Repeated Quantum Interactions and Unitary Random Walks

Stéphane ATTAL[1] and Ameur DHAHRI[2]

[1] Université de Lyon, Université de Lyon 1
Institut Camille Jordan, U.M.R. 5208
21 av Claude Bernard
69622 Villeubanne cedex, France

[2] Facultad de Matemáticas
Campus San Joaquín
Avenida Vicuña Mackenna 4860
Santiago, Chile

Abstract

Among the discrete evolution equations describing a quantum system $\mathcal{H}_S$ undergoing repeated quantum interactions with a chain of exterior systems, we study and characterize those which are directed by classical random variables in $\mathbb{R}^N$. The characterization we obtain is entirely algebraical in terms of the unitary operator driving the elementary interaction. We show that the solutions of these equations are then random walks on the group $U(\mathcal{H}_0)$ of unitary operators on $\mathcal{H}_0$.

1 Introduction

In the article [AP], Attal and Pautrat have explored the Hamiltonian description of a quantum system undergoing repeated interactions with a chain of quantum systems. They have shown that these "deterministic" dynamics give rise to quantum stochastic differential equations in the continuous limit. Since that result, some interest has been found in the repeated quantum interaction model in itself (cf [AJ1], [AJ2], [BJM1], [BJM2], [BP]) and several physical works are in progress on that subject (for example [KP]). These repeated interaction models are interesting for several reasons:

– they provide a quantum dynamic which is at the same time Hamiltonian and Markovian,

– they allow to implement easily the dissipation for a quantum system, in particular they are practical models for simulation,

– they exactly correspond to actual physical situations, in which a particle, or a field, is undergoing repeated interactions with another system (see e.g. [Har]),

– they are exactly the physical situations in which are performed indirect measurements of a quantum system and give rise to the so-called "quantum trajectories" (see e.g. [Pel]).

The probabilistic nature of the continuous limit found by Attal and Pautrat is not due to the passage to the limit, it is already built-in the Hamiltonian dynamics of repeated quantum interactions (it is actually built-in the axioms of quantum mechanics).

The evolution equations describing the repeated quantum interactions are purely deterministic but they already show up terms which can be interpreted as "discrete-time quantum noises". The point with these discrete quantum noises is that sometimes they may give rise to classical noises. That is, some linear combinations of these quantum noises happen to be mutually commuting families of Hermitian operators, hence they simultaneously diagonalize and they can be represented as classical stochastic processes.

In the other cases, that is, with different combinations of the quantum noises, no classical process emerges and the dynamics of repeated quantum interactions are purely quantum.

The aim of the article is to characterise algebraically, on the Hamiltonian, the case when the dynamics are classically driven.

The article is organised as follows. We first (Section 2) present the physical and mathematical setups for the repeated quantum interactions. In Section 3 we introduce the basic algebraic tool: the *obtuse random walks* which are an appropriate "basis" of random walks adapted to this language. We then explore and characterise the unitary random walks which emerge classically from the repeated quantum interactions (Section 4). We specialize in Section 5 our result to the one dimensional case which already shows up a non-trivial structure. Finally, the last section is devoted to physical examples; we exhibit explicit Hamiltonians giving rise to classical dynamics.

2 Repeated Quantum Interactions

2.1 The Physical Model

Repeated quantum interaction models are physical models developed by Attal and Pautrat in [AP] which consist in describing the Hamiltonian dynamics of a quantum system undergoing a sequence of interactions with an environment made of a chain of identical systems. These models were developed for they furnish a toy model for a quantum dissipative system, they are at the same time Hamiltonian and Markovian, they spontaneously give rise to quantum stochastic differential equations in the continuous time limit. Let us describe precisely the physical and the mathematical setup of these models.

We consider a reference quantum system with state space $\mathcal{H}_0$, which we shall call the *small system* (even if it is not that small!). Another system $\mathcal{H}_E$, called the *environment* is made up of a chain of identical copies of a quantum system $\mathcal{H}$, that is,

$$\mathcal{H}_E = \bigotimes_{n\in\mathbb{N}^*} \mathcal{H}$$

where the countable tensor product is understood in a sense that we shall make precise later.

The dynamics in between $\mathcal{H}_0$ and $\mathcal{H}_E$ is driven as follows. The small system $\mathcal{H}_0$ interacts with the first copy $\mathcal{H}$ of the chain during an interval $[0, h]$ of time and following an Hamiltonian H on $\mathcal{H}_0 \otimes \mathcal{H}$. That is, the two systems evolve together following the unitary operator

$$U = e^{-ihH}\,.$$

After this first interaction, the small system $\mathcal{H}_0$ stops interacting with the first copy and starts an interaction with the second copy which was left unchanged until then. This second interaction follows the same unitary operator U. And so on, the small system $\mathcal{H}_0$ interacts repeatedly with the elements of the chain one after the other, following the same unitary evolution U.

Let us give a mathematical setup to this repeated quantum interaction model.

2.2 The Mathematical Setup

Let $\mathcal{H}_0$ and $\mathcal{H}$ be two separable Hilbert spaces (in the following, for our probabilistic interpretations, the space $\mathcal{H}$ will be chosen to be finite dimensional). We choose a fixed orthonormal basis $\{X^n; n \in \mathcal{N} \cup \{0\}\}$ where $\mathcal{N} = \mathbb{N}^*$ or

$\{1, \ldots, N\}$ depending on wether $\mathcal{H}$ is infinite dimensional or not (note the particular role played by the vector X^0 in our notation). We consider the Hilbert space

$$T\Phi = \bigotimes_{n \in \mathbb{N}^*} \mathcal{H}$$

where this countable tensor product is understood with respect to the stabilizing sequence $(X^0)_{n \in \mathbb{N}^*}$. This is to say that an orthonormal basis of $T\Phi$ is made of the vectors

$$X_\sigma = \bigotimes_{n \in \mathbb{N}^*} X_n^{i_n}$$

where $\sigma = (i_n)_{n \in \mathbb{N}^*}$ runs over the set $\mathcal{P}$ of all sequences in $\mathcal{N} \cap \{0\}$ with only a finite number of terms different from 0.

Let U be a fixed unitary operator on $\mathcal{H}_0 \otimes \mathcal{H}$. We denote by U_n the natural ampliation of U to $\mathcal{H}_0 \otimes T\Phi$ where U_n acts as U on the tensor product of $\mathcal{H}_0$ and the n-th copy of $\mathcal{H}$ and U act as the identity of the other copies of $\mathcal{H}$. In our physical model, the operator U_n is the unitary operator expressing the result of the n-th interaction. We also define

$$V_n = U_n \, U_{n-1} \ldots U_1,$$

with the convention $V_0 = I$. Physically, V_n is clearly the unitary operator expressing the transformation of the whole system after the n first interactions.

Define the elementary operators a_j^i, $i, j \in \mathcal{N} \cap \{0\}$ on $\mathcal{H}$ by

$$a_j^i X^k = \delta_{i,k} \, X^j \, .$$

We denote by $a_j^i(n)$ their natural ampliation to $T\Phi$ acting on the n-th copy of $\mathcal{H}$ only. That is, if $\sigma = (i_n)_{n \in \mathbb{N}^*}$

$$a_j^i(n) X_\sigma = \delta_{i,i_n} \, X_{\sigma \setminus \{i_n\} \cup \{j\}} \, .$$

One can easily prove (in the finite dimensional case this is obvious, in the infinite dimensional case it is an exercise) that U can always be written as

$$U = \sum_{i,j \in \mathcal{N} \cup \{0\}} U_j^i \otimes a_j^i$$

for some bounded operators U_j^i on $\mathcal{H}_0$ such that:

– the series above is strongly convergent,

– $\sum_{k \in \mathcal{N} \cup \{0\}} (U_i^k)^* \, U_j^k = \sum_{k \in \mathcal{N} \cup \{0\}} U_j^k \, (U_i^k)^* = \delta_{i,j} \, I$.

With this representation for U, it is clear that the operator U_n, representing the n-th interaction, is given by

$$U_n = \sum_{i,j \in \mathcal{N} \cup \{0\}} U^i_j \otimes a^i_j(n)\,.$$

With these notations, the sequence (V_n) of unitary operators describing the n first repeated interactions can be represented as follows:

$$\begin{aligned} V_{n+1} &= U_{n+1}\, V_n \\ &= \sum_{i,j \in \mathcal{N} \cup \{0\}} U^i_j \otimes a^i_j(n+1) V_n\,. \end{aligned}$$

But, inductively, the operator V_n acts only on the n first sites of the chain $T\Phi$, whereas the operators $a^i_j(n+1)$ act on the $(n+1)$-th site only. Hence they commute. In the following, we shall drop the $\otimes$ symbols, identifying operators like $a^i_j(n+1)$ with $I_{\mathcal{H}_0} \otimes a^i_j(n+1)$. This gives finally

$$V_{n+1} = \sum_{i,j \in \mathcal{N} \cup \{0\}} U^i_j\, V_n\, a^i_j(n+1)\,. \tag{1}$$

In Quantum Probability Theory, the operators $a^i_j(n)$ have a particular interpretation, they are *discrete-time quantum noises*, they describe the different types of basic innovations that can be brought by the environment when interacting with the small system. See [At] for complete details on that theory, the understanding of which is not necessary here.

The only important point to understand at that stage is the following. In some cases the above equation (1) corresponds to an equation driven by a *classical noise*, i.e. driven by a *random walk*. This is what we shall describe in the next section.

3 Classical Random Walks

In order to understand the link that may exist between the discrete-time quantum noises a^i_j and classical random walks, one needs to pass through a particular family of random walks, the *obtuse random walks*. Defined by Attal and Emery in [A-E], these random walks constitute a kind of basis of all the random walks in $\mathbb{R}^N$. Let us describe them.

3.1 Obtuse Random Walks in $\mathbb{R}^N$

Let X be a random variable in $\mathbb{R}^N$ taking $N+1$ values $v_0, ..., v_N$ with respective probabilities $p_0, ..., p_N$ such that $p_i > 0$, $\forall i \in \{0, 1, ..., N\}$. The canonical space of X is the triple $(A,\, \mathcal{A},\, P)$, where $A = \{0, 1, ..., N\}$, $\mathcal{A}$ is the $\sigma-$field of subsets of A and P is the probability measure given by $P(\{i\}) = p_i$. Hence for all $i \in \{0, 1, ..., N\}$ we have $X(i) = v_i$ and $P(X = v_i) = P(\{i\}) = p_i$.

We say that such a random variable X is *centered* if $\mathbb{E}(X) = 0$ (as a vector of $\mathbb{R}^N$). We say that X is *normalized* if $Cov(X) = I$ (as a $N \times N$-matrix).

Let us denote by $X^1, ..., X^N$ the coordinates of X in the canonical basis of $\mathbb{R}^N$ and define the random variable X^0 on $(A,\, \mathcal{A},\, P)$ given by $X^0(i) = 1,\ \forall i \in A$. Let us introduce the random variables $\widetilde{X}^i$ defined by

$$\widetilde{X}^i(j) = \sqrt{p_j}\, X^i(j)\,,$$

for all $i,\, j \in \{0, 1, ..., N\}$. We then have the following easy characterization (cf [A-E]).

Proposition 3.1 *The following assertions are equivalent:*

1) *The random variable X is centered and normalized,*

2) *The family $v_0, ..., v_N$ of values of X satisfies $< v_i\,,\, v_j >= -1$, for all $i \neq j$ and the probabilities p_i's are given by*

$$p_i = \frac{1}{1 + ||v_i||^2}, \quad \text{for all } i \in \{0, 1, ..., N\}\,,$$

3) *The matrix $(\widetilde{X}^0, \widetilde{X}^1, ..., \widetilde{X}^N)$ is unitary.*

A family of $N + 1$ vectors in $\mathbb{R}^N$ satisfying the above condition

$$< v_i,\, v_j >= -1\,,$$

for all $i \neq j$, is called an *obtuse system* in [A-E]. Hence, a random variable X satisfying one of the above condition is called an *obtuse random variable.*

Note that, as a corollary of the above proposition, the random variables $X^0, X^1, ..., X^N$ are linearly independent and hence they form an orthonormal basis of $L^2(A,\, \mathcal{A},\, P)$. In particular, for every $i, j \in \{1, \ldots, N\}$ the random variable $X^i X^j$ can be decomposed into

$$X^i X^j = \sum_{k=0}^{N} T_k^{ij} X^k \tag{2}$$

for some real coefficients T_k^{ij}. The familly of such coefficients forms a so-called 3-tensor, that is they are the coordinates of a linear mapping T from $\mathbb{R}^N$ to $M_N(\mathbb{R})$.

We say that a 3-tensor T is *sesqui-symmetric* if the two following assumptions are satisfied:

i) $(i,\, j,\, k) \longmapsto T^{ij}_k$ is symmetric,

ii) $(i,\, j,\, l,\, m) \longmapsto \sum_{k=1}^N T^{ij}_k T^{lm}_k + \delta_{ij}\delta_{lm}$ is symmetric.

Using the commutativity and the associativity of the product X^iX^j it is easy to prove the following (cf [A-E]).

Theorem 3.2 *If X is a centered and normalized random variable in $\mathbb{R}^N$, taking exactly $N+1$ values, then there exists a sesqui-symmetric 3-tensor T such that*

$$X \otimes X = I + T(X)\,.$$

In the following, by an *obtuse random walk* we mean a sequence $(X_p)_{p\in\mathbb{N}}$ of independent copies of a given obtuse random variable X. Actually, the random walk is the sequence made of the partial sums $\sum_{p\leq n} X_p$, but we shall not make any distinction between the two processes in the terminology.

3.2 More General Random Variables

We claimed above that obtuse random variables are a kind of basis for the random variables in $\mathbb{R}^N$ in general. Let us make precise here what we mean by that.

First of all, a remark on the number $N+1$ of values attached to X in $\mathbb{R}^N$. If one had asked that X takes less than $N+1$ values in $\mathbb{R}^N$ (k, say) and be centered and normalized too, it is not difficult to show that X is actually taking values on a proper subspace of $\mathbb{R}^N$, with dimension $k-1$. For example, a centered, normalized random variable in $\mathbb{R}^2$ which takes only two different values, is living on a line.

Now, if Y is a random variable in $\mathbb{R}^N$ taking k different possible values $w_1, \ldots, w_k$, with probability $p_1, \ldots, p_k$ and $k > n+1$. Consider an obtuse random variable X in $\mathbb{R}^{k-1}$ taking values $v_1, \ldots, v_k$ with the same probabilities $p_1, \ldots, p_k$ as those of Y. We have seen that the coordinate random variables $X^1, \ldots, X^{k-1}$, together with the deterministic random variable X^0, form an orthonormal basis of $L^2(A, \mathcal{A}, P)$. As a consequence, we can represent each of the coordinates of Y as

$$Y^i = \sum_{j=0}^{k-1} \alpha^i_j\, X^j\,.$$

Hence we have a simple representation of Y in terms of a given obtuse random variable X.

3.3 Connecting With the Discrete Quantum Noises

The obtuse random walks admit a very simple and natural representation in terms of the operators $a^i_j(n)$ defined in Section 2.2.

Let X be an obtuse random variable in $\mathbb{R}^N$. On the product space $(A^{\mathbb{N}}, \mathcal{A}^{\otimes\mathbb{N}}, P^{\otimes\mathbb{N}})$ we define a sequence $(X_p)_{p\in\mathbb{N}}$ of independent, identically distributed, random variables, each with the same law as X.

Consider the space $T\Phi(X) = L^2(A^{\mathbb{N}}, \mathcal{A}^{\otimes\mathbb{N}}, P^{\otimes\mathbb{N}})$ and the random variables

$$X_A = \prod_{(p,i)\in A} X^i(p)\,,$$

where A is any sequence in $\{0,1,...,N\}$ with only finitely many terms different from 0.

The following result is also easy to prove (cf [At]).

Proposition 3.3 *The random variables X_A, where A runs over the sequences in $\{0,1,...,N\}$ with only finitely many terms different from 0, form an orthonormal basis of $T\Phi(X)$.*

In particular we see that there exists a very natural Hilbert space isomorphism between the space $T\Phi(X)$ and the chain $T\Phi$ constructed in Section 2.2, over the space $\mathcal{H} = \mathbb{C}^{N+1}$.

At this point we need to stop for a discussion. Consider the situation where we have a probability space $(\Omega, \mathcal{F}, P)$ and some random variables $X, Y, ... \in L^2(\Omega, \mathcal{F}, P)$, together with a unitary isomorphism U from $L^2(\Omega, \mathcal{F}, P)$ to some abstract Hilbert space $\mathcal{H}$. One can wonder, when carrying $L^2(\Omega, \mathcal{F}, P)$ to $\mathcal{H}$ through U, where the probabilistic informations about the random variables (such as laws, independance, ...) appear in $\mathcal{H}$.

Certainly not through the images UX, UY, ... of the random variables $X, Y, ...$, because, via a unitary isomorphism, they can be sent on any vector of $\mathcal{H}$ (with same norm). Hence UX, as an element of $\mathcal{H}$ contains no information at all about the probabilistic properties of X.

Consider now the operator $\mathcal{M}_X$ of multiplication by X on $L^2(\Omega, \mathcal{F}, P)$:

$$\begin{array}{rccc} \mathcal{M}_X & : & \mathrm{Dom}\,\mathcal{M}_X \subset L^2(\Omega, \mathcal{F}, P) & \to & L^2(\Omega, \mathcal{F}, P) \\ & & F & \mapsto & XF\,. \end{array}$$

This operator contains all the information about X. From it one can compute easily all the probabilistic properties of X, for example the law:

$$\mathbb{E}[f(X)] = \langle\, \mathbb{1}\,,\ f(\mathcal{M}_X)\mathbb{1}\,\rangle\,,$$

the independance:

$$\begin{aligned}&\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]\\ \Leftrightarrow&\langle\, \mathbb{1}\,,\, f(\mathcal{M}_X)g(\mathcal{M}_Y)\mathbb{1}\,\rangle = \langle\, \mathbb{1}\,,\, f(\mathcal{M}_X)\mathbb{1}\,\rangle\,\langle\, \mathbb{1}\,,\, g(\mathcal{M}_Y)\mathbb{1}\,\rangle\,.\end{aligned}$$

and so on ... Now, when transporting these operators through the isomorphism U, we lose no information about $X, Y, \ldots$. For example, put $\mathbb{X} = U\mathcal{M}_X U^*$ and $\Psi = U\mathbb{1}$, then $\mathbb{X}$ is a self-adjoint operator on $\mathcal{H}$, hence it admits a bounded functional calculus and we have, for example

$$\langle\, \Psi\,,\, f(\mathbb{X})\Psi\,\rangle_{\mathcal{H}} = \mathbb{E}[f(X)]\,.$$

In the same way, we can translate all the probabilistic properties of X on $\mathcal{H}$. Actually, there is no way to differentiate the operator $\mathbb{X}$ from the actual random variable X.

Regarding this discussion back to our setup, one can consider the operator $\mathcal{M}_{X^i(p)}$ of multiplication by the random variable X^i_p on $T\Phi(X)$. This self-adjoint operator contains all the probabilistic information associated to the random variable X^i_p, it admits the same functional calculus, etc ... it is the actual representative of the random variable X^i_p in this Hilbert space setup.

As each of the probabilistic space $T\Phi(X)$ is made isomorphic to $T\Phi$ we can naturally wonder what happens to the operators $\mathcal{M}_{X^i(p)}$ through this identification. The answer is surprisingly simple (cf [At]).

Theorem 3.4 *Let X be an obtuse random variable in $\mathbb{R}^N$ and let $(X_p)_{p\in\mathbb{N}}$ be the associated random walk on the canonical space $T\Phi(X)$. Let T be the sesqui-symmetric 3-tensor associated to X. If we denote by U the natural unitary isomorphism from $T\Phi(X)$ to $T\Phi$, then for all $p \in \mathbb{N}$, $i \in \{1, ..., N\}$ we have*

$$U\mathcal{M}_{X^i_p}U^* = a^0_i(p) + a^i_0(p) + \sum_{j,l=1}^{N} T^{jl}_i a^j_l(p)\,.$$

Here we are! By a simple linear combination of the basic matrices $a^i_j(p)$ one can reproduce any random variable on $\mathbb{R}^N$.

Coming back to the evolution equation (1), we see basically that two different cases may appear.

First case: the coefficients U^i_j of the basic unitary matrix U are such that Equation 1 reduces to something like

$$V_{n+1} = AV_n + \sum_{i=1}^{N} B_i V_n \mathcal{M}_{X^i_p}\,.$$

This means that this operator-valued evolution equation, when transported back to $T\Phi(X)$ is an operator-valued (actually unitary operator-valued) equation driven by a random walk $(X_p)_{p\in\mathbb{N}}$. It is a random walk on $U(N)$.

Second case: there is no such arrangement in Equation 1, this means it is purely quantum, it cannot be expressed via classical noises, only quantum noises.

Our aim, in the rest of the article is to characterize completely those unitary operators U which give rise to a classically driven evolution (first case).

4 Random Walks on $U(\mathcal{H}_0)$

In this section we work on the state space

$$T\Phi = \bigotimes_{n\in\mathbb{N}^*} \mathbb{C}^{N+1}\,.$$

We consider a fixed obtuse random variable X, with values $v_1, \ldots, v_N$ and with associated 3-tensor T. We identify the operator

$$a_i^0(p) + a_0^i(p) + \sum_{j,l=1}^{N} T_i^{jl} a_l^j(p)$$

with the random variable X_p^i and we denote it by X_p^i, instead of $\mathcal{M}_{X_p^i}$. Recall that X_p^0 is the constant random variable equal to 1, hence as a multiplication operator on $T\Phi$ it coincides with the identity operator I.

In the following we extend the coefficients of the 3-tensor T to the set $\{0, 1, \ldots, N\}$. This extension is achieved by assigning the following values:

$$T_0^{ij} = T_j^{i0} = T_j^{0i} = \delta_{i,j}\,.$$

With that extension, the second sesqui-symmetric relation for T is written simply

$$ii) \quad (i,\, j,\, l,\, m) \longmapsto \sum_{k=0}^{N} T_k^{ij} T_k^{lm} \ \text{ is symmetric.}$$

Recall the discrete time evolution equation (1) associated to the repeated quantum interactions:

$$V_{n+1} = \sum_{i,\,j=0}^{N} U_j^i\, V_n\, a_j^i(n+1)\,,$$

with the convention $V_0 = I$.

Proposition 4.1 *The discrete-time evolution equation (1) can be written as*

$$V_{n+1} = \sum_{i=0}^{N} B_i \, V_n \, X^i_{n+1} \, ,$$

for some operators B_k on $\mathcal{H}_0$, if and only if the coefficients U^i_j are of the form

$$U^i_j = \sum_{k=0}^{N} T^{ij}_k B_k \, . \tag{3}$$

Proof Let us prove first the sufficient direction. If U is of the form (3) then

$$\begin{aligned} V_{n+1} &= \sum_{i,j=0}^{N} U^i_j V_n \, a^i_j(n+1) \\ &= U^0_0 V_n \, a^0_0(n+1) + \sum_{i=1}^{N} U^i_0 V_n \, a^i_0(n+1) + \sum_{i=1}^{N} U^0_i V_n \, a^0_i(n+1) + \\ &\quad + \sum_{i,j=1}^{N} U^i_j V_n \, a^i_j(n+1) \, . \end{aligned}$$

The relation (3) implies in particular $U^0_0 = B_0$ and $U^0_i = U^i_0 = B_i$. This gives

$$\begin{aligned} V_{n+1} &= B_0 V_n \, a^0_0(n+1) + \sum_{i=1}^{N} B_i V_n \left(a^i_0(n+1) + a^0_i(n+1)\right) + \\ &\quad + \sum_{k=1}^{N} \sum_{i,j=1}^{N} T^{ij}_k B_k V_n \, a^i_j(n+1) + \sum_{i=0}^{N} B_0 V_n \, a^i_i(n+1) \\ &= B_0 V_n + \sum_{k=1}^{N} B_k V_n \left[a^k_0(n+1) + a^0_k(n+1) + \sum_{i,j=1}^{N} T^{ij}_k a^i_j(n+1)\right] \\ &= B_0 V_n + \sum_{k=1}^{N} B_k V_n \, X^k_{n+1} \\ &= \sum_{k=0}^{N} B_k V_n \, X^k_{n+1} \, . \end{aligned}$$

This gives the requiered result in one direction. The converse is easy to prove by reversing all the arguments above. □

Now, consider the operators

$$W_l = \sum_{i=0}^{N} v_l^i B_i\,,$$

with the convention $v_k^0 = 1$, for all $k \in \{0, 1, ..., N\}$. Our purpose in the sequel is to prove that these operators are unitary if and only if the evolution operator U is unitary. Here is the first step.

Proposition 4.2 *If U is a unitary operator, then for all $l \in \{0, 1, ..., N\}$ the operator W_l is unitary.*

Proof We have

$$W_l W_l^* = \sum_{i,j=0}^{N} v_l^i v_l^j B_i B_j^*\,.$$

But the relation (2) implies immediately

$$v_l^i v_l^j = \sum_{m=0}^{N} T_m^{ij} v_l^m\,.$$

Hence, we get

$$\begin{aligned}
W_l W_l^* &= \sum_{i,j,m=0}^{N} T_m^{ij} v_l^m B_i B_j^* \\
&= \sum_{j,m=0}^{N} v_l^m \Big(\sum_{i=0}^{N} T_m^{ij} B_i\Big) B_j^* \\
&= \sum_{j,m=0}^{N} v_l^m U_m^j U_j^{0*} \\
&= \sum_{m=0}^{N} v_l^m \Big(\sum_{j=0}^{N} U_m^j U_j^{0*}\Big) \\
&= \sum_{m=0}^{N} v_l^m \Big(\sum_{j=0}^{N} \delta_{m0} I\Big) \\
&= v_l^0 I = I\,.
\end{aligned}$$

This completes the proof. □

Now, our aim is to prove the converse of Proposition 4.2. In order to achieve this, we need to express the coefficients U_j^i of U in terms of the operators W_l. This is the aim of the following two lemmas.

Lemma 4.3 *For all $i \in \{0, 1, ..., N\}$ we have*

$$B_i = \sum_{l=0}^{N} p_l \, v_l^i \, W_l \, .$$

Proof We have

$$\begin{aligned} \sum_{l=0}^{N} p_l \, v_l^i \, W_l &= \sum_{l=0}^{N} p_l \, v_l^i \, \big(\sum_{j=0}^{N} v_l^j \, B_j \big) \\ &= \sum_{j=0}^{N} B_j \big(\sum_{l=0}^{N} p_l \, v_l^i v_l^j \big) \\ &= \sum_{j=0}^{N} B_j \, \mathbb{E}(X^i X^j) \\ &= \sum_{j=0}^{N} B_j \delta_{ij} = B_i \, . \end{aligned}$$

This ends the proof. □

Lemma 4.4 *For all $l, \; k \in \{0, 1, ..., N\}$ we have*

$$U_l^k = \sum_{i=0}^{N} p_i \, v_i^k v_i^l \, W_i \, .$$

Proof Recall that we have

$$U_l^k = \sum_{j=0}^{N} T_j^{kl} B_j$$

and

$$v_i^l v_i^k = \sum_{j=0}^{N} T_j^{kl} v_i^j . \tag{4}$$

By using Lemma 4.3 and relation (4), we get

$$\begin{aligned} U_l^k &= \sum_{i,j=0}^{N} p_i \, T_j^{kl} \, v_i^j \, W_i \\ &= \sum_{i=0}^{N} p_i \, W_i \big(\sum_{j=0}^{N} T_j^{kl} \, v_i^j \big) \\ &= \sum_{i=0}^{N} p_i \, v_i^k v_i^l \, W_i . \end{aligned}$$

□

As a corollary of the two above lemmas, we prove the following.

Proposition 4.5 *If all the operators W_i, for $i \in \{0,1,...,N\}$, are unitary then the operator U is unitary.*

Proof We have

$$\begin{aligned}
\sum_{k=0}^N (U_k^l)(U_m^k)^* &= \sum_{i,j,k=0} p_i p_j\, v_i^k v_j^k v_i^l v_j^m\, W_i W_j^* \\
&= \sum_{i,k=0}^N p_i^2\, (v_i^k)^2 v_i^l v_i^m\, I + \sum_{i,j,k=0,\, i\neq j}^N p_i p_j\, v_i^k v_j^k v_i^l v_j^m\, W_i W_j^* \\
&= \sum_{i=0}^N p_i \big(p_i(||v_i||^2+1)\big) v_i^l v_i^m\, I + \\
&\quad + \sum_{i,j=0,\, i\neq j}^N p_i p_j \big(\sum_{k=0}^N v_i^k v_j^k\big) v_i^l v_j^m\, W_i W_j^* \\
&= \sum_{i=0}^N p_i \big(p_i(||v_i||^2+1)\big) v_i^l v_i^m\, I + \\
&\quad + \sum_{i,j=0, i\neq j}^N p_i p_j (< v_i,\, v_j > +1) v_i^l v_j^m\, W_i W_j^*\,.
\end{aligned}$$

But recall that, by Proposition 3.1, we have $p_i(||v_i||^2+1) = 1$ and $< v_i,\, v_j >= -1$ for all $i \neq j$. Therefore we get

$$\sum_{k=0}^N (U_k^l)(U_m^k)^* = \mathbb{E}(X^l X^m) I = \delta_{ml} I\,.$$

We have proved the unitary character of U. □

Altogether we have proved the following result, which resumes all the results obtained above.

Theorem 4.6 *Let X be an obtuse random walk in $\mathbb{R}^N$, with values $v_0, \ldots, v_N$, with probabilities $p_0, \ldots, p_N$ and with associated 3-tensor T. Let $(X_p)_{p\in\mathbb{N}}$ be its associated obtuse random walk. Then the repeated quantum interaction evolution equation*

$$V_{n+1} = \sum_{i,j=0}^N U_j^i V_n\, a_j^i(n+1)$$

takes the form

$$V_{n+1} = \sum_{k=0}^{N} B_k V_n X_{n+1}^k$$

if and only if there exists unitary operators W_i, $i \in \{0, \dots, N\}$, on $\mathcal{H}_0$ such that the coefficients U_j^i of U are of the form

$$U_l^k = \sum_{i=0}^{N} p_i \, v_i^k v_i^l \, W_i \, .$$

In that case, the coefficients B_k above are given by

$$B_k = \sum_{l=0}^{N} p_l \, v_l^k \, W_l \, .$$

When the conditions above are satisfied, the evolution equation

$$V_{n+1} = \sum_{k=0}^{N} B_k V_n X_{n+1}^k$$

is, when seen in the space $T\Phi(X)$, an operator-valued evolution equation, driven by a random walk. It is natural to wonder what kind of stochastic process it gives rise to.

Theorem 4.7 *As a random sequence in $U(\mathcal{H}_0)$, the solution of the equation*

$$V_{n+1} = \sum_{k=0}^{N} B_k V_n X_{n+1}^k$$

is an homogeneous Markov chain on $U(N)$ (actually a standard random walk), described as follows: $V_0 = I$ almost surely and V_{n+1} takes one of the values $W_i V_n$, $i \in \{0, 1, ..., N\}$, with respective probability p_i, independently of V_n.

Proof Assume V_n is given, depending on the random variables $X_1, \dots, X_n$ only. Then the random variable X_{n+1} is independent and $X_{n+1}^i = v_l^i$, with probability p_l. Therefore, with probability p_l we get

$$V_{n+1} = \sum_{i=0}^{N} B_i v_l^i \, V_n = W_l V_n \, .$$

This proves the result. □

5 The Case $N = 1$

In order to illustrate the results of the previous section, we detail here the situation in the case $N = 1$.

Consider the set $\Omega = \{0,\, 1\}^{\mathbb{N}}$, equipped with the σ-field $\mathcal{F}$ generated by finite cylinders. We denote by ν_n the coordinate mappings, for all $n \in \mathbb{N}$, that is $\nu_n(\omega) = \omega(n)$.

For $p \in]0, 1[$ and $q = 1 - p$, we define the probability measure μ_p on $(\Omega,\, \mathcal{F})$ which makes $(\nu_n)_{n\in\mathbb{N}}$ to be a sequence of independent, identically distributed, Bernoulli random variables with law $p\delta_1 + q\delta_0$. We denote by $\mathbb{E}_p$ the expectation with respect to μ_p.

Define the random variables

$$X_n = \frac{\nu_n - p}{\sqrt{pq}}\,.$$

They satisfy $\mathbb{E}_p[X_n] = 0$ and $\mathbb{E}_p[X_n^2] = 1$, hence they are obtuse random variables in $\mathbb{R}$. They take the two values $v_0 = \sqrt{q/p}$ and $v_1 = -\sqrt{p/q}$ with respective probabilities p and q.

The 3-tensor T associated to X is easy to determine. Indeed, one can easily check the following multiplication formula.

Proposition 5.1 *We have*

$$X_n^2 = 1 + c_p X_n,$$

where $c_p = \frac{q-p}{\sqrt{pq}}$.

This means that the 3-tensor in this context, which is a constant, is $T = c_p$.

In this context also, note that the space $T\Phi(X)$ is the space $L^2(\Omega,\, \mathcal{F},\, \mu_p)$, whereas the space $T\Phi$ is $\otimes_{i\in\mathbb{N}}\, \mathbb{C}^2$. As an application of Theorem 3.4, the operator of multiplication by X_n on $T\Phi(X)$ is represented on $T\Phi$ as

$$M^p_{X_n} = a^0_1(n) + a^1_0(n) + c_p a^1_1(n)\,.$$

Here we are, we have put all the corresponding notations. We can apply Theorem 4.6 to this particular case.

Theorem 5.2 *Consider the obtuse random walk* $(X_n)_{n\in\mathbb{N}}$ *on* $\mathbb{R}$*, as described above. Then the repeated quantum interaction evolution equation*

$$V_{n+1} = \sum_{i,j=0}^{N} U^i_j V_n\, a^i_j(n+1)$$

takes the form

$$V_{n+1} = B_0 V_n + B_1 V_n X_{n+1}$$

if and only if there exist 2 unitary operators W_0 and W_1 on $\mathcal{H}_0$ such that

$$U = \begin{pmatrix} pW_0 + qW_1 & \sqrt{pq}\,(W_0 - W_1) \\ \sqrt{pq}\,(W_0 - W_1) & qW_0 + pW_1 \end{pmatrix} . \tag{5}$$

In that case, the coefficients B_i above are given by

$$B_0 = U^0_0 \,, \quad B_1 = U^0_1 = U^1_0 \,.$$

The random sequence $(V_n)_{n\in\mathbb{N}}$ is defined by $V_0 = I$ and

$$V_{n+1} = \begin{cases} W_0\, V_n & \text{with probability } p \\ W_1\, V_n & \text{with probability } q \,. \end{cases}$$

6 Some Physical Examples

We end up this article with a few physical examples in order to illustrate our results. For simplicity we stick to the case $N = 1$, that is, we are dealing with two-dimensional pieces of environment.

For a total Hamiltonian between the small system $\mathcal{H}_S$ and one piece $\mathcal{H}$ of the environment, we are considering typical Hamiltonians of the form

$$H_{\text{tot}} = H_S \otimes I + I \otimes H + \sum_i (V_i \otimes a^0_i + V^*_i \otimes a^i_0) + \sum_{i,j} D_{i,j} \otimes a^i_j$$

where $D^*_{i,j} = D_{j,i}$.

In our two-dimensional setup we consider an Hamiltonian of the form

$$H_{\text{tot}} = H_S \otimes I + V \otimes a^0_1 + V^* \otimes a^1_0 + D \otimes a^1_1 \,.$$

Let $p \in (0, 1)$ and put $c_p = (q - p)/\sqrt{pq}$, then in the case

$$V = V^*, \qquad D = c_p V,$$

the Hamiltonian is the block-matrix

$$\begin{pmatrix} H_S & V \\ V & H_S + c_p V \end{pmatrix} .$$

If furthermore we assume that H_S and V commute then, by an easy computation, we get

$$U = e^{-ihH_{\text{tot}}} = \begin{pmatrix} pW_0 + qW_1 & \sqrt{pq}(W_0 - W_1) \\ \sqrt{pq}(W_0 - W_1) & qW_0 + pW_1 \end{pmatrix} ,$$

with

$$W_0 = e^{-ih\left(H_S + \sqrt{\frac{q}{p}} V\right)} \qquad \text{and} \qquad W_1 = e^{-ih\left(H_S - \sqrt{\frac{p}{q}} V\right)} .$$

That is to say U is of the form (5). The repeated interaction dynamics associated to this Hamiltonian are driven by a classical sequence of Bernoulli random variables with parameter p. In particular the repeated interaction unitary operators V_n follow a Bernoulli random walk on $U(2)$ with jumps W_0 and W_1 as described above.

In other words, let $(\varepsilon_n)_{n\in\mathbb{N}^*}$ be a sequence of identically distributed, independant Bernoulli random variables, taking the values $\sqrt{q/p}$ with probability p and $-\sqrt{p/q}$ with probability $q = 1-p$. Let $X_n = \sum_{k=1}^n \varepsilon_k$ be the associated random walk. Then

$$V_n = e^{-ih(nH_S + X_n V)} .$$

In more general situations, for example when H_S does not commute with V the computations are in general very difficult to handle, at least explicitely. One case can be computed with great generality, it is the case of small time interactions, that is, for h very small. Assume, for example that we have a total Hamiltonian of the form

$$H_{\text{tot}} = H_S \otimes I + \frac{1}{\sqrt{h}} \left(V \otimes a_1^0 + V \otimes a_0^1 \right)$$

with $V = V^*$.

Note that H_{tot} depends on h too. Indeed, when considering the limit $h \to 0$, that is, passing from repeated interactions to continuous interactions, we have to reinforce the strength of the interactions between the two systems. This is achieved by renormalizing the field operators a_0^1 and a_1^0 by a factor $1/\sqrt{h}$. For a complete discussion on this limit and renormalization, see [AP].

The following discussion is written in a"non-rigorous" style, but all the arguments below can be easily justified (same reference).

Up to terms which are all $o(h)$ we then have

$$U = e^{-ihH_{\text{tot}}} = \begin{pmatrix} I - ihH_S - \frac{1}{2}hV^2 & -i\sqrt{h}V \\ -i\sqrt{h}V & I - ihH_S - \frac{1}{2}hV^2 \end{pmatrix} .$$

Putting

$$W_0 = I - ihH_S - \frac{1}{2}hV^2 - i\sqrt{h}V \qquad \text{and} \qquad W_1 = I - ihH_S - \frac{1}{2}hV^2 + i\sqrt{h}V ,$$

we see that U is under the form (5) for a symmetric Bernoulli random walk (i.e. $p = 1/2$). Note that W_0 and W_1 here are unitary up to $o(h)$ again, that is, $W_i^* W_i = I + o(h)$.

Let (ε_n) be a sequence of independent symmetric Bernoulli random variables, then the sequence (V_n) of unitary operators implementing the repeated interactions associated to the above Hamiltonian is given by

$$V_n = \prod_{k=1}^{n} \left(I - ihH_S - \frac{1}{2}hV^2 + i\sqrt{h}\,\varepsilon_k\,V \right)$$

or else, by the evolution equation

$$V_{n+1} - V_n = (-iH_S - \frac{1}{2}V^2)hV_n + i\sqrt{h}\,V\,V_n\,\varepsilon_{n+1}$$

which, in the continuous limit $h \to 0$ converges to a Schrödinger equation perturbed by a Brownian motion term

$$dV_t = (-iH_S - \frac{1}{2}V^2)V_t\,dt + iV\,V_t\,dW_t\,.$$

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