Quantum Noises

Stéphane ATTAL

Institut Camille Jordan
Université Claude Bernard Lyon 1
21 av Claude Bernard
69622 Villeurbanne Cedex, France
E-mail: `attal@math.univ-lyon1.fr`
Home page: `http://math.univ-lyon1.fr/ attal`

1 Introduction

In the Markovian approach of quantum open systems, the environment acting on a simple quantum system is unknown, or is not being given a model. The only effective data that the physicists deal with is the evolution of the simple quantum system. This evolution shows up the fact that the system is not isolated and is dissipating.

One of the question one may ask then is wether one can give a model for the environment and its action that gives an account of this effective evolution. One way to answer that question is to describe the exterior system as a noise, a quantum random effect of the environment which perturbs the Hamiltonian evolution of the small system.

This approach is a quantum version of what has been exposed in L. Rey-Bellet's courses: a dissipative dynamical evolution on some system is represented as resulting of the evolution of a closed but larger system, in which part of the action is represented by a noise, a Brownian motion for example.

This is the aim of R. Rebolledo's course and F. Fagnola's course, following this one in this volume, to show up how the dissipative quantum systems can be dilated into a closed evolution driven by quantum noises. But before hands, the mathematical theory of these quantum noises needs to be developed. This theory is not an obvious extension of the classical theory of stochastic processes and stochastic integration. It needs its own developments, where the fact that we are dealing with unbounded operators calls for being very careful with domain constraints.

On the other hand, the quantum theory of noise is somehow easier than the classical one, it can be described in a very natural way, it contains very natural physical interpretations, it is deeply connected to the classical theory of noises. This is the aim of this course to develop the theory of quantum noises and quantum stochastic integration, to connect it with its classical counterpart, while trying to keep it connected with some physical intuition.

The intuitive construction of this theory and its final rules, such as the quantum Itô formula, are not very difficult to understand, but the whole precise mathematical theory is really much more difficult and subtle, it needs quite long and careful developments. We have tried to be as precise as possible in this course, the most important proofs are there, but we have tried to keep it reasonable in size and to always preserves the intuition all along the constructions, without getting lost in long expositions of technical details.

The theory of quantum noises and quantum stochastic integration was started in quantum physics with the notion of quantum Langevin equations (see for example [1], [21], [22]). They have been given many different meanings in terms of several definitions of quantum noises or quantum Brownian motions (for example [23], [25], [24]). One of the most developed and useful mathematical languages developed for that purpose is the quantum stochastic calculus of Hudson and Parthasarathy and their quantum stochastic differential equations ([25]). The quantum Langevin equations they allow to consider have been used very often to model typical situations of quantum open systems: continual quantum measurement ([12], [14]), quantum optics ([19], [20], [13]), electronic transport ([16]), thermalization ([8], [28], [27]), repeated quantum interactions ([8], [11]). This theory can be found much more developed in the books [2], [30] and [29].

The theory of quantum noises and quantum stochastic integration we present in this course is rather different from the original approach of Hudson and Parthasarathy. It is an extention of it, essentially developed by the author, which presents several advantages: it gives a maximal definition of quantum stochastic integrals in terms of domains, it admits a very intuitive approach in terms of discrete approximations with spin chains, it gives a natural language for connecting this quantum theory of noises to the classical one. This is the point of view we adopt all along this course, the main reference we follow here is [2].

2 Discrete time

2.1 Repeated quantum interactions

We first motivate the theory of quantum noises and quantum stochastic differential equations through a family of physical examples: the continuous time limit of repeated quantum interactions. This physical context is sufficiently wide to be of real interest in many applications, but it is far from being the only motivation for the introduction of quantum noises. We present it here for it appears to be an illuminating application in the context of these volumes. The approach presented in this section has been first developed in [11].

We consider a *small quantum system* $\mathcal{H}_0$ (a finite dimensional Hilbert space in this course, but the infinite dimensional case can also be handled) and another quantum system $\mathcal{H}$ which represents a piece of the environment: a measurement apparatus, an incoming photon, a particle ... or any other system which is going to interact with the small system. We assume that these two systems are coupled and interact during a small interval of time of length h. That is, on the space $\mathcal{H}_0 \otimes \mathcal{H}$ we have an Hamiltonian H which describes the interaction, the evolution is driven by the unitary operator $U = e^{ihH}$. An initial state $\rho \otimes \omega$ for the system is thus transformed into

$$U^*(\rho \otimes \omega)U.$$

After this time h the two systems are separated and another copy of $\mathcal{H}$ is presented before $\mathcal{H}_0$ in order to interact with it, following the same unitary operator U. And so on, for an arbitrary number of interactions. One can think of several sets of examples where this situation arises: in repeated quantum measurement, where a family of identical measurement devices is repeatedly presented before the system $\mathcal{H}_0$ (or one single device which is refreshened after every use); in quantum optics, where a sequence of independent atoms arrive one after the other to interact with $\mathcal{H}_0$ (a cavity with a strong electromagnetic field) for a short time; a particle is having a succession of chocs with a gas of other particles ...

In order to describe the first two interactions we need to consider the space $\mathcal{H}_0 \otimes \mathcal{H} \otimes \mathcal{H}$. We put U_1 to be the operator acting as U on the tensor product of $\mathcal{H}_0$ with the first copy of $\mathcal{H}$ and which acts as the identity on the second copy of $\mathcal{H}$. We put U_2 to be the operator acting as U on the tensor product of $\mathcal{H}_0$ with the second copy of $\mathcal{H}$ and which acts as the identity on the first copy of $\mathcal{H}$.

For an initial state $\rho \otimes \omega \otimes \omega$, say, the state after the first interaction is

$$U_1(\rho \otimes \omega \otimes \omega)U_1^*$$

and after the second interaction is

$$U_2U_1(\rho \otimes \omega \otimes \omega)U_1^*U_2^*.$$

It is now easy to figure out what the setup should be for an indefinite number of repeated interactions: we consider the state space

$$\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}} \mathcal{H},$$

(this countable tensor product will be made more precise later on). For every $n \in \mathbb{N}$, the operator U_n is the copy of the operator U but acting on the tensor product of $\mathcal{H}_0$ with the n-th copy of $\mathcal{H}$, it acts as the identity on the other copies of $\mathcal{H}$. Let

$$V_n = U_n \dots U_2U_1,$$

then the result of the n-th measurement on the initial state

$$\rho \otimes \bigotimes_{n\in\mathbb{N}} \omega$$

is given by the state

$$V_n \left(\rho \otimes \bigotimes_{n\in\mathbb{N}} \omega\right) V_n^*.$$

Note that the V_n are solution of

$$V_{n+1} = U_{n+1} V_n,$$

with $V_0 = I$. This way, the V_n's describe the Hamiltonian evolution of the repeated quantum interactions. It is more exactly a time-dependent Hamiltonian evolution, it can also be seen as a Hamiltonian evolution in interaction picture.

We wish to pass to the limit $h \to 0$, that is, to pass to the limit from repeated interactions to continuous interactions. Our model of repeated interactions can be considered as a toy model for the interaction with a quantum field, we now want to pass to a more realistic model: a continuous quantum field.

We will not obtain a non trivial limit if no assumption is made on the Hamiltonian H. Clearly, it will need to satisfy some normalization properties with respect to the parameter h. As we will see later, this situation is somehow like for the central limit theorem: if one considers a Bernoulli random walk with time step h and if one tries to pass to the limit $h \to 0$ then one obtains 0; the only scale of normalization of the walk which gives a non trivial limit (namely the Brownian motion) is obtained when scaling the random walk by $\sqrt{h}$. Here, in our context we can wonder what are the scaling properties that the Hamiltonian should satisfy and what type of limit evolution we shall get for V_n.

Note that the evolution $(V_n)_{n\in\mathbb{N}}$ is purely Hamiltonian, in particular it is completely deterministic, the only ingredient here being the Hamiltonian operator H which drives everything in this setup.

At the end of this course, we will be able to give a surprising result: under some renormalization conditions on H, in the continuous limit, we obtain a limit evolution equation for $(V_t)_{t\in\mathbb{R}^+}$ which is a Schrödinger evolution perturbed by quantum noise terms, a *quantum Langevin equation*.

The point with that result is that it shows that these quantum noise terms are spontaneously produced by the limit equation and do not arise by an assumption or a model made on the interaction with the field. The limit quantum Langevin equation is really the effective continuous limit of the Hamiltonian description of the repeated quantum interactions.

We shall illustrate this theory with a very basic example. Assume $\mathcal{H}_0 = \mathcal{H} = \mathbb{C}^2$ that is, both are two-level systems with basis states Ω (the fundamental state) and X (the excited state). Their interaction is described as follows: if the states of the two systems are the same (both fondamental or both excited) then nothing happens, if they are different (one fundamental and the other one excited) then they can either be exchanged or stay as they are. Following this description, in the basis $\{\Omega\otimes\Omega, \Omega\otimes X, X\otimes\Omega, X\otimes X\}$ we take the unitary operator U to be of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.2 The Toy Fock space

The spin chain structure

We start with a description of the structure of the chain $\otimes_{\mathbb{N}}\mathcal{H}$ in the case where $\mathcal{H} = \mathbb{C}^2$. This is the simplest case, but it contains all the ideas. We shall later indicate how the theory is to be changed when $\mathcal{H}$ is larger (even infinite dimensional).

In every copy of $\mathbb{C}^2$ we choose the same orthonormal basis $\{\Omega, X\}$, representing fundamental or excited states. An orthonormal basis of the space $T\Phi = \otimes_{\mathbb{N}}\mathbb{C}^2$ is given by the set

$$\{X_A; A \in \mathcal{P}_{\mathbb{N}}\}$$

where $\mathcal{P}_{\mathbb{N}}$ is the set of finite subsets of $\mathbb{N}$ and X_A denotes the tensor product

$$X_{i_1} \otimes \ldots \otimes X_{i_n}$$

where $A = \{i_1, \ldots, i_n\}$ and the above vector means we took tensor products of X in each of copies number i_k with Ω in all the other copies. If $A = \emptyset$, we put $X_\emptyset = \Omega$, that is, the tensor product of Ω in each copy of $\mathbb{C}^2$. This is to say that the countable tensor product above has been constructed as associated to the stabilizing sequence $(\Omega)_{n\in\mathbb{N}}$.

Note that any element f of $T\Phi$ is of the form

$$f = \sum_{A\in\mathcal{P}_{\mathbb{N}}} f(A)\, X_A$$

with

$$||f||^2 = \sum_{A\in\mathcal{P}_{\mathbb{N}}} |f(A)|^2 < \infty.$$

The space $T\Phi$ defined this way is called the *Toy Fock space* .

This particular choice of a basis gives $T\Phi$ a particular structure. If we denote by $T\Phi_{i]}$ the space generated by the X_A such that $A \subset \{0, \ldots, i\}$ and by $T\Phi_{[j}$ the one generated by the X_A such that $A \subset \{j, j+1, \ldots\}$, we get an obvious natural isomorphism between $T\Phi$ and $T\Phi_{i-1]} \otimes T\Phi_{[i}$ given by

$$[f \otimes g](A) = f\left(A \cap \{0, \ldots, i-1\}\right)\, g\left(A \cap \{i, \ldots\}\right).$$

Operators on the spin chain

We consider the following basis of matrices on $\mathbb{C}^2$:

$$a^\times = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$a^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$a^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$a^\circ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For every $\varepsilon = \{\times, +, -, \circ\}$, we denote by a_n^ε the operator which acts on $\mathrm{T}\Phi$ as a^ε on the copy number n of $\mathbb{C}^2$ and the identity elsewhere. On the basis X_A this gives

$$\begin{aligned} a_n^\times X_A &= X_A \, \mathbb{1}_{n \notin A} \\ a_n^+ X_A &= X_{A \cup \{n\}} \, \mathbb{1}_{n \notin A} \\ a_n^- X_A &= X_{A \setminus \{n\}} \, \mathbb{1}_{n \in A} \\ a_n^\circ X_A &= X_A \, \mathbb{1}_{n \in A}. \end{aligned}$$

Note that the von Neumann algebra generated by all the operators a_n^ε is the whole of $\mathcal{B}(\mathrm{T}\Phi)$, for there is no non-trivial subspace of $\mathrm{T}\Phi$ which is invariant under this algebra. But this kind of theorem does not help much to give an explicit representation of a given bounded operator H on $\mathrm{T}\Phi$ in terms of the operators a_n^ε. There are two concrete ways of representing an (eventually unbounded) operator on $\mathrm{T}\Phi$ in terms of these basic operators. The first one is a representation as a kernel

$$H = \sum_{\mathcal{P}^3} k(A, B, C) \, a_A^+ \, a_B^\circ \, a_C^-$$

where $a_A^\varepsilon = a_{i_1}^\varepsilon \, \ldots \, a_{i_n}^\varepsilon$ if $A = \{i_1 \, \ldots \, i_n\}$.

Note that the term $a^\times$ does not appear in the above kernel. The reason is that $a^\times + a^\circ$ is the identity operator and introducing the operator $a^\times$ in the above representation will make us lose the uniqueness of the above representation. Note that a° is not necessary either for it is equal to $a^+ a^-$. But if we impose the sets A, B, C to be two by two disjoint then the above representation is unique.

We shall not discuss much this kind of representation here, but better a different kind of representation (which one can derive from the above kernel representation by grouping the terms in 3 packets depending on which set A, B or C contains $\max A \cup B \cup C$). This is the so-called *integral representation*:

$$H = \sum_{\varepsilon=+,\circ,-} \sum_{i\in\mathbb{N}} H_i^\varepsilon \, a_i^\varepsilon \tag{1}$$

where the H_i^ε are operators acting on $T\!\!\Phi_{i-1]}$ only (and as the identity on $T\!\!\Phi_{[i}$). This kind of representation will be of great interest for us in the sequel.

For the existence of such a representation we have very mild conditions, even for unbounded H ([32]).

Theorem 2.1. *If the orthonormal basis $\{X_A, A \in \mathcal{P}_{\mathbb{N}}\}$ belongs to* $\operatorname{Dom} H \cap \operatorname{Dom} H^*$ *then there exists a unique integral representation of H of the form* (1). ■

One important point needs to be understood at that stage. The integral representation of a single operator H as in (1) makes use of only 3 of the four matrices a_i^ε. The reason is the same as for the kernel representation above: the sum $a_i^\circ + a_i^\times$ is the identity operator I, if we allow $a_i^\times$ to appear in the representation, we lose uniqueness. But, very often one has to consider processes of operators, that is, families $(H_i)_{i\in\mathbb{N}}$ of operators on $T\!\!\Phi_{i]}$ respectively. In that case, the fourth family is necessary and we get representations of the form

$$H_i = \sum_{\varepsilon=+,\circ,-,\times} \sum_{j\leq i} H_j^\varepsilon \, a_j^\varepsilon. \tag{2}$$

One interesting point with the integral representations is that they are stable under composition. The integral representation of a composition of integral representations is given by the *discrete quantum Itô formula*, which is almost straightforward if we forget about details on the domain of operators.

Theorem 2.2 (Discrete quantum Itô formula). *If*

$$H_i = \sum_{\varepsilon=+,\circ,-,\times} \sum_{j\leq i} H_j^\varepsilon \, a_j^\varepsilon$$

and

$$K_i = \sum_{\varepsilon=+,\circ,-,\times} \sum_{j\leq i} K_j^\varepsilon \, a_j^\varepsilon$$

are operators on $T\!\!\Phi_{i]}$, indexed by $i \in \mathbb{N}$, then we have the following "integration by part formula":

$$H_i K_i = \sum_{\varepsilon=+,\circ,-,\times} \sum_{j\leq i} H_{j-1} K_j^\varepsilon \, a_j^\varepsilon + \sum_{\varepsilon=+,\circ,-,\times} \sum_{j\leq i} H_j^\varepsilon K_{j-1} \, a_j^\varepsilon + \\ + \sum_{\varepsilon,\nu=+,\circ,-,\times} \sum_{j\leq i} H_j^\varepsilon K_j^\nu \, a_j^\varepsilon a_j^\nu$$

where the products $a^\varepsilon a^\nu$ are given by the following table

	a^+	a^-	a°	$a^\times$
a^+	0	a°	0	a^+
a^-	$a^\times$	0	a^-	0
a°	a^+	0	a°	0
$a^\times$	0	a^-	0	$a^\times$

Note the following two particular cases of the above formula, which will be of many consequences for the probabilistic interpretations of quantum noises.

The Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a^+ + a^-$$

satisfies

$$\sigma_x^2 = 1. \tag{3}$$

The matrix

$$X_\lambda = \sigma_x + \lambda a^\circ$$

satisfies

$$X_\lambda^2 = I + \lambda X_\lambda. \tag{4}$$

These two very simple matrix relations are actually the discrete version of the famous relations

$$(dW_t)^2 = dt$$

and

$$(dX_t)^2 = dt + \lambda\, dX_t$$

which characterise the Brownian motion $(W_t)_{t\geq 0}$ and the compensated Poisson process $(X_t)_{t\geq 0}$,with intensity λ. We shall give a rigourous meaning to these affirmations in section 6.4.

Probabilistic interpretations

In this section, we describe the *probabilistic interpretations* of the space $T\Phi$ and of its basic operators.

We realize a Bernoulli random walk on its canonical space. Let $\Omega = \{0,1\}^{\mathbb{N}}$ and $\mathcal{F}$ be the σ-field generated by finite cylinders. One denotes by ν_n the coordinate mapping : $\nu_n(\omega) = \omega_n$, for all $n \in \mathbb{N}$. Let $p \in]0,1[$ and $q = 1-p$. Let μ_p be the probability measure on $(\Omega, \mathcal{F})$ which makes the sequence $(\nu_n)_{n\in\mathbb{N}}$ a sequence of independent, identically distributed Bernoulli random variables with law $p\delta_1 + q\delta_0$. Let $\mathbb{E}_p[\,\cdot\,]$ denote the expectation with respect to μ_p. We have $\mathbb{E}_p[\nu_n] = \mathbb{E}_p[\nu_n^2] = p$. Thus the random variables

$$X_n = \frac{\nu_n - p}{\sqrt{pq}}\ ,$$

satisfy the following:

i) they are independent,

ii) they take the value $\sqrt{q/p}$ with probability p and $-\sqrt{p/q}$ with probability q,

iii) $\mathbb{E}_p[X_n] = 0$ and $\mathbb{E}_p[X_n^2] = 1$.

Let $T\Phi_p$ denote the space $L^2(\Omega, \mathcal{F}, \mu_p)$. We define particular elements of $T\Phi_p$ by

$$\begin{cases} X_\emptyset = \mathbb{1}, & \text{in the sense } X_\emptyset(\omega) = 1 \text{ for all } \omega \in \Omega \\ X_A = X_{i_1} \cdots X_{i_n}, & \text{if } A = \{i_1 \dots i_n\} \text{ is any finite subset of } \mathbb{N}. \end{cases}$$

Recall that $\mathcal{P}_{\mathbb{N}}$ denotes the set of finite subsets of $\mathbb{N}$. From *i)* and *iii)* above it is clear that $\{X_A \ ; \ A \in \mathcal{P}_{\mathbb{N}}\}$ is an orthonormal set of vectors in $T\Phi_p$.

Proposition 2.3. *The family $\{X_A; A \in \mathcal{P}_{\mathbb{N}}\}$ is an orthonormal basis of $T\Phi_p$.*

Proof. We only have to prove that $\{X_A, A \in \mathcal{P}_{\mathbb{N}}\}$ forms a total set in $T\Phi_p$. In the same way as for the X_A, define

$$\begin{cases} \nu_\emptyset = \mathbb{1} \\ \nu_A = \nu_{i_1} \cdots \nu_{i_n} \quad \text{for } A = \{i_1 \dots i_n\}. \end{cases}$$

It is sufficient to prove that the set $\{\nu_A \ ; \ A \in \mathcal{P}_{\mathbb{N}}\}$ is total. The space $(\Omega, \mathcal{F}, \mu_p)$ can be identified to $\big([0,1], \mathcal{B}([0,1]), \widetilde{\mu}_p\big)$ for some probability measure $\widetilde{\mu}_p$, via the base 2 decomposition of real numbers. Note that

$$\nu_n(\omega) = \omega_n = \begin{cases} 1 & \text{if } \omega_n = 1 \\ 0 & \text{if } \omega_n = 0 \end{cases}$$

thus $\nu_n(\omega) = \mathbb{1}_{\omega_n = 1}$. As a consequence $\nu_A(\omega) = \mathbb{1}_{\omega_{i_1}=1} \cdots \mathbb{1}_{\omega_{i_n}=1}$. Now let $f \in T\Phi_p$ be such that $\langle f, \nu_A \rangle = 0$ for all $A \in \mathcal{P}_{\mathbb{N}}$. Let $I = \big[k2^{-n}, (k+1)2^{-n}\big]$ be a dyadic interval with $k < 2^n$. The base 2 decomposition of $k2^{-n}$ is of the form $(\alpha_1 \dots \alpha_n, 0, 0, \dots)$. Thus

$$\int_I f(\omega)\, d\widetilde{\mu}_p(\omega) = \int_{[0,1]} f(\omega) \mathbb{1}_{\omega_1 = \alpha_1} \cdots \mathbb{1}_{\omega_n = \alpha_n}\, d\widetilde{\mu}_p(\omega) \ .$$

The function $\mathbb{1}_{\omega_1=\alpha_1} \cdots \mathbb{1}_{\omega_n=\alpha_n}$ can be clearly written as a linear combination of the ν_A. Thus $\int_I f \ d\widetilde{\mu}_p = 0$. The integral of f vanishes on every dyadic interval, thus on all intervals. It is now easy to conclude that $f \equiv 0$. ■

We have proved that every element $f \in T\Phi_p$ admits a unique decomposition

$$f = \sum_{A \in \mathcal{P}_{\mathbb{N}}} f(A) X_A \tag{5}$$

with

$$\|f\|^2 = \sum_{A \in \mathcal{P}_{I\!N}} |f(A)|^2 < \infty \; . \tag{6}$$

This means that there exists a natural isomorphism between $T\!\Phi$ and $T\!\Phi_p$ which consists in identifying the natural orthonormal basis $\{X_A; A \in \mathcal{P}_{I\!N}\}$ of both space. For each $p \in]0,1[$, the space $T\!\Phi_p$ is called the *p-probabilistic interpretation* of $T\!\Phi$. That is, it gives an interpretation of $T\!\Phi$ in terms of a probabilistic space: it is the canonical space associated to the Bernoulli random walk with parameter p.

Identifying the basis element $X_{\{n\}}$ of $T\!\Phi$ with the random variable $X_n \in T\!\Phi_p$, as elements of some Hilbert spaces, does not give much information on the probabilistic nature of X_n. One cannot read this way the distribution of X_n or its independence with respect to other X_m's, ... The only way to represent the random variable $X_n \in T\!\Phi_p$ with all its probabilistic structure, inside the structure of $T\!\Phi$, is to consider the operator of *multiplication* by X_n acting on $T\!\Phi_p$ and to represent it as a self-adjoint operator in $T\!\Phi$ through the above natural isomorphism. When knowing the multiplication operator by X_n one knows all the probabilistic information on the random variable X_n. One cannot make the difference between the multiplication operator by X_n pushed on $T\!\Phi$ and the "true" random variable X_n in $T\!\Phi_p$.

Let us compute this multiplication operator by X_n. The way we have chosen the basis of $T\!\Phi_p$ makes the product being determined by the value of X_n^2, $n \in I\!N$. Indeed, if $n \notin A$ then $X_n X_A = X_{A\cup\{n\}}$.

Proposition 2.4. *In $T\!\Phi_p$ we have*

$$X_n^2 = 1 + c_p X_n$$

where $c_p = (q-p)/\sqrt{pq}$. Furthermore $p \mapsto c_p$ is a one to one application from $]0,1[$ to $I\!R$.

Proof.

$$\begin{aligned} X_n^2 &= \frac{1}{pq}(\nu_n^2 + p^2 - 2p\nu_n) = \frac{1}{pq}\left(p^2 + (1{-}2p)\nu_n\right) \\ &= \frac{1}{pq}\left(p^2 + (q{-}p)\nu_n\right) = 1 + \frac{p^2{-}qp}{qp} + \frac{q-p}{qp}\nu_n \\ &= 1 - \frac{pc_p}{\sqrt{pq}} + \frac{c_p}{\sqrt{pq}}\nu_n = 1 + c_p\frac{\nu_n - p}{\sqrt{pq}}. \qquad \blacksquare \end{aligned}$$

The above formula determines an associative product on $T\!\Phi$ which is called the *p-product*. The operator of p-multiplication by X_n in $T\!\Phi$ is the exact representation of the random variable X_n in the p-probabilistic interpretation. By means of all these p-multiplication operators we are able to put in a single structure a whole continuum of probabilistic situations that had no relation whatsoever: the canonical Bernoulli random walks with parameter p, for every $p \in]0,1[$. What's more we get a very simple represention of these multiplication operators.

Proposition 2.5. *The operator $M^p_{X_n}$ of p-multiplication by X_n on $T\Phi$ is given by*

$$M^p_{X_n} = a^+_n + a^-_n + c_p a^\circ_n.$$

Proof.

$$\begin{aligned} X_n X_A &= X_{A\cup\{n\}} 1\!\!1_{n\notin A} + X_{A\setminus\{n\}}(1 + c_p X_n) 1\!\!1_{n\in A} \\ &= a^+_n X_A + a^-_n X_A + c_p a^\circ_n X_A. \qquad \blacksquare \end{aligned}$$

This result is amazing in the sense that the whole continuum of different probabilistic situations, namely $T\Phi_p$, $p \in]0,1[$, can be represented in $T\Phi$ by means of very simple linear combinations of only 3 differents operators!

2.3 Higher multiplicities

In the case where $\mathcal{H}$ is not $\mathbb{C}^2$ but $\mathbb{C}^{N+1}$, or any separable Hilbert space, the above presentation is changed as follows. Let us consider the case $\mathbb{C}^{N+1}$ (the infinite dimensional case can be easily derived from it).

Each copy of $\mathbb{C}^{N+1}$ is considered with the same fixed orthonormal basis $\{\Omega, X_1, \dots, X_N\}$. We shall sometimes write $X_0 = \Omega$. The space

$$T\Phi = \bigotimes_{k\in\mathbb{N}} \mathbb{C}^{N+1}$$

has a natural orthonormal basis X_A indexed by the subsets

$$A = \{(n_1, i_1), \dots, (n_k, i_k)\}$$

of $\mathbb{N} \otimes \{1, \dots, N\}$, such that the n_j's are different. This is the so-called *Toy Fock space with multiplicity N*.

The basis for the matrices on $\mathbb{C}^{N+1}$ is the usual one:

$$a^i_j X_k = \delta_{ik} X_j$$

for all $i, j, k = 0, \dots, N$. We also have their natural ampliations to $T\Phi$: $a^i_j(k)$, $k \in \mathbb{N}$.

We now develop the probabilistic interpretations of the space $T\Phi$ in the case of multiplicity higher than 1. Their structure is very rich and interesting, but it is not used in the rest of this course. The reader is advised to skip that part at first reading.

Let X be a random variable in $\mathbb{R}^N$ which takes exactly $N+1$ different values $v_1, \dots, v_{N+1}$ with respective probability $\alpha_1, \dots, \alpha_{N+1}$ (all different from

0 by hypothesis). We assume, for simplicity, that X is defined on its canonical space $(A, \mathcal{A}, P)$, that is, $A = \{1, \ldots, N+1\}$, $\mathcal{A}$ is the σ-field of subsets of A, the probability measure P is given by $P(\{i\}) = \alpha_i$ and X is given by $X(i) = v_i$, for all $i = 1, \ldots, N+1$.

Such a random variable X is called *centered and normalized* if $\mathbb{E}[X] = 0$ and $\mathrm{Cov}(X) = I$.

A family of elements $v_1, \ldots, v_{N+1}$ of $\mathbb{R}^N$ is called an *obtuse system* if

$$< v_i \, , \, v_j > = -1$$

for all $i \neq j$.

We consider the coordinates $X_1, \ldots, X_N$ of X in the canonical basis of $\mathbb{R}^N$, together with the random variable Ω on $(A, \mathcal{A}, P)$ which is deterministic always equal to 1. We put $\widetilde{X}_i$ to be the random variable $\widetilde{X}_i(j) = \sqrt{\alpha_j}\, X_i(j)$ and $\widetilde{\Omega}(j) = \sqrt{\alpha_j}$. For any element $v = (a_1, \ldots, a_N)$ of $\mathbb{R}^N$ we put $\widehat{v} = (1, a_1, \ldots, a_N) \in \mathbb{R}^{N+1}$. The following proposition is rather straightforward and left to the reader.

Proposition 2.6. *The following assertions are equivalent.*

i) X is centered and normalized.

ii) The $(N+1) \times (N+1)$-matrix $(\widetilde{\Omega}, \widetilde{X}_1, \ldots, \widetilde{X}_N)$ is unitary.

iii) The $(N+1) \times (N+1)$-matrix $(\sqrt{\alpha_1}\, \widehat{v}_1, \ldots, \sqrt{\alpha_{N+1}}\, \widehat{v}_{N+1})$ is unitary.

iv) The family $v_1, \ldots, v_{N+1}$ is an obtuse system of $\mathbb{R}^N$ and

$$\alpha_i = \frac{1}{1 + ||v_i||^2}.$$

Let T be a 3-tensor in $\mathbb{R}^N$, that is (at least, this is the way we interpret them here), a linear mapping from $\mathbb{R}^N$ to $M_N(\mathbb{R})$. We denote by T_k^{ij} the coefficients of T in the canonical basis of $\mathbb{R}^N$, that is,

$$(T(x))_{i,j} = \sum_{k=1}^{N} T_k^{ij} x_k.$$

Such a 3-tensor T is called *sesqui-symmetric* if

i) $(i, j, k) \longmapsto T_k^{ij}$ is symmetric

ii) $(i, j, l, m) \longmapsto \sum_k T_k^{ij} T_k^{lm} + \delta_{ij}\delta_{lm}$ is symmetric.

Theorem 2.7. *If X is a centered and normalized random variable in $\mathbb{R}^N$, taking exactly $N+1$ values, then there exists a unique sesqui-symmetric 3-tensor T such that*

$$X \otimes X = I + T(X). \tag{7}$$

Proof. By Proposition 2.6, the matrix $(\sqrt{\alpha_1}\,\widehat{v}_1, \ldots, \sqrt{\alpha_{N+1}}\,\widehat{v}_{N+1})$ is unitary. In particular the matrix $(\widehat{v}_1, \ldots, \widehat{v}_{N+1})$ is invertible. But the lines of this matrix are the values of the random variables $\Omega, X_1, \ldots, X_N$. As a consequence, these $N+1$ random variables are linearly independent. They thus form a basis of $L^2(A, \mathcal{A}, P)$ which is a $N+1$ dimensional space.

The random variable $X_i X_j$ belongs to $L^2(A, \mathcal{A}, P)$ and can thus be written uniquely as

$$X_i X_j = \sum_{k=0}^{N} T_k^{ij} X_k$$

where X_0 denotes Ω and for some real coefficients T_k^{ij}, $k = 0, \ldots, N$, $i, j = 1, \ldots N$. The fact that $\mathbb{E}[X_k] = 0$ and $\mathbb{E}[X_i X_j] = \delta_{ij}$ implies $T_0^{ij} = \delta_{ij}$. This gives the representation (7).

The fact that the 3-tensor T associated to the above coefficients T_k^{ij}, $i, j, k = 1, \ldots N$, is sesqui-symmetric is an easy consequence of the fact that the expressions $X_i X_j$ are symmetric in i, j and $X_i(X_j X_m) = (X_i X_j) X_m$ for all i, j, m. We leave this to the reader. ∎

The following theorem is an interesting characterization of the sesqui-symmetric tensors. The proof of this result is far from obvious, but as we shall not need it we omit the proof and convey the interested reader to read the proof in [6], Theorem 2, p. 268-272.

Theorem 2.8. *The formulas*

$$S = \{x \in \mathbb{R}^N ; x \otimes x = I + T(x)\}.$$

and

$$T(y) = \sum_{x \in S} p_x < x\,,\, y > x \otimes x,$$

where $p_x = 1/(1+||x||^2)$*, define a bijection between the set of sesqui-symmetric 3-tensor* T *on* $\mathbb{R}^N$ *and the set of obtuse systems* S *in* $\mathbb{R}^N$*.*

Now we wish to consider the random walks (or more exactly the sequences of independent copies of induced by obtuse random variables). That is, on the probability space $(A^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}}, P^{\otimes \mathbb{N}})$, we consider a sequence $(X(n))_{n \in \mathbb{N}^*}$ of independent random variables with the same law as a given obtuse random variable X (once again, the use of the terminology "random walk" is not correct here in the sense that it usually refers to the *sum* of these independent random variables, but we shall anyway use it here as it is shorter and essentially means the same).

For any $A \in \mathcal{P}_n$ we define the random variable

$$X_A = \prod_{(n,i) \in A} X_i(n)$$

with the convention

$$X_\emptyset = \mathbb{1}.$$

Proposition 2.9. *The family $\{X_A; A \in \mathcal{P}_{\mathbb{N}}\}$ forms an orthonormal basis of the space $L^2(A^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}}, P^{\otimes \mathbb{N}})$.*

Proof. For any $A, B \in \mathcal{P}_n$ we have

$$< X_A \, , \, X_B > = \mathbb{E}[X_A X_B] = \mathbb{E}[X_{A\Delta B}]\mathbb{E}[X^2_{A\cap B}]$$

by the independence of the $X(n)$. For the same reason, the first term $\mathbb{E}[X_{A\Delta B}]$ gives 0 unless $A\Delta B = \emptyset$, that is $A = B$. The second term $\mathbb{E}[X^2_{A\cap B}]$ is then equal to $\prod_{(n,i)\in A} \mathbb{E}[X_i(n)^2] = 1$. This proves the orthonormal character of the family $\{X_A; A \in \mathcal{P}_n\}$.

Let us now prove that it generates a dense subspace of $L^2(A^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}}, P^{\otimes \mathbb{N}})$. Had we considered random walks indexed by $\{0, \ldots, M\}$ instead of $\mathbb{N}$, it would be clear that the X_A, $A \subset \{0, \ldots, M\}$ form an orthonormal basis of $L^2(A^M, \mathcal{A}^{\otimes M}, P^{\otimes M})$, for the dimensions coincide. Now a general element f of $L^2(A^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}}, P^{\otimes \mathbb{N}})$ can be easily approximated by a sequence $(f_M)_M$ such that $f_M \in L^2(A^M, \mathcal{A}^{\otimes M}, P^{\otimes M})$, for all M, by taking conditional expectations on the trajectories of X up to time M. ■

For every obtuse random variable X, we thus obtain a Hilbert space $T\Phi(X) = L^2(A^{\mathbb{N}}, \mathcal{A}^{\otimes \mathbb{N}}, P^{\otimes \mathbb{N}})$, with a natural orthonormal basis $\{X_A; A \in \mathcal{P}_{\mathbb{N}}\}$ which emphasizes the independence of the $X(n)$'s. In particular there is a natural isomorphism between all the spaces $T\Phi(X)$ which consists in identifying the associated bases. In the same way, all these canonical spaces $T\Phi(X)$ of obtuse random walks are naturally isomorphic to the Toy Fock space $T\Phi$ with multiplicity N (again by identifying their natural orthonormal bases).

In the same way as in multiplicity 1 we compute the representation of the multiplication operator by $X_i(k)$ in $T\Phi$.

Theorem 2.10. *Let X be an obtuse random variable, let $(X(k))_{k\in\mathbb{N}}$ be the associated random walk on the canonical space $T\Phi(X)$. Let T be the sesqui-symmetric 3-tensor associated to X by Theorem 2.7. Let U be the natural unitary isomorphism from $T\Phi(X)$ to $T\Phi$. Then, for all $k \in \mathbb{N}, i = \{1, \ldots, n\}$ we have*

$$U\mathcal{M}_{X_i(k)}U^* = a^0_i(k) + a^i_0(k) + \sum_{j,l} T^{jl}_i \, a^j_l(k).$$

Proof. It suffices to compute the action of $X_i(k)$ on the basis elements X_A, $A \in \mathcal{P}_n$. We get

$$
\begin{aligned}
X_i(k)X_A &= 1\!\!1_{(k,\cdot)\notin A} X_i(k)X_A + \sum_j 1\!\!1_{(k,j)\in A} X_i(k) X_A \\
&= 1\!\!1_{(k,\cdot)\notin A} X_{A\cup\{(k,i)\}} + \sum_j 1\!\!1_{(k,j)\in A} X_i(k) X_j(k) X_{A\setminus\{(k,j)\}} \\
&= 1\!\!1_{(k,\cdot)\notin A} X_{A\cup\{(k,i)\}} + \sum_j 1\!\!1_{(k,j)\in A} (\delta_{ij} + \sum_l T^{ij}_l X_l(k)) X_{A\setminus\{(k,j)\}} \\
&= 1\!\!1_{(k,\cdot)\notin A} X_{A\cup\{(k,i)\}} + 1\!\!1_{(k,i)\in A} X_{A\setminus(k,i)} + \\
&\quad + \sum_j \sum_l 1\!\!1_{(k,j)\in A} T^{ij}_l X_{A\setminus\{(k,j)\}\cup\{(k,i)\}}
\end{aligned}
$$

and we recognize the formula for

$$
a^0_i(k)X_A + a^i_0(k)X_A + \sum_{k,l} T^{ij}_l a^j_l(k)X_A. \qquad \blacksquare
$$

This ends the section on the discrete time setting for quantum noises. We shall come back to it later when using it to approximate the Fock space structure.

3 Itô calculus on Fock space

3.1 The continuous version of the spin chain: heuristics

We now present the structure of the continuous version of $T\!\!\Phi$. By a continuous version of the spin chain we mean a Hilbert space which should be of the form

$$
\Phi = \bigotimes_{I\!\!R^+} \mathbb{C}^2.
$$

We first start with a heuristical discussion in order to make out an idea of how this space should be defined. We mimick, in a continuous time version, the structure of $T\!\!\Phi$.

The countable orthonormal basis $X_A, A \in \mathcal{P}_{I\!\!N}$ is replaced by a continuous orthonormal basis $d\chi_\sigma$, $\sigma \in \mathcal{P}$, where $\mathcal{P}$ is the set of finite subsets of $I\!\!R^+$. With the same idea as for $T\!\!\Phi$, this means that each copy of $\mathbb{C}^2$ is equipped with an orthonormal basis $\Omega, d\chi_t$ (where t is the parameter attached to the copy we are looking at). The orthonormal basis $d\chi_\sigma$ is the one obtained by specifying a finite number of sites $t_1, \ldots, t_n$ which are going to be excited, the other ones being in the fundamental state Ω.

The representation of an element f of $T\!\!\Phi$:

$$
\begin{aligned}
f &= \sum_{A\in\mathcal{P}_{I\!\!N}} f(A)\, X_A \\
||f||^2 &= \sum_{A\in\mathcal{P}_{I\!\!N}} |f(A)|^2
\end{aligned}
$$

is replaced by an integral version of it in Φ:

$$f = \int_{\mathcal{P}} f(\sigma)\, d\chi_\sigma,$$
$$||f||^2 = \int_{\mathcal{P}} |f(\sigma)|^2 \, d\sigma,$$

where, in the last integral, the measure $d\sigma$ is a "Lebesgue measure" on $\mathcal{P}$, that we shall explain later.

A good basis of operators acting on Φ can be obtained by mimicking the operators a_n^ε of $T\Phi$. Here we have a set of infinitesimal operators da_t^ε acting on the copy t of $\mathbb{C}^2$ by

$$\begin{aligned}
da_t^\times\, \Omega &= dt\, \Omega \quad &\text{and} \quad da_t^\times\, d\chi_t &= 0,\\
da_t^+\, \Omega &= d\chi_t \quad &\text{and} \quad da_t^+\, d\chi_t &= 0,\\
da_t^-\, \Omega &= 0 \quad &\text{and} \quad da_t^-\, d\chi_t &= dt\, \Omega,\\
da_t^\circ\, \Omega &= 0 \quad &\text{and} \quad da_t^\circ\, d\chi_t &= d\chi_t.
\end{aligned}$$

In the basis $d\chi_\sigma$, this means

$$\begin{aligned}
da_t^\times\, d\chi_\sigma &= d\chi_\sigma\, dt\, \mathbb{1}_{t\notin\sigma}\\
da_t^+\, d\chi_\sigma &= d\chi_{\sigma\cup\{t\}}\, \mathbb{1}_{t\notin\sigma}\\
da_t^-\, d\chi_\sigma &= d\chi_{\sigma\setminus\{t\}}\, dt\, \mathbb{1}_{t\in\sigma}\\
da_t^\circ\, d\chi_\sigma &= d\chi_\sigma\, \mathbb{1}_{t\in\sigma}.
\end{aligned}$$

3.2 The Guichardet space

We now describe a setting in which the above heuristic discussion is made rigorous.

Notations

Let $\mathcal{P}$ denote the set of finite subsets of $\mathbb{R}^+$. That is, $\mathcal{P} = \cup_n \mathcal{P}_n$ where $\mathcal{P}_0 = \{\emptyset\}$ and $\mathcal{P}_n$ is the set of n elements subsets of $\mathbb{R}^+$, $n \geq 1$. By ordering elements of a $\sigma = \{t_1, t_2 \dots t_n\} \in \mathcal{P}_n$ we identify $\mathcal{P}_n$ with $\Sigma_n = \{0 < t_1 < t_2 < \cdots < t_n\} \subset (\mathbb{R}^+)^n$. This way $\mathcal{P}_n$ inherits the measured space structure of $(\mathbb{R}^+)^n$. By putting the Dirac measure $\delta_\emptyset$ on $\mathcal{P}_0$, we define a σ-finite measured space structure on $\mathcal{P}$ (which coincides with the n-dimensional Lebesgue measure on each $\mathcal{P}_n$) whose only atom is $\{\emptyset\}$. The elements of $\mathcal{P}$ are denoted with small greek letters $\sigma, \omega, \tau, \dots$ the associated measure is denoted $d\sigma$, $d\omega$, $d\tau \dots$ (with in mind that $\sigma = \{t_1 < t_2 < \cdots < t_n\}$ and $d\sigma = dt_1 dt_2 \cdots dt_n$).

The space $L^2(\mathcal{P})$ defined this way is naturally isomorphic to the symmetric Fock space $\Phi = \Gamma_s(L^2(\mathbb{R}^+))$. Indeed, $L^2(\mathcal{P}) = \bigoplus_n L^2(\mathcal{P}_n)$ is isomorphic to $\bigoplus_n L^2(\Sigma_n)$ (with $\Sigma_0 = \{\emptyset\}$) that is Φ by identifying the space $L^2(\Sigma_n)$ to the space of symmetric functions in $L^2((I\!R^+)^n)$. In order to be really clear, the isomorphism between Φ and $L^2(\mathcal{P})$ can be explicitly written as:

$$\begin{aligned} V : \Phi &\longrightarrow L^2(\mathcal{P}) \\ f &\longmapsto Vf \end{aligned}$$

where $f = \sum_n f_n$ and

$$[Vf](\sigma) = \begin{cases} f_0 & \text{if } \sigma = \emptyset \\ f_n(t_1 \dots t_n) & \text{if } \sigma = \{t_1 < \dots < t_n\}. \end{cases}$$

Let us fix some notations on $\mathcal{P}$. If $\sigma \neq \emptyset$ we put $\vee\sigma = \max\sigma$, $\sigma- = \sigma\backslash\{\vee\sigma\}$. If $t \in \sigma$ then $\sigma \setminus t$ denotes $\sigma \setminus \{t\}$. If $\{t \notin \sigma\}$ then $\sigma \cup t$ denotes $\sigma \cup \{t\}$. If $0 \leq s \leq t$ then

$$\begin{aligned} \sigma_{s)} &= \sigma \cap [0, s[\\ \sigma_{(s,t)} &= \sigma \cap]s, t[\\ \sigma_{(t} &= \sigma \cap]t, +\infty[\ . \end{aligned}$$

We also put

$$1\!\!1_{\sigma \leq t} = \begin{cases} 1 & \text{if } \sigma \subset [0, t] \\ 0 & \text{otherwise.} \end{cases}$$

If $0 \leq s \leq t$ then

$$\begin{aligned} \mathcal{P}^{s)} &= \{\sigma \in \mathcal{P}; \sigma \subset [0, s[\} \\ \mathcal{P}^{(s,t)} &= \{\sigma \in \mathcal{P}; \sigma \subset]s, t[\} \\ \mathcal{P}^{(t} &= \{\sigma \in \mathcal{P}; \sigma \subset]t, +\infty[\} \ . \end{aligned}$$

Finally, $\#\sigma$ denotes the cardinal of σ.

If we put $\Phi_{t]} = \Gamma_s(L^2([0, t]))$, $\Phi_{[t} = \Gamma_s(L^2([t, +\infty[)$ and so on ... we clearly have

$$\begin{aligned} \Phi_{s]} &\simeq L^2(\mathcal{P}^{s)}) \\ \Phi_{[s,t]} &\simeq L^2(\mathcal{P}^{(s,t)}) \\ \Phi_{[t} &\simeq L^2(\mathcal{P}^{(t}) \ . \end{aligned}$$

In the following we make several identifications:

• Φ is not distinguished from $L^2(\mathcal{P})$ (and the same holds for $\Phi_{s]}$ and $L^2(\mathcal{P}^{s)})$, etc...)

• $L^2(\mathcal{P}^{s)})$, $L^2(\mathcal{P}^{(s,t)})$ and $L^2(\mathcal{P}^{(t})$ are seen as subspaces of $L^2(\mathcal{P})$: the subspace of $f \in L^2(\mathcal{P})$ such that $f(\sigma) = 0$ for all σ such that $\sigma \not\subset [0, s]$ (*resp.* $\sigma \not\subset [s, t]$, *resp.* $\sigma \not\subset [t, +\infty[$).

A particular family of elements of Φ is of great use: the space of *coherent vectors*. For every $h \in L^2(I\!R^+)$, consider the element $\varepsilon(h)$ of Φ defined by

$$[\varepsilon(h)](\sigma) = \prod_{s\in\sigma} h(s)$$

with the convention that the empty product is equal to 1. They satisfy the relation

$$\langle \varepsilon(h)\,,\, \varepsilon(k)\rangle = e^{\langle h\,,\,k\rangle}.$$

The linear space $\mathcal{E}$ generated by these vectors is dense in Φ and any finite family of distinct coherent vectors is linearly free.

If $\mathcal{M}$ is any dense subset of $L^2(\mathbb{R}^+)$ then $\mathcal{E}(\mathcal{M})$ denotes the linear space spaned by the vectors $\varepsilon(h)$ such that $h \in \mathcal{M}$. This forms a dense subspace of Φ.

The *vacuum* element of Φ is the element Ω given by

$$\Omega(\sigma) = \mathbb{1}_{\sigma=\emptyset}.$$

The $\mathfrak{L}$-Lemma

The following lemma is a very important and useful combinatoric result that we shall use quite often in the sequel.

Theorem 3.1 ($\mathfrak{L}$-Lemma). *Let f be a measurable positive (resp. integrable) function on $\mathcal{P}\times\mathcal{P}$. Define a function g on $\mathcal{P}$ by*

$$g(\sigma) = \sum_{\alpha\subset\sigma} f(\alpha, \sigma\setminus\alpha)\ .$$

Then g is measurable positive (resp. integrable) and

$$\int_{\mathcal{P}} g(\sigma)\ d\sigma = \int_{\mathcal{P}\times\mathcal{P}} f(\alpha,\beta)\ d\alpha\ d\beta\ .$$

Proof. By density arguments one can restrict ourselves to the case where $f(\alpha,\beta) = h(\alpha)k(\beta)$ and where $h = \varepsilon(u)$ and $k = \varepsilon(v)$ are coherent vectors. In this case one has

$$\begin{aligned}\int_{\mathcal{P}\times\mathcal{P}} f(\alpha,\beta)\ d\alpha\ d\beta &= \int_{\mathcal{P}} \varepsilon(u)(\alpha)\ d\alpha \int_{\mathcal{P}} \varepsilon(v)(\beta)\ d\beta \\ &= e^{\int_0^\infty u(s)\ ds} e^{\int_0^\infty v(s)\ ds}\ \text{(take } u,v\in L^1\cap L^2(\mathbb{R}^+))\end{aligned}$$

and

$$\begin{aligned}\int_{\mathcal{P}} \sum_{\alpha\subset\sigma} f(\alpha,\sigma\setminus\alpha)\ d\sigma &= \int_{\mathcal{P}} \sum_{\alpha\subset\sigma}\prod_{s\in\alpha} u(s) \prod_{s\in\sigma\setminus\alpha} v(s)\ d\sigma \\ &= \int_{\mathcal{P}} \prod_{s\in\sigma}(u(s)+v(s))\ d\sigma = e^{\int_0^\infty u(s)+v(s)\ ds}\ . \qquad \blacksquare\end{aligned}$$

In the same way as for the Toy Fock space we have a natural isomorphism between Φ and $\Phi_{t]} \otimes \Phi_{[t}$.

Theorem 3.2. *The mapping:*

$$\begin{aligned} \Phi_{t]} \otimes \Phi_{[t} &\longrightarrow \Phi \\ f \otimes g &\longmapsto h \end{aligned}$$

where $h(\sigma) = f(\sigma_{t)})g(\sigma_{(t})$ *defines an isomorphism between* $\Phi_{t]} \otimes \Phi_{[t}$ *and* Φ.

Proof.

$$\begin{aligned} \int_{\mathcal{P}} |h(\sigma)|^2 \, d\sigma &= \int_{\mathcal{P}} |f(\sigma_{t)})|^2 |g(\sigma_{(t})|^2 \, d\sigma \\ &= \int_{\mathcal{P}} \sum_{\alpha \subset \sigma} \mathbb{1}_{\alpha \subset [0,t]} \mathbb{1}_{\sigma \setminus \alpha \subset [t,+\infty[} |f(\alpha)|^2 |g(\sigma \setminus \alpha)|^2 \, d\sigma \\ &= \int_{\mathcal{P}} \int_{\mathcal{P}} \mathbb{1}_{\alpha \subset [0,t]} \mathbb{1}_{\beta \subset [t,+\infty[} |f(\alpha)|^2 |g(\beta)|^2 \, d\alpha \, d\beta \\ &\qquad\qquad \text{(by the } \textstyle\sum\!\!\!\!\int\text{-Lemma)} \\ &= \int_{\mathcal{P}^{t)}} |f(\alpha)|^2 \, d\alpha \int_{\mathcal{P}^{(t}} |g(\beta)|^2 \, d\beta \\ &= \|f \otimes g\|^2. \qquad \blacksquare \end{aligned}$$

3.3 Abstract Itô calculus on Fock space

We are now ready to define the main ingredients of our structure: several differential and integral operators on the Fock space.

Projectors

For all $t > 0$ define the operator P_t from Φ to Φ by

$$[P_t f](\sigma) = f(\sigma) \mathbb{1}_{\sigma \subset [0,t]}.$$

It is clear that P_t is the orthogonal projector from Φ onto $\Phi_{t]}$.

For $t = 0$ we define P_0 by

$$[P_0 f](\sigma) = f(\emptyset) \mathbb{1}_{\sigma = \emptyset}$$

which is the orthogonal projection onto $L^2(\mathcal{P}_0) = \mathbb{C}\mathbb{1}$ where $\mathbb{1}$ is the vacuum of Φ: ($\mathbb{1}(\sigma) = \mathbb{1}_{\sigma=\emptyset}$).

Gradients

For all $t \in \mathbb{R}^+$ and all f in Φ define the following function on $\mathcal{P}$:

$$[D_t f](\sigma) = f(\sigma \cup t)\mathbb{1}_{\sigma \subset [0,t]}.$$

The first natural question is: for which f does $D_t f$ lie in $\Phi = L^2(\mathcal{P})$?

Proposition 3.3. *For all $f \in \Phi$, we have*

$$\int_0^\infty \int_{\mathcal{P}} |[D_t f](\sigma)|^2 \, d\sigma \, dt = \|f\|^2 - |f(\emptyset)|^2.$$

Proof. This is again an easy application of the $\not{\Sigma}$-Lemma:

$$\begin{aligned}
&\int_0^\infty \int_{\mathcal{P}} |f(\sigma \cup t)|^2 \mathbb{1}_{\sigma \subset [0,t]} \, d\sigma \, dt \\
&\qquad = \int_{\mathcal{P}} \int_{\mathcal{P}} |f(\alpha \cup \beta)|^2 \mathbb{1}_{\#\beta = 1} \mathbb{1}_{\alpha \subset [0, \vee \beta]} \, d\alpha \, d\beta \\
&\qquad = \int_{\mathcal{P}} \sum_{\alpha \subset \sigma} |f(\alpha \cup \sigma \setminus \alpha)|^2 \mathbb{1}_{\#(\sigma \setminus \alpha) = 1} \mathbb{1}_{\alpha \subset [0, \vee(\sigma \setminus \alpha)]} \, d\sigma \\
&\qquad = \int_{\mathcal{P} \setminus \mathcal{P}_0} \sum_{t \in \sigma} |f(\sigma)|^2 \mathbb{1}_{\sigma \setminus t \subset [0,t]} \, d\sigma \quad \text{(this forces } t \text{ to be } \vee\sigma) \\
&\qquad = \int_{\mathcal{P} \setminus \mathcal{P}_0} |f(\sigma)|^2 \, d\sigma = \|f\|^2 - |f(\emptyset)|^2. \qquad \blacksquare
\end{aligned}$$

This proposition implies the following: for all f in Φ, for almost all $t \in \mathbb{R}^+$ (the negligible set depends on f), the function $D_t f$ belongs to $L^2(\mathcal{P})$. Hence for *all* f in Φ, *almost all* t, $D_t f$ is an element of Φ. Nevertheless, D_t is not a well-defined operator from Φ to Φ. The only operators which can be well defined are either

$$\begin{aligned}
D : L^2(\mathcal{P}) &\longrightarrow L^2(\mathcal{P} \times \mathbb{R}^+) \\
f &\longmapsto \big((\sigma, t) \mapsto D_t f(\sigma)\big)
\end{aligned}$$

which is a partial isometry; or the regularised operators D_h, for $h \in L^2(\mathbb{R}^+)$:

$$[D_h f](\sigma) = \int_0^\infty h(t)[D_t f](\sigma) \, dt.$$

But, anyway, in this course we will treat the D_t's as linear operators defined on the whole of Φ. This, in general, poses no problem; one just has to be careful in some particular situations.

Integrals

A family $(g_t)_{t\geq 0}$ of elements of Φ is said to be an *Itô integrable process* if the following holds:

i) $f \mapsto \|g_t\|$ is measurable

ii) $g_t \in \Phi_{t]}$ for all t

iii) $\int_0^\infty \|g_t\|^2 \; dt < \infty$.

If $g. = (g_t)_{t\geq 0}$ is an Itô integrable process, define

$$[\mathcal{I}(g.)](\sigma) = \begin{cases} 0 & \text{if } \sigma = \emptyset \\ g_{\vee\sigma}(\sigma -) & \text{if } \sigma \neq \emptyset. \end{cases}$$

Proposition 3.4. *For all Itô integrable process* $g. = (g_t)_{t\geq 0}$ *one has*

$$\int_{\mathcal{P}} |[\mathcal{I}(g.)](\sigma)|^2 \; d\sigma = \int_0^\infty \|g_t\|^2 \; dt < \infty.$$

Proof. Another application of the $\not\Sigma$-Lemma (Exercise). ∎

Hence, for all Itô integrable process $g. = (g_t)_{t\geq 0}$, the function $\mathcal{I}(g.)$ defines an element of Φ, the *Itô integral* of the process $g.$.

Recall the operator $D : L^2(\mathcal{P}) \to L^2(\mathcal{P} \times I\!R^+)$ from last subsection.

Proposition 3.5.

$$\mathcal{I} = D^*.$$

Proof.

$$\begin{aligned} \langle f, \mathcal{I}(g.)\rangle &= \int_{\mathcal{P}\setminus\mathcal{P}_0} \overline{f}(\sigma) g_{\vee\sigma}(\sigma -) \; d\sigma \\ &= \int_0^\infty \int_{\mathcal{P}} \overline{f}(\sigma \cup t) g_t(\sigma) \mathbb{1}_{\sigma\subset[0,t]} \; d\sigma \; dt \; (\not\Sigma\text{-Lemma}) \\ &= \int_0^\infty \int_{\mathcal{P}} [\overline{D_t f}](\sigma) g_t(\sigma) \; d\sigma \; dt \\ &= \int_0^\infty \langle D_t f, g_t\rangle \; dt. \quad \blacksquare \end{aligned}$$

The abstract Itô integral is a true integral

We are going to see that the Itô integral defined above can be interpreted as a true integral $\int_0^\infty g_t \; d\chi_t$ with respect to some particular family $(\chi_t)_{t\geq 0}$ in Φ.

For all $t \in I\!R^+$, define the element χ_t of Φ by

$$\begin{cases} \chi_t(\sigma) = 0 & \text{if } \#\sigma \neq 1 \\ \chi_t(s) = \mathbb{1}_{[0,t]}(s). \end{cases}$$

This family of elements of Φ has some very particular properties. The main one is the following: not only does χ_t belong to $\Phi_{t]}$ for all $t \in \mathbb{R}^+$, but also $\chi_t - \chi_s$ belongs to $\Phi_{[s,t]}$ for all $s \leq t$ (this is very easy to check from the definition). We will see later that, in some sense, $(\chi_t)_{t\geq 0}$ is the only process to satisfy this property.

Let us take an Itô integrable process $(g_t)_{t\geq 0}$ which is *simple*, that is, constant on intervals:

$$g_t = \sum_i g_{t_i} \mathbb{1}_{[t_i,t_{i+1}[}(t).$$

Define $\int_0^\infty g_t d\chi_t$ to be $\sum_i g_{t_i} \otimes (\chi_{t_{i+1}} - \chi_{t_i})$ (recall that $g_{t_i} \in \Phi_{t_i]}$ and $\chi_{t_{i+1}} - \chi_{t_i} \in \Phi_{[t_i,t_{i+1}]} \subset \Phi_{(t_i})$. We have

$$\begin{aligned}
\left[\int_0^\infty g_t \ d\chi_t\right](\sigma) &= \sum_i [g_{t_i} \otimes (\chi_{t_{i+1}} - \chi_{t_i})](\sigma) \\
&= \sum_i g_{t_i}(\sigma_{t_i)})(\chi_{t_{i+1}} - \chi_{t_i})(\sigma_{(t_i}) \\
&= \sum_i g_{t_i}(\sigma_{t_i)}) \mathbb{1}_{\#\sigma_{(t_i}=1} \mathbb{1}_{\vee\sigma_{(t_i}\in]t_i,t_{i+1}]} \\
&= \sum_i g_{t_i}(\sigma_{t_i)}) \mathbb{1}_{\sigma-\subset[0,t_i]} \mathbb{1}_{\vee\sigma\in]t_i,t_{i+1}]} \\
&= \sum_i g_{t_i}(\sigma-) \mathbb{1}_{\vee\sigma\in]t_i,t_{i+1}]} \\
&= \sum_i g_{\vee\sigma}(\sigma-) \mathbb{1}_{\vee\sigma\in]t_i,t_{i+1}]} \\
&= g_{\vee\sigma}(\sigma-).
\end{aligned}$$

Thus for simple Itô-integrable processes we have proved that

$$\mathcal{I}(g.) = \int_0^\infty g_t \ d\chi_t. \tag{8}$$

But because of the isometry formula of Proposition 3.4 we have

$$||\mathcal{I}(g.)||^2 = \left|\left|\int_0^\infty g_t \ d\chi_t\right|\right|^2 = \int_0^\infty \|g_t\|^2 \ dt.$$

So one can pass to the limit from simple Itô integrable processes to Itô integrable processes in general and extend the definition of this integral $\int_0^\infty g_t \ d\chi_t$. As a result, (8) holds for every Itô integrable process $(g_t)_{t\geq 0}$. So from now on we will denote the Itô integral by

$$\int_0^\infty g_t \ d\chi_t.$$

Fock space predictable representation property

If f belongs to Φ, Proposition 3.3 shows that $(D_t f)_{t\geq 0}$ is an Itô integrable process. Let us compute $\int_0^\infty D_t f\ d\chi_t$:

$$\begin{aligned}
\left[\int_0^\infty D_t f\ d\chi_t\right](\sigma) &= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ [D_{\vee\sigma} f](\sigma-) & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma - \cup\vee\sigma)\mathbb{1}_{\sigma-\subset[0,\vee\sigma]} & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & \text{if } \sigma = \emptyset \\ f(\sigma) & \text{otherwise} \end{cases} \\
&= f(\sigma) - [P_0 f](\sigma).
\end{aligned}$$

This computation together with Propositions 3.3 and 3.4 give the following fundamental result.

Theorem 3.6 (Fock space predictable representation property). *For every $f \in \Phi$ we have the representation*

$$f = P_0 f + \int_0^\infty D_t f\ d\chi_t \tag{9}$$

and

$$\|f\|^2 = |P_0 f|^2 + \int_0^\infty \|D_t f\|^2\ dt. \tag{10}$$

The representation (9) *of f as a sum of a constant and an Itô integral is unique.*

The norm identity (10) *polarizes as follows*

$$\langle f, g\rangle = \overline{P_0 f} P_0 g + \int_0^\infty \langle D_t f, D_t g\rangle\ dt$$

for all $f, g \in \Phi$.

Proof. The only point remaining to be proved is the uniqueness property. If $f = c + \int_0^\infty g_t\ d\chi_t$ then $P_0 f = P_0 c + P_0 \int_0^\infty g_t\ d\chi_t = c$. Hence $\int_0^\infty g_t\ d\chi_t = \int_0^\infty D_t f\ d\chi_t$ that is, $\int_0^\infty (g_t - D_t f)\ d\chi_t = 0$. This implies $\int_0^\infty \|g_t - D_t f\|^2\ dt = 0$ thus the result. ■

Fock space chaotic expansion property

Let h_1 be an element of $L^2(\mathbb{R}^+) = L^2(\mathcal{P}_1)$, we can define

$$\int_0^\infty h_1(t)\Omega\, d\chi_t$$

which we shall simply denote by $\int_0^\infty h_1(t)\,d\chi_t$. Note that the element f of Φ that we obtain this way is given by

$$f(\sigma) = \begin{cases} 0 & \text{if } \#\sigma \neq 1 \\ h_1(s) & \text{if } \sigma = \{s\}. \end{cases}$$

That is, we construct this way all the elements of the first particle space of Φ.

For $h_2 \in L^2(\mathcal{P}_2)$ we want to define

$$\int_{0\leq s_1\leq s_2} h_2(s_1, s_2)\,d\chi_{s_1}\,d\chi_{s_2}$$

where we again omit to Ω-symbol. This can be done in two ways:

• either by starting with simple h_2's and defining the iterated integral above as being

$$\sum_{s_j} \sum_{t_i\leq s_j} h_2(t_i, s_j)(\chi_{t_{i+1}} - \chi_{t_i})(\chi_{s_{j+1}} - \chi_{s_j}).$$

One proves easily (exercise) that the norm^2 of the expression above is exactly

$$\int_{0\leq s_1\leq s_2} |h_2(s_1, s_2)|^2\; ds_1\; ds_2;$$

so one can pass to the limit in order to define $\int_{0\leq s_1\leq s_2} h_2(s_1, s_2)\; d\chi_{s_1}\; d\chi_{s_2}$ for any $h_2 \in L^2(\mathcal{P}_2)$.

• or one says that $g = \int_{0\leq s_1\leq s_2} h_2(s_1, s_2)\; d\chi_{s_1}\; d\chi_{s_2}$ is the only $g \in \Phi$ such that the continuous linear form

$$\begin{aligned} \lambda : \Phi &\longrightarrow \mathbb{C} \\ f &\longmapsto \int_{0\leq s_1\leq s_2} \overline{f}(\{s_1, s_2\}) h_2(s_1, s_2)\; ds_1\; ds_2 \end{aligned}$$

is of the form $\lambda(f) = \langle f, g\rangle$.

The two definitions coincide (exercise). The element of Φ which is formed this way is just the element of the second particle space associated to the function h_2.

In the same way, for $h_n \in L^2(\mathcal{P}_n)$ one defines

$$\int_{0\leq s_1\leq\cdots\leq s_n} h_n(s_1 \ldots s_n)\; d\chi_{s_1} \cdots d\chi_{s_n}.$$

We get

$$\begin{aligned} \Big\langle \int_{0\leq s_1\leq\cdots\leq s_n} & h_n(s_1 \ldots s_n)\,d\chi_{s_1} \ldots d\chi_{s_n}, \\ & \int_{0\leq s_1\leq\cdots\leq s_m} k_m(s_1 \ldots s_m)\,d\chi_{s_1} \cdots\, d\chi_{s_m} \Big\rangle \\ &= \delta_{n,m} \int_{0\leq s_1\leq\cdots\leq s_n} \overline{h}_n(s_1 \ldots s_n) k_n(s_1 \ldots s_n)\,ds_1 \cdots ds_n \end{aligned}$$

For $f \in L^2(\mathcal{P})$ we define

$$\int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma = f(\emptyset)\mathbb{1} + \sum_n \int_{0\le s_1\le\cdots\le s_n} f(\{s_1\dots s_n\})\ d\chi_{s_1}\cdots d\chi_{s_n}.$$

Theorem 3.7 (Fock space chaotic representation property). *For all $f \in \Phi$ we have*

$$f = \int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma.$$

Proof. For $g \in \Phi$ we have by definition

$$\begin{aligned}&\langle g, \int_{\mathcal{P}} f(\sigma)\ d\chi_\sigma\rangle \\ &= \overline{g(\emptyset)}f(\emptyset) + \sum_n \int_{0\le s_1\le\cdots\le s_n} \overline{g}(\{s_n\dots s_n\})f(\{s_n\dots s_n\})\ ds_1\cdots ds_n \\ &= \langle g, f\rangle.\end{aligned}$$

(Details are left to the reader). ■

$(\chi_t)_{t\ge0}$ is the only independent increment curve in Φ

We have seen that $(\chi_t)_{t\ge0}$ is a family in Φ satisfying

i) $\chi_t \in \Phi_{t]}$ for all $t \in \mathbb{R}^+$;

ii) $\chi_t - \chi_s \in \Phi_{[s,t]}$ for all $0 \le s \le t$.

These properties where fundamental ingredients for defining our Itô integral. We can naturally wonder if there are any other families $(Y_t)_{t\ge0}$ in Φ satisfying these two properties?

If one takes $a(\cdot)$ to be a function on $\mathbb{R}^+$, and $h \in L^2(\mathbb{R}^+)$ then $Y_t = a(t)\mathbb{1} + \int_0^t h(s)\ d\chi_s$ clearly satisfies *i)* and *ii)*. But clearly, apart from multiplying every terms by a scalar factor, these families $(Y_t)_{t\ge0}$ do not change the notions of Itô integrals. The claim now is that the above families $(Y_t)_{t\ge0}$ are the only possible ones.

Theorem 3.8. *If $(Y_t)_{t\ge0}$ is a vector process on Φ satisfying i) and ii) then there exist $a : \mathbb{R}^+ \to \mathbb{C}$ and $h \in L^2(\mathbb{R}^+)$ such that*

$$Y_t = a(t)\mathbb{1} + \int_0^t h(s)\ d\chi_s.$$

Proof. Let $a(t) = P_0Y_t$. Then $\widetilde{Y}_t = Y_t - a(t)\mathbb{1}$, $t \in \mathbb{R}^+$, satisfies *i)* and *ii)* with $\widetilde{Y}_0 = 0$ (for $Y_0 = P_0Y_0 = P_0(Y_t - Y_0) + P_0Y_0 = P_0Y_t$). We can now drop the $\sim$ symbol and assume $Y_0 = 0$. Now note that $P_sY_t = P_sY_s + P_s(Y_t - Y_s) = P_sY_s = Y_s$. This implies easily (exercise) that the chaotic expansion of Y_t is of the form:

$$Y_t = \int_{\mathcal{P}} \mathbb{1}_{\mathcal{P}^{t)}}(\sigma) y(\sigma) \; d\chi_\sigma \; .$$

If $\#\sigma \geq 2$, for example $\sigma = \{t_1 < t_2 < \cdots < t_n\}$, let $s < t$ be such that $t_1 < s < t_n < t$. Then

$$(Y_t - Y_s)(\sigma) = 0 \;\text{ for }\; Y_t - Y_s \in \Phi_{[s,t]} \;\text{ and }\; \sigma \not\subset [s,t] \; .$$

Furthermore

$$Y_s(\sigma) = P_s Y_s(\sigma) = \mathbb{1}_{\sigma \subset [0,s]} Y_s(\sigma) = 0 \; .$$

Thus $Y_t(\sigma) = 0$, for any $\sigma \in \mathcal{P}$ with $\#\sigma \geq 2$, any $t \in \mathbb{R}^+$. This means that $Y_t = \int_0^t y(s) \; d\chi_s$. ■

Higher multiplicities

When considering the Fock space $\Gamma_s(L^2(\mathbb{R}^+;\mathbb{C}^n))$ or $\Gamma_s(L^2(\mathbb{R}^+;\mathcal{G}))$ for some separable Hilbert space $\mathcal{G}$, we speak of Fock space with multiplicity n or infinite multiplicity.

The Guichardet space is then associated to the the set $\mathcal{P}_n$ of finite subsets of $\mathcal{P}$ but whose elements are given a label, a color, in $\{1,\ldots,n\}$. This is also equivalent to giving oneself a family of n disjoint subsets of $\mathbb{R}^+$: $\sigma = (\sigma_1,\ldots,\sigma_n)$. The norm on that Fock space is then

$$||f||^2 = \int_{\mathcal{P}_n} |f(\sigma)|^2 \; d\sigma$$

with obvious notations.

The universal curve $(\chi_t)_{t\geq 0}$ is replaced by a family $(\chi_t^i)_{t\geq 0}$ defined by

$$\chi_t^i(\sigma) = \begin{cases} \mathbb{1}_{[0,t]}(s) & \text{if } \sigma_j = \emptyset \text{ for all } j \neq i \text{ and } \sigma_i = \{s\} \\ 0 & \text{otherwise.} \end{cases}$$

The Fock space predictable representation is now of the form

$$f = P_0 f + \sum_i \int_0^\infty D_s^i f \, d\chi_s^i$$

with

$$||f||^2 = |P_0 f|^2 + \sum_i \int_0^\infty \left|\left|D_s^i f\right|\right|^2 ds$$

and

$$[D_s^i f](\sigma) = f(\sigma \cup \{s\}_i) \mathbb{1}_{\sigma \subset [0,s]}$$

with the notation $\{s\}_i = (\emptyset,\ldots,\emptyset,\{s\},\emptyset,\ldots,\emptyset) \in \mathcal{P}_n$.

The quantum noises are $a^i_j(t)$, labelled by $i, j = 0, 1, \dots, n$, with the formal table

$$\begin{aligned} da^0_0(t)\,\mathbb{1} &= dt\mathbb{1} \\ da^0_0(t)\,d\chi^k_t &= 0 \\[1em] da^0_i(t)\,\mathbb{1} &= d\chi^i_t \\ da^0_i(t)\,d\chi^k_t &= 0 \\[1em] da^i_0(t)\,\mathbb{1} &= 0 \\ da^i_0(t)\,d\chi^k_t &= \delta_{ki}\,dt\mathbb{1} \\[1em] da^i_j(t)\,\mathbb{1} &= 0 \\ da^i_j(t)\,d\chi^k_t &= \delta_{ki}\,d\chi^j_t. \end{aligned}$$

3.4 Probabilistic interpretations of Fock space

In this section we present the general theory of probabilistic interpretations of Fock space. This section is not really necessary to understand the rest of the course, but the ideas coming from these notions underly the whole work.

This section needs some knowledge in the basic elements of stochastic processes, martingales and stochastic integrals. Some of that material can be found in L. Rey-Bellet's first course in this volume.

Chaotic expansions

We consider a martingale $(x_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$. We take $(\mathcal{F}_t)_{t\geq 0}$ to be the natural filtration of $(x_t)_{t\geq 0}$ (the filtration is made complete and right continuous) and we suppose that $\mathcal{F} = \mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$. Such a martingale is called *normal* if $(x_t^2 - t)_{t\geq 0}$ is still a martingale for $(\mathcal{F}_t)_{t\geq 0}$. This is equivalent to saying that $\langle x, x\rangle_t = t$ for all $t \geq 0$, where $\langle\cdot\,,\cdot\rangle$ denotes the probabilistic angle bracket.

A normal martingale is said to satisfy the *predictable representation property* if all $f \in L^2(\Omega, \mathcal{F}, P)$ can be written as a stochastic integral

$$f = \mathbb{E}[f] + \int_0^\infty h_s\; dx_s$$

for a $(\mathcal{F}_t)_{t\geq 0}$-predictable process $(h_t)_t \geq 0$. Recall that

$$\mathbb{E}[|f|^2] = |\mathbb{E}[f]|^2 + \int_0^\infty \mathbb{E}[|h_s|^2]\; ds$$

that is, in the $L^2(\Omega)$-norm notation:

$$\|f\|^2 = |\mathbb{E}[f]|^2 + \int_0^\infty \|h_s\|^2 \, ds \ .$$

Recall that if f_n is a function in $L^2(\Sigma_n)$, where $\Sigma_n = \{0 \leq t_1 < t_2 < \cdots < t_n \in \mathbb{R}^n\} \subset (\mathbb{R}^+)^n$ is equipped with the restriction of the n-dimensional Lebesgue measure, one can define an element $I_n(f_n) \in L^2(\Omega)$ by

$$I_n(f_n) = \int_{\Sigma_n} f_n(t_1 \ldots t_n) \, dx_{t_1} \cdots dx_{t_n}$$

which is defined, with the help of the Itô isometry formula, as an iterated stochastic integral satisfying

$$\|I_n(f_n)\|^2 = \int_{\Sigma_n} |f_n(t_1 \ldots t_n)|^2 \, dt_1 \cdots dt_n \ .$$

It is also important to recall that

$$\langle I_n(f_n), I_m(f_m)\rangle = 0 \ \text{ if } \ n \neq m \ .$$

The *chaotic space* of $(x_t)_{t\geq 0}$, denoted $CS(x)$, is the sub-Hilbert space of $L^2(\Omega)$ made of the random variables $f \in L^2(\Omega)$ which can be written as

$$f = \mathbb{E}[f] + \sum_{n=1}^{\infty} \int_{\Sigma_n} f_n(t_1 \ldots t_n) \, dx_{t_1} \cdots dx_{t_n} \tag{11}$$

for some $f_n \in L^2(\Sigma_n)$, $n \in \mathbb{N}^*$, such that

$$\|f\|^2 = |\mathbb{E}[f]|^2 + \sum_{n=1}^{\infty} \int_{\Sigma_n} |f_n(t_1 \ldots t_n)|^2 \, dt_1 \cdots dt_n < \infty \ .$$

When $CS(x)$ is the whole of $L^2(\Omega)$ one says that x satisfies the *chaotic representation property*. The decomposition of f as in (11) is called the *chaotic expansion* of f.

Note that the chaotic representation property implies the predictable representation property for if f can be written as in (11) then, by putting h_t to be

$$h_t = f_1(t) + \sum_{n=1}^{\infty} \int_{\Sigma_n} \mathbb{1}[0,t](t_n) f_{n+1}(t_1 \ldots t_n, t) \, dx_{t_1} \cdots dx_{t_n}$$

we have

$$f = \mathbb{E}[f] + \int_0^\infty h_t \, dx_t \ .$$

The cases where $(x_t)_{t\geq 0}$ is the Brownian motion, the compensated Poisson process or the Azéma martingale with coefficient $\beta \in [-2, 0]$, are examples of normal martingales which possess the chaotic representation property.

Isomorphism with Φ

Let us consider a normal martingale $(x_t)_{t\geq 0}$ with the predictable representation property and its chaotic space $CS(x) \subset L^2(\Omega, \mathcal{F}, P)$.

By identifying a function $f_n \in L^2(\Sigma_n)$ with a symmetric function $\tilde{f}_n$ on $(\mathbb{R}^+)^n$, one can identify $L^2(\Sigma_n)$ with $L^2_{\mathrm{sym}}((\mathbb{R}^+)^n) = L^2(\mathbb{R}^+)^{\odot n}$ (with the correct symmetric norm: $\|\tilde{f}_n\|^2_{L^2(\mathbb{R}^+)^{\odot n}} = n!\|\tilde{f}_n\|^2_{L^2(\mathbb{R}^+)^{\otimes n}}$ if one puts $\tilde{f}_n$ to be $\frac{1}{n!}$ times the symmetric expansion of f_n). It is now clear that $CS(x)$ is naturally isomorphic to the symmetric Fock space

$$\Phi = \Gamma(L^2(\mathbb{R}^+)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^+)^{\odot n} .$$

The isomorphism can be explicitly written as follows:

$$\begin{aligned} U_x : \Phi &\longrightarrow CS(x) \\ f &\longmapsto Uf \end{aligned}$$

where $f = \sum_n f_n$ with $f_n \in L^2(\mathbb{R}^+)^{\odot n}$, $n \in \mathbb{N}$, and

$$U_x f = f_0 + \sum_{n=1}^{\infty} n! \int_0^t f_n(t_1 \dots t_n)\ dx_{t_1} \cdots dx_{t_n} .$$

If $f = \mathbb{E}[f] + \sum_{n=1}^{\infty} \int_0^t f_n(t_1 \dots t_n)\ dx_{t_1} \cdots dx_{t_n}$ is an element of $CS(x)$, then $U_x^{-1} f = \sum_n g_n$ with $g_0 = \mathbb{E}[f]$ and $g_n = \frac{1}{n!} f_n$ symmetrized.

These isomorphisms are called the *probabilistic interpretations* of Φ. One may speak of *Brownian interpretation*, or *Poisson interpretation* ...

Structure equations

If $(x_t)_{t\geq 0}$ is a normal martingale, with the predictable representation property and if x_t belongs to $L^4(\Omega)$, for all t, then $([x,x]_t - \langle x,x\rangle_t)_{t\geq 0}$ is a $L^2(\Omega)$-martingale; so by the predictable representation property there exists a predictable process $(\psi_t)_{t\geq 0}$ such that

$$[x,x]_t - \langle x,x\rangle_t = \int_0^t \psi_s\ dx_s$$

that is,

$$[x,x]_t = t + \int_0^t \psi_s\ dx_s$$

or else

$$d[x,x]_t = dt + \psi_t\ dx_t . \tag{12}$$

This equation is called a *structure equation* for $(x_t)_{t\geq 0}$. One has to be careful that, in general, there can be many structure equations describing the same solution $(x_t)_{t\geq 0}$; there also can be several solutions (in law) to some structure equations.

What can be proved is the following:

• when $\psi_t \equiv 0$ for all t then the only solution (in law) of (12) is the Brownian motion;

• when $\psi_t \equiv c$ for all t then the only solution (in law) of (12) is the compensated Poisson process with intensity $1/c^2$;

• when $\psi_t = \beta x_{t-}$ for all t, then the only solution (in law) of (12) is the Azéma martingale with parameter β.

The importance of structure equations appears when one considers products within two different probabilistic interpretations. For exemple, let f, g be two elements of Φ and let $U_w f$ and $U_w g$ be their image in the Brownian motion interpretation $(w_t)_t \geq 0$. That is, $U_w f$ and $U_w g$ are random variables in the canonical space $L^2(\Omega)$ of the Brownian motion. They admit a natural product, as random variables: $U_w f\cdot U_w g$. If the resulting random variable is still an element of $L^2(\Omega)$ (for example if f and g are coherent vectors) then we can pull back the resulting random variable to the space Φ:

$$U_w^{-1}(U_w f\cdot U_w g).$$

This operation defines an associative product on Φ:

$$f *_w g = U_w^{-1}(U_w f\cdot U_w g)$$

called the *Wiener product*.

We could have done the same operations with the Poisson interpretation:

$$f *_p g = U_p^{-1}(U_p f\cdot U_p g),$$

this gives the *Poisson product* on Φ. One can also define an Azéma product.

The point is that one always obtains different products on Φ when considering different probabilistic interpetations. This comes frome the fact that all probabilistic interpretations of Φ have the same angle bracket $\langle x,x\rangle_t = t$ but not the same square bracket: $[x,x]_t = t + \int_0^t \psi_s\, dx_s$. The product of two random variables makes use of the square bracket: if $f = \mathbb{E}[f] + \int_0^\infty h_s\, dx_s$ and $g = \mathbb{E}[g] + \int_0^\infty k_s\, dx_s$, if $f_s = \mathbb{E}[f|\mathcal{F}_s]$ and $g_s = \mathbb{E}[g|\mathcal{F}_s]$ for all $s \geq 0$ then

$$\begin{aligned}
fg &= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty f_s k_s\, dx_s + \int_0^\infty g_s h_s\, dx_s + \int_0^\infty h_s k_s d[x,x]_s \\
&= \mathbb{E}[f]\mathbb{E}[g] + \int_0^\infty f_s k_s\, dx_s + \int_0^\infty g_s h_s\, dx_s + \int_0^\infty h_s k_s ds + \\
&\quad + \int_0^\infty h_s k_s \psi_s\, dx_s\ .
\end{aligned}$$

For example if one takes the element χ_t of Φ, we have

$$U_w\chi_t = \int_0^\infty \mathbb{1}_{[0,t]}(s)\ dw_s = w_t \text{ the Brownian motion itself} \tag{13}$$

and

(14)

$$U_p\chi_t = \int_0^\infty \mathbb{1}_{[0,t]}(s)\ dx_s = x_t \text{ the compensated Poisson process itself.} \tag{15}$$

So, as $w_t^2 = 2\int_0^t w_s\ dw_s + t$ and $x_t^2 = 2\int_0^t x_s\ dx_s + t + x_t$, we have

$$\chi_t *_w \chi_t = t + 2\int_0^t \chi_s\, d\chi_s \tag{16}$$

and

$$\chi_t *_p \chi_t = t + 2\int_0^t \chi_s\, d\chi_s + \chi_t. \tag{17}$$

We get two different elements of Φ.

Probabilistic interpretations of the abstract Itô calculus

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t \geq 0, P, (x_t)_{t\geq 0})$ be a probabilistic interpretation of the Fock space Φ. Via the isomorphism described above, the space $\Phi_{t]}$ interprets as the space of $f \in CS(x.)$ whose chaotic expansion contains only functions with support included in $[0,t]$; that is, the space $CS(x) \cap L^2(\mathcal{F}_t)$. So when the chaotic expansion property holds we have $\Phi_{t]} \simeq L^2(\mathcal{F}_t)$ and thus P_t is nothing but $\mathbb{E}[\cdot\,|\mathcal{F}_t]$ (the conditional expectation) when interpreted in $L^2(\Omega)$.

The process $(\chi_t)_{t\geq 0}$ interprets as a process of random variables whose chaotic expansion is given by

$$\chi_t = \int_0^\infty \mathbb{1}_{[0,t]}(s)\ dx_s = x_t.$$

So, in any probabilistic interpretation $(\chi_t)_{t\geq 0}$ becomes the noise $(x_t)_{t\geq 0}$ itself (Brownian motion, compensated Poisson process, Azéma martingale,...). $(\chi_t)_{t\geq 0}$ is the "universal" noise, seen in the Fock space Φ.

As we have proved that the Itô integral $\mathcal{I}(g.)$ on Φ is the L^2-limit of the Riemann sums $\sum_i g_{t_i}(\chi_{t_{i+1}} - \chi_{t_i})$, it is clear that in $L^2(\Omega)$, the Itô integral interprets as the usual Itô integral with respect to $(x_t)_{t\geq 0}$.

One remark is necessary here. When one writes the approximation of the Itô integral $\int_0^\infty g_s\, dx_s$ as $\sum_i g_{t_i}(x_{t_{i+1}} - x_{t_i})$ there appear products $(g_{t_i} \cdot (x_{t_{i+1}} - x_{t_i}))$, so this notion seems to depend on the probabilistic interpretation of Φ. The point in that the product $g_{t_i} \cdot (x_{t_{i+1}} - s_{t_i})$ is not really a product. By this we mean that the Itô formula for this product does not involve any bracket term:

$$g_{t_i}(x_{t_{i+1}} - x_{t_i}) = \int_{t_i}^{t_{i+1}} g_{t_i}\ dx_s$$

so it gives rise to the same formula whatever is the probabilistic interpretation $(x_t)_{t\geq 0}$. Only is involved the tensor product structure: $\Phi \simeq \Phi_{t_i]} \otimes \Phi_{[t_i}$; the product $g_{t_i}(x_{t_{i+1}} - x_{t_i})$ is only a tensor product $g_{t_i} \otimes (x_{t_{i+1}} - x_{t_i})$ in this structure. This tensor product structure is common to all the probabilistic interpretations.

We have seen that $\int_0^\infty g_t\ d\chi_t$ interprets as the usual Itô integral $\int_0^\infty g_t\ d\chi_t$ in any probabilistic interpretation $(x_t)_{t\geq 0}$. Thus the representation

$$f = P_0 f + \int_0^\infty D_s f\ d\chi_x$$

of Theorem 3.6 is just a Fock space expression of the predictable representation property. The process $(D_t f)_{t\geq 0}$ is then interpreted as the predictable process that represents f in his predictable representation.

4 Quantum stochastic calculus

We now leave the probabilistic intepretations of the Fock space and we enter into the theory of quantum noises itself, with its associated theory of integration.

4.1 An heuristic approach to quantum noise

Adaptedness

When trying to define "quantum stochastic integrals" of operators on Φ, mimicking integral representations such as in the Toy Fock space, we have to consider integrals of the form

$$\int_0^t H_s\, dMs$$

where $(H_t)_{t\geq 0}$ and $(M_t)_{t\geq 0}$ are families of operators on Φ.

The first natural idea is to consider approximations of the above by Riemann sums:

$$\sum_i H_{t_i}\left(M_{t_{i+1}} - M_{t_i}\right),$$

but, immediatly, this kind of definition faces two difficulties:

i) The operators we are going to consider are not in general bounded and therefore the above sum may lead us to serious domain problems.

ii) The operators we consider H_s, M_s need not commute in general and we can naturally wonder why we should not give the preference to sums like

$$\sum_i \left(M_{t_{i+1}} - M_{t_i}\right) H_{t_i},$$

or even more complicated forms.

This means that, at this stage of the theory, we have to make concessions: we cannot integrate any operator process with respect to any operator process. But this should not be a surprise, already in the classical theory of stochastic calculus one can only integrate predictable processes against semimartingales.

The first step in this integration theory consists is obtained by applying, at the operator level, a construction similar to the one of the Itô integral on Φ with respect to $(\chi_t)_{t\geq 0}$. Indeed, recall the decomposition of the Fock space Φ:

$$\Phi = \Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$$

for all $s \leq t$.

If there exist operator families $(X_t)_{t\geq 0}$ on Φ with the property that $X_t - X_s$ acts on $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$ as $I \otimes K_{s,t} \otimes I$ and if we consider operator families $(H_t)_{t\geq 0}$ such that H_t is of the form $H_t \otimes I$ on $\Phi_{t]} \otimes \Phi_{[t}$ then the Riemann sums

$$\sum_i H_{t_i} \left(X_{t_{i+1}} - X_{t_i}\right)$$

are well-defined and unambiguous for the products

$$H_{t_i} \left(X_{t_{i+1}} - X_{t_i}\right)$$

are not true compositions of operators anymore but just tensor products (just like for vectors in the Itô integral):

$$(H_{t_i} \otimes I) \left(I \otimes \left(X_{t_{i+1}} - X_{t_i}\right) \otimes I\right) = H_{t_i} \otimes \left(X_{t_{i+1}} - X_{t_i}\right).$$

In particular, there are no more domain problem added by the composition of operators, no more commutation problem.

Families of operators of the form $H_t \otimes I$ on $\Phi_{t]} \otimes \Phi_{[t}$ are obvious to construct. They are called *adapted processes of operators*. The true definition of adapted processes of operators, in the case of unbounded operators, are actually not that simple. They are exactly what is stated above, in the spirit, but this requires a more careful definition. We do not develop these refinements in this course (see [9]).

The existence of non-trivial operator families $(X_t)_{t\geq 0}$ on Φ with the property that $X_t - X_s$ acts on $\Phi_{s]} \otimes \Phi_{[s,t]} \otimes \Phi_{[t}$ as $I \otimes K_{s,t} \otimes I$ for all $s \leq t$ is not so clear.

The three quantum noises: heuristics

We call "*quantum noise*" any processes of operators on Φ, say $(X_t)_{t\geq 0}$, such that, for all $t_i \leq t_{i+1}$, the operator $X_{t_{i+1}} - X_{t_i}$ acts as $I \otimes k \otimes I$ on $\Phi_{t_i]} \otimes \Phi_{[t_i,t_{i+1}]} \otimes \Phi_{[t_{i+1}}$.

Let us consider the operator $dX_t = X_{t+dt} - X_t$. It acts only on $\Phi_{[t,t+dt]}$. The chaotic representation property of Fock space (Theorem 3.7) shows that this part of the Fock space is generated by the vacuum $\mathbb{1}$ and by $d\chi_t = \chi_{t+dt} - \chi_t$. Hence dX_t is determined by its value on $\mathbb{1}$ and on $d\chi_t$. These values have to remain in $\Phi_{[t,t+dt]}$ and to be integrators also, that is $d\chi_t$ or $dt\mathbb{1}$ (denoted dt). As a consequence the only *irreducible* noises are given by

	$d\chi_t$	$\mathbb{1}$
da_t°	$d\chi_t$	0
da_t^-	dt	0
da_t^+	0	$d\chi_t$
$da_t^\times$	0	dt

These are four noises and not three as announced, but we shall see later that $da_t^\times$ is just dtI.

The three quantum noises: serious business

Recall the definitions of creation, annihilation and differential second quantization operators on Φ. For any $h \in L^2(\mathbb{R}^+)$, any operator H on $L^2(\mathbb{R}^+)$ and any symmetric tensor product $u_1 \circ \ldots \circ u_n$ in Φ we put

$$\begin{aligned}
a^+(h)u_1 \circ \ldots \circ u_n &= h \circ u_1 \circ \ldots \circ u_n \\
a^-(h)u_1 \circ \ldots \circ u_n &= \sum_{i=1}^n <h\,,\, u_i> u_1 \circ \ldots \circ \widehat{u_i} \circ \ldots \circ u_n \\
\Lambda(H)u_1 \circ \ldots \circ u_n &= \sum_{i=1}^n u_1 \circ \ldots \circ Hu_i \circ \ldots \circ u_n.
\end{aligned}$$

In the case where $H = \mathcal{M}_h$ is the multiplication operator by h we write $a^\circ(h)$ for $\Lambda(H)$.

An easy computation shows that $a^+(h), a^-(h), a^\circ(h)$ are closable operators whose domain contains $\mathcal{E}$, the space of coherent vectors, and which satisfy

$$\begin{aligned}
<\varepsilon(u)\,,\, a^+(h)\varepsilon(v)> &= \int_0^\infty \overline{u}(s)\, h(s)\, ds <\varepsilon(u)\,,\, \varepsilon(v)> \\
<\varepsilon(u)\,,\, a^-(h)\varepsilon(v)> &= \int_0^\infty \overline{v}(s)\, h(s)\, ds <\varepsilon(u)\,,\, \varepsilon(v)> \\
<\varepsilon(u)\,,\, a^\circ(h)\varepsilon(v)> &= \int_0^\infty \overline{u}(s)v(s)\, h(s)\, ds <\varepsilon(u)\,,\, \varepsilon(v)>.
\end{aligned}$$

One can also easily obtain the above explicit formulas:

$$\begin{aligned} \big[a^+(h)f\big] &= \sum_{s\in\sigma} h(s)\, f(\sigma \setminus s) \\ \big[a^-(h)f\big] &= \int_0^\infty h(s)\, f(\sigma \cup s)\, ds \\ [a^\circ(h)f] &= \sum_{s\in\sigma} h(s)\, f(\sigma). \end{aligned}$$

For any $t \in \mathbb{R}^+$ and any $\varepsilon = +, -, \circ$ we put $a^\varepsilon(h_{t]}) = a^\varepsilon(h\mathbb{1}_{[0,t]})$. It is then easy to check from the definitions that any operator process $(X_t)_{t\geq 0}$ of the form

$$X_t = a(t)I + a^+(f_{t]}) + a^-(g_{t]}) + a^\circ(k_{t]})$$

is a quantum noise (in order to avoid domain problems we have to ask that h, k belong to $L^2(\mathbb{R}^+)$ and k belongs to $L^\infty(\mathbb{R}^+)$).

The previous heuristical discussion seems to says that they should be the only ones. This result is intuitively simple, but its proof is not so simple (see [17]), we do not develop it here.

Theorem 4.1. *A family of closable operators $(X_t)_{t\geq 0}$ defined on $\mathcal{E}$ is a quantum noise if and only if there exist a function a on $\mathbb{R}^+$, functions $f, g \in L^2(\mathbb{R}^+)$ and a function $k \in L^\infty(\mathbb{R}^+)$ such that*

$$X_t = a(t)I + a^+(f_{t]}) + a^-(g_{t]}) + a^\circ(k_{t]})$$

for all t. ∎

Putting $a^\varepsilon_t = a^\varepsilon(\mathbb{1}_{[0,t]})$, we can see that all the quantum noises are determined by the four processes $(a^\varepsilon_t)_{t\geq 0}$, for $\varepsilon = +, -, \circ, \times$ where we have put $a^\times_t = tI$.

It is with respect to these four operator processes that the quantum stochastic integrals are defined.

4.2 Quantum stochastic integrals

Heuristic approach

Let us now formally consider a quantum stochastic integral

$$T_t = \int_0^t H_s da^\varepsilon_s$$

with respect to one of the four above noises. Let it act on a vector process

$$f_t = P_t f = \int_0^t D_s f \; d\chi_s \quad \text{(we omit the expectation } P_0 f \text{ for the moment).}$$

The result is a process of vectors $(T_t f_t)_{t\geq 0}$ in Φ. We claim that one can expect the family $(T_t f_t)_{t\geq 0}$ to satisfy an Itô-like integration by part formula:

$$\begin{aligned} d(T_t f_t) &= T_t df_t + (dT_t) f_t + (dT_t)(df_t) \\ &= T_t (D_t f \; d\chi_t) + (H_t da_t^\varepsilon) f_t + (H_t da_t^\varepsilon)(D_t f \; d\chi_t). \end{aligned}$$

There are three reasons for that claim:

i) This is the continuous version of the quantum Itô formula obtained in discrete time (Theorem 2.2).

ii) Quantum stochastic calculus contains in particular the classical one, it should then satisfy the same kind of Itô integration by part formula.

ii) More convincing: if one considers an operator process $(H_t)_{t\geq 0}$ which is simple (i.e. constant by intervals) and a vector process $(D_t f)_{t\geq 0}$ which is simple too, then the integrated form of the above identity is exactly true (Exercise for very motivated readers!).

In the tensor product structure $\Phi = \Phi_{t]} \otimes \Phi_{[t}$ this formula writes

$$d(T_t f_t) = (T_t \otimes I)(D_t f \otimes d\chi_t) + (H_t \otimes da_t^\varepsilon)(f_t \otimes \mathbb{1}) + (H_t \otimes da_t^\varepsilon)(D_t f \otimes d\chi_t),$$

that is,

$$d(T_t f_t) = T_t D_t f_t \otimes d\chi_t + H_t f_t \otimes da_t^\varepsilon \mathbb{1} + H_t D_t f \otimes da_t^\varepsilon \; d\chi_t. \tag{18}$$

In the right hand side one sees three terms; the first one always remains and is always the same. The other two depend on the heuristic table satisfied by the quantum noises. Integrating (18) and using the quantum noise table one gets

$$T_t f_t = \int_0^t T_s D_s f \; d\chi_s + \begin{cases} \int_0^t H_s D_s f \; d\chi_s & \text{if } \varepsilon = 0 \\ \int_0^t H_s P_s f \; d\chi_s & \text{if } \varepsilon = + \\ \int_0^t H_s D_s f \; ds & \text{if } \varepsilon = - \\ \int_0^t H_s P_s f \; ds & \text{if } \varepsilon = \times. \end{cases} \tag{19}$$

A correct definition

We want to exploit formula (19) as a definition of the quantum stochastic integrals $T_t = \int_0^t H_s da_s^\varepsilon$.

Let $(H_t)_{t\geq 0}$ be an adapted process of operators on Φ, let $(T_t)_{t\geq 0}$ be another one. One says that (19) is *meaningful* for a given $f \in \Phi$ if

- $P_t f \in \operatorname{Dom} T_t$;
- $D_s f \in \operatorname{Dom} T_s$, $s \leq t$ and $\int_0^t \|T_s D_s f\|^2 \; ds < \infty$;
- $\begin{cases} \text{if } \varepsilon = \circ, D_s f \in \operatorname{Dom} H_s, s \leq t \text{ and } \int_0^t \|H_s D_s f\|^2 \; ds < \infty \\ \text{if } \varepsilon = +, P_s f \in \operatorname{Dom} H_s, s \leq t \text{ and } \int_0^t \|H_s P_s f\|^2 \; ds < \infty \\ \text{if } \varepsilon = -, D_s f \in \operatorname{Dom} H_s, s \leq t \text{ and } \int_0^t \|H_s D_s f\| \; ds < \infty \\ \text{if } \varepsilon = \times, P_s f \in \operatorname{Dom} H_s, s \leq t \text{ and } \int_0^t \|H_s P_s f\| \; ds < \infty \end{cases}$

One says that (19) is *true* if the equality holds.

A subspace $\mathcal{D} \subset \Phi$ is called an *adapted domain* if for all $f \in \mathcal{D}$ and all (almost all) $t \in \mathbb{R}^+$, one has

$$P_t f \ \text{ and } \ D_t f \in \mathcal{D}\ .$$

There are many examples of adapted domains. All the domains we shall meet during this course are adapted:

• $\mathcal{D} = \Phi$ itself is adapted;

• $\mathcal{D} = \mathcal{E}$ is adapted; even more $\mathcal{D} = \mathcal{E}(\mathcal{M})$ is adapted once $\mathbb{1}_{[0,t]}\mathcal{M} \subset \mathcal{M}$ for all t.

• The space of *finite particles* $\Phi_f = \big\{f \in L^2(\mathcal{P});\ f(\sigma) = 0$ for $\#\sigma > N$, for some $N \in \mathbb{N}\big\}$ is adapted.

• All the Fock scales $\Phi^{(a)} = \big\{f \in L^2(\mathcal{P});\ \int_{\mathcal{P}} a^{\#\sigma}|f(\sigma)|^2\ d\sigma < \infty\big\}$, for $a \geq 1$, are adapted.

• Maassen's space of test vectors: $\big\{f \in L^2(\mathcal{P});\ f(\sigma) = 0$ for $\#\sigma \not\subset [0, T]$,for some $T \in \mathbb{R}^+$, and $|f(\sigma)| \leq CM^{\#\sigma}$ for some $C, M\big\}$ is adapted.

The above equation (19) is the definition of the quantum stochastic integrals that we shall follow and apply along this course. The definition is exactly formulated as follows.

Let $(H_t)_{t\geq 0}$ be an adapted process of operators defined on an adapted domain $\mathcal{D}$. One says that a process $(T_t)_{t\geq 0}$ is the *quantum stochastic integral*

$$T_t = \int_0^t H_s\ da_s^\varepsilon$$

on the domain $\mathcal{D}$, if (19) is meaningfull and true for all $f \in \mathcal{D}$.

We now have to give at least one criterion for the existence of a solution to equation (19). When considering the domain $\mathcal{E}$ there is a simple characterization.

Theorem 4.2. *Let* $(H_t)_{t\geq 0}$ *be an adapted process of operators defined on* $\mathcal{E}$. *If for every* $u \in L^2(\mathbb{R}^+)$ *and every* $t \in \mathbb{R}^+$ *we have*

$$\begin{cases} \displaystyle\int_0^t |u(s)|^2\ ||H_s\varepsilon(u)||^2\, ds < \infty & \text{if } \varepsilon = \circ \\ \displaystyle\int_0^t ||H_s\varepsilon(u)||^2\, ds < \infty & \text{if } \varepsilon = + \\ \displaystyle\int_0^t |u(s)|\ ||H_s\varepsilon(u)||\ ds < \infty & \text{if } \varepsilon = \\ \displaystyle\int_0^t ||H_s\varepsilon(u)||\ ds < \infty & \text{if } \varepsilon = \times \end{cases}$$

is satisfied. Then the corresponding equation (19) *for*

$$\int_0^t H_s \, da_s^\varepsilon$$

admits a unique solution on $\mathcal{E}$ which satisfies

$$< \varepsilon(u)\,,\, \int_0^t H_s \, da_s^\varepsilon \, \varepsilon(v) > = \begin{cases} \int_0^t \overline{u}(s)v(s) < \varepsilon(u)\,,\, H_s\varepsilon(v) > ds & \text{if } \varepsilon = \circ \\ \int_0^t \overline{u}(s) < \varepsilon(u)\,,\, H_s\varepsilon(v) > ds & \text{if } \varepsilon = + \\ \int_0^t v(s) < \varepsilon(u)\,,\, H_s\varepsilon(v) > ds & \text{if } \varepsilon = - \\ \int_0^t < \varepsilon(u)\,,\, H_s\varepsilon(v) > ds & \text{if } \varepsilon = \times. \end{cases} \tag{20}$$

Furthermore, any operator T_t which satisfies (20) *for some $(H_t)_{t\geq 0}$ is of the form $T_t = \int_0^t H_s \, da_s^\varepsilon$ in the sense of the definition* (19).

Proof. Let $(H_t)_{t\geq 0}$ be an adapted process satisfying the above condition for some ε. We shall prove that (19) admits a unique solution by using a usual Picard method. Let us write it for the case $\varepsilon = \circ$ and leave the three other cases to the reader.

For $u \in L^2(\mathbb{R}^+)$, one can easily check that $D_t\varepsilon(u) = u(t)\varepsilon(u_{t]})$ for almost all t, where $u_{t]}$ means $u\mathbb{1}_{[0,t]}$. This means that, in order to construct the desired quantum stochastic integral on $\mathcal{E}$, we have to solve the equation

$$T_t\varepsilon(u_{t]}) = \int_0^t u(s)T_s\varepsilon(u_{s]}) \; d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]}) \; d\chi_s. \tag{21}$$

Let $x_t = T_t\varepsilon(u_{t]})$, $t \geq 0$. We have to solve

$$x_t = \int_0^t u(s)x_s \; d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]}) \; d\chi_s.$$

Put $x_t^0 = \int_0^t u(s)H_s\varepsilon(u_{s]}) \; d\chi_s$ and

$$x_t^{n+1} = \int_0^t u(s)x_s^n \; d\chi_s + \int_0^t u(s)H_s\varepsilon(u_{s]}) \; d\chi_s.$$

Let $y_t^0 = x_t^0$ and $y_t^{n+1} = x_t^{n+1} - x_t^n = \int_0^t u(s)y_s^n \; d\chi_s$. We have

$$\begin{aligned}
\|y_t^{n+1}\|^2 &= \int_0^t |u(s)|^2 \|y_s^n\|^2 \, ds \\
&= \int_0^t \int_0^{t_2} |u(t_1)|^2 |u(t_2)|^2 \|y_{t_1}^{n-1}\|^2 \, dt_1 \, dt_2 \\
&\qquad \vdots \\
&= \int_{0\le t_1\le\cdots\le t_n\le t} |u(t_1)|^2 \cdots |u(t_n)|^2 \|y_{t_1}^0\|^2 \, dt_1 \cdots dt_n \\
&= \int_{0\le t_1\le\cdots\le t_n\le t} |u(t_1)|^2 \cdots |u(t_n)|^2 \int_0^{t_1} |u(s)|^2 \|H_s \varepsilon(u_{s]})\|^2 \, ds \, dt_1 \cdots dt_n \\
&\le \int_0^t |u(s)|^2 \|H_s \varepsilon(u_{s]})\|^2 \, ds \; \frac{\left(\int_0^t |u(s)|^2 \, ds\right)^n}{n!}.
\end{aligned}$$

From this estimate one easily sees that the sequences

$$x_t^n = \sum_{k=0}^n y_t^k, \quad n \in \mathbb{N}, \quad t \in \mathbb{R}^+$$

are Cauchy sequences in Φ. Let us call $x_t = \lim_{n\to+\infty} x_t^n$. One also easily sees, from the same estimate, that

$$\int_0^t |u(s)|^2 \|x_s\|^2 \, ds < \infty \quad \text{for all} \quad t \in \mathbb{R}^+.$$

Passing to the limit in equality (21), one gets

$$x_t = \int_0^t u(s) x_s \, d\chi_s + \int_0^t u(s) H_s \varepsilon(u_{s]}) \, d\chi_s.$$

Define operators T_t on $\Phi_{t]}$ (more precisely on $\mathcal{E} \cap \Phi_{t]}$) by putting $T_t \varepsilon(u_{t]}) = x_t$. We leave to the reader to check that this defines (by linear extension) an operator on $\mathcal{E} \cap \Phi_{t]}$ (use the fact that any finite family of coherent vectors is free). Extend the operator T_t to $\mathcal{E}$ by adaptedness:

$$T_t \varepsilon(u) = T_t \varepsilon(u_{t]}) \otimes \varepsilon(u_{[t}).$$

We thus get a solution to (19). Uniqueness is easily obtained by Gronwall's lemma.

Let us now prove that this solution satisfies the announced identity. We have

$$\begin{aligned}
\langle \varepsilon(v_{t]}), T_t \varepsilon(u_{t]}) \rangle = \int_0^t \overline{v}(s) u(s) \langle \varepsilon(v_{t]}), T_s \varepsilon(u_{t]}) \rangle \, ds& \\
+ \int_0^t \overline{v}(s) u(s) \langle \varepsilon(v_{s]}), H_s \varepsilon(u_{s]}) \rangle \, ds.&
\end{aligned}$$

Put $\alpha_t = \langle \varepsilon(v_{t]}), T_t \varepsilon(u_{t]})\rangle$, $t \in \mathbb{R}^+$. We have

$$\alpha_t = \int_0^t \overline{v}(s)u(s)\alpha_s \ ds + \int_0^t \overline{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s \varepsilon(u_{s]})\rangle \ ds$$

that is,

$$\frac{d}{dt}\alpha_t = \overline{v}(t)u(t)\alpha_t + \overline{v}(t)u(t)\langle \varepsilon(v_{t]}), H_t \varepsilon(u_{t]})\rangle.$$

Or else

$$\begin{aligned}
\alpha_t &= e^{\int_0^t \overline{v}(s)u(s) \ ds} \int_0^t \overline{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s \varepsilon(u_{s]})\rangle e^{-\int_0^s \overline{v}(k)u(k) \ dk} \ ds \\
&= \int_0^t \overline{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s \varepsilon(u_{s]})\rangle e^{-\int_s^t \overline{v}(k)u(k) \ dk} \ ds \\
&= \int_0^t \overline{v}(s)u(s)\langle \varepsilon(v_{s]}), H_s \varepsilon(u_{s]})\rangle \langle \varepsilon(v_{[s,t]}), \varepsilon(u[s,t])\rangle \ ds \\
&= \int_0^t \overline{v}(s)u(s)\langle \varepsilon(v_{t]}), H_s \varepsilon(u_{t]})\rangle \ ds \quad \text{(by adaptedness).}
\end{aligned}$$

The converse direction is easy to obtain by reversing the above arguments. ■

Let us now see how equation (19) can provide a solution on Φ_f, the space of finite particles. We still only take the example $T_t = \int_0^t H_s \ da_s^\circ$ (the reader may easily check the other three cases). The following computation are only made algebraically, without taking much care about integrability or domain problems. We have the equation

$$T_t f_t = \int_0^t T_s D_s f \ d\chi_s + \int_0^t H_s D_s f \ d\chi_s.$$

Let $f = \mathbb{1}$. This implies (as $D_t \mathbb{1} = 0$ for all t)

$$T_t \mathbb{1} = 0 \ .$$

Let $f = \int_0^\infty f_1(s) \ d\chi_s$ for $f_1 \in L^2(\Sigma_1)$. We have

$$\begin{aligned}
T_t f_t &= \int_0^t T_s f_1(s) \mathbb{1} \ d\chi_s + \int_0^t H_s f_1(s) \mathbb{1} \ d\chi_s \\
&= 0 + \int_0^t f_1(s) H_s \mathbb{1} \ d\chi_s.
\end{aligned}$$

Let $f = \int_{0 \le t_1 \le t_2} f_2(t_1, t_2) \ d\chi_{t_1} \ d\chi_{t_2}$ for $f_2 \in L^2(\Sigma_2)$. We have

$$\begin{aligned}
T_t f_t &= \int_0^t T_s \int_0^s f_2(t_1, s) \ d\chi_{t_1} \ d\chi_s + \int_0^t H_s \int_0^s f_2(t_1, s) \ d\chi_{t_1} \ d\chi_s \\
&= \int_0^t \int_0^s f_2(u, s) H_u \mathbb{1} \ d\chi_u \ d\chi_s + \int_0^t H_s \int_0^s f_2(u, s) \ d\chi_u \ d\chi_s.
\end{aligned}$$

This way, one sees that, by induction on the chaoses, one can derive the action of T_t on Φ_f.

Let us now give the formulas for the formal adjoint of a quantum stochastic integral. We do not discuss here the very difficult problem of the domain of the adjoint of a quantum stochastic integral and the fact that it is a quantum stochastic integral or not. In the case of the domain $\mathcal{E}$ the reader may easily derive conditions for this adjoint to exist on $\mathcal{E}$.

$$\begin{aligned}
\left(\int_0^\infty H_s\, da_s^\circ\right)^* &= \int_0^\infty H_s^*\, da_s^\circ \\
\left(\int_0^\infty H_s\, da_s^+\right)^* &= \int_0^\infty H_s^*\, da^- \\
\left(\int_0^\infty H_s\, da_s^-\right)^* &= \int_0^\infty H_s^*\, da_s^+ \\
\left(\int_0^\infty H_s\, da_s^\times\right)^* &= \int_0^\infty H_s^*\, da_s^\times
\end{aligned}$$

We now have a useful theorem which often helps to extend the domain of a quantum stochastic integral when it is already defined on $\mathcal{E}$.

Theorem 4.3 (Extension theorem). *If $(T_t)_{t\geq 0}$ is an adapted process of operators on Φ which admits an integral representation on $\mathcal{E}$ and such that the adjoint process $(T_t^*)_{t\geq 0}$ admits an integral representation on $\mathcal{E}$. Then the integral representations of $(T_t)_{t\geq 0}$ and $(T_t^*)_{t\geq 0}$ can be extended everywhere equation* (19) *is meaningful.*

Before proving this theorem, we shall maybe be clear about what it exactly means. The hypotheses are that:

- $T_f = \int_0^t H_s^\circ\ da_s^\circ + \int_0^t H_s^+\ da_s^+ + \int_0^t H_s^-\ da_s^- + \int_0^t H_s^\times\ da_s^\times$ on $\mathcal{E}$. This in particular means that

$$\int_0^t |u(s)|^2 \|H_s^\circ \varepsilon(u_{s]})\|^2 + \|H_s^+ \varepsilon(u_{s]})\|^2 + |u(s)|\ \|H_s^- \varepsilon(u_{s]})\| + \|H_s^\times \varepsilon(u_{s]})\|\ ds$$

is finite for all $t \in I\!R^+$, all $u \in L^2(I\!R^+)$.

- The assumption on the adjoint simply means that

$$\int_0^t |u(s)|^2 \|H_s^{0*} \varepsilon(u_{s]})\|^2 + \|H_s^{-*} \varepsilon(u_{s]})\|^2 + |u(s)|\ \|H_s^{+*} \varepsilon(u_{s]})\| \\ + \|H_s^{\times *} \varepsilon(u_{s]})\|\ ds < \infty$$

for all $t \in I\!R^+$ and all u in $L^2(I\!R^+)$.

The conclusion is that for all $f \in \Phi$, such that equation (19) is meaningful (for $(T_t)_{t\geq 0}$ or for $(T_t^*)_{t\geq 0}$), then the equality (19) will be valid.

Let us take an example. Let $J_t\varepsilon(u) = \varepsilon(-u_{t]}) \otimes \varepsilon(u_{[t})$. It is an adapted process of operators on Φ which is made of unitary operators, and $J_t^2 = I$. We leave as an exercise to check the following points.

- The quantum stochastic integral $B_t = \int_0^t J_s \; da_s^-$ is well defined on $\mathcal{E}$, the quantum stochastic integral $B_t^* = \int_0^t J_s da_s^+$ is well defined on $\mathcal{E}$ and is the adjoint of B_t (on $\mathcal{E}$);
- We have $J_t = I - 2\int_0^t J_s \; da_s^\circ$;
- If $X_t = -2\int_0^t X_s \; da_s^\circ$ then $X_t \equiv 0$ for all t.
- Altogether this gives

$$B_t J_t + J_t B_t = 0 \; ;$$

- We conclude that $B_t B_t^* + B_t^* B_t = tI$.

The last identity shows that B_t is a bounded operator with norm smaller that $\sqrt{t}$.

Now, we know that, for all $f \in \mathcal{E}$ we have

$$B_t f_t = \int_0^t B_s D_s f \; d\chi_s + \int_0^t J_s D_s f \; ds. \tag{22}$$

We know that the adjoint of B_t can be represented as a Quantum stochastic integral on $\mathcal{E}$. Hence the hypotheses of the Extension Theorem hold.

For which $f \in \Phi$ do we have all the terms of equation (19) being well defined? The results above easily show that for *all* $f \in \Phi$ the quantities $B_t f_t$, $\int_0^t B_s D_s f \; d\chi_s$, $\int_0^t J_s D_s f \; ds$ are well defined. Hence the extension theorem says that equation (19) is valid for all $f \in \Phi$. The same holds for B_t^*. The integral representation of $(B_t)_{t\geq 0}$ (and $(B_t^*)_{t\geq 0}$) is valid on *all* Φ.

Let us now prove the extension theorem.

Proof. Let $f \in \Phi$ be such that all the terms of equation (19) are meaningful. Let $(f_n)_n$ be a sequence in $\mathcal{E}$ which converges to f. Let $g \in \mathcal{E}$. We have

$$\begin{aligned}
&\Big|\langle g, T_t f_t - \int_0^t T_s D_s f \; d\chi_s - \int_0^t H_s^{\circ} D_s f \; d\chi_s \\
&\qquad\qquad - \int_0^t H_s^{+} P_s f \; d\chi_s - \int_0^t H_s^{-} D_s f \; ds - \int_0^t H_s^{\times} P_s f \; ds\rangle\Big| \\
&\quad \leq \quad |\langle g, T_t P_t (f - f_n)\rangle| + \Big|\langle g, \int_0^t T_s D_s (f - f_n) \; d\chi_s\rangle\Big| \\
&\qquad + \Big|\langle g, \int_0^t H_s^{\circ} D_s (f - f_n) \; d\chi_s\rangle\Big| + \Big|\langle g, \int_0^t H_s^{\times} P_s (f - f_n) \; ds\rangle\Big| \\
&\qquad + |\langle g, \int_0^t H_s^{-} D_s (f - f_n) \; ds\rangle| + |\langle g, \int_0^t H_s^{\times} P_s (f - f_n) \; ds\rangle| \\
&\quad \leq \quad \|T_t^* g\| \; \|f - f_n\| + \int_0^t |\langle T_s^* D_s g, D_s (f - f_n)\rangle| \; ds \\
&\qquad + \int_0^t |\langle H_s^{\circ *} D_s g, D_s (f - f_n)\rangle| \; ds + \int_0^t |\langle H_s^{+*} D_s g, P_s (f - f_n)\rangle| \; ds \\
&\qquad + \int_0^t |\langle H_s^{-*} g, D_s (f - f_n)\rangle| \; ds + \int_0^t |\langle H_s^{\times *} g, P_s (f - f_n)\rangle| \; ds \\
&\quad \leq \quad \Big[\|T_t^* g\| + \int_0^t \|T_s^* D_s g\|^2 \; ds + \int_0^t \|H_s^{\circ *} D_s g\|^2 \; ds + \int_0^t \|H_s^{+*} D_s g\| \; ds \\
&\qquad + \int_0^t \|H_s^{-*} g\|^2 \; ds + \int_0^t \|H_s^{\times *} g\| \; ds\Big] \|f - f_n\| \; .
\end{aligned}$$

The theorem is proved. ∎

Quantum stochastic integrals satisfy a quantum Itô formula, that is, they are stable under composition and the integral representation of the composition is given by a Itô-like integration by part formula.

The complete quantum Itô formula with correct domain assumptions is a rather heavy theorem. We shall give a complete statement of it later on, but for the moment we state under a form which is sufficient for many applications.

Let

$$T_t = \int_0^t H_s \, da_s^{\varepsilon}$$

and

$$S_t = \int_0^t K_s \, da_s^{\nu}$$

be two quantum stochastic integral processes. Then, on any domain where each term is well defined we have

$$T_t S_t = \int_0^t T_s \, dS_s + \int_0^t dT_s \, S_s + \int_0^t dT_s \, dS_s$$

in the sense

$$T_tS_t = \int_0^t T_sK_s\, da_s^\nu + \int_0^t H_sS_s\, da_s^\varepsilon + \int_0^t H_sK_s\, da_s^\varepsilon da_s^\nu$$

where the quadratic terms $da_s^\varepsilon da_s^\nu$ are given by the following Itô table:

	da^+	da^-	da°	$da^\times$
da^+	0	0	0	0
da^-	$da^\times$	0	da^-	0
da°	da^+	0	da°	0
$da^\times$	0	0	0	0

This quantum Itô formula will be proved in section 5.1 in the case of quantum stochastic integrals having the whole of Φ as a domain.

Maximal solution

This section is not necessary for the understanding of the rest of the course, it is addressed to readers motivated by fine domain problems on quantum stochastic integrals.

We have not yet discussed here the existence of solution to equation (19) in full generality. That is, for a given adapted process of operators $(H_t)_{t\geq 0}$ and a given $\varepsilon \in \{+, -, \circ, \times\}$ we consider the associated equation (19). We then wonder

i) if there always exists a solution $(T_t)_{t\geq 0}$;

ii) if the solution is always unique;

iii) on which maximal domain that solution is defined.

The complete answer to these three questions has been given in [9]. It is a long and difficult result for which we need to completely revisit the whole theory of quantum stochastic calculus and the notion of adaptedness. Here we shall just give the main result.

For $\sigma = \{t_1 < \ldots < t_n\} \in \mathcal{P}$ we put

$$D_\sigma = D_{t_1} \ldots D_{t_n}$$

with $D_\emptyset = I$.

Consider the following operators on Φ

$$\left[\Lambda_t^\circ(H.)f\right](\sigma) = \sum_{s\in\sigma_{t]}} \left[H_s D_s D_{\sigma_{(s}} f\right](\sigma_{s)})$$
$$\left[\Lambda_t^+(H.)f\right](\sigma) = \sum_{s\in\sigma_{t]}} \left[H_s P_s D_{\sigma_{(s}} f\right](\sigma_{s)})$$
$$\left[\Lambda_t^-(H.)f\right](\sigma) = \int_0^t \left[H_s D_s D_{\sigma_{(s}} f\right](\sigma_{s)})\, ds$$
$$\left[\Lambda_t^\times(H.)f\right](\sigma) = \int_0^t \left[H_s P_s D_{\sigma_{(s}} f\right](\sigma_{s)})\, ds$$

together with their maximal domain $\operatorname{Dom}\Lambda_t^\varepsilon(H.)$, that is, the space of $f \in \Phi$ such that the above expression is well-defined and square integrable as a function of σ.

We then have the following complete characterization (see [2]).

Theorem 4.4. *For every adapted process $(H_t)_{t\geq 0}$ of operators on Φ and every $\varepsilon \in \{\circ, +, -, \times\}$, the following assertions are equivalent.*

i) $(T_t)_{t\geq 0}$ is a solution of the equation (19).

ii) $(T_t)_{t\geq 0}$ is the restriction of $\Lambda_t^\varepsilon(H.)$ to a stable subspace of $\operatorname{Dom}\Lambda_t^\varepsilon(H.)$.

This result means that with the above formulas and above domains we have

i) the explicit action of any quantum stochastic integral on any vector of its domain

ii) the maximal domain of that operator

iii) the right to use equation (19) on that domain without restriction (every term is well-defined).

4.3 Back to probabilistic interpretations

Multiplication operators

Consider a probabilistic interpretation $(\Omega, \mathcal{F}, P, (x_t)_{t\geq 0})$ of the Fock space, which is described by a structure equation

$$d[x,x]_t = dt + \psi_t\; dx_t.$$

The operator M_{x_t} on Φ of multiplication by x_t (for this interpretation) is a particular operator on Φ. It is adapted at time t. The process $(M_{x_t})_{t\geq 0}$ is an adapted process of operators on Φ. Can we represent this process as a sum of quantum stochastic integrals?

If one denotes by M_{ψ_t} the operator of multiplication by ψ_t (for the $(x_t)_{t\geq 0}$-product again) we have the following:

Theorem 4.5.

$$M_{x_t} = a_t^+ + a_t^- + \int_0^t M_{\psi_t}\ da_t^\circ.$$

Proof. Let us be clear about domains: the domain of M_{x_t} is exactly the space of $f \in \Phi$ such that $x_t \cdot U_x f$ belongs to $L^2(\Omega)$ (recall that U_x is the isomorphism $U_x : \Phi \to L^2(\Omega)$).

Let us now go to the proof of the theorem. We have

$$x_t f = \int_0^\infty x_{s\wedge t} D_s f\ dx_s + \int_0^t P_s f\ dx_s + \int_0^t D_s f\ ds + \int_0^t \psi_s D_s f\ dx_s$$

by the usual Itô formula. That is, on Φ

$$M_{x_t} f = \int_0^\infty M_{x_{s\wedge t}} D_s f\ d\chi_s + \int_0^t P_s f\ d\chi_s + \int_0^t D_s f\ ds + \int_0^t M_{\psi_s} D_s f\ d\chi_s$$

which is exactly equation (19) for the quantum stochastic process $X_t = a_t^+ + a_t^- + \int_0^t M_{\psi_t}\ da_t^\circ$. ■

In particular we have obtained the following very important results.

- The multiplication operator by the Brownian motion is $a_t^+ + a_t^-$.
- The multiplication operator by compensated Poisson process is $a_t^+ + a_t^- + a_t^\circ$.
- The multiplication operator by the β-Azéma martingale is the unique solution of

$$X_t = a_t^+ + a_t^- + \int_0^t \beta X_s\ da_s^\circ.$$

Once again, as in the discrete time setup, we have obtained in a single structure, the Fock space Φ, a very simple way to represent many different classical noises that have nothing to do together. Furthermore their representation is obtained by very simple combinations of the three quantum noises. The three quantum noises appear as very natural (their form, together with the process $a_t^\times$, a kind of basis for local operator processes on the continuous tensor product structure of Φ), and they constitute basic bricks from which one can recover the main classical noises.

5 The algebra of regular quantum semimartingales

In this section we present several developments of the definitions of quantum stochastic integrals. These developments make great use of the versatility of our definitions, in particular the fact that quantum stochasitc integrals can a priori be defined on any kind of domain.

5.1 Everywhere defined quantum stochastic integrals

A true quantum Itô formula

With our definition of quantum stochastic integrals defined on any stable domain, we may meet quantum stochastic integrals that are defined on the whole of Φ. Let us recall a few facts. An adapted process of bounded operators $(T_t)_{t\geq 0}$ on Φ is said to have the integral representation

$$T_t = \sum_{\varepsilon=\{0,+,-,\times\}} \int_0^t H_s^\varepsilon \; da_s^\varepsilon$$

on the *whole of* Φ if, for all $f \in \Phi$ one has

$$\int_0^t \|T_s D_s f\|^2 + \|H_s^\circ D_s f\|^2 + \|H_s^+ P_s f\|^2 + \|H_s^- D_s f\| + \|H_s^\times P_s f\| \; ds < \infty$$

for all $t \in \mathbb{R}^+$ (the H_t^ε are bounded operators) and

$$T_t P_t f = \int_0^t T_s D_s f \; d\chi_s + \int_0^t H_s^\circ D_s f \; d\chi_s + \int_0^t H_s^+ P_s f \; d\chi_s + \int_0^t H_s^- D_s f \; ds + \int_0^t H_s^\times P_s f \; ds.$$

If we have two such processes $(S_t)_{t\geq 0}$ and $(T_t)_{t\geq 0}$ one can compose them. As announced previously with the quantum Itô formula, the resulting process $(S_t T_t)_{t\geq 0}$ is also representable as a sum of quantum stochastic integrals on the whole of Φ.

Theorem 5.1. *If $T_t = \sum_\varepsilon \int_0^t H_s^\varepsilon \; da_s^\varepsilon$ and $S_t = \sum_\varepsilon \int_0^t K_s^\varepsilon \; da_s^\varepsilon$ are everywhere defined quantum stochastic integrals, then $(S_t T_t)_{t\geq 0}$ is everywhere representable as a sum of quantum stochastic integrals:*

$$\begin{aligned} S_t T_t = &\int_0^t (S_s H_s^\circ + K_s^\circ T_s + K_s^\circ H_s^\circ) \; da_s^\circ + \int_0^t (S_s H_s^+ + K_s^+ T_s + K_s^\circ H_s^+) \; da_s^+ \\ &+ \int_0^t (S_s H_s^- + K_s^- T_s + K_s^- H_s^\circ) \; da_s^- + \int_0^t (S_s H_s^\times + K_s^\times T_s + K_s^- H_s^+) \; da_s^\times . \end{aligned}$$

Before proving this theorem we will need the following preliminary result.

Lemma 5.2. *Let $g_t = \int_0^t v_s \, ds$ be an adapted process of vectors of Φ, with $\int_0^t \|v_s\| \, ds < \infty$ for all t. Let $(S_t)_{t\geq 0}$ be as in Theorem 5.1. Then*

$$S_t g_t = \int_0^t S_s v_s \, ds + \int_0^t K_s^+ g_s \, d\chi_s + \int_0^t K_s^\times g_s \, ds.$$

Proof. As S_t is bounded we have (details are left to the reader)

$$\begin{aligned}
S_t g_t &= S_t \int_0^t v_s \ ds = \int_0^t S_t v_s \ ds \\
&= \int_0^t S_t (P_0 v_s + \int_0^s D_u v_s \ d\chi_u) \ ds \\
&= \int_0^t S_t P_0 v_s \ ds + \int_0^t \Big[\int_0^s S_u D_u v_s \ d\chi_u + \int_0^s K_u^\circ D_u v_s \ d\chi_u \\
&\qquad + \int_0^s K_u^- D_u v_s \ ds + \int_0^t K_u^+ P_u \int_0^s D_v v_s \ d\chi_v \ d\chi_u \\
&\qquad + \int_0^t K_u^\times P_u \int_0^s D_v v_s \ d\chi_v \ du \Big] \ ds \\
&= \int_0^t S_t P_0 v_s \ ds + \int_0^t \Big[S_s \int_0^s D_u v_s \ d\chi_u + \int_s^t K_u^+ \int_0^s D_v v_s d\chi_v \ d\chi_u \\
&\qquad + \int_s^t K_u^\times \int_0^s D_v v_s \ d\chi_v \ du \Big] \ ds \\
&= \int_0^t S_s v_s \ ds + \int_0^t \int_s^t K_u^+ v_s \ d\chi_u \ ds + \int_0^t \int_s^t K_u^\times v_s \ du \ ds \\
&= \int_0^t S_s v_s \ ds + \int_0^t \int_0^u K_u^+ v_s \ ds \ d\chi_u + \int_0^t \int_0^u K_u^\times v_s \ ds \ du \\
&= \int_0^t S_s v_s \ ds + \int_0^t K_u^+ \int_0^u v_s \ ds \ d\chi_u + \int_0^t K_u^\times \int_0^u v_s \ ds \ du \\
&= \int_0^t S_s v_s \ ds + \int_0^t K_u^+ g_u \ d\chi_u + \int_0^t K_u^\times g_u \ du.
\end{aligned}$$

This proves the Lemma. ■

We now prove the theorem.

Proof. We just compute the composition, using Lemma 5.2

$$\begin{aligned}
T_t f_t = \int_0^t T_s D_s f \ d\chi_s + \int_0^t H_s^\circ D_s f \ d\chi_s + \int_0^t H_s^+ P_s f \ d\chi_s \\
+ \int_0^t H_s^- D_s f \ ds + \int_0^t H_s^\times P_s f \ ds.
\end{aligned}$$

Hence

$$\begin{aligned}
S_tT_tf_t &= \int_0^t S_s\big[T_sD_sf + H^\circ_sD_sf + H^+_sP_sf\big]\ d\chi_s \\
&\quad + \int_0^t K^\circ_s\big[T_sD_sf + H^\circ_sD_sf + H^+_sP_sf\big]\ d\chi_s + \int_0^t K^-_s\big[T_sD_sf + H^\circ_sD_sf + H^+_sP_sf\big]\ ds \\
&\quad + \int_0^t K^+_s\Big[\int_0^s T_uD_uf\ d\chi_u + \int_0^s H^\circ_uD_uf\ d\chi_u + \int_0^s H^+_uP_uf\ d\chi_u\Big]\ d\chi_s \\
&\quad + \int_0^t K^\times_s\Big[\int_0^s T_uD_uf\ d\chi_u + \int_0^s H^\circ_uD_uf\ d\chi_u + \int_0^s H^+_uP_uf\ d\chi_u\Big]\ du \\
&\quad + \int_0^t S_s\big[H^-_sD_sf + H^\times_sP_sf\big]\ ds + \int_0^t K^+_s\Big[\int_0^s H^-_uD_uf + \int_0^s H^\times_uP_uf\ du\Big]\ d\chi_s \\
&\quad + \int_0^t K^\times_s\Big[\int_0^s H^-_uD_uf + \int_0^s H^\times_uP_uf\ du\Big]\ ds \\
&= \int_0^t S_sT_sD_sf\ d\chi_s + \int_0^t \big[S_sH^\circ_s + K^\circ_sT_s + K^\circ_sH^\circ_s\big]D_sf\ d\chi_s \\
&\quad + \int_0^t\big[S_sH^+_s + K^+_sT_s + K^\circ_sH^+_s\big]P_sf\ d\chi_s + \int_0^t\big[S_sH^-_s + K^-_sT_s + K^-_sH^\circ_s\big]D_sf\ ds \\
&\quad + \int_0^t \big[S_sH^\times_s + K^\times_sT_s + K^-_sH^+_s\big]P_sf\ ds\ .
\end{aligned}$$

This proves the theorem. ∎

A family of examples

We have seen $B_t = \int_0^t J_s\ da^-_s$ as an example of everywhere defined quantum stochastic integrals. This example belongs to a larger family of examples which we shall present here.

Let $\mathcal{S}$ be the set of *bounded* adapted processes of operators $(T_t)_{t\geq 0}$ on Φ such that

$$T_t = \sum_\varepsilon \int_0^t H^\varepsilon_s\ da^\varepsilon_s \quad \text{on} \quad \mathcal{E},$$

all the operators H^ε_s being bounded and

$$\begin{cases} t \mapsto \|H^\circ_t\| \in L^\infty_{\text{loc}}(\mathbb{R}^+) \\ t \mapsto \|H^+_t\| \in L^2_{\text{loc}}(\mathbb{R}^+) \\ t \mapsto \|H^-_t\| \in L^2_{\text{loc}}(\mathbb{R}^+) \\ t \mapsto \|H^\times_t\| \in L^1_{\text{loc}}(\mathbb{R}^+)\ . \end{cases}$$

With these conditions, we claim that $t \mapsto \|T_t\|$ has to be in $L^\infty_{\text{loc}}(\mathbb{R}^+)$. Indeed, the operator $\int_0^t H^\times_s\ da^\times_s$ satisfies

$$\int_0^t H^\times_s\ da^\times_s f = \int_0^t H^\times_s f\ ds$$

hence it is a bounded operator, with norm dominated by $\int_0^t \|H_s^\times\|\, ds$, which is a locally bounded function of t. The difference $M_t = T_t - \int_0^t H_s^\times\, da_s^\times$ is thus a martingale of bounded operators, that is $P_S M_t P_s = M_s P_s$ for all $s \le t$. But as M is a martingale we have $\|M_s f_s\| = \|P_s M_t f_s\| \le \|M_t f_s\|$ for $s \le t$. Hence $t \mapsto \|M_t\|$ is locally bounded. Thus, so is $t \mapsto \|T_t\|$.

With all these informations, it is easy to check that the integral representation of $(T_t)_{t\ge 0}$, as well as the one of $(T_t^*)_{t\ge 0}$, can be extended on the whole of Φ by the extension Theorem (Theorem 4.3).

5.2 The algebra of regular quantum semimartingales

It is an algebra

As all elements of $\mathcal{S}$ are everywhere defined quantum stochastic integrals, one can compose them and use the quantum Itô formula (Theorem 5.1.

Theorem 5.3. *$\mathcal{S}$ is a $*$-algebra for the adjoint and composition operations.*

Proof. Let

$$T_t = \int_0^t H_s^\circ\, da_s^\circ + \int_0^t H_s^+\, da_s^+ + \int_0^t H_s^-\, da_s^- + \int_0^t H_s^\times\, da_s^\times$$

and

$$S_t = \int_0^t K_s^\circ\, da_s^\circ + \int_0^t K_s^+\, da_s^+ + \int_0^t K_s^-\, da_s^- + \int_0^t K_s^\times\, da_s^\times$$

be two elements of $\mathcal{S}$. The adjoint process $(T_t^*)_t \ge 0$ is given by

$$T_t^* = \int_0^t H_s^{\circ *}\, da_s^\circ + \int_0^t H_s^{-*}\, da_s^+ + \int_0^t H_s^{+*}\, da_s^- + \int_0^t H_s^{\times *}\, da_s^\times \ .$$

It is straightforward to check that it belongs to $\mathcal{S}$. The Itô formula for the composition of two elements of $\mathcal{S}$ gives

$$\begin{aligned} S_t T_t = &\int_0^t \left[S_s H_s^\circ + K_s^\circ T_s + K_s^\circ H_s^\circ\right]\, da_s^\circ \\ &+ \int_0^t \left[S_s H_s^+ + K_s^+ T_s + K_s^\circ H_s^+\right]\, da_s^+ \\ &+ \int_0^t \left[S_s H_s^- + K_s^- T_s + K_s^- H_s^\circ\right]\, da_s^- \\ &+ \int_0^t \left[S_s H_s^\times + K_s^\times T_s + K_s^- H_s^+\right]\, da_s^\times . \end{aligned}$$

From the conditions on the maps $t \mapsto \|S_t\|$, $t \mapsto \|T_t\|$, $t \mapsto \|K_t^\varepsilon\|$ and $t \mapsto \|H_t^\varepsilon\|$, it is easy to check that the coefficients in the representation of $(S_t T_t)_t \ge$

0 are bounded operators that satisfy the norm conditions for being in $\mathcal{S}$. For example, the coefficient of $da_t^\times$ satisfies

$$\begin{aligned}\int_0^t &\|S_s H_s^\times + K_s^\times T_s + K_s^- H_s^+\| \, ds \\ &\leq \sup_{s\leq t} \|S_s\| \int_0^t \|H_s^\times\| \, ds + \sup_{s\leq t} \|T_s\| \int_0^t \|K_s^\times\| \, ds \\ &\quad + \Big(\int_0^t \|K_s^-\|^2 \, ds\Big)^{1/2} \Big(\int_0^t \|H_s^+\|^2 \, ds\Big)^{1/2}\end{aligned}$$

hence it is locally integrable. ∎

Thus $\mathcal{S}$ is a nice space of quantum semimartingales that one can compose without bothering about any domain problem, one can pass to the adjoint, one can use formula (19) on the whole of Φ.

A characterization

A problem comes from the definition of $\mathcal{S}$. Indeed, it is in general difficult to know if a process of operators is representable as quantum stochastic integrals; it is even more difficult to know the regularity of its coefficients. We know that $\mathcal{S}$ is not empty, as it contains $B_t = \int_0^t J_s \, da_s^-$ that we have met above. It is natural to wonder how large that space is. It is natural to seek for a characterization of $\mathcal{S}$ that depends only on the process $(T_t)_t \geq 0$.

One says that a process $(T_t)_t \geq 0$ of *bounded* adapted operators is a *regular quantum semimartingale* is there exists a locally integrable function h on $\mathbb{R}$ such that for all $r \leq s \leq t$, all $f \in \mathcal{E}$ one has (where $f_r = P_r f$)

i) $\|T_t f_r - T_s f_r\|^2 \leq \|f_r\|^2 \int_s^t h(u) \, du$;

ii) $\|T_t^* f_r - T_s^* f_r\|^2 \leq \|f_r\|^2 \int_s^t h(u) \, du$;

iii) $\|P_s T_t f_r - T_s f_r\| \leq \|f_r\| \int_s^t h(u) \, du$.

Theorem 5.4. *A process $(T_t)_t \geq 0$ of bounded adapted operators is a regular quantum semimartingale if and only if it belongs to $\mathcal{S}$.*

Proof. Showing that elements of $\mathcal{S}$ satisfy the three estimates that define regular quantum semimartingales is straightforward. We leave it as an exercise.

The interesting part is to show that a regular quantum semimartingale is representable as quantum stochastic integrals and belongs to $\mathcal{S}$. We will only sketch that proof, as the details are rather long and difficult to develop.

Let $x_t = T_t f_r$ for $t \geq r$ (r is fixed, t varies). It is an adapted process of vectors on Φ. It satisfies

$$\|P_s x_t - x_s\| \le \|f_r\| \int_s^t h(u)\ du.$$

This condition is a Hilbert space analogue of a condition in classical probability that defines particular semimartingales: the quasimartingales. O. Enchev [18] has provided a Hilbert space extension of this result and we can deduce from his result that $(x_t)_{t\ge r}$ can be written

$$x_t = m_t + \int_0^t k_s\ ds$$

where m is a martingale in Φ ($P_s m_t = m_s$) and h is an adapted process in Φ such that $\int_0^t \|k_s\|\ ds < \infty$.

Thus $P_s x_t - x_s = \int_s^t P_s k_u\ du$ and we have

$$\Big\| \int_0^t P_s k_u\ du \Big\| \le \|f_r\| \int_0^t h(u)\ du, \quad \text{for all} \quad r \le s \le t.$$

Actually k_u depends linearly on f_r. The inequality above implies (difficult exercise) that

$$\|k_u(f_r)\| \le \|f_r\| h(u)\ .$$

Hence k_u is a bounded operator on $\Phi_{u]}$, we extend it as a bounded adapted operator $H_u^\times$.

Let $M_t = T_t - \int_0^t H_u^\times\ da_u^\times$, $t \in I\!R^+$. It is easy to check, from what we have already done, that $(M_t)_t \ge 0$ is a martingale of bounded operator (Hint: compute $P_s M_t f_r - M_s f_r$). It is easy to check that $(M_t)_t \ge 0$ also satisfies the conditions *i)* and *ii)* of the definition of regular quantum semimartingales, with another function h, say h'.

Now, let $(y_t)_{t\ge r}$ be $(M_t f_r)_{t\ge r}$. It is a martingale of vectors in Φ. Thus it can be represented as

$$y_t - y_s = \int_s^t \xi_u\ d\chi_u.$$

The vector ξ_u depends linearly on f_r and we have

$$\int_0^t \|\xi_u(f_r)\|^2\ du \le \|f_r\|^2 \int_s^t h'(u)\ du \quad (\text{by } i)).$$

Thus ξ_u extends to a unique adapted bounded operator H_u^+ on Φ. Doing the same with $(M_t^* f_r)_{t\ge r}$ gives an adapted process of operators (bounded): $(H_u^-)_{u\ge 0}$.

Let $f \in \Phi$, let $f_t = P_t f$ and define

$$X_t f_t = T_t f_t - \int_0^t T_s D_s f\ d\chi_s - \int_0^t H_s^+ P_s f\ d\chi_s - \int_0^t H_s^- D_s f\ ds - \int_0^t H_s^\times P_s f\ ds.$$

One easily checks that each X_t commutes with all the P_u's, $u \in \mathbb{R}^+$. Let us consider a bounded operator H on Φ such that $P_u H = H P_u$ for all $u \in \mathbb{R}^+$. Notice that for almost all t, all $a \le t \le b$, all f one has

$$D_t H(P_b f - P_a f) = D_t P_b H f - D_t P_a H f = D_t H f$$

for $D_t P_s = \begin{cases} D_t & \text{if } t \le s \\ 0 & \text{if } t > s. \end{cases}$

Define $\widetilde{H}_t^\circ$ by

$$\widetilde{H}_t^\circ f_t = D_t \int_a^b P_u f \, d\chi_u - H f_t \quad \text{for any} \quad a \le t \le b \; .$$

By computing $\int_a^b \|\widetilde{H}_t^\circ f_t\|^2 \, dt$ one easily checks that $\widetilde{H}_t^\circ$ is bounded with locally bounded norm. Moreover we have

$$Hg = \int_0^\infty H D_s f \, d\chi_s + \int_0^\infty \widetilde{H}_s^\circ D_s f \, d\chi_s \; .$$

That is exactly $H = \int_0^\infty \widetilde{H}_s^\circ \, da_s^\circ$.

Actually we have (almost) proved the following nice characterization.

Theorem 5.5. *Let T be a bounded operator on Φ. The following are equivalent.*

i) $TP_t = P_t T$ for all $t \in \mathbb{R}^+$;

ii) $T = \lambda I + \int_0^\infty H_s \, da_s^\circ$ on the whole of Φ.

Applying this to X_t, we finally get, putting $H_s^\circ = \widetilde{H}_s^\circ + X_s$

$$\begin{aligned} T_t f_t = \int_0^t T_s D_s f \, d\chi_s + \int_0^t H_s^\circ D_s f \, d\chi_s + \int_0^t H_s^+ P_s f \, d\chi_s \\ + \int_0^t H_s^- D_s f \, ds + \int_0^t H_s^\times P_s f \, ds. \end{aligned}$$

This is equation (19) for the announced integral representation. ■

6 Approximation by the toy Fock space

In this section, we are back to the spin chain setting. As announced in the first section of this course, we will show that the toy Fock space $T\Phi$ can be embedded into the Fock space Φ in such a way that it constitutes an approximation of it and of its basic operators.

6.1 Embedding the toy Fock space into the Fock space

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ be a partition of $\mathbb{R}^+$ and $\delta(\mathcal{S}) = \sup_i |t_{i+1} - t_i|$ be the diameter of $\mathcal{S}$. For $\mathcal{S}$ being fixed, define $\Phi_i = \Phi_{[t_i, t_{i+1}]}$, $i \in \mathbb{N}$. We then have $\Phi \simeq \bigotimes_{i\in\mathbb{N}} \Phi_i$. For all $i \in \mathbb{N}$, define

$$\begin{aligned} X_i &= \frac{\chi_{t_{i+1}} - \chi_{t_i}}{\sqrt{t_{i+1} - t_i}} \in \Phi_i \ , \\ a_i^- &= \frac{a_{t_{i+1}}^- - a_{t_i}^-}{\sqrt{t_{i+1} - t_i}} \circ P_{1]} \ , \\ a_i^\circ &= a_{t_{i+1}}^\circ - a_{t_i}^\circ \ , \\ a_i^+ &= P_{1]} \circ \frac{a_{t_{i+1}}^+ - a_{t_i}^+}{\sqrt{t_{i+1} - t_i}} \ , \end{aligned}$$

where $P_{1]}$ is the orthogonal projection onto $L^2(\mathcal{P}_1)$ and where the above definition of a_i^+ is understood to be valid on Φ_i only, with a_i^+ being the identity operator I on the other Φ_j's (the same is automatically true for a_i^-, a_i°).

Proposition 6.1. *With the above notations we have*

$$\begin{cases} a_i^- X_i = \mathbb{1} \\ a_i^- \mathbb{1} = 0 \end{cases}$$
$$\begin{cases} a_i^\circ X_i = X_i \\ a_i^\circ \mathbb{1} = 0 \end{cases}$$
$$\begin{cases} a_i^+ X_i = 0 \\ a_i^+ \mathbb{1} = X_i \end{cases} .$$

Proof. As $a_t^- \mathbb{1} = a_t^\circ \mathbb{1} = 0$ it is clear that $a_i^- \mathbb{1} = a_i^\circ \mathbb{1} = 0$. Furthermore, $a_t^+ \mathbb{1} = \chi_t$ thus

$$a_i^+ \mathbb{1} = P_{1]} \frac{\chi_{t_{i+1}} - \chi_{t_i}}{\sqrt{t_{i+1} - t_i}} = X_i \ .$$

Furthermore, by (19) we have

$$\begin{aligned}
a_i^- X_i &= \frac{1}{t_{i+1}-t_i}\left(a^-_{t_{i+1}} - a^-_{t_i}\right)\int_{t_i}^{t_{i+1}} \mathbb{1}\, d\chi_t \\
&= \frac{1}{t_{i+1}-t_i}\left[\int_{t_i}^{t_{i+1}} \left(a^-_t - a^-_{t_i}\right)\mathbb{1}\, d\chi_t + \int_{t_i}^{t_{i+1}} \mathbb{1}\, dt\right] \\
&= \frac{1}{t_{i+1}-t_i}\left(0 + t_{i+1} - t_i\right) = \mathbb{1}\ ; \\
a_i^\circ X_i &= \frac{1}{t_{i+1}-t_i}\left(a^\circ_{t_{i+1}} - a^\circ_{t_i}\right)\int_{t_i}^{t_{i+1}} \mathbb{1}\, d\chi_t \\
&= \frac{1}{t_{i+1}-t_i}\left[\int_{t_i}^{t_{i+1}} \left(a^\circ_t - a^\circ_{t_i}\right)\mathbb{1}\, d\chi_t + \int_{t_i}^{t_{i+1}} \mathbb{1}\, d\chi_t\right] \\
&= \frac{1}{t_{i+1}-t_i}\left(\chi_{t_{i+1}} - \chi_{t_i}\right) = X_i\ ; \\
a_i^+ X_i &= \frac{1}{t_{i+1}-t_i} P_{1]}\left(a^+_{t_{i+1}} - a^+_{t_i}\right)\int_{t_i}^{t_{i+1}} \mathbb{1}\, d\chi_t \\
&= \frac{1}{t_{i+1}-t_i} P_{1]}\left[\int_{t_i}^{t_{i+1}} \left(a^+_t - a^+_{t_i}\right)\mathbb{1}\, d\chi_t + \int_{t_i}^{t_{i+1}}\int_{t_i}^{t} \mathbb{1}\, d\chi_s\, d\chi_t\right] \\
&= \frac{2}{t_{i+1}-t_i} P_{1]}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t} \mathbb{1}\, d\chi_s\, d\chi_t \\
&= 0\ .
\end{aligned}$$

These are the announced relations. ∎

Thus the action of the operators a_i^ε on the X_i and on $\mathbb{1}$ is similar to the action of the corresponding operators on the toy Fock spaces. We are now going to construct the toy Fock space inside Φ. We are still given a fixed partition $\mathcal{S}$. Define $T\Phi(\mathcal{S})$ to be the space of vectors $f \in \Phi$ which are of the form

$$f = \sum_{A \in \mathcal{P}_{\mathbb{N}}} f(A) X_A$$

(with $\|f\|^2 = \sum_{A\in\mathcal{P}_{\mathbb{N}}} |f(A)|^2 < \infty$). The space $T\Phi(\mathcal{S})$ can be clearly identified to the toy Fock space $\mathbb{T}\Phi$; the operators a_i^ε, $\varepsilon \in \{+,-,0\}$, act on $T\Phi(\mathcal{S})$ exactly in the same way as the corresponding operators on $\mathbb{T}\Phi$. We have completely embedded the toy Fock space into the Fock space.

6.2 Projections on the toy Fock space

Let $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ be a fixed partition of $\mathbb{R}^+$. The space $T\Phi(\mathcal{S})$ is a closed subspace of Φ. We denote by $\mathbb{E}\left[\cdot\, / \mathcal{F}(\mathcal{S})\right]$ the operator of orthogonal projection from Φ onto $T\Phi(\mathcal{S})$.

Proposition 6.2. *If* $\mathcal{S} = \{0 = t_0 < t_1 < \cdots < t_n < \cdots\}$ *and if* $f \in \Phi$ *is of the form*

$$f = \int_{0<s_1<\cdots<s_m} f(s_1,\ldots,s_m) d\chi_{s_1}\cdots d\chi_{s_m}$$

then

$$\mathbb{E}\big[f/\mathcal{F}(\mathcal{S})\big] = \sum_{i_1<\cdots<i_m\in\mathbb{N}} \frac{1}{\sqrt{t_{i_1+1}-t_{i_1}}\cdots\sqrt{t_{i_m+1}-t_{i_m}}} \int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}} f(s_1,\ldots,s_m)\ ds_1\cdots ds_m\ X_{i_1}\cdots X_{i_m}\ .$$

Proof. The quantity f_n on the right handside of the above identity is clearly an element of $T\Phi(\mathcal{S})$. We have, for $A=\{i_1\ldots i_k\}$

$$\begin{aligned}\langle f, X_A\rangle &= \\ &= \frac{\delta_{k,m}}{\sqrt{t_{i_1+1}-t_{i_1}}\cdots\sqrt{t_{i_m+1}-t_{i_m}}}\Big\langle \int_{0<s_1<\cdots<s_m} f(s_1,\ldots,s_m)\ d\chi_{s_1}\cdots d\ \chi_{s_m}, \\ &\int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}} \mathbb{1}\ d\chi_{s_1}\cdots d\chi_{s_m}\Big\rangle \\ &= \frac{\delta_{k,m}}{\sqrt{t_{i_1+1}-t_{i_1}}\cdots\sqrt{t_{i_m+1}-t_{i_m}}}\int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}} \overline{f}(s_1,\ldots,s_m)\ ds_1\cdots ds_m\ .\end{aligned}$$

On the other hand we have

$$\begin{aligned}\langle f_n, X_A\rangle &= \delta_{k,m}\frac{1}{(t_{i_1+1}-t_{i_1})^{3/2}\cdots(t_{i_m+1}-t_{i_m})^{3/2}} \\ &\times\int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}}\overline{f}(s_1,\ldots,s_m)\ ds_1\cdots ds_m\big\|\big(\chi_{t_{i_1+1}}-\chi_{t_{i_1}}\big)-\big(\chi_{t_{i_m+1}}-\chi_{t_{i_m}}\big)\big\|^2 \\ &= \delta_{k,m}\frac{1}{\sqrt{t_{i_1+1}-t_{i_1}}\cdots\sqrt{t_{i_m+1}-t_{i_m}}} \\ &\qquad\times\int_{t_{i_1}}^{t_{i_1+1}}\cdots\int_{t_{i_m}}^{t_{i_m+1}}\overline{f}(s_1,\ldots,s_m)\ ds_1\cdots ds_m\ .\end{aligned}$$

This proves our proposition. ■

The following identities could also have been used as natural definitions of the operators a_i^ε on $T\Phi(\mathcal{S})$.

Proposition 6.3. *For any partition $\mathcal{S}$ and any $f\in\mathcal{D}$ we have*

$$a_i^\circ\ \mathbb{E}\big[f/\mathcal{F}(\mathcal{S})\big] = \mathbb{E}\big[\big(a^\circ_{t_{i+1}} - a^\circ_{t_i}\big)f/\mathcal{F}(\mathcal{S})\big]$$

$$\sqrt{t_{i+1}-t_i}\ a_i^\pm\ \mathbb{E}\big[f/\mathcal{F}(\mathcal{S})\big] = \mathbb{E}\big[\big(a^\pm_{t_{i+1}} - a^\pm_{t_i}\big)f/\mathcal{F}(\mathcal{S})\big]\ .$$

Proof. Let us take f of the form

$$f = \int_{0<s_1<\cdots<s_m} f(s_1, \ldots, s_m)\; d\chi_{s_1} \cdots d\chi_{s_m}\ .$$

Then

$$\big(a^\circ_{t_{i+1}} - a^\circ_{t_i}\big) f = \int_{0<s_1<\cdots<s_m} \big|\{s_1, \ldots, s_m\} \cap [t_i, t_{i+1}]\big|\; f(s_1, \ldots, s_m)\; d\chi_{s_1} \cdots d\chi_{s_m}$$

$$\begin{aligned}
&\mathbb{E}\big[(a^\circ_{t_{i+1}} - a^\circ_{t_i}) f / \mathcal{F}(\mathcal{S})\big] \\
&\qquad = \sum_{j_1<\cdots<j_m \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_m+1} - t_{j_m}}} \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_m}}^{t_{j_m+1}} \\
&\qquad \times \big|\{s_1, \ldots, s_m\} \cap [t_i, t_{i+1}]\big|\; f(s_1, \ldots, s_m)\; ds_1 \cdots ds_m\; X_{j_1} \cdots X_{j_m} \\
&\qquad = \sum_{j_1<\cdots<j_m \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_m+1} - t_{j_m}}} \mathbb{1}_{i \in \{j_1 \ldots j_m\}} \\
&\qquad \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_m}}^{t_{j_m+1}} f(s_1, \ldots, s_m)\; ds_1 \cdots ds_m\; X_{j_1} \cdots X_{j_m} \\
&\qquad = a^\circ_i \sum_{j_1<\cdots<j_m \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_m+1} - t_{j_m}}} \\
&\qquad \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_m}}^{t_{j_m+1}} f(s_1, \ldots, s_m)\; ds_1 \cdots ds_m\; X_{j_1} \cdots X_{j_m} \\
&\qquad = a^\circ_i\; \mathbb{E}\big[f / \mathcal{F}(\mathcal{S})\big]\ .
\end{aligned}$$

In the same way

$$\big(a^-_{t_{i+1}} - a^-_{t_i}\big) f = \int_{0<s_1<\cdots<s_{m-1}} \int_{t_i}^{t_{i+1}} f\big(\{s_1, \ldots, s_{m-1}\} \cup s\big)\; ds\; d\chi_{s_1} \cdots d\chi_{s_{m-1}}$$

$$
\begin{aligned}
&\mathbb{E}\big[(a^-_{t_{i+1}} - a^-_{t_i}) f / \mathcal{F}(\mathcal{S})\big] \\
&\quad = \sum_{j_1 < \cdots < j_{m-1} \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_{m-1}+1} - t_{j_{m-1}}}} \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_{m-1}}}^{t_{j_{m-1}+1}} \int_{t_i}^{t_{i+1}} \\
&\quad \times f\big(\{s_1, \dots, s_{m-1}\} \cup s\big)\ ds\ ds_1 \cdots ds_{m-1}\ X_{j_1} \cdots X_{j_{m-1}} \\
&\quad = \sum_{j_1 < \cdots < j_{m-1} \in \mathbb{N}} \sum_{k=0}^{m-1} \mathbb{1}_{0 < j_1 < \cdots < j_k < i < j_{k+1} < \cdots < j_{m-1}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_{m-1}+1} - t_{j_{m-1}}}} \\
&\quad \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_k}}^{t_{j_k+1}} \int_{t_i}^{t_{i+1}} \int_{t_{j_{k+1}}}^{t_{j_{k+1}+1}} \cdots \int_{t_{j_{m-1}}}^{t_{j_{m-1}+1}} f(s_1, \dots, s_k, s, s_{k+1} \dots s_{m-1}) \\
&\quad \times ds_1 \cdots ds_k\ ds\ ds_{k+1} \cdots ds_{m-1}\ X_{j_1} \cdots X_{j_{m-1}} \\
&\quad = \sqrt{t_{i+1} - t_i} \sum_{j_1 < \cdots < j_m \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_m+1} - t_{j_m}}} \\
&\quad \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_m}}^{t_{j_m+1}} f(s_1, \dots, s_m)\ ds_1 \cdots ds_m\ \mathbb{1}_{i \in \{j \dots j_m\}} X_{j_1} \cdots \widehat{X}_i \cdots X_{j_m} \\
&\quad = \sqrt{t_{i+1} - t_i}\ a^-_i\ \mathbb{E}\big[f / \mathcal{F}(\mathcal{S})\big]\ .
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\big(a^+_{t_{i+1}} - a^+_{t_i}\big) f = &\sum_{k=0}^{n} \int_{0 < s_1 < \cdots < s_k < s < s_{k+1} < \cdots < s_m} \mathbb{1}_{[t_i, t_{i+1}]}(s) \\
&\times f(s_1, \dots, s_m)\ d\chi_{s_1} \cdots d\chi_{s_k}\ d\chi_s\ d\chi_{s_{k+1}} \cdots d\chi_{s_m}\ .
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\big[(a^+_{t_{i+1}} - a^+_{t_i}) f / \mathcal{F}(\mathcal{S})\big] \\
&\quad = \sum_{j_1 < \cdots < j_{m+1} \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_{m+1}+1} - t_{j_{m+1}}}} \sum_{k=0}^{n} \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_{m+1}}}^{t_{j_{m+1}+1}} \\
&\quad \times \mathbb{1}_{[t_i, t_{i+1}]}(t_{j_{k+1}}) f(s_1, \dots, \widehat{s_{k+1}} \dots s_{m+1})\ ds_1 \cdots ds_{m+1}\ X_{j_1} \cdots X_{j_{m+1}} \\
&\quad = \sum_{j_1 < \cdots < j_{m+1} \in \mathbb{N}} \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_{m+1}+1} - t_{j_{m+1}}}}\ \mathbb{1}_{i \in \{j_1 \dots j_{m+1}\}} \\
&\quad \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_{m+1}}}^{t_{j_m+1}} f(s_1, \dots, \widehat{s}_i \dots s_{m+1})\ ds_1 \cdots ds_{m+1}\ X_{j_1} \cdots X_{j_{m+1}} \\
&\quad = \sum_{j_1 < \cdots < j_m \in \mathbb{N}} \sqrt{t_{i+1} - t_i}\ \frac{1}{\sqrt{t_{j_1+1} - t_{j_1}} \cdots \sqrt{t_{j_m+1} - t_{j_m}}} \\
&\quad \int_{t_{j_1}}^{t_{j_1+1}} \cdots \int_{t_{j_m}}^{t_{j_m+1}} f(s_1, \dots, s_m)\ ds_1 \cdots ds_m\ \mathbb{1}_{i \notin \{j_1 \dots j_m\}} X_{j_1} \cdots X_{j_m} X_i \\
&\quad = \sqrt{t_{i+1} - t_i}\ \ a^+_i\ \mathbb{E}\big[f / \mathcal{F}(\mathcal{S})\big]\ .
\end{aligned}
$$

We have proved all the announced relations. ■

6.3 Approximations

We are now going to prove that the Fock space Φ and its basic operators a_t^+, a_t^-, a_t° can be approximated by the toy Fock spaces $T\Phi(\mathcal{S})$ and their basic operators a_i^+, a_i^-, a_i°.

We are given a refining sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of partitions whose diameter $\delta(\mathcal{S}_n)$ tends to 0 when n tends to $+\infty$. Let $T\Phi(n) = T\Phi(\mathcal{S}_n)$ and $P_n = \mathbb{E}[\cdot/\mathcal{F}(S_n)]$, for all $n\in\mathbb{N}$.

Theorem 6.4.

i) For every $f\in\Phi$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ such that $f_n\in T\Phi(n)$, for all $n\in\mathbb{N}$, and $(f_n)_{n\in\mathbb{N}}$ converges to f in Φ.

ii) If $\mathcal{S}_n = \{0 = t_0^n < t_1^n < \cdots < t_k^n < \cdots\}$, then for all $t\in\mathbb{R}^+$, the operators $\sum_{i;t_i^n\le t} a_i^\circ$, $\sum_{i;t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^-$ and $\sum_{i;t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^+$ converge strongly on $\mathcal{D}$ to a_t°, a_t^- and a_t^+ respectively.

iii) With the same notations as in ii), for all $t\in\mathbb{R}^+$, the operators $\sum_{i;t_i^n\le t} a_i^\circ\ P_n$, $\sum_{i;t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^-\ P_n$ and $\sum_{i;t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^+\ P_n$ converge strongly on $\mathcal{D}$ to a_t°, a_t^- and a_t^+ respectively.

Proof. i) As the $\mathcal{S}_n$ are refining then the $(P_n)_n$ forms an increasing family of orthogonal projection in Φ. Let $P_\infty = \bigvee_n P_n$. Clearly, for all $s \le t$, we have that $\chi_t - \chi_s$ belongs to $\mathrm{Ran}P_\infty$. But by the construction of the Itô integral and by Theorem 5, we have that the $\chi_t - \chi_s$ generate Φ. Thus $P_\infty = I$. Consequently if $f\in\Phi$, the sequence $f_n = P_n f$ satisfies the statements.

ii) The convergence of $\sum_{i,t_i^n\le t} a_i^\circ$ and $\sum_{i,t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^-$ to a_t° and a_t^- respectively is clear from the definitions. Let us check the case of a^+. We have, for $f\in\mathcal{D}$

$$\Big[\sum_{i;t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^+ f\Big](\sigma) = \sum_{i;t_i^n\le t}\mathbb{1}_{|\sigma\cap[t_i^n,t_{i+1}^n]|=1}\sum_{s\in\sigma\cap[t_i^n,t_{i+1}^n]} f(\sigma\setminus\{s\})\ .$$

Put $t^n = \inf\left\{t_i^n\in\mathcal{S}_n\ ; t_i^n \ge t\right\}$. We have

$$\begin{aligned}
&\Big\|\sum_{i;t_i^n\le t}\sqrt{t_{i+1}^n - t_i^n}\,a_i^+ - a_t^+ f\Big\|^2 \\
&\quad= \int_{\mathcal{P}}\Big|\sum_{i;t_i^n\le t}\mathbb{1}_{|\sigma\cap[t_i^n,t_{i+1}^n]|=1}\sum_{s\in\sigma\cap[t_i^n,t_{i+1}^n]} f(\sigma\setminus\{s\}) - \sum_{s\in\sigma\cap[0,t]} f(\sigma\setminus\{s\})\Big|^2\,d\sigma \\
&\quad\le 2\int_{\mathcal{P}}\Big|\sum_{s\in\sigma\cap[t,t]} f(\sigma\setminus\{s\})\Big|^2\,d\sigma + \\
&\qquad + 2\int_{\mathcal{P}}\Big|\sum_{i;t_i^n\le t}\mathbb{1}_{|\sigma\cap[t_i^n,t_{i+1}^n]|\ge 2}\sum_{s\in\sigma\cap[t_i^n,t_{i+1}^n]} f(\sigma\setminus\{s\})\Big|^2\,d\sigma.
\end{aligned}$$

For any fixed σ, the terms inside each of the integrals above converge to 0 when n tends to $+\infty$. Furthermore we have, for n large enough,

$$\begin{aligned}\int_{\mathcal{P}} \Big| \sum_{s\in\sigma\cap[t,t^n]} f(\sigma\setminus\{s\})\Big|^2 \,d\sigma &\leq \int_{\mathcal{P}} |\sigma| \sum_{\substack{s\in\sigma\\ s\leq t+1}} |f(\sigma\setminus\{s\})|^2 \,d\sigma \\ &= \int_0^{t+1}\int_{\mathcal{P}} (|\sigma|+1)|f(\sigma)|^2 \,d\sigma\, ds \\ &\leq (t+1)\int_{\mathcal{P}} (|\sigma|+1)|f(\sigma)|^2 \,d\sigma\end{aligned}$$

which is finite for $f\in\mathcal{D}$;

$$\begin{aligned}&\int_{\mathcal{P}} \Big| \sum_{i;t_i^n\leq t} \mathbb{1}_{|\sigma\cap[t_i^n,t_{i+1}^n]|\geq 2} \sum_{s\in\sigma\cap[t_i^n,t_{i+1}^n]} f(\sigma\setminus\{s\})\Big|^2 \,d\sigma \\ &\quad\leq \int_{\mathcal{P}} \Big(\sum_{i;t_i^n\leq t} \mathbb{1}_{|\sigma\cap[t_i^n,t_{i+1}^n]|\geq 2} \Big| \sum_{s\in\sigma\cap[t_i^n,t_{i+1}^n]} f(\sigma\setminus\{s\})\Big|\Big)^2 \,d\sigma \\ &\quad\leq \int_{\mathcal{P}} \Big(\sum_{i;t_i^n\leq t} \sum_{s\in\sigma\cap[t_i^n,t_{i+1}^n]} |f(\sigma\setminus\{s\})|\Big)^2 \,d\sigma \\ &\quad= \int_{\mathcal{P}} \Big(\sum_{\substack{s\in\sigma\\ s\leq t^n}} |f(\sigma\setminus\{s\})|\Big)^2 \,d\sigma \\ &\quad= \int_{\mathcal{P}} |\sigma| \sum_{\substack{s\in\sigma\\ s\leq t^n}} |f(\sigma\setminus\{s\})|^2 \,d\sigma \\ &\quad\leq (t+1)\int_{\mathcal{P}} (|\sigma|+1)\big|f(\sigma)\big|^2 \,d\sigma\end{aligned}$$

in the same way as above. So we can apply Lebesgue's theorem. This proves *ii)*.

iii) By Proposition 6.3, we have for all $f\in\mathcal{D}$

$$\sum_{i;t_i^n\leq t} \sqrt{t_{i+1}^n - t_i^n}\; a_i^+\; P_n f = P_n a_{t^n}^+ f\ .$$

Consequently

$$\begin{aligned}&\Big\| \sum_{i;t_i^n\leq t} \sqrt{t_{i+1}^n - t_i^n}\; a_i^+\; P_n f - a_t^+ f\Big\|^2 \\ &\qquad\leq 2\big\|a_t^+ f - P_n a_t^+ f\big\|^2 + 2\big\|P_n(a_t^+ f - a_{t^n}^+ f)\big]\big\|^2 \\ &\qquad\leq 2\big\|a_t^+ f - P_n a_t^+ f\big\|^2 + 2\big\|a_t^+ f - a_{t^n}^+ f\big\|^2\end{aligned}$$

which tends to 0 as n tends to $+\infty$.

The cases of a° and a^- are obtained in the same way. ■

6.4 Probabilistic interpretations

Recall that the operator of multiplication by the Brownain motion in the Fock space Φ is

$$W_t = a_t^+ + a_t^-$$

and the operator of Poisson multiplication by the Poisson process is

$$N_t = a_t^+ + a_t^- + a_t^\circ + tI \ .$$

Let us consider an approximation of the Fock space Φ by toy Fock spaces $T\Phi(n)$, $n \in \mathbb{N}$.

Theorem 6.5. *On $T\Phi(n)$, let $X_i = a_i^+ + a_i^-$, $i \in \mathbb{N}$. Then, for all $t \in \mathbb{R}^+$; we have that*

$$\sum_{i; t_i \le t} \sqrt{t_{i+1} - t_i}\ X_i$$

converges strongly to W_t.

Proof. The proof is immediate from Theorem 6.4. ■

Let $\mathcal{S}_n = \{i/n \ ; \ i \in \mathbb{N}\}$.

Theorem 6.6. *On $T\Phi(n)$, let $X_i = a_i^+ + a_i^- + c_n a_i^\circ$, $i \in \mathbb{N}$ be associated to the coefficient $p_n = 1/n$. Then, for all $t \in \mathbb{R}^+$, we have that*

$$\frac{1}{\sqrt{n}} \sum_{i; t_i \le t} X_i$$

converges strongly to $X_t = N_t - tI$, the operator of multiplication by the compensated Poisson process.

Proof. If $p_n = 1/n$, then $q_n = 1 - 1/n$ and $c_n = \frac{1-2/n}{\sqrt{1/n - 1/n^2}} = \frac{n-2}{\sqrt{n-1}}$. Thus $c_n/\sqrt{n}$ converges to 1. Now,

$$\begin{aligned}
\frac{1}{\sqrt{n}} \sum_{i; t_i \le t} X_i &= \sum_{i; t_i \le t} \frac{1}{\sqrt{n}}\ a_i^+ + \frac{1}{\sqrt{n}}\ a_i^- + \frac{c_n}{\sqrt{n}}\ a_i^\circ \\
&= \sum_{i; t_i \le t} \sqrt{t_{i+1} - t_i}(a_i^+ + a_i^-) + \frac{c_n}{\sqrt{n}} \sum_{i; t_i \le t} a_i^\circ
\end{aligned}$$

which clearly converges to $a_t^+ + a_t^- + a_t^\circ$ by Theorem 6.4 ■

6.5 The Itô tables

This section is heuristic, but it gives a good idea of why the discrete quantum Itô table is a discrete approximation of the usual one, though they seem different. Let $\mathcal{S}_n = \{i/n \ ; \ i \in \mathbb{N}\}$. Let $\tilde{a}_i^+ = 1/\sqrt{n}\ a_i^+$, $\tilde{a}_i^- = 1/\sqrt{n}\ a_i^-$ and $\tilde{a}_i^\circ = a_i^\circ$. The Theorem 6.4 shows that $\tilde{a}_i^\varepsilon$ is a good approximation of da_t^ε, where $t = t_i$. Now the discrete Itô table becomes

$\to$	$\tilde{a}_i^+$	$\tilde{a}_i^-$	$\tilde{a}_i^\circ$
$\tilde{a}_i^+$	0	$\frac{1}{n}\tilde{a}_i^\circ$	0
$\tilde{a}_i^-$	$\frac{1}{n}I - \frac{1}{n}\tilde{a}_i^\circ$	0	$\tilde{a}_i^-$
$\tilde{a}_i^\circ$	$\tilde{a}_i^+$	0	$\tilde{a}_i^\circ$.

But

1) $\frac{1}{n}\tilde{a}_i^\circ$ is not an infinitesimal for $\sum_{i;t_i\leq t}\frac{1}{n}\tilde{a}_i^\circ$ is almost $\frac{1}{n}a_t^\circ$ which converges to 0. Thus $\frac{1}{n}\tilde{a}_i^\circ$ can be considered to be 0 in this table;

2) $\frac{1}{n}I$ is simply $dt\ I$, that is $(t_{i+1} - t_i)I$. Thus at the limit this table becomes

$\to$	da_t^+	da_t^-	da_t°
da_t^+	0	0	0
da_t^-	$dt\ I$	0	da_t^-
da_t°	da_t^+	0	da_t° .

That is, the usual Itô table.

These heuristic arguments have been made rigourous in [33].

7 Back to repeated interactions

We are now ready to come back to repeated quantum interactions and to give an idea of what happens in the limit $h \to 0$.

Recall our evolution equation on $\mathcal{H}_0 \otimes \bigotimes_{\mathbb{N}} \mathbb{C}^{N+1}$:

$$V_{n+1} = U_{n+1} V_n \tag{23}$$

of section I.

7.1 Unitary dilations of completely positive semigroups

In this section, we will show that equations such as (23) appear naturally in a general setup and allow one to obtain natural unitary dilations of completely positive semigroups in discrete time.

Consider a discrete semigroup $(P_n)_{n\in\mathbb{N}}$ of completely positive maps on $\mathcal{B}(\mathcal{H}_0)$, that is,

$$P_n(X) = \ell^n(X)$$

where ℓ is a completely positive, weakly continuous map on $\mathcal{B}(\mathcal{H}_0)$.

In the sequel we always assume that $\ell(I) = I$. By Kraus' theorem (see [30], Proposition 29.8) this means that ℓ is of the form

$$\ell(X) = \sum_{i=0}^{N} V_i^* X V_i$$

for some N and some family (V_i) of bounded operators on $\mathcal{H}_0$ such that $\sum_i V_i^* V_i = I$. Of course the indexation is *a priori* indifferent to the specificity of the value $i = 0$. The special role played by one of the values will appear later on.

Let $\mathbb{E}_0$ be the partial trace on $\mathcal{H}_0$ defined by

$$<\phi\,,\,\mathbb{E}_0(H)\psi> = <\phi\otimes\Omega\,,\,H\psi\otimes\Omega>$$

for all $\phi, \psi \in \mathcal{H}_0$ and every operator H on $\mathcal{H}_0 \otimes \mathrm{T}\Phi$.

Theorem 7.1. *For any completely positive map*

$$\ell(X) = \sum_{i=0}^{N} V_i^* X V_i$$

on $\mathcal{B}(\mathcal{H}_0)$ there exists a unitary operator $\mathbb{L}$ on $\mathcal{H}_0 \otimes \mathbb{C}^{N+1}$ such that the associated unitary family of automorphisms

$$j_n(H) = u_n^* H u_n$$

(where u_n is associated to $\mathbb{L}$ by (23)) satisfies

$$\mathbb{E}_0(j_n(X\otimes I)) = P_n(X),$$

for all $n \in \mathbb{N}$.

Proof. Consider a decomposition of $\mathcal{L}$ of the form

$$\ell(X) = \sum_{i=0}^{N} V_i^* X V_i$$

for a familly (V_i) of bounded operators on $\mathcal{H}_0$ such that $\sum_{i=0}^N V_i^* V_i = I$.

We claim that there exists a unitary operator $\mathbb{L}$ on $\mathcal{H}_0 \otimes \mathbb{C}^{N+1}$ of the form

$$\mathbb{L} = \begin{pmatrix} V_1 & \ldots & \ldots \\ V_2 & \ldots & \ldots \\ \vdots & \vdots & \vdots \\ V_N & \ldots & \ldots \end{pmatrix}.$$

Indeed, the condition $\sum_{i=0}^N V_i^* V_i = I$ guarantees that the m first columns of $\mathbb{L}$ (where $m = \dim \mathcal{H}_0$) constitute an orthonormal family of $\mathcal{H}_0 \otimes \mathbb{C}^{N+1}$. We can thus fill the matrix by completing it into an orthonormal basis of $\mathcal{H}_0 \otimes \mathbb{C}^{N+1}$; this makes out a unitary, $(N+1) \times (N+1)$ matrix $\mathbb{L}$ on $\mathcal{H}_0$, of which we denote the coefficients by $\left(A_j^i\right)_{i,j=0,\ldots,N}$; with this notation we have for all i, $A_0^i = V_{i+1}$. To this matrix $\mathbb{L}$ we associate a family $(\mathbb{L}_i)_{i \geq 0}$ of ampliations as explained in section

Now, for every operator H on $\mathcal{H}_0 \otimes \mathbb{C}^{N+1}$, put

$$j_n(H) = u_n^* H u_n.$$

It satisfies

$$j_{n+1}(H) = u_n^* \mathbb{L}_{n+1}^* \, H \, \mathbb{L}_{n+1} u_n.$$

We consider this relation for an operator H of the form $H = X \otimes I$, where X is an operator on $\mathcal{H}_0$. Write $\mathbb{L}_{n+1}^*(X \otimes I)\mathbb{L}_{n+1}$ in $(\mathcal{H}_0 \otimes \mathrm{T}\Phi_{n]}) \otimes \mathbb{C}_{n+1}^{N+1}$; it is simply

$$\mathbb{L}_{n+1}^*(X \otimes I)\mathbb{L}_{n+1} =$$
$$= \begin{pmatrix} (A_0^0)^* & (A_0^1)^* & \ldots \\ (A_1^0)^* & (A_1^1)^* & \ldots \\ \vdots & \vdots & \vdots \\ (A_N^0)^* & (A_N^1)^* & \ldots \end{pmatrix} \begin{pmatrix} X & 0 & \ldots & 0 \\ 0 & X & \ldots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \ldots & X \end{pmatrix} \begin{pmatrix} A_0^0 & A_1^0 & \ldots \\ A_0^1 & A_1^1 & \ldots \\ \vdots & \vdots & \vdots \\ A_0^N & A_1^N & \ldots \end{pmatrix}$$

which is easily seen to be the matrix $\left(B_j^i(X)\right)_{i,j=0,\ldots,N}$ with

$$B_j^i(X) = \sum_{k=0}^N (A_j^k)^* \, X \, A_i^k.$$

Note that, more precisely, the operator $\mathbb{L}_{n+1}^*(X \otimes I)\mathbb{L}_{n+1}$ is written in $(\mathcal{H}_0 \otimes \mathrm{T}\Phi_{n]}) \otimes \mathbb{C}_{n+1}^{N+1}$ as the matrix $(B_j^i(X) \otimes I)_{i,j=0,\ldots,N}$. The operator u_n, in turn, acts only on $\mathcal{H}_0 \otimes \mathbb{C}_n^{N+1}$, so that $u_n^* \mathbb{L}_{n+1}^*(X \otimes I)\mathbb{L}_{n+1} u_n$ can be written in $(\mathcal{H}_0 \otimes \mathrm{T}\Phi_{n]}) \otimes \mathbb{C}_{n+1}^{N+1}$ as $\left(u_n^* \left(B_j^i(X) \otimes I\right) u_n\right)_{i,j=0,\ldots,N}$; simply put, we have proved that

$$\left(j_{n+1}(X \otimes I)\right)_j^i = j_n(B_j^i(X) \otimes I)$$

because both terms act as the identity beyond $(\mathcal{H}_0 \otimes \mathrm{T}\Phi_{n]}) \otimes \mathbb{C}^{N+1}_{n+1}$.

Consider now $T_n(X) = \mathbb{E}_0(j_n(X \otimes I))$. We have

$$\begin{aligned}
< \phi\,,\, T_{n+1}(X)\psi > &= < \phi \otimes \Omega\,,\, j_{n+1}(X \otimes I)\psi \otimes \Omega > \\
&= < \phi\,,\, (j_{n+1}(X \otimes I))^0_0\,\psi > \\
&= < \phi\,,\, j_n(B^0_0(X) \otimes I)\psi > \\
&= < \phi\,,\, T_n(B^0_0(X))\psi >;
\end{aligned}$$

now remember that for all i, $A^i_0 = V_{i+1}$. This implies that $B^0_0(X) = \ell(X)$. The above proves that $T_{n+1}(X) = T_n(\ell(X))$ for any n and the theorem follows. ■

7.2 Convergence to Quantum Stochastic Differential Equations

We now describe the convergence of these discrete time evolutions to continuous time ones.

Quantum stochastic differential equations

We do not develop here the whole theory of Q.S.D.E., this will be done in F. Fagnola's course much more precisely, but we just give an idea of what they are.

Quantum stochastic differential equations are equations of the form

$$dU_t = \sum_{i,j} L^i_j U_t \, da^i_j(t), \tag{22}$$

with initial condition $U_0 = I$. The above equation has to be understood as an integral equation

$$U_t = I + \int_0^t \sum_{i,j} L^i_j U_t \, da^i_j(t),$$

for operators on $\mathcal{H}_0 \otimes \Phi$, the operators L^i_j being bounded operators on $\mathcal{H}_0$ alone which are ampliated to $\mathcal{H}_0 \otimes \Phi$.

The main motivation and application of that kind of equation is that it gives an account of the interaction of the small system $\mathcal{H}_0$ with the bath Φ in terms of quantum noise perturbation of a Schödinger-like equation. Indeed, the first term of the equation

$$dU_t = L^0_0 U_t \, dt + \ldots$$

describes the induced dynamics on the small system, all the other terms are quantum noises terms.

One of the main application of equations such as (22) is that they give explicit constructions of unitary dilations of semigroups of completely positive maps of $\mathcal{B}(\mathcal{H}_0)$ (see [H-P]). Let us only recall one of the main existence, uniqueness and boundedness theorems connected to equations of the form (22). The literature is huge about those equations; we refer to [Par] for the result we mention here.

Theorem 7.2. *If $\mathcal{H}_0$ is finite dimensional then the quantum stochastic differential equation*

$$dU_t = \sum_{i,j} L^i_j U_t \, da^i_j(t),$$

with $U_0 = I$, admits a unique solution defined on the space of coherent vectors.

The solution $(U_t)_{t\geq 0}$ is made of unitary operators if and only if there exist, on $\mathcal{H}_0$, a bounded self-adjoint H, bounded operators S^i_j, $i,j = 1,\dots,N$, such that the matrix $(S^i_j)_{i,j}$ is unitary, and bounded operators L_i, $i = 1,\dots,N$ such that, for all $i,j = 1,\dots,N$

$$\begin{aligned}
L^0_0 &= -(iH + \frac{1}{2}\sum_k L^*_k L_k) \\
L^0_i &= L_i \\
L^i_0 &= -\sum_k L^*_k S^k_j \\
L^i_j &= S^i_j - \delta_{ij} I.
\end{aligned}$$

If the operators L^i_j are of this form then the unitary solution $(U_t)_{t\geq 0}$ of the above equation exists even if $\mathcal{H}_0$ is only assumed to be separable.

Convergence theorems

In this section we study the asymptotic behaviour of the solutions of an equation

$$u_{n+1} = \mathbb{L}_{n+1} u_n;$$

if the matrix $\mathbb{L}(h)$ converges (with a particular normalization) as h tends to zero and prove that, in the limit, the solutions of such equations converge to solutions of quantum stochastic differential equations of the form (22). Notice that we no longer assume that $\mathbb{L}(h)$ has been conveniently constructed for our needs; in particular $\mathbb{L}$ is not assumed to be unitary.

Let h be a parameter in $\mathbb{R}^+$, which is thought of as representing a small time interval. Let $\mathbb{L}(h)$ be an operator on $\mathcal{H}_0 \otimes \mathbb{C}^{N+1}$, with coefficients $\mathbb{L}^i_j(h)$ as a $(N+1)\times(N+1)$ matrix of operators on $\mathcal{H}_0$. Let $u_n(h)$ be the associated solution of

$$u_{n+1}(h) = \mathbb{L}_{n+1}(h)\, u_n(h).$$

In the following we will drop dependency in h and write simply $\mathbb{L}$ or u_n. Besides, we denote

$$\varepsilon_{ij} = \frac{1}{2}(\delta_{0i} + \delta_{0j})$$

for all $i, j = 0, \ldots, N$.

Theorem 7.3. *Assume that there exist operators L^i_j on $\mathcal{H}_0$ such that*

$$\lim_{h \to 0} \frac{\mathbb{L}^i_j(h) - \delta_{ij} I}{h^{\varepsilon_{ij}}} = L^i_j$$

for all $i, j = 0, \ldots, N$, where convergence is in operator norm. Assume that the quantum stochastic differential equation

$$dU_t = \sum_{i,j} L^i_j U_t \, da^i_j(t)$$

with initial condition $U_0 = I$ has a solution $(U_t)_{t \geq 0}$ which is a process of bounded operators with a locally uniform norm bound.

Then, for all t, for every ϕ, ψ in $L^\infty([0,t])$, the quantity

$$< a \otimes \varepsilon(\phi)\, ,\, \mathbb{E}_{\mathcal{S}} u_{[t/h]} \mathbb{E}_{\mathcal{S}}\, b \otimes \varepsilon(\psi) >$$

converges to

$$< a \otimes \varepsilon(\phi)\, ,\, U_t\, b \otimes \varepsilon(\psi) >$$

when h goes to 0.

Moreover, the convergence is uniform for a, b in any bounded ball of $\mathcal{K}$, uniform for t in a bounded interval of $\mathbb{R}_+$.

If furthermore $||u_k||$ is locally uniformly bounded in the sense that, for any t in $\mathbb{R}_+$, $\{||u_k(h)||\, ,\ k \leq t/h\}$ is bounded for any h, then $u_{[t/h]}$ converges weakly to U_t on all $\mathcal{H}_0 \otimes \Phi$.

Remarks

– This is where we particularize the index zero : the above hypotheses of convergence simply mean that, among the coefficients of $\mathbb{L}$,

$(\mathbb{L}^0_0(h) - I)/h$ converges,

$\mathbb{L}^i_j(h)/\sqrt{h}$ converges if either i or j is zero,

$\mathbb{L}^i_j(h) - \delta_{i,j}$ converges if neither i nor j is zero

and we recover the fact that the 0 index must relate to the small system, on which the considered time scale is different from the time scale of the reservoir.

– The assumption that $\mathcal{H}_0$ is finite dimensional is only needed in order to ensure that the quantum stochastic differential equation has a solution; if for example the L^i_j's are of the form described in Theorem 8 then the separability of $\mathcal{H}_0$ is enough.

For our example where $\mathbb{L}$ is given by

$$\mathbb{L} = \begin{pmatrix} \cos\alpha & 0 & 0 & -\sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix}$$

with $\alpha = \sqrt{h}$, since for all h the matrix $\mathbb{L}(h)$ is unitary, we get that for all t, $u_{[t/h]}$ converges strongly to U_t where $(U_t)_{t\in\mathbb{R}_+}$ is the solution of

$$dU_t = -\frac{1}{2}V^*V\,U_t\,dt + VU_t\,da^0_1(t) - V^*U_t\,da^1_0(t)$$

with $V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$; this is the evolution associated to the spontaneous decay into the ground state in the Wigner-Weisskopf model for the two-level atom.

8 Bibliographical comments

The mathematical theory of quantum stochastic calculus was first developed by Hudson and Parthasarathy [25]. They defined quantum stochastic integrals on the space of coherent vectors. They also defined and solved the first class of quantum Langevin equations. They finaly proved that quantum Langevin equations allow to construct unitary dilations of any completely positive semigroups. No need to say that this article is a fundamental one, which started a whole theory.

An extension of their quantum stochastic calculus, trying to go further than the domain of coherent vectors was proposed by Belavkin and later by Lindsay ([15], [26]). Their definitions were making use of the Malliavin gradient (and was constrained by its domain) and the Skorohod integral.

The definition of quantum stochastic integrals as in subsection 4.2 is due to Attal and Meyer ([10] and later developed in [5]). The main point with that approach was the absence of arbitrary domain constraints. The discovery of the quantum semimartingales by Attal in [4] was a direct consequence of that approach and of and anterior work of Parthasarathy and Sinha on regular quantum martingales ([31]).

The maximal definition of quantum stochastic integrals and unification of the different approaches, as in subsection 4.2 was given by Attal and Lindsay ([9]).

The theorem showing rigorously that there are only 3 quantum noises is due to Coquio ([17]).

The notion of Toy Fock space with its probabilistic interpretations in terms of random walks was developed by Meyer ([29]). The concrete realization of the Toy Fock as a subspace and an approximation of Φ is due to Attal ([2]) and has been developed much further by Pautrat ([32], [33]). These different works led to the proof of the convergence of repeated interactions to quantum stochastic differential equation ([11]).

References

1. L. Accardi, A. Frigerio, Y.G. Lu, *Quantum Langevin equation in the weak coupling limit*, Quantum probability and applications V (Heidelberg, 1988), p. 1–16, Lecture Notes in Math., 1442, Springer, Berlin, 1990.
2. S. Attal, *Quantum Noise Theory*, book to appear, Springer Verlag.
3. S. Attal, *Approximating the Fock space with the toy Fock space* Séminaire de Probabilités, XXXVI, p. 477–491, Lecture Notes in Math., 1801, Springer Verlag, Berlin, 2003.
4. S. Attal, *An algebra of non-commutative bounded semimartingales. Square and angle quantum brackets*, Journal of Functional Analysis 124 (1994), no. 2, p. 292–332.
5. S. Attal, P.-A. Meyer, *Interprétation probabiliste et extension des intégrales stochastiques non commutatives*, Séminaire de Probabilités, XXVII, p. 312–327, Lecture Notes in Math., 1557, Springer Verlag, Berlin, 1993.
6. S. Attal, M. Emery, *Equations de structure pour des martingales vectorielles* Séminaire de Probabilités, XXVIII, p. 256–278, Lecture Notes in Math., 1583, Springer, Berlin, 1994.
7. S. Attal, A. Joye, *Weak coupling limit for repeated quantum interactions*, preprint.
8. S. Attal, A. Joye, *The Langevin equation for a quantum heat bath*, preprint.
9. S. Attal, J.M. Lindsay, *Quantum stochastic calculus with maximal operator domains*, The Annals of Probability 32 (2004), p. 488–529.
10. S. Attal, P.A. Meyer, *Interprétation probabiliste et extension des intégrales stochastiques non commutatives*, Séminaire de Probabilités XXVII, Lect. Notes in Math. 1557, p. 312–327.
11. S. Attal, Y. Pautrat, *From repeated to continuous quantum interactions*, Annales Henri Poincaré (Physique Théorique), to appear (2005).
12. A. Barchielli, *Measurement theory and stochastic differential equations in quantum mechanics*, Physical Review A (3) 34 (1986), no. 3,P. 1642–1649.
13. A. Barchielli, *Some stochastic differential equations in quantum optics and measurement theory: the case of counting processes*, Stochastic evolution of quantum states in open systems and in measurement processes (Budapest, 1993), p. 1–14, World Sci. Publishing, River Edge, NJ, 1994.
14. A. Barchielli, V.P. Belavkin, *Measurements continuous in time and a posteriori states in quantum mechanics*, Journal of Physics A 24 (1991), no. 7,P. 1495–1514.
15. V.P. Belavkin, *A quantum non-adapted Itô formula and non stationnary evolutions in Fock scale*, Quantum Probability and Related Topics VI, World Scientific (1991), p. 137-180.
16. J. Bellissard, R. Rebolledo, D. Spehner, W. von Waldenfels, *The quantum flow of electronic transport*, preprint.

17. A. Coquio, *Why are there only three quantum noises?*, Probability Theory and Related Fields 118 (2000), p. 349–364.
18. O. Enchev, *Hilbert-space-valued quasimartingales*, Boll. Un. Mat. Ital. B (7) 2 (1988), p. 19–39.
19. F. Fagnola, R. Rebolledo, *A view on stochastic differential equations derived from quantum optics*, Stochastic models (Guanajuato, 1998),P. 193–214, Aportaciones Mat. Investig., 14, Soc. Mat. Mexicana, Mxico, 1998.
20. F. Fagnola, R. Rebolledo, C. Saavedra, *Quantum flows associated to master equations in quantum optics*, Journal of Mathematical Physics 35 (1994), no. 1, P. 1–12.
21. G. W. Ford, M. Kac, P. Mazur, *Statistical mechanics of assemblies of coupled oscillators*, Journal of Mathematical Physisics 6 (1965),p. 504–515.
22. G. W. Ford, J. T. Lewis, R. F. O'Connell, *Quantum Langevin equation*, Physical Review A (3), vol 37 (1988), no. 11, p. 4419–4428.
23. C. W. Gardiner, P. Zoller, *Quantum noise. A handbook of Markovian and non-Markovian quantum stochastic methods with applications to quantum optics.*, Second edition. Springer Series in Synergetics. Springer-Verlag, Berlin, 2000.
24. H. Grabert, P. Schramm, G-L. Ingold, *Quantum Brownian motion: the functional integral approach*, Physical Reports 168 (1988), no. 3, p. 115–207.
25. R.L. Hudson, K.R. Parthasarathy, *Quantum Itô's formula and stochastic evolutions* Communications in Mathematical Physics, 93 (1984), P. 301–323.
26. J.M. Lindsay, *Quantum and non-causal stochastic calculus*, Probability Theory and Related Fields 97 (1993), p. 65-80.
27. J.M. Lindsay, H. Maassen, *Stochastic calculus for quantum Brownian motion of nonminimal variance—an approach using integral-sum kernel operators*, Mark Kac Seminar on Probability and Physics Syllabus 1987–1992 (Amsterdam, 1987–1992), 97–167, CWI Syllabi, 32, Math. Centrum, Centrum Wisk. Inform., Amsterdam, 1992.
28. H. Maassen, P. Robinson, *Quantum stochastic calculus and the dynamical Stark effect*, Reports in Math. Phys. 30 (1991), 185–203.
29. P.-A. Meyer, *Quantum Probability for Probabilists*, 2nd edition, Lecture Notes in mathematics 1538, Springer-Verlag, Berlin, 1993.
30. K.R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, Monographs in Mathematics 85, Birkhäuser (1992).
31. K.R. Parthasarathy, K.B. Sinha *Stochastic integral representation of bounded quantum martingales on Fock space*, Journal of Functional Analysis, 67 (1986), p. 126-151.
32. Y. Pautrat, *Integral and kernel representation of operators on the infinite dimensional Toy Fock space*, Seminaire de Probabilités, Springer Lecture Notes in Mathematics (to appear).
33. Y. Pautrat, *Pauli matrices and quantum Itô formula*, Mathematical Physics, Analysis and Geometry (to appear).