(\mathcal{H})^n$ let P be the semi-norm defined by

$$P(A_1, \dots, A_n) = \sum_{i=1}^{n} ||A_i x_i||\,.$$

On the diagonal of $\mathcal{B}(\mathcal{H})^n$ we define the linear form $\widetilde{\Psi}$ by $\widetilde{\Psi}(A,\dots,A)=\Psi(A)$. We then have $\left|\widetilde{\Psi}(A,\dots,A)\right| \leq P(A,\dots,A)$. By Hahn-Banach, there exists a linear form Ψ on $\mathcal{B}(\mathcal{H})^n$ which extends $\widetilde{\Psi}$ and such that

$$|\Psi(A_1,\dots,A_n)| \leq P(A_1,\dots,A_n).$$

Let Ψ_k be the linear form on $\mathcal{B}(\mathcal{H})$ defined by

$$\Psi_k(A)=\Psi(0,\dots,0,A,0,\dots,0). \qquad (A \text{ is at the } k\text{-th place})$$

Then $|\Psi_k(A)| \leq ||Ax_k||$ for every A. Every vector $y \in \mathcal{H}$ can be written as Ax_k for some $A \in \mathcal{B}(\mathcal{H})$. The linear form $Ax_k \mapsto \Psi_k(A)$ is thus well-defined and continuous on $\mathcal{H}$. By Riesz theorem there exists a $y_k \in \mathcal{H}$ such that $\Psi_k(A) = <y_k\,,\, Ax_k>$. We have proved that

$$\Psi(A)=\sum_{i=1}^{n} <y_k\,,\, Ax_k>.$$

Thus Ψ is weakly continuous.

iii) is an easy consequence of ii) and of the geometric form of Hahn-Banach theorem. ■

Another topology is of importance for us, the *σ-weak topology.* It is the one determined by the semi-norms

$$p_{(x_n)_{n\in\mathbb{N}},(y_n)_{n\in\mathbb{N}}}(A)=\sum_{n=0}^{\infty}|<x_n\,,\, Ay_n>|$$

where $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ run over all sequences in $\mathcal{H}$ such that $\sum_n ||x_n||^2<\infty$ and $\sum_n ||y_n||^2<\infty$.

Let $\mathcal{T}(\mathcal{H})$ denote the Banach space of trace class operators on $\mathcal{H}$, equiped with the trace norme $||H||_1 = \mathrm{tr}\,|H|$, where $|H|=\sqrt{H^*H}$.

Theorem 2.2 – *The Banach space $\mathcal{B}(\mathcal{H})$ is the topological dual of $\mathcal{T}(\mathcal{H})$ thanks to the duality*

$$(A,T)\mapsto tr(AT),$$

$A \in \mathcal{B}(\mathcal{H})$, $T \in \mathcal{T}(\mathcal{H})$. Furthermore the $*$-weak topology on $\mathcal{B}(\mathcal{H})$ associated to this duality is the σ-weak topology.

Proof

The inequality $|\mathrm{tr}(AT)| \leq ||A||\,||T||_1$ proves that $\mathcal{B}(\mathcal{H})$ is included in the topological dual of $\mathcal{T}(\mathcal{H})$. Conversely, let ω be an element of the dual of $\mathcal{T}(\mathcal{H})$. Consider the rank one operators $E_{\xi,\nu} = |\,\xi\,><\,\nu\,|$. One easily checks that $||E_{\xi,\nu}||_1 = ||\xi||\,||\nu||$. Thus $|\omega(E_{\xi,\nu})| \leq ||\omega||\,||\xi||\,||\nu||$. By Riesz theorem there exists an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\omega(E_{\xi,\nu} = <\nu\,,\, A\xi>$. The linear form $\mathrm{tr}(A\,\cdot)$ then coincides with ω on rank one projectors. One concludes that they coincide on $\mathcal{T}(\mathcal{H})$ by density of finite rank operators. This proves the announced duality.

The $*$-weak topology associated to this duality is defined by the seminorms

$$P_T(A) = \mathrm{tr}(AT)$$

where T runs over $\mathcal{T}(\mathcal{H})$. But every trace class operator T writes

$$T = \sum_{n=0}^{\infty} \lambda_n \, | \, \xi_n >< \nu_n |$$

for some orthonormed systems $(\nu_n)_{n\in\mathbb{N}}$, $(\xi_n)_{n\in\mathbb{N}}$ and some absolutely summable sequence of complex numbers $(\lambda_n)_{n\in\mathbb{N}}$. Thus

$$\mathrm{tr}(AT) = \sum_{n=0}^{\infty} \lambda_n < \nu_n \, , \, A\Xi_n >$$

and the seminorms P_T are equivalent to those defining the σ-weak topology. ■

Corollary 2.3 – *Every σ-weakly continuous linear form on $\mathcal{B}(\mathcal{H})$ is of the form*

$$A \mapsto tr(AT)$$

for some $T \in \mathcal{T}(\mathcal{H})$. ■

We can now put the first definition of a von Neumann algebra.

A *von Neumann algebra* is a C^*-algebra acting on $\mathcal{H}$ which contains a unit I and which is weakly (strongly) closed.

Of course the whole of $\mathcal{B}(\mathcal{H})$ is the first example of a von Neumann algebra.

Another example, which is actually the archetype of commutative von Neumann algebra, is obtained when considering a measured space (X, μ), with a σ-finite measure μ. The $*$-algebra $L^\infty(X, \mu)$ acts on $\mathcal{H} = L^2(X, \mu)$ by multiplication. One can assume that X is locally compact. The C^*-algebra $C_0(X)$ also acts on $\mathcal{H}$. But every function $f \in L^\infty(X, \mu)$ is almost sure limit of a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_0(X)$. By dominated convergence, the space $L^\infty(X, \mu)$ is included in the weak closure of $C_0(X)$. But as $L^\infty(X, \mu)$ is also equal to its weak closure, we have that $L^\infty(X, \mu)$ is the weak closure of $C_0(X)$. We have proved that $L^\infty(X, \mu)$ is a von Neumann algebra and we have obtained it as the weak closure of some C^*-algebra.

2.2 Commutant

Let $\mathcal{M}$ be a subset of $\mathcal{B}(\mathcal{H})$. We put

$$\mathcal{M}' = \{B \in \mathcal{B}(\mathcal{H}); BM = MB \text{ for all } M \in \mathcal{M}\}.$$

The space $\mathcal{M}'$ is called the *commutant* of $\mathcal{M}$. We also define

$$\mathcal{M}'' = (\mathcal{M}')', \ldots, \mathcal{M}^{(n)} = (\mathcal{M}^{(n-1)})', \ldots$$

Proposition 2.4 – *For every subset $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ we have*

i) $\mathcal{M}'$ is weakly closed;

ii) $\mathcal{M}' = \mathcal{M}''' = \mathcal{M}^{(5)} = \dots$
and $\mathcal{M} \subset \mathcal{M}'' = \mathcal{M}^{(4)} = \dots$

Proof

i) If $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}'$ which converges weakly to A in $\mathcal{B}(\mathcal{H})$ then for all $B \in \mathcal{M}$ and all $x, y \in \mathcal{H}$ we have

$$|< x\,, (AB - BA)y >| \le |< x\,, (A - A_n)By >| + |< x\,, B(A - A_n)y >| \to_{n\to\infty} 0.$$

Thus A belongs to $\mathcal{M}'$.

ii) If B belongs to $\mathcal{M}'$ and A belongs to $\mathcal{M}$ then $AB = BA$, thus A belongs to $(\mathcal{M}')' = \mathcal{M}''$. This proves the inclusion $\mathcal{M} \subset \mathcal{M}''$. But note that if $\mathcal{M}_1 \subset \mathcal{M}_2$ then clearly $\mathcal{M}_2' \subset \mathcal{M}_1'$. Applying this to the previous inclusion gives $\mathcal{M}''' \subset \mathcal{M}'$. But as $\mathcal{M}'''$ is also equal to $(\mathcal{M}')''$ we should also have the converse inclusion to hold true. This means $\mathcal{M}' = \mathcal{M}'''$. We now conclude easily. ■

Proposition 2.5 – *Let $\mathcal{M}$ be a self-adjoint subset of $\mathcal{B}(\mathcal{H})$. Let $\mathcal{E}$ be a closed subspace of $\mathcal{H}$ and P be the orthogonal projector onto $\mathcal{E}$. Then $\mathcal{E}$ is invariant under $\mathcal{M}$ (in the sense $M\mathcal{E} \subset \mathcal{E}$ for all $M \in \mathcal{M}$) if and only if $P \in \mathcal{M}'$.*

Proof

The space $\mathcal{E}$ is invariant under $M \in \mathcal{M}$ if and only $MP = PMP$. Thus if $\mathcal{E}$ is invariant under $\mathcal{M}$ we have $MP = PMP$ for all $M \in \mathcal{M}$. Applying the involution on this equality and using the fact that $\mathcal{M}$ is self-adjoint, gives $PM = PMP$ for all $M \in \mathcal{M}$. Finally $PM = MP$ for all $M \in \mathcal{M}$ and P belongs to $\mathcal{M}'$. The converse is obvious. ■

Theorem 2.6 [Von Neumann density theorem] – *Let $\mathcal{M}$ be a sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$ which contains the identity I. Then $\mathcal{M}$ is weakly (strongly) dense in $\mathcal{M}''$.*

Proof

Let $B \in \mathcal{M}''$. Let $x_1, \dots, x_n \in \mathcal{H}$. Let

$$V = \{A \in \mathcal{B}(\mathcal{H}); ||(A - B)x_i|| < \varepsilon,\ i = 1, \dots, n\}$$

be a strong neighborhood of B. It is sufficient to show that V intersects $\mathcal{M}$. One can assume B to be self-adjoint as it can always be decomposed as a linear combination of two self-adjoint operators which also belong to $\mathcal{M}''$.

Let $\widetilde{\mathcal{H}} = \oplus_{i=1}^n \mathcal{H}$ and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widetilde{\mathcal{H}})$ be given by $\pi(A) = \oplus_{i=1}^n A$. Let $x = \{x_1, \dots, x_n\} \in \widetilde{\mathcal{H}}$. Let P be the orthogonal projection from $\widetilde{\mathcal{H}}$ onto the closure of $\pi(\mathcal{M})x = \{\pi(A)x; A \in \mathcal{M}\} \subset \widetilde{\mathcal{H}}$. By Proposition 2.5 we have that P belongs to $\pi(\mathcal{M})'$.

If one identifies $\mathcal{B}(\widetilde{\mathcal{H}})$ to $M_n(\mathcal{B}(\mathcal{H}))$ it is easy to see that $\pi(\mathcal{M})' = M_n(\mathcal{M}')$ and $\pi(\mathcal{M}'') \subset M_n(\mathcal{M}')'$ (be aware that the prime symbols above are relative to different operator spaces!).

This means that $\pi(B)$ belong to $\pi(\mathcal{M}'') \subset M_n(\mathcal{M}')' = \pi(\mathcal{M})''$. In particular B commutes with $P \in \pi(\mathcal{M})'$. This means that the space $\overline{\pi(\mathcal{M})x}$ is invariant under $\pi(B)$. In particular

$$\pi(B)\,(\pi(I)x) = \begin{pmatrix} Bx_1 \\ \vdots \\ Bx_n \end{pmatrix}$$

belongs to $\overline{\pi(\mathcal{M})x}$. This means that there exists a $A \in \mathcal{M}$ such that $||(B-A)x_i||$ is small for all $i = 1, \dots n$. Thus A belongs to $\mathcal{M} \cap V$. ■

As immediate corollary we have a characterization of von Neumann algebras.

Corollary 2.7 [Bicommutant theorem] – *Let $\mathcal{M}$ be a sub-$*$-algebra of $\mathcal{B}(\mathcal{H})$ which contains I. Then the following assertions are equivalent.*

i) $\mathcal{M}$ is weakly (strongly) closed.

ii) $\mathcal{M} = \mathcal{M}''$.

As I always belong to $\mathcal{M}''$, we have that a C^*-algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$.

■

2.3 Predual, normal states

Let $\mathcal{M}$ be a von Neumann algebra. Put $\mathcal{M}_1 = \{M \in \mathcal{M}; ||M|| \leq 1\}$. It is a weakly closed subset of the unit ball of $\mathcal{B}(\mathcal{H})$ which is weakly compact. Thus $\mathcal{M}_1$ is weakly compact. Note that the weak topology and the σ-weak topology coincide on $\mathcal{M}_1$ (exercise).

We denote by $\mathcal{M}_*$ the space of weakly (σ-weakly) continuous linear forms on $\mathcal{M}_1$. The space $\mathcal{M}_*$ is called the *predual* of $\mathcal{M}$, for a reason that will appear clear in next proposition. If Ψ belongs to $\mathcal{M}_*$ then $\Psi(\mathcal{M}_1)$ is compact in $\mathbb{C}$, thus Ψ is norm continuous. Thus $\mathcal{M}_*$ is a subspace of $\mathcal{M}^*$ the topological dual of $\mathcal{M}$.

Proposition 2.6 –

i) $\mathcal{M}_*$ is closed in $\mathcal{M}^*$, it is thus a Banach space.

ii) $\mathcal{M}$ is the dual of $\mathcal{M}_*$.

Proof

i) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_*$ which converges to a f in $\mathcal{M}^*$, that is

$$\sup_{||A||=1} |f_n(A) - f(A)| \longrightarrow_{n\to\infty} 0.$$

We want to show that f belongs to $\mathcal{M}_*$, that is f is weakly continuous on $\mathcal{M}_1$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}_1$ which converges weakly to $A \in \mathcal{M}_1$. Then

$$\begin{aligned} |f(A_n) - f(A)| &\leq |f(A_n) - f_m(A_n)| + |f_m(A) - f(A)| + |f_m(A_n) - f_m(A)| \\ &\leq 2 \sup_{||B||=1} |f_m(B) - f(B)| + |f_m(A_n) - f_m(A)| \end{aligned}$$

$$\to_{n\to\infty} 2 \sup_{||B||=1} |f_m(B) - f(B)|$$
$$\to_{m\to\infty} 0.$$

This proves i).

ii) For a $A \in \mathcal{M}$ we put

$$||A||_{du} = \sup_{\omega \in \mathcal{M}_*;||\omega||=1} |\omega(A)|$$

the norm of A for the duality announced in the statement of ii). Clearly we have $||A||_{du} \leq ||A||$.

For $x, y \in \mathcal{H}$ we denote by $\omega_{x,y}$ the linear form $A \mapsto <y\,,\, Ax>$ on $\mathcal{B}(\mathcal{H})$ and $\omega_{x,y|\mathcal{M}}$ the restriction of $\omega_{x,y}$ to $\mathcal{M}$. We have

$$||A|| = \sup_{||x||=||y||=1} |<y\,,\, Ax>| \leq \sup_{\omega=\omega_{x,y};||\omega||=1} |\omega(A)| \leq ||A||_{du}.$$

Thus $\mathcal{M}$ is indeed identified linearly and isometrically to a subspace of $(\mathcal{M}_*)^*$. We just have to prove that this identification is onto.

Let ϕ be a continuous linear form on $\mathcal{M}_*$. Let $\phi'(x, y) = \phi(\omega_{x,y|\mathcal{M}})$. Then ϕ' is a continuous sesquilinear form on $\mathcal{H}$, it is thus of the form $\phi'(x, y) = <y\,,\, Ax>$ for some $A \in \mathcal{B}(\mathcal{H})$.

If T' is a self-adjoint element of $\mathcal{M}'$ then $\omega_{T'x,y|\mathcal{M}} = \omega_{x,T'y|\mathcal{M}}$ and

$$<AT'x\,,\, y> = <T'Ax\,,\, y>$$

for all $x, y \in \mathcal{H}$. Thus A belong to $\mathcal{M}'' = \mathcal{M}$.

As $\omega_{x,y}(A) = <y\,,\, Ax> = \phi'(x, y) = \phi(\omega_{x,y|\mathcal{M}})$ then the image of A in $(\mathcal{M}_*)^*$ coincides with ϕ at least on the $\omega_{x,y}$. Now, it remains to show that this is sufficient for A and ϕ to coincide everywhere. That is, we have to prove that an element a of $(\mathcal{M}_*)^*$ which vanishes on all the $\omega_{x,y}$ is null. But all the elements of $\mathcal{M}_*$ are linear forms ω of the form $\omega(A) = \mathrm{tr}(\rho A)$ for some trace class operator ρ. As every trace class operator ρ writes as

$$\rho = \sum_n \lambda_n \,|x_n\rangle\langle x_n|$$

for some orthonormal basis $(x_n)_{n\in\mathbb{N}}$ and some summable sequence $(\lambda_n)_{n\in\mathbb{N}}$, we have that

$$\omega = \sum_n \lambda_n \,\omega_{x_n,x_n}$$

where the series above is convergent in $\mathcal{M}_*$. One concludes easily. ■

The two main examples of von Neumann algebra have well-known preduals. Indeed, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$ then $\mathcal{M}_* = \mathcal{T}(\mathcal{H})$ the space of trace class operators.

If $\mathcal{M} = L^\infty(X, \mu)$ then $\mathcal{M}_* = L^1(X, \mu)$.

Theorem 2.7 [Sakai theorem] – *A C^*-algebra is a von Neumann algebra if and only if it is the dual of some Banach space.*

Admitted. ∎

A state on a von Neumann algebra $\mathcal{M}$ is called *normal* if it is σ-weakly continuous. The following characterization is now straightforward.

Theorem 2.8 – *On a von Neumann algebra $\mathcal{M}$, for a state ω on $\mathcal{M}$, the following assertions are equivalent.*

i) The state ω is normal

ii) There exists a positive, trace class operator ρ on $\mathcal{H}$ such that $\mathrm{tr}\rho = 1$ *and*

$$\omega(A) = \mathrm{tr}(\rho A)$$

for all $A \in \mathcal{M}$.

∎

3. MODULAR THEORY OF VON NEUMANN ALGEBRAS

3.1 The modular operators

The starting point here is a couple $(\mathcal{M}, \omega)$, where $\mathcal{M}$ is a von Neumann algebra acting on some Hilbert space, ω is a normal faithful state on $\mathcal{M}$. Recall that ω is then of the form

$$\omega(A) = \mathrm{Tr}(\rho A)$$

for a strictly positive ρ, with $\mathrm{Tr}\rho = 1$.

Let us consider the G.N.S. representation of $(\mathcal{M}, \omega)$. That is, a triple $(\mathcal{H}, \Pi, \Omega)$ such that

i) Π is a morphism from $\mathcal{M}$ to $\mathcal{B}(\mathcal{H})$.

ii) $\omega(A) = <\Omega\,, \Pi(A)\Omega>$

iii) $\Pi(\mathcal{M})\Omega$ is dense in $\mathcal{H}$.

From now on, we omit to mention the representation Π and identify $\mathcal{M}$ and $\mathcal{M}'$ with $\Pi(\mathcal{M})$ and $\Pi(\mathcal{M}')$. We thus write $\omega(A) = <\Omega\,, A\Omega>$.

Proposition 3.1 – *The vector Ω is cyclic and separating for $\mathcal{M}$ and $\mathcal{M}'$.*

Proof

Ω is cyclic for $\mathcal{M}$ by iii) above. Let us see that it is separating for $\mathcal{M}$. If $A \in \mathcal{M}$ is such that $A\Omega = 0$ then $\omega(A) = 0$, but as ω is faithful this implies $A = 0$.

Let us now see that these properties of Ω on $\mathcal{M}$ imply the same ones on $\mathcal{M}'$. If A' belongs to $\mathcal{M}'$ and $A'\Omega = 0$ then $A'B\Omega = BA'\Omega = 0$ for all $B \in \mathcal{M}$. Thus A' vanishes on a dense subspace of $\mathcal{H}$, it is thus the null operator. This proves that Ω is separating for $\mathcal{M}'$.

Finally, let P' be the orthogonal projector onto the space $\mathcal{M}'\Omega$. As it is the projection onto a $\mathcal{M}'$-invariant space, it belongs to $(\mathcal{M}')' = \mathcal{M}$. But $P\Omega = \Omega$ and thus $(I - P)\Omega = 0$. As Ω is separating for $\mathcal{M}$ this implies $I - P = 0$ and Ω is cyclic for $\mathcal{M}'$. ■

As a consequence the (**anti-linear**) operators

$$\begin{array}{rccc} S_0 : & \mathcal{M}\Omega & \longrightarrow & \mathcal{M}\Omega \\ & A\Omega & \longmapsto & A^*\Omega \end{array}$$

$$\begin{array}{rccc} F_0 : & \mathcal{M}'\Omega & \longrightarrow & \mathcal{M}'\Omega \\ & B\Omega & \longmapsto & B^*\Omega \end{array}$$

are well-defined (by the separability of Ω) on dense domains.

Proposition 3.2 – *The operators S_0 and F_0 are closable and $\overline{F}_0 = S_0^*$, $\overline{S}_0 = F_0^*$.*

Proof

For all $A \in \mathcal{M}$, $B \in \mathcal{M}'$ we have

$$< B\Omega\,,\, S_0 A\Omega > = < B\Omega\,,\, A^*\Omega > = < A\Omega\,,\, B^*\Omega > = < A\Omega\,,\, F_0 B\Omega >.$$

This proves that $F_0 \subset S_0^*$ and $S_0 \subset F_0^*$. The operators S_0 and F_0 are thus closable.

Let us show that $\overline{F}_0 = S_0^*$. Actually it is sufficient to show that $S_0^* \subset \overline{F}_0$. Let $x \in \operatorname{Dom} S_0^*$ and $y = S_0^* x$. For any $A \in \mathcal{M}$ we have

$$< A\Omega\,,\, y > = < A\Omega\,,\, S_0^* x > = < x\,,\, S_0 A\Omega > = < x\,,\, A^*\Omega >.$$

If we define the operators Q_0 and Q_0^+ by

$$\begin{array}{rcccl} Q_0 & : & A\Omega & \longmapsto & Ax \\ Q_0^+ & : & A\Omega & \longmapsto & Ay \end{array}$$

we then have

$$\begin{aligned} < B\Omega\,,\, Q_0 A\Omega > &= < B\Omega\,,\, Ax > = < A^* B\Omega\,,\, x > \\ &= < y\,,\, B^* A\Omega > = < By\,,\, A\Omega > \\ &= < Q_0^+ B\Omega\,,\, A\Omega >. \end{aligned}$$

This proves that $Q_0^+ \subset Q_0^*$ and Q_0 is closable. Let $Q = \overline{Q}_0$. Note that we have

$$Q_0 AB\Omega = ABx = AQ_0 B\Omega.$$

This proves that $Q_0 A = AQ_0$ on $\operatorname{Dom} Q_0$ and thus $AQ \subset QA$ for all $A \in \mathcal{M}$. This means that Q is *affiliated* to $\mathcal{M}'$, that is, it fails from belonging to $\mathcal{M}'$ only by the fact it is an unbounded operator; but every bounded function of Q is thus in $\mathcal{M}'$. In particular, if $Q = U\,|Q|$ is the polar decomposition of Q then U belongs to $\mathcal{M}'$ and the spectral projections of $|Q|$ also belong to $\mathcal{M}'$.

Let $E_n = \mathbb{1}_{[0,n]}(|Q|)$. The operator $Q_n = UE_n\,|Q|$ thus belongs to $\mathcal{M}'$ and

$$\begin{aligned} Q_n\Omega &= UE_n\,|Q|\,\Omega = UE_n U^* U\,|Q|\,\Omega \\ &= UE_n U^* Q_0 \Omega = UE_n U^* x. \end{aligned}$$

Furthermore we have

$$Q_n^*\Omega = E_n\,|Q|\,U^*\Omega = E_n Q_0^+\Omega = E_n y.$$

This way UE_nU^*x belongs to $\operatorname{Dom} F_0$ and $F_0(UE_nU^*x) = E_n y$. But E_n tends to I and UU^* is the orthogonal projector onto $\operatorname{Ran} Q$, which contains x.

Finally, we have proved that $x \in \operatorname{Dom} \overline{F}_0$ and $\overline{F}_0 x = y = S_0^* x$. That is, $S_0^* \subset \overline{F}_0$.

The other case is treated similarly. ■

We now put $S = \overline{S}_0$ anf $F = \overline{F}_0$.

Lemma 3.3 – *We have*

$$S = S^{-1}.$$

Proof

Let $z \in \operatorname{Dom} S^*$. We have

$$< S_0 A\Omega\,,\, S^* z > = < A^*\Omega\,,\, S_0^* z > = < z\,,\, S_0 A^*\Omega > = < z\,,\, A\Omega >.$$

Thus S^*z belongs to $\mathrm{Dom}\, S_0^* = S^*$ and $(S^*)^2 z = z$.

Let $y \in \mathrm{Dom}\, S$ and $z \in \mathrm{Dom}\, S^*$, we have $S^*z \in \mathrm{Dom}\, S^*$ and

$$< S^*z\,,\, Sy > = < y\,,\, (S^*)^2 z > = < y\,,\, z >.$$

This means that Sy belongs to $\mathrm{Dom}\, S^{**} = \mathrm{Dom}\, S$ and $S^2 y = S^{**}Sy = y$.

We have proved that $\mathrm{Dom}\, S^2 = \mathrm{Dom}\, S$ and $S^2 = I$ on $\mathrm{Dom}\, S$. ■

We had proved in Proposition 3.2 that $F = S^*$. Thus the operators FS and SF are (self-adjoint) positive. The operators F and S have their range equal to their domain, they are invertible and equal to their inverse.

Let $\Delta = FS = S^*S$. Then Δ is invertible, with inverse $\Delta^{-1} = SF = SS^*$.

As S, Δ and thus $\Delta^{1/2}$ have a dense range then the partial anti-isometry J such that

$$S = J(S^*S)^{1/2}$$

(modular decomposition of S) is an anti-isometry from $\mathcal{H}$ to $\mathcal{H}$.

Furthermore

$$S = J\Delta^{1/2} = (SS^*)^{1/2}J = \Delta^{-1/2}J.$$

Let x belong to $\mathrm{Dom}\, S$. Then

$$x = S^2 x = J\Delta^{1/2}\Delta^{-1/2}Jx = J^2 x$$

and thus $J^2 = I$.

Note the following relations

$$\begin{aligned} S &= J\Delta^{1/2} \\ F = S^* &= \Delta^{1/2}J \\ \Delta^{-1} &= J\Delta J. \end{aligned}$$

The operator Δ has a spectral measure (E_λ). Thus the operator $\Delta^{-1} = J\Delta J$ has the spectral measure $(JE_\lambda J)$. Let f be a bounded Borel function, we have

$$\begin{aligned} < f(\Delta^{-1})x\,,\, x > &= \int \overline{f}(\lambda)\, d< JE_\lambda Jx\,,\, x > \\ &= \int \overline{f}(\lambda)\, d< Jx\,,\, E_\lambda Jx > \\ &= \int \overline{f}(\lambda)\, d< E_\lambda Jx\,,\, Jx > \\ &= < f(\Delta)Jx\,,\, Jx > \\ &= < Jx\,,\, \overline{f}(\Delta)Jx > \\ &= < J\overline{f}(\Delta)Jx\,,\, x >. \end{aligned}$$

This proves

$$f(\Delta^{-1}) = J\overline{f}(\Delta)J.$$

In particular

$$\Delta^{it} = J\Delta^{it}J$$
$$\Delta^{it}J = J\Delta^{it}.$$

Finally note that $S\Omega = F\Omega = \Omega$ and thus $\Delta\Omega = FS\Omega = \Omega$ which finally gives

$$\Delta^{1/2}\Omega = \Omega.$$

Let us now resume the situation we have already described.

Theorem 3.4 – *There exists an anti-unitary operator J from $\mathcal{H}$ to $\mathcal{H}$ and an (unbounded) invertible, positive operator Δ such that*

$$\Delta = FS,\ \ \Delta^{-1} = SF,\ \ J^2 = I$$
$$S = J\Delta^{1/2} = \Delta^{-1/2}J$$
$$F = J\Delta^{-1/2} = \Delta^{1/2}J$$
$$J\Delta^{it} = \Delta^{-it}J$$
$$J\Omega = \Delta\Omega = \Omega.$$

■

The operator Δ is called the *modular operator* and J is the *modular conjugation.*

It is interesting to note the following. If the state ω were *tracial*, that is, $\omega(AB) = \omega(BA)$ for all A, B, we would have

$$||S_0A\Omega||^2 = ||A^*\Omega||^2 = < A^*\Omega\,,\ A^*\Omega > = \omega(AA^*) = \omega(A^*A) = ||A\Omega||^2.$$

Thus S_0 would be an isometry and thus

$$S = J = F$$
$$\Delta = I.$$

3.2 The modular group

Let $A, B, C \in \mathcal{M}$. We have

$$SASBC\Omega = SAC^*B^*\Omega = BCA^*\Omega = BSAC^*\Omega = BSASC\Omega.$$

This proves that B and SAS commute. Thus SAS is affiliated to $\mathcal{M}'$.

Let us assume for a moment that Δ **is bounded**. In that case the operators $\Delta^{-1} = J\Delta J$, S and F are also bounded.

We have seen that

$$S\mathcal{M}S \subset \mathcal{M}'$$
$$F\mathcal{M}'F \subset \mathcal{M}.$$

This way we have

$$\Delta\mathcal{M}\Delta^{-1} = \Delta^{1/2}JJ\Delta^{1/2}\mathcal{M}\Delta^{-1/2}JJ\Delta^{-1/2}$$
$$= FS\mathcal{M}SF \subset F\mathcal{M}'F \subset \mathcal{M}.$$

We also have
$$\Delta^n \mathcal{M} \Delta^{-n} \subset \mathcal{M}$$
for all $n \in I\!N$.

For any $A \in \mathcal{M}$, $A' \in \mathcal{M}'$, the function
$$f(z) = ||\Delta||^{-2z} < \phi\,,\, [\Delta^z A \Delta^{-z}, A'] \psi >$$
is analytic on $\mathbb{C}$. It vanishes for $z = 0, 1, 2, ...$

As $\left|\left|\Delta^{-1}\right|\right| \, ||J\Delta J|| = ||\Delta||$ we have
$$|f(z)| = O\left(||\Delta||^{-2\Re z} \, (||\Delta||^{|\Re z|})^2\right) = O(1)$$
when $\Re z > 0$.

By Carlson's theorem we have $f(z) = 0$ for all $z \in \mathbb{C}$. Thus
$$\Delta^z \mathcal{M} \Delta^{-z} \subset \mathcal{M}'' = \mathcal{M}$$
for all $z \in \mathbb{C}$. But
$$\mathcal{M} = \Delta^z (\Delta^{-z} \mathcal{M} \Delta^z) \Delta^{-z} \subset \Delta^z \mathcal{M} \Delta^{-z}$$
and finally
$$\Delta^z \mathcal{M} \Delta^{-z} = \mathcal{M}.$$
Furthermore
$$J\mathcal{M}J = J\Delta^{1/2} \mathcal{M} \Delta^{-1/2} J = S\mathcal{M}S \subset \mathcal{M}'$$
$$J\mathcal{M}'J = J\Delta^{-1/2} \mathcal{M} \Delta^{1/2} J = F\mathcal{M}F \subset \mathcal{M}.$$
We have proved
$$J\mathcal{M}J = \mathcal{M}'.$$

The results we have obtained here are fundamental and extend to the case when Δ is unbounded. This is what the following theorem says. We do not prove it as it implies pages of difficult analytic considerations. We hope that the above computations make it credible.

Theorem 3.5 [Tomita-Takesaki's theorem] – *In any case we have*
$$J\mathcal{M}J = \mathcal{M}'$$
$$\Delta^{it} \mathcal{M} \Delta^{-it} = \mathcal{M}.$$

■

Put
$$\sigma_t(A) = \Delta^{it} A \Delta^{-it}.$$
This defines a one parameter group of automorphisms of $\mathcal{M}$.

Proposition 3.6 – *We have, for all $A, B \in \mathcal{M}$*
$$\omega(A\sigma_t(B)) = \omega(\sigma_{t+i}(B)A). \qquad (1)$$

Proof

$$\begin{aligned}
<\Omega\,,\,A\Delta^{it}B\Delta^{-it}\Omega> &= <\Delta^{-it}A^*\Omega\,,\,B\Omega> \\
&= <\Delta^{-it-1/2}A^*\Omega\,,\,\Delta^{1/2}B\Omega> \\
&= <\Delta^{-it-1}\Delta^{1/2}A^*\Omega\,,\,\Delta^{1/2}B\Omega> \\
&= <J\Delta^{-it+1}J\Delta^{1/2}A^*\Omega\,,\,\Delta^{1/2}B\Omega> \\
&= <J\Delta^{1/2}B\Omega\,,\,\Delta^{-it+1}J\Delta^{1/2}A^*\Omega> \\
&= <B^*\Omega\,,\,\Delta^{-it+1}A\Omega> \\
&= <\Omega\,,\,B\Delta^{-i(t+i)}A\Omega> \\
&= <\Omega\,,\,\Delta^{i(t+i)}B\Delta^{-i(t+i)}A\Omega> \\
&= \omega(\sigma_{t+i}(B)A).
\end{aligned}$$

■

It is interesting to relate the above equality with the following result.

Proposition 3.7 – *Let ω be a state of the form*

$$\omega(A) = \mathrm{Tr}(\rho A)$$

on $\mathcal{B}(\mathcal{K})$ for some trace-class positive ρ with $\mathrm{Tr}\rho = 1$. Let (σ_t) be the following group of automorphisms of $\mathcal{B}(\mathcal{K})$:

$$\sigma_t(A) = e^{itH}Ae^{-itH}$$

for some self-adjoint operator H on $\mathcal{K}$. Then the following assertions are equivalent.

i) For all $A, B \in \mathcal{B}(\mathcal{K})$, all $t \in \mathbb{R}$ and a fixed $\beta \in \mathbb{R}$ we have

$$\omega(A\sigma_t(B)) = \omega(\sigma_{t-\beta i}(B)A).$$

ii) ρ is given by

$$\rho = \frac{1}{Z}e^{-\beta H},$$

where $Z = \mathrm{Tr}(\exp(-\beta H))$.

Proof

ii) implies i): We compute directly

$$\begin{aligned}
\omega(A\sigma_t(B)) &= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}Ae^{itH}Be^{-itH}) \\
&= \frac{1}{Z}\mathrm{Tr}(Ae^{itH}Be^{(-it-\beta)H}) \\
&= \frac{1}{Z}\mathrm{Tr}(Ae^{-\beta H}e^{(it+\beta)H}Be^{(-it-\beta)H}) \\
&= \frac{1}{Z}\mathrm{Tr}(e^{-\beta H}e^{(it+\beta)H}Be^{(-it-\beta)H}A) \\
&= \omega(\sigma_{t-\beta i}(B)A).
\end{aligned}$$

i) implies ii): We have

$$\begin{aligned}\mathrm{Tr}(AB\rho) &= \mathrm{Tr}(\rho AB) = \omega(AB)\\ &= \omega(\sigma_{-\beta i}(B)A) = \mathrm{Tr}(\rho e^{\beta H} B e^{-\beta H} A) = \mathrm{Tr}(A\rho e^{\beta H} B e^{-\beta H}).\end{aligned}$$

As this is valid for any A we conclude that

$$B\rho = \rho e^{\beta H} B e^{-\beta H}$$

for all B. This means

$$B\left(\rho e^{\beta H}\right) = \left(\rho e^{\beta H}\right) B.$$

As this is valid for all B we conclude that $\rho \exp(\beta H)$ is a multiple of the identity. This gives ii). ■

Another very interesting result to add to Proposition 3.6 is that the modular group is the only one to perform the relation (1).

Theorem 3.8 – *$\sigma.$ is the only automorphism group to satisfy (1) on $\mathcal{M}$ for the given state ω.*

Proof

Let $\tau_{.}$ be another automorphism group on $\mathcal{M}$ which satisfies (1). Define the operators U_t by

$$U_t A\Omega = \tau_t(A)\Omega.$$

Then U_t is unitary for

$$\begin{aligned}\|U_t A\Omega\|^2 &= <\tau_t(A)\Omega\,,\,\tau_t(A)\Omega> = <\Omega\,,\,\tau_t(A^*A)\Omega>\\ &= \omega(\tau_t(A^*A)) = \omega(\tau_{t+i}(I)A^*A)\\ &= \omega(A^*A) = \|A\Omega\|^2.\end{aligned}$$

The family $U.$ is clearly a group, it is thus of the form $U_t = \exp itM$ for a self-adjoint operator M.

Note that $U_t\Omega = \Omega$ and thus $M\Omega = 0$.

Let A, B be entire elements for $\tau.$, then the relation $\omega(\tau_i(B)A) = \omega(AB)$ implies

$$\begin{aligned}<B^*\Omega\,,\,\Delta A\Omega> &= <\Delta^{1/2}B^*\Omega\,,\,JJ\Delta^{1/2}A\Omega>\\ &= <A^*\Omega\,,\,B\Omega>\\ &= \omega(AB)\\ &= \omega(\tau_i(B)A)\\ &= <\Omega\,,\,e^{-M}Be^{M}A\Omega>\\ &= <B^*\Omega\,,\,e^{M}A\Omega>.\end{aligned}$$

This means

$$\Delta = e^M$$

and $\tau = \sigma$. ■

3.3 Self-dual cone and standard form

We put

$$\mathcal{P} = \overline{\{AJAJ\Omega; A \in \mathcal{M}\}}.$$

Proposition 3.9 –

i) $\mathcal{P} = \overline{\Delta^{1/4}\mathcal{M}_+\Omega} = \overline{\Delta^{-1/4}\mathcal{M}'_+\Omega}$ *and thus* $\mathcal{P}$ *is a convex cone.*

ii) $\Delta^{it}\mathcal{P} = \mathcal{P}$ *for all* t.

iii) If f *is of positive type then* $f(\log\Delta)\mathcal{P} \subset \mathcal{P}$.

iv) If $\xi \in \mathcal{P}$ *then* $J\xi = \xi$.

v) If $A \in \mathcal{M}$ *then* $AJAJ\mathcal{P} \subset \mathcal{P}$.

Proof

i) Let $\mathcal{M}_0$ be the $*$-algebra of elements of $\mathcal{M}$ which are *entire* for the modular group σ. (that is, $t \mapsto \sigma_t(A)$ admits an analytic extension). We shall admit here that $\mathcal{M}_0$ is σ-weakly dense in $\mathcal{M}$.

For every $A \in \mathcal{M}_0$ we have

$$\begin{aligned}\Delta^{1/4}AA^*\Omega &= \sigma_{-i/4}(A)\sigma_{i/4}(A)^*\Omega\\ &= \sigma_{-i/4}(A)J\Delta^{1/2}\sigma_{i/4}(A)\Omega\\ &= \sigma_{-i/4}(A)J\sigma_{-i/4}(A)J\Omega\\ &= BJBJ\Omega\end{aligned}$$

where $B = \sigma_{-i/4}(A)$. By $\sigma_{-i/4}(\mathcal{M}_0) = \mathcal{M}_0$ and by the density of $\mathcal{M}_0$ in $\mathcal{M}$ we have

$$BJBJ\Omega \in \overline{\Delta^{1/4}\mathcal{M}_+\Omega} \subset \overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}}$$

for all $B \in \mathcal{M}$. Thus

$$\mathcal{P} \subset \overline{\Delta^{1/4}\mathcal{M}_+\Omega} \subset \overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}}.$$

Conversely, $\mathcal{M}_0^+\Omega$ is dense in $\overline{\mathcal{M}_+\Omega}$. Let $\psi \in \overline{\mathcal{M}_+\Omega}$. There exists a sequence $(A_n) \subset \mathcal{M}_0^+$ such that $A_n\Omega \to \psi$. We know by the above that $\Delta^{1/4}A_n\Omega$ belongs to $\mathcal{P}$. But

$$J\Delta^{1/2}A_n\Omega = A_n\Omega \to \psi = J\Delta^{1/2}\psi$$

and thus

$$\left|\left|\Delta^{1/4}(\psi - A_n\Omega)\right|\right|^2 = <\psi - A_n\Omega\,,\, \Delta^{1/2}(\psi - A_n\Omega)> \to 0.$$

Thus $\Delta^{1/4}\psi$ belongs to $\mathcal{P}$ and $\overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}} \subset \mathcal{P}$.

This proves the first equality of i). The second one is treated exactly in the same way.

ii) We have

$$\Delta^{it}\Delta^{1/4}\mathcal{M}_+\Omega = \Delta^{1/4}\Delta^{it}\mathcal{M}_+\Omega = \Delta^{1/4}\sigma_t(\mathcal{M}_+)\Omega = \Delta^{1/4}\mathcal{M}_+\Omega.$$

iii) If f is of positive type then f is the Fourier transform of some positive, finite, Borel measure μ on $\mathbb{R}$. In particular

$$f(\log\Delta) = \int \Delta^{it}\,d\mu(t).$$

One concludes with ii) now.

iv) $JAJAJ\Omega = JAJA\Omega = AJAJ\Omega$.

v) $AJAJBJBJ\Omega = ABJAJJBJ\Omega = ABJABJ\Omega$. ∎

Theorem 3.10 –

i) P is self-dual*, that is $\mathcal{P} = \mathcal{P}^\vee$ where*

$$\mathcal{P}^\vee = \{x \in \mathcal{H}; < y\,,\, x > \geq 0, \forall y \in \mathcal{P}\}.$$

ii) $\mathcal{P}$ is pointed*, that is,*

$$\mathcal{P} \cap (-\mathcal{P}) = \{0\}.$$

iii) If $J\xi = \xi$ then ξ admits a unique decomposition as $\xi = \xi_1 - \xi_2$ with $\xi_1, \xi_2 \in \mathcal{P}$ and ξ_1 orthogonal to ξ_2.

iv) The linear span of $\mathcal{P}$ is the whole of $\mathcal{H}$.

Proof

i) If $A \in \mathcal{M}_+$ and $A' \in \mathcal{M}'_+$ then

$$< \Delta^{1/4} A\Omega\,,\, \Delta^{-1/4} A'\Omega > = < A\Omega\,,\, A'\Omega > = < \Omega\,,\, A^{1/2} A' A^{1/2}\Omega > \geq 0.$$

Thus $\mathcal{P}$ is included in $\mathcal{P}^\vee$.

Conversely, if $\xi \in \mathcal{P}^\vee$, that is $< \xi\,,\, \nu > \geq 0$ for all $\nu \in \mathcal{P}$, we put

$$\xi_n = f_n(\log \Delta)\xi$$

where $f_n(x) = \exp(-x^2/2n^2)$. Then ξ_n belongs to $\cap_{\alpha \in \mathbb{C}} \operatorname{Dom} \Delta^\alpha$ and ξ_n converges to ξ. We know that $f_n(\log \Delta)\nu$ belongs to $\mathcal{P}$ and thus

$$< \xi_n\,,\, \nu > = < \xi\,,\, f_n(\log \Delta)\nu > \geq 0.$$

Let $A \in \mathcal{M}_+$ then $\Delta^{1/4} A\Omega$ belongs to $\mathcal{P}$ and

$$< \Delta^{1/4}\xi_n\,,\, A\Omega > = < \xi_n\,,\, \Delta^{1/4} A\Omega > \geq 0.$$

Thus $\Delta^{1/4}\xi_n$ belongs to $\overline{\mathcal{M}_+\Omega}^\vee$ which coincides with $\overline{\mathcal{M}'_+\Omega}$ (admitted). This finally gives that ξ_n belongs to $\Delta^{-1/4}\overline{\mathcal{M}'_+\Omega} \subset \mathcal{P}$. This proves i).

ii) If $\xi \in \mathcal{P} \cap (-\mathcal{P}) = \mathcal{P} \cap (-\mathcal{P}^\vee)$ then $< \xi\,,\, -\xi > \geq 0$ and $\xi = 0$.

iii) If $J\xi = \xi$ then, as $\mathcal{P}$ is convex and closed, there exists a unique $\xi_1 \in \mathcal{P}$ such that

$$||\xi - \xi_1|| = \inf\{||\xi - \nu||\,;\, \nu \in \mathcal{P}\}.$$

We put $\xi_2 = \xi_1 - \xi$. Let $\nu \in \mathcal{P}$ and $\lambda > 0$. Then $\xi_1 + \lambda\nu$ belongs to $\mathcal{P}$ and

$$||\xi - \xi_1||^2 \leq ||\xi_1 + \lambda\nu - \xi||^2.$$

That is $||\xi_2||^2 \leq ||\xi_2 + \lambda\nu||^2$, or else $\lambda^2 ||\nu||^2 + 2\lambda\Re < \xi_2\,,\, \nu > \geq 0$. This implies that $\Re < \xi_2\,,\, \nu >$ is positive. But as $J\xi_2 = \xi_2$ and $J\nu = \nu$ then

$$< \xi_2\,,\, \nu > = < J\xi_2\,,\, J\nu > = \overline{< \xi_2\,,\, \nu >}.$$

That is $< \xi_2\,,\, \nu > \geq 0$ and $\xi_2 \in \mathcal{P}^\vee = \mathcal{P}$.

iv) If ξ is orthogonal to the linear span of $\mathcal{P}$ then ξ belongs to $\mathcal{P}^\vee = \mathcal{P}$. thus $<\xi\,,\,\xi> = 0$ and $\xi = 0$. ■

Theorem 3.11 [Universality] –

1) If $\xi \in \mathcal{P}$ then ξ is cyclic for $\mathcal{M}$ if and only if it is separating for $\mathcal{M}$.

2) If $\xi \in \mathcal{P}$ is cyclic for $\mathcal{M}$ then $J_\xi, \mathcal{P}_\xi$ associated to $(\mathcal{M}, \xi)$ satisfy

$$J_\xi = J \quad \textit{and} \quad \mathcal{P}_\xi = \mathcal{P}.$$

Proof

1) If ξ is cyclic for $\mathcal{M}$ then $J\xi$ is cyclic for $\mathcal{M}' = J\mathcal{M}J$ and thus $\xi = J\xi$ is separating for $\mathcal{M}$. And conversely.

2) Define as before (the closed version of)

$$\begin{array}{rcccc} S_\xi & : & A\xi & \longmapsto & A^*\xi \\ F_\xi & : & A'\xi & \longmapsto & A'^*\xi. \end{array}$$

We have

$$\begin{aligned} JF_\xi JA\xi &= JF_\xi JAJ\xi \\ &= J(JAJ)^*\xi \\ &= A^*\xi \\ &= S_\xi A\xi. \end{aligned}$$

This proves that $S_\xi \subset JF_\xi J$. By a symmetric argument $F_\xi \subset JS_\xi J$ and thus $JS_\xi = F_\xi J$.

Note that

$$(JS_\xi)^* = S_\xi^* J = F_\xi J = JS_\xi.$$

This means that JS_ξ is self-adjoint. Let us prove that it is positive. We have

$$<A\xi\,,\,JS_\xi A\xi> = <A\xi\,,\,JA^*\xi> = <\xi\,,\,A^*JA^*\xi>$$

which is a positive quantity for ξ and A^*JA^*J belong to $\mathcal{P}$. This proves the positivity of JS_ξ.

We have

$$S_\xi = J_\xi \Delta_\xi^{1/2} = J(JS_\xi).$$

By uniqueness of the polar decomposition we must have $J = J_\xi$.

Finally, we have that $\mathcal{P}_\xi$ is generated by the $AJ_\xi AJ_\xi\xi = AJAJ\xi$. But as ξ belongs to $\mathcal{P}$ we have that $AJAJ\xi$ belongs to $\mathcal{P}$ and thus $\mathcal{P}_\xi \subset \mathcal{P}$. Finally, $\mathcal{P} = \mathcal{P}^\vee \subset \mathcal{P}_\xi^\vee = \mathcal{P}_\xi$ and $\mathcal{P} = \mathcal{P}_\xi$. ■

The following theorem is very usefull and powerfull, but its proof is very long, tedious and cannot be resumed, thus we prefer not enter into it and give the result as it is (cf [B-R], p. 108-117).

For every $\xi \in \mathcal{P}$ one can define a particular normal positive form

$$\omega_\xi(A) = <\xi\,,\,A\xi>$$

on $\mathcal{M}$. That is, $\xi \in \mathcal{M}_{*+}$.

Theorem 3.12 –

1) For every $\omega \in \mathcal{M}_{*+}$ there exists a unique $\xi \in \mathcal{P}$ such that

$$\omega = \omega_\xi .$$

2) The mapping $\xi \longmapsto \omega_\xi$ is an homeomorphism and

$$||\xi - \nu||^2 \leq ||\omega_\xi - \omega_\nu||^2 \leq ||\xi - \nu|| \, ||\xi + \nu|| \, .$$

■

We denote by $\omega \longmapsto \xi(\omega)$ the inverse mapping of $\xi \longmapsto \omega_\xi$.

Corollary 3.13 – *There exists a unique unitary representation*

$$\alpha \in Aut(\mathcal{M}) \longmapsto U_\alpha$$

of the group of $*$-automorphisms of $\mathcal{M}$ on $\mathcal{H}$, such that

i) $U_\alpha A U_\alpha^* = \alpha(A)$, for all $A \in \mathcal{M}$,

ii) $U_\alpha \mathcal{P} \subset \mathcal{P}$ and, moreover,

$$U_\alpha \xi(\omega) = \xi(\alpha^{-1^*}(\omega))$$

for all $\omega \in \mathcal{M}_{*+}$ and where $(\alpha^*\omega)(A) = \omega(\alpha(A))$.

iii) $[U_\alpha, J] = 0$.

Proof

Let $\alpha \in \mathrm{Aut}(\mathcal{M})$. Let $\xi \in \mathcal{P}$ be the representant of the state

$$A \longmapsto <\Omega \, , \, \alpha^{-1}(A)\Omega>.$$

That is,

$$<\xi \, , \, A\xi> = <\Omega \, , \, \alpha^{-1}(A)\Omega>.$$

In particular ξ is separating for $\mathcal{M}$ and hence cyclic. Define the operator

$$UA\Omega = \alpha(A)\xi.$$

We have

$$||UA\Omega||^2 = <\xi \, , \, \alpha(A^*A)\xi> = <\Omega \, , \, A^*A\Omega> = ||A\Omega||^2.$$

Thus U is unitary. In particular

$$U^* A\xi = \alpha^{-1}(A)\Omega.$$

Now, for $A, B \in \mathcal{M}$ we have

$$UAU^*B\xi = UA\alpha^{-1}(B)\Omega = \alpha(A\alpha^{-1}(B))\xi = \alpha(A)B\xi$$

and

$$\alpha(A) = UAU^*.$$

We have proved the existence of the unitary representation.

Note that

$$\begin{aligned} SU^*A\xi &= S\alpha^{-1}(A)\Omega \\ &= \alpha^{-1}(A)^*\Omega \\ &= \alpha^{-1}(A^*)\Omega \\ &= U^*A^*\xi \\ &= U^*S_\xi A\xi. \end{aligned}$$

Hence by closure

$$J\Delta^{1/2}U^* = U^*J_\xi\Delta_\xi^{1/2} = U^*J\Delta_\xi^{1/2}.$$

That is

$$UJU^*U\Delta^{1/2}U^* = J\Delta_\xi^{1/2}.$$

By uniqueness of the polar decomposition we must have $UJU^* = J$. This gives iii).

For $A \in \mathcal{M}$ we have

$$UAJAJ\Omega = \alpha(A)J\alpha(A)J\xi.$$

Since ξ belongs to $\mathcal{P}$ we deduce

$$U\mathcal{P} = \mathcal{P}.$$

If $\phi \in \mathcal{M}_{*+}$ we have

$$\begin{aligned} < U\xi(\phi)\,,\, AU\xi(\phi) > &= < \xi(\phi)\,,\, U^*AU\xi(\phi) > \\ &= < \xi(\phi)\,,\, \alpha^{-1}(A)\xi(\phi) > \\ &= \phi(\alpha^{-1}(A)) \\ &= (\alpha^{-1^*}(\phi))(A) \\ &= < \xi(\alpha^{-1^*}(\phi))\,,\, A\xi(\alpha^{-1^*}(\phi)) >. \end{aligned}$$

By uniqueness of the representing vector in $\mathcal{P}$

$$U(\alpha)\xi(\phi) = \xi(\alpha^{-1^*}(\phi)).$$

This gives ii) and also the uniqueness of the unitary representation. ■